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Composing generic linearly perturbed mappings and immersions/injections

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Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday

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Abstract. Let N (resp., U) be a manifold (resp., an open subset of $\mathbb{R}^m$). Let $f : N \to U$ and $F : U \to \mathbb{R}^\ell$ be an immersion and a C^∞ mapping, respectively. Generally, the composition $F \circ f$ does not necessarily yield a mapping transverse to a given subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$. Nevertheless, in this paper, for any $\mathcal{A}^1$-invariant fiber, we show that composing generic linearly perturbed mappings of F and the given immersion f yields a mapping transverse to the subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the given fiber. Moreover, we show a specialized transversality theorem on crossings of compositions of generic linearly perturbed mappings of a given mapping $F : U \to \mathbb{R}^\ell$ and a given injection $f : N \to U$. Furthermore, applications of the two main theorems are given.

1. Introduction.

Throughout this paper, let ℓ, m and n stand for positive integers. In this paper, unless otherwise stated, all manifolds and mappings belong to class C^∞ and all manifolds are without boundary. Let $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$, U and $F : U \to \mathbb{R}^\ell$ be a linear mapping, an open subset of $\mathbb{R}^m$ and a mapping, respectively.

Set

$$F_\pi = F + \pi.$$

Here, the mapping π in $F_\pi = F + \pi$ is restricted to U.

Let $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the space consisting of all linear mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$. Remark that we have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$. An n-dimensional manifold is denoted by N. For a given mapping $f : N \to U$, a property of mappings $F_\pi \circ f : N \to \mathbb{R}^\ell$ will be said to be true for a *generic mapping* if there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has the property. In the case $F = 0$, by John Mather, for a given embedding $f : N \to \mathbb{R}^m$, a generic mapping $\pi \circ f : N \to \mathbb{R}^\ell$ $(m > \ell)$ is investigated in the celebrated paper [**10**]. The main theorem in [**10**] yields many applications. On the other hand, in this paper, for a given immersion or a given injection $f : N \to U$, a

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generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is investigated, where ℓ is an arbitrary positive integer which may possibly satisfy $m \leq \ell$.

The main purpose of this paper is to show two main theorems (Theorems 1 and 2 in Section 2) and to give some of their applications. The first main theorem (Theorem 1) is as follows. Let $f : N \to U$ (resp., $F : U \to \mathbb{R}^\ell$) be an immersion (resp., a mapping). Then, generally, the composition $F \circ f$ does not necessarily yield a mapping transverse to a given subfiber-bundle of the jet bundle $J^1(N, \mathbb{R}^\ell)$. Nevertheless, Theorem 1 asserts that for any $\mathcal{A}^1$-invariant fiber, a generic mapping $F_\pi \circ f$ yields a mapping transverse to the subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the given fiber. The second main theorem (Theorem 2) is a specialized transversality theorem on crossings of a generic mapping $F_\pi \circ f$, where $f : N \to U$ is a given injection and $F : U \to \mathbb{R}^\ell$ is a given mapping.

For a given immersion (resp., injection) $f : N \to U$, the following (1)–(4) (resp., (5)) are obtained as applications of Theorem 1 (resp., Theorem 2).

(1) If $(n, \ell) = (n, 1)$, then a generic function $F_\pi \circ f : N \to \mathbb{R}$ is a Morse function.

(2) If $(n, \ell) = (n, 2n-1)$ and $n \geq 2$, then any singular point of a generic mapping $F_\pi \circ f : N \to \mathbb{R}^{2n-1}$ is a singular point of Whitney umbrella.

(3) If $\ell \geq 2n$, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion.

(4) A generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has corank at most k singular points (for the definition of corank at most k singular points, see Subsection 5.1), where k is the maximum integer satisfying $(n - v + k)(\ell - v + k) \leq n$ ($v = \min\{n, \ell\}$).

(5) If $\ell > 2n$, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is injective.

Moreover, by combining the assertions (3) and (5), for a given embedding $f : N \to U$, the following assertion (6) is obtained.

(6) If $\ell > 2n$ and N is compact, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an embedding.

In Section 2, some standard definitions are reviewed, and the two main theorems (Theorems 1 and 2) are stated. Section 3 (resp., Section 4) is devoted to the proof of Theorem 1 (resp., Theorem 2). In Section 5, the assertions (1)–(6) above are shown. Moreover, in Section 6, as further applications, the two main theorems are adapted to quadratic mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$ of a special type called "generalized distance-squared mappings" (for the precise definition of generalized distance-squared mappings, see Section 6). Since some corollaries in this paper (the assertion (6) in Section 1, Corollary 7 in Section 5 and Corollary 9 in Section 6) are also obtained by using the main theorem in [**4**], which is an improvement of the main theorem in [**10**], for the sake of readers' convenience, Section 7 explains the main theorems in [**4**] and [**10**] as an appendix.

2. Preliminaries and the statements of Theorems 1 and 2.

Let N and P be manifolds. Firstly, we recall the definition of transversality.

DEFINITION 1. Let W be a submanifold of P. Let $g : N \to P$ be a mapping.

1. We say that $g : N \to P$ is *transverse* to W *at* q if $g(q) \notin W$ or in the case of $g(q) \in W$, the following holds:

$$dg_q(T_qN) + T_{g(q)}W = T_{g(q)}P.$$

2. We say that $g : N \to P$ is *transverse* to W if for any $q \in N$, the mapping g is transverse to W at q.

We say that $g : N \to P$ is $\mathcal{A}$*-equivalent* to $h : N \to P$ if there exist diffeomorphisms $\Phi : N \to N$ and $\Psi : P \to P$ such that $g = \Psi \circ h \circ \Phi^{-1}$.

Let $J^r(N, P)$ be the space of r-jets of mappings of N into P. For a given mapping $g : N \to P$, the mapping $j^r g : N \to J^r(N, P)$ is defined by $q \mapsto j^r g(q)$ (for details on the space $J^r(N, P)$ or the mapping $j^r g : N \to J^r(N, P)$, see for example, [**3**]).

For the statement and the proof of Theorem 1, it is sufficient to consider the case of $r = 1$ and $P = \mathbb{R}^\ell$. Let $\{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ be a coordinate neighborhood system of N. Let $\Pi : J^1(N, \mathbb{R}^\ell) \to N \times \mathbb{R}^\ell$ be the natural projection defined by $\Pi(j^1 g(q)) = (q, g(q))$. Let $\Phi_\lambda : \Pi^{-1}(U_\lambda \times \mathbb{R}^\ell) \to \varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times J^1(n, \ell)$ be the homeomorphism defined by

$$\Phi_\lambda \left(j^1 g(q)\right) = \left(\varphi_\lambda(q), g(q), j^1(\psi_\lambda \circ g \circ \varphi_\lambda^{-1} \circ \widetilde{\varphi}_\lambda)(0)\right),$$

where $J^1(n, \ell) = \{j^1 g(0) \mid g : (\mathbb{R}^n, 0) \to (\mathbb{R}^\ell, 0)\}$ and $\widetilde{\varphi}_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ (resp., $\psi_\lambda : \mathbb{R}^m \to \mathbb{R}^m$) is the translation defined by $\widetilde{\varphi}_\lambda(0) = \varphi_\lambda(q)$ (resp., $\psi_\lambda(g(q)) = 0$). Then, $\{(\Pi^{-1}(U_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)\}_{\lambda\in\Lambda}$ is a coordinate neighborhood system of $J^1(N, \mathbb{R}^\ell)$. A subset X of $J^1(n, \ell)$ is said to be $\mathcal{A}^1$*-invariant* if for any $j^1 g(0) \in X$, and for any two germs of diffeomorphisms $H : (\mathbb{R}^\ell, 0) \to (\mathbb{R}^\ell, 0)$ and $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, we have $j^1(H \circ g \circ h^{-1})(0) \in X$. Let X be an $\mathcal{A}^1$-invariant submanifold of $J^1(n, \ell)$. Set

$$X(N, \mathbb{R}^\ell) = \bigcup_{\lambda\in\Lambda} \Phi_\lambda^{-1} \left(\varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times X\right).$$

Then, the set $X(N, \mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the fiber X such that

$$\begin{aligned} \operatorname{codim} X(N, \mathbb{R}^\ell) &= \dim J^1(N, \mathbb{R}^\ell) - \dim X(N, \mathbb{R}^\ell) \\ &= \dim J^1(n, \ell) - \dim X \\ &= \operatorname{codim} X. \end{aligned}$$

Then, the first main theorem in this paper is the following.

THEOREM 1. *Let N be a manifold of dimension n. Let f be an immersion of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If X is an $\mathcal{A}^1$-invariant submanifold of $J^1(n, \ell)$, then there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to the submanifold $X(N, \mathbb{R}^\ell)$.*

Now, in order to state the second main theorem (Theorem 2), we will prepare some definitions. Set $N^{(s)} = \{(q_1, q_2, \ldots, q_s) \in N^s \mid q_i \neq q_j \ (i \neq j)\}$. Notice that $N^{(s)}$ is an open submanifold of N^s. For any mapping $g : N \to P$, let $g^{(s)} : N^{(s)} \to P^s$ be the mapping defined by

$$g^{(s)}(q_1, q_2, \ldots, q_s) = (g(q_1), g(q_2), \ldots, g(q_s)).$$

Set $\Delta_s = \{(y, \ldots, y) \in P^s \mid y \in P\}$. It is clearly seen that Δ_s is a submanifold of P^s such that

$$\operatorname{codim} \Delta_s = \dim P^s - \dim \Delta_s = (s-1)\dim P.$$

Definition 2. Let g be a mapping of N into P. Then, g is called a *mapping with normal crossings* if for any positive integer s ($s \geq 2$), the mapping $g^{(s)} : N^{(s)} \to P^s$ is transverse to the submanifold Δ_s.

For any injection $f : N \to \mathbb{R}^m$, set

$$s_f = \max\left\{ s \;\middle|\; \forall (q_1, q_2, \ldots, q_s) \in N^{(s)}, \dim \sum_{i=2}^{s} \mathbb{R}\overrightarrow{f(q_1)f(q_i)} = s-1 \right\}.$$

Since the mapping f is injective, we get $2 \leq s_f$. Since $f(q_1), f(q_2), \ldots, f(q_{s_f})$ are points of $\mathbb{R}^m$, it follows that $s_f \leq m+1$. Thus, we have

$$2 \leq s_f \leq m+1.$$

Furthermore, in the following, for a set X, we denote the number of its elements (or its cardinality) by $|X|$. Then, the second main theorem in this paper is the following.

Theorem 2. *Let N be a manifold of dimension n. Let f be an injection of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Then, there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any s $(2 \leq s \leq s_f)$, the mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to the submanifold Δ_s. Moreover, if the mapping F_π satisfies that $|F_\pi^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$, then $F_\pi \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings.*

The following well known result is important for the proofs of Theorems 1 and 2.

Lemma 1 ([**1**], [**10**]). *Let N, P, Z be manifolds, and let W be a submanifold of P. Let $\Gamma : N \times Z \to P$ be a mapping. If Γ is transverse to W, then there exists a subset Σ of Z with Lebesgue measure zero such that for any $p \in Z - \Sigma$, the mapping $\Gamma_p : N \to P$ is transverse to W, where $\Gamma_p(q) = \Gamma(q, p)$.*

Remark 1. 1. We explain the advantage that the domain of the mapping F is an arbitrary open set. Suppose that $U = \mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$ be the mapping defined by $x \mapsto |x|$. Since F is not differentiable at $x = 0$, we cannot apply Theorems 1 and 2 to the mapping $F : \mathbb{R} \to \mathbb{R}$.

On the other hand, if $U = \mathbb{R} - \{0\}$, then Theorems 1 and 2 can be applied to the restriction $F|_U$.

2. There is a case of $s_f = 3$ as follows. If $n+1 \leq m$, $N = S^n$ and $f : S^n \to \mathbb{R}^m$ is the inclusion $f(x) = (x, 0, \ldots, 0)$, then it is easily seen that $s_f = 3$. Indeed, suppose that there exists a point $(q_1, q_2, q_3) \in (S^n)^{(3)}$ such that $\dim \sum_{i=2}^{3} \mathbb{R}\overrightarrow{f(q_1)f(q_i)} = 1$. Then, since the number of the intersections of $f(S^n)$ and a straight line of $\mathbb{R}^m$ is at most two, this contradicts the assumption. Thus, we get $s_f \geq 3$. From $S^1 \times \{0\} \subset f(S^n)$, it follows that $s_f < 4$, where $0 = \underbrace{(0, \ldots, 0)}_{(m-2)\text{-tuple}}$. Hence, we have $s_f = 3$.
3. The essential idea for the proofs of Theorems 1 and 2 is to apply Lemma 1, and it is almost similar to the idea of the proofs of main results in [**8**]. Nevertheless, the two main theorems in this paper are drastically improved. As an effect of the improvement, many applications are obtained by the two main theorems (for the applications, see Sections 5 and 6).

3. Proof of Theorem 1.

Let $(\alpha_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ be a representing matrix of a linear mapping $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$. Set $F_\alpha = F_\pi$, and we have

$$F_\alpha(x) = \left(F_1(x) + \sum_{j=1}^{m} \alpha_{1j} x_j, F_2(x) + \sum_{j=1}^{m} \alpha_{2j} x_j, \ldots, F_\ell(x) + \sum_{j=1}^{m} \alpha_{\ell j} x_j \right), \tag{3.1}$$

where $F = (F_1, F_2, \ldots, F_\ell)$, $\alpha = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1m}, \ldots, \alpha_{\ell 1}, \alpha_{\ell 2}, \ldots, \alpha_{\ell m}) \in (\mathbb{R}^m)^\ell$ and $x = (x_1, x_2, \ldots, x_m)$. For a given immersion $f : N \to U$, the mapping $F_\alpha \circ f : N \to \mathbb{R}^\ell$ is given as follows:

$$F_\alpha \circ f = \left(F_1 \circ f + \sum_{j=1}^{m} \alpha_{1j} f_j, F_2 \circ f + \sum_{j=1}^{m} \alpha_{2j} f_j, \ldots, F_\ell \circ f + \sum_{j=1}^{m} \alpha_{\ell j} f_j \right), \tag{3.2}$$

where $f = (f_1, f_2, \ldots, f_m)$. Since we have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$, in order to prove Theorem 1, it is sufficient to show that there exists a subset Σ with Lebesgue measure zero of $(\mathbb{R}^m)^\ell$ such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $j^1(F_\alpha \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to the given submanifold $X(N, \mathbb{R}^\ell)$.

Now, let $\Gamma : N \times (\mathbb{R}^m)^\ell \to J^1(N, \mathbb{R}^\ell)$ be the mapping defined by

$$\Gamma(q, \alpha) = j^1(F_\alpha \circ f)(q).$$

If the mapping Γ is transverse to the submanifold $X(N, \mathbb{R}^\ell)$, then from Lemma 1, it follows that there exists a subset Σ of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $\Gamma_\alpha : N \to J^1(N, \mathbb{R}^\ell)$ ($\Gamma_\alpha = j^1(F_\alpha \circ f)$) is transverse to the submanifold $X(N, \mathbb{R}^\ell)$. Thus, in order to finish the proof of Theorem 1, it is sufficient to show that if $\Gamma(\widetilde{q}, \widetilde{\alpha}) \in X(N, \mathbb{R}^\ell)$, then the following holds:

$$d\Gamma_{(\widetilde{q},\widetilde{\alpha})}(T_{(\widetilde{q},\widetilde{\alpha})}(N\times(\mathbb{R}^m)^\ell))+T_{\Gamma(\widetilde{q},\widetilde{\alpha})}X(N,\mathbb{R}^\ell)=T_{\Gamma(\widetilde{q},\widetilde{\alpha})}J^1(N,\mathbb{R}^\ell). \tag{3.3}$$

As in Section 2, let $\{(U_\lambda,\varphi_\lambda)\}_{\lambda\in\Lambda}$ (resp., $\{(\Pi^{-1}(U_\lambda\times\mathbb{R}^\ell),\Phi_\lambda)\}_{\lambda\in\Lambda}$) be a coordinate neighborhood system of N (resp., $J^1(N,\mathbb{R}^\ell)$). There exists a coordinate neighborhood $\big(U_{\widetilde{\lambda}}\times(\mathbb{R}^m)^\ell,\varphi_{\widetilde{\lambda}}\times id\big)$ containing the point $(\widetilde{q},\widetilde{\alpha})$ of $N\times(\mathbb{R}^m)^\ell$, where id is the identity mapping of $(\mathbb{R}^m)^\ell$ into $(\mathbb{R}^m)^\ell$, and the mapping $\varphi_{\widetilde{\lambda}}\times id: U_{\widetilde{\lambda}}\times(\mathbb{R}^m)^\ell\to\varphi_{\widetilde{\lambda}}(U_{\widetilde{\lambda}})\times(\mathbb{R}^m)^\ell$ $(\subset\mathbb{R}^n\times(\mathbb{R}^m)^\ell)$ is defined by $\big(\varphi_{\widetilde{\lambda}}\times id\big)(q,\alpha)=\big(\varphi_{\widetilde{\lambda}}(q),id(\alpha)\big)$. There exists a coordinate neighborhood $\big(\Pi^{-1}(U_{\widetilde{\lambda}}\times\mathbb{R}^\ell),\Phi_{\widetilde{\lambda}}\big)$ containing the point $\Gamma(\widetilde{q},\widetilde{\alpha})$ of $J^1(N,\mathbb{R}^\ell)$. Let $t=(t_1,t_2,\ldots,t_n)\in\mathbb{R}^n$ be a local coordinate on $\varphi_{\widetilde{\lambda}}(U_{\widetilde{\lambda}})$ containing $\varphi_{\widetilde{\lambda}}(\widetilde{q})$. Then, the mapping Γ is locally given by the following:

$$\begin{aligned}
&(\Phi_{\widetilde{\lambda}}\circ\Gamma\circ(\varphi_{\widetilde{\lambda}}\times id)^{-1})(t,\alpha)\\
&=(\Phi_{\widetilde{\lambda}}\circ j^1(F_\alpha\circ f)\circ\varphi_{\widetilde{\lambda}}^{-1})(t)\\
&=\Bigg(t,(F_\alpha\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})(t),\\
&\quad\frac{\partial(F_{\alpha,1}\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})}{\partial t_1}(t),\frac{\partial(F_{\alpha,1}\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})}{\partial t_2}(t),\ldots,\frac{\partial(F_{\alpha,1}\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})}{\partial t_n}(t),\\
&\quad\frac{\partial(F_{\alpha,2}\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})}{\partial t_1}(t),\frac{\partial(F_{\alpha,2}\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})}{\partial t_2}(t),\ldots,\frac{\partial(F_{\alpha,2}\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})}{\partial t_n}(t),\\
&\quad\cdots\cdots\cdots,\\
&\quad\frac{\partial(F_{\alpha,\ell}\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})}{\partial t_1}(t),\frac{\partial(F_{\alpha,\ell}\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})}{\partial t_2}(t),\ldots,\frac{\partial(F_{\alpha,\ell}\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})}{\partial t_n}(t)\Bigg)\\
&=\Bigg(t,(F_\alpha\circ f\circ\varphi_{\widetilde{\lambda}}^{-1})(t),\\
&\quad\frac{\partial F_1\circ\widetilde{f}}{\partial t_1}(t)+\sum_{j=1}^m\alpha_{1j}\frac{\partial\widetilde{f}_j}{\partial t_1}(t),\frac{\partial F_1\circ\widetilde{f}}{\partial t_2}(t)+\sum_{j=1}^m\alpha_{1j}\frac{\partial\widetilde{f}_j}{\partial t_2}(t),\ldots,\frac{\partial F_1\circ\widetilde{f}}{\partial t_n}(t)+\sum_{j=1}^m\alpha_{1j}\frac{\partial\widetilde{f}_j}{\partial t_n}(t),\\
&\quad\frac{\partial F_2\circ\widetilde{f}}{\partial t_1}(t)+\sum_{j=1}^m\alpha_{2j}\frac{\partial\widetilde{f}_j}{\partial t_1}(t),\frac{\partial F_2\circ\widetilde{f}}{\partial t_2}(t)+\sum_{j=1}^m\alpha_{2j}\frac{\partial\widetilde{f}_j}{\partial t_2}(t),\ldots,\frac{\partial F_2\circ\widetilde{f}}{\partial t_n}(t)+\sum_{j=1}^m\alpha_{2j}\frac{\partial\widetilde{f}_j}{\partial t_n}(t),\\
&\quad\cdots\cdots\cdots,\\
&\quad\frac{\partial F_\ell\circ\widetilde{f}}{\partial t_1}(t)+\sum_{j=1}^m\alpha_{\ell j}\frac{\partial\widetilde{f}_j}{\partial t_1}(t),\frac{\partial F_\ell\circ\widetilde{f}}{\partial t_2}(t)+\sum_{j=1}^m\alpha_{\ell j}\frac{\partial\widetilde{f}_j}{\partial t_2}(t),\ldots,\frac{\partial F_\ell\circ\widetilde{f}}{\partial t_n}(t)+\sum_{j=1}^m\alpha_{\ell j}\frac{\partial\widetilde{f}_j}{\partial t_n}(t)\Bigg),
\end{aligned}$$

where $F_\alpha=(F_{\alpha,1},F_{\alpha,2},\ldots,F_{\alpha,\ell})$ and $\widetilde{f}=(\widetilde{f}_1,\widetilde{f}_2,\ldots,\widetilde{f}_m)=(f_1\circ\varphi_{\widetilde{\lambda}}^{-1},f_2\circ\varphi_{\widetilde{\lambda}}^{-1},\ldots,f_m\circ\varphi_{\widetilde{\lambda}}^{-1})=f\circ\varphi_{\widetilde{\lambda}}^{-1}$. The Jacobian matrix of the mapping Γ at $(\widetilde{q},\widetilde{\alpha})$ is the following:

$$J\Gamma_{(\widetilde{q},\widetilde{\alpha})}=\left(\begin{array}{c|cccc}
E_n & 0 & \cdots & \cdots & 0\\
\hline
 & * & \cdots & \cdots & *\\
 & {}^t(Jf_{\widetilde{q}}) & & 0 & \\
* & & {}^t(Jf_{\widetilde{q}}) & & \\
 & & & \ddots & \\
 & & 0 & & {}^t(Jf_{\widetilde{q}})
\end{array}\right)_{(t,\alpha)=(\varphi_{\widetilde{\lambda}}(\widetilde{q}),\widetilde{\alpha})},$$

where E_n is the $n \times n$ unit matrix and $Jf_{\widetilde{q}}$ is the Jacobian matrix of the mapping f at $\widetilde{q}$. Note that ${}^t(Jf_{\widetilde{q}})$ is the transpose of the matrix $Jf_{\widetilde{q}}$ and that there are ℓ copies of ${}^t(Jf_{\widetilde{q}})$ in the above description of $J\Gamma_{(\widetilde{q},\widetilde{\alpha})}$. Since $X(N,\mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N,\mathbb{R}^\ell)$ with the fiber X, it is clear that in order to show (3.3), it suffices to prove that the matrix M_1 given below has rank $n+\ell+n\ell$:

$$M_1 = \left(\begin{array}{c|cccc} E_{n+\ell} & * & \cdots & \cdots & * \\ \hline & {}^t(Jf_{\widetilde{q}}) & & 0 & \\ 0 & & {}^t(Jf_{\widetilde{q}}) & & \\ & & & \ddots & \\ & & 0 & & {}^t(Jf_{\widetilde{q}}) \end{array}\right)_{(t,\alpha)=(\varphi_{\widetilde{\lambda}}(\widetilde{q}),\widetilde{\alpha})},$$

where $E_{n+\ell}$ is the $(n+\ell)\times(n+\ell)$ unit matrix. Note that there are ℓ copies of ${}^t(Jf_{\widetilde{q}})$ in the above description of M_1. Notice that for any i $(1 \le i \le m\ell)$, the $(n+\ell+i)$-th column vector of M_1 coincides with the $(n+i)$-th column vector of $J\Gamma_{(\widetilde{q},\widetilde{\alpha})}$. Since the mapping f is an immersion $(n \le m)$, we have that the rank of the matrix M_1 is equal to $n+\ell+n\ell$. Hence, we have (3.3). □

4. Proof of Theorem 2.

By the same method as in the proof of Theorem 1, set $F_\alpha = F_\pi$, where F_α is given by (3.1) in Section 3. For a given injection $f : N \to U$, the mapping $F_\alpha \circ f : N \to \mathbb{R}^\ell$ is given by the same expression as (3.2). Since we have the natural identification $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$, in order to show that there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, and for any s $(2 \le s \le s_f)$, the mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to the submanifold Δ_s, it is sufficient to show that there exists a subset Σ of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, and for any s $(2 \le s \le s_f)$, the mapping $(F_\alpha \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to Δ_s.

Now, let s be a positive integer satisfying $2 \le s \le s_f$. Let $\Gamma : N^{(s)} \times (\mathbb{R}^m)^\ell \to (\mathbb{R}^\ell)^s$ be the mapping defined by

$$\Gamma(q_1,q_2,\dots,q_s,\alpha) = ((F_\alpha \circ f)(q_1),(F_\alpha \circ f)(q_2),\dots,(F_\alpha \circ f)(q_s)).$$

If for any positive integer s $(2 \le s \le s_f)$, the mapping Γ is transverse to Δ_s, then from Lemma 1, it follows that for any positive integer s $(2 \le s \le s_f)$, there exists a subset Σ_s of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma_s$, the mapping $\Gamma_\alpha : N^{(s)} \to (\mathbb{R}^\ell)^s$ $(\Gamma_\alpha = (F_\alpha \circ f)^{(s)})$ is transverse to Δ_s. Then, set $\Sigma = \bigcup_{s=2}^{s_f} \Sigma_s$. It is clearly seen that Σ is a subset of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero. Therefore, it follows that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, and for any s $(2 \le s \le s_f)$, the mapping $\Gamma_\alpha : N^{(s)} \to (\mathbb{R}^\ell)^s$ $(\Gamma_\alpha = (F_\alpha \circ f)^{(s)})$ is transverse to Δ_s.

Hence, for the proof, it is sufficient to show that for any positive integer s $(2 \le s \le s_f)$, if $\Gamma(\widetilde{q},\widetilde{\alpha}) \in \Delta_s$ $(\widetilde{q} = (\widetilde{q}_1,\widetilde{q}_2,\dots,\widetilde{q}_s))$, then the following holds:

$$d\Gamma_{(\widetilde{q},\widetilde{\alpha})}(T_{(\widetilde{q},\widetilde{\alpha})}(N^{(s)} \times (\mathbb{R}^m)^\ell)) + T_{\Gamma(\widetilde{q},\widetilde{\alpha})}\Delta_s = T_{\Gamma(\widetilde{q},\widetilde{\alpha})}(\mathbb{R}^\ell)^s. \quad (4.1)$$

Let $\{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ be a coordinate neighborhood system of N. There exists a coordinate neighborhood $(U_{\widetilde{\lambda}_1}\times U_{\widetilde{\lambda}_2}\times\cdots\times U_{\widetilde{\lambda}_s}\times(\mathbb{R}^m)^\ell, \varphi_{\widetilde{\lambda}_1}\times\varphi_{\widetilde{\lambda}_2}\times\cdots\times\varphi_{\widetilde{\lambda}_s}\times id)$ containing the point $(\widetilde{q},\widetilde{\alpha})$ of $N^{(s)}\times(\mathbb{R}^m)^\ell$, where id is the identity mapping of $(\mathbb{R}^m)^\ell$ into $(\mathbb{R}^m)^\ell$, and the mapping $\varphi_{\widetilde{\lambda}_1}\times\varphi_{\widetilde{\lambda}_2}\times\cdots\times\varphi_{\widetilde{\lambda}_s}\times id : U_{\widetilde{\lambda}_1}\times U_{\widetilde{\lambda}_2}\times\cdots\times U_{\widetilde{\lambda}_s}\times(\mathbb{R}^m)^\ell\to(\mathbb{R}^n)^s\times(\mathbb{R}^m)^\ell$ is defined by $(\varphi_{\widetilde{\lambda}_1}\times\varphi_{\widetilde{\lambda}_2}\times\cdots\times\varphi_{\widetilde{\lambda}_s}\times id)(q_1,q_2,\ldots,q_s,\alpha)=(\varphi_{\widetilde{\lambda}_1}(q_1),\varphi_{\widetilde{\lambda}_2}(q_2),\ldots,\varphi_{\widetilde{\lambda}_s}(q_s),id(\alpha))$. Let $t_i=(t_{i1},t_{i2},\ldots,t_{in})$ be a local coordinate around $\varphi_{\widetilde{\lambda}_i}(\widetilde{q}_i)$ $(1\le i\le s)$. Then, the mapping Γ is locally given by the following:

$$
\begin{aligned}
&\Gamma\circ\left(\varphi_{\widetilde{\lambda}_1}\times\varphi_{\widetilde{\lambda}_2}\times\cdots\times\varphi_{\widetilde{\lambda}_s}\times id\right)^{-1}(t_1,t_2,\ldots,t_s,\alpha)\\
=&\left((F_\alpha\circ f\circ\varphi_{\widetilde{\lambda}_1}^{-1})(t_1),(F_\alpha\circ f\circ\varphi_{\widetilde{\lambda}_2}^{-1})(t_2),\ldots,(F_\alpha\circ f\circ\varphi_{\widetilde{\lambda}_s}^{-1})(t_s)\right)\\
=&\Bigg(F_1\circ\widetilde{f}(t_1)+\sum_{j=1}^m\alpha_{1j}\widetilde{f}_j(t_1),F_2\circ\widetilde{f}(t_1)+\sum_{j=1}^m\alpha_{2j}\widetilde{f}_j(t_1),\ldots,F_\ell\circ\widetilde{f}(t_1)+\sum_{j=1}^m\alpha_{\ell j}\widetilde{f}_j(t_1),\\
&F_1\circ\widetilde{f}(t_2)+\sum_{j=1}^m\alpha_{1j}\widetilde{f}_j(t_2),F_2\circ\widetilde{f}(t_2)+\sum_{j=1}^m\alpha_{2j}\widetilde{f}_j(t_2),\ldots,F_\ell\circ\widetilde{f}(t_2)+\sum_{j=1}^m\alpha_{\ell j}\widetilde{f}_j(t_2),\\
&\cdots\cdots\cdots,\\
&F_1\circ\widetilde{f}(t_s)+\sum_{j=1}^m\alpha_{1j}\widetilde{f}_j(t_s),F_2\circ\widetilde{f}(t_s)+\sum_{j=1}^m\alpha_{2j}\widetilde{f}_j(t_s),\ldots,F_\ell\circ\widetilde{f}(t_s)+\sum_{j=1}^m\alpha_{\ell j}\widetilde{f}_j(t_s)\Bigg),
\end{aligned}
$$

where $\widetilde{f}(t_i)=(\widetilde{f}_1(t_i),\widetilde{f}_2(t_i),\ldots,\widetilde{f}_m(t_i))=(f_1\circ\varphi_{\widetilde{\lambda}_i}^{-1}(t_i),f_2\circ\varphi_{\widetilde{\lambda}_i}^{-1}(t_i),\ldots,f_m\circ\varphi_{\widetilde{\lambda}_i}^{-1}(t_i))$ $(1\le i\le s)$. For simplicity, set $t=(t_1,t_2,\ldots,t_s)$ and $z=(\varphi_{\widetilde{\lambda}_1}\times\varphi_{\widetilde{\lambda}_2}\times\cdots\times\varphi_{\widetilde{\lambda}_s})(\widetilde{q}_1,\widetilde{q}_2,\ldots,\widetilde{q}_s)$.

The Jacobian matrix of the mapping Γ at $(\widetilde{q},\widetilde{\alpha})$ is the following:

$$
J\Gamma_{(\widetilde{q},\widetilde{\alpha})}=\left(\begin{array}{c|c} * & B(t_1)\\ * & B(t_2)\\ \vdots & \vdots\\ * & B(t_s)\end{array}\right)_{(t,\alpha)=(z,\widetilde{\alpha})},
$$

where

$$
B(t_i)=\left.\begin{pmatrix}\boldsymbol{b}(t_i) & & & 0\\ & \boldsymbol{b}(t_i) & & \\ & & \ddots & \\ 0 & & & \boldsymbol{b}(t_i)\end{pmatrix}\right\}\ \ell\text{ rows}
$$

and $\boldsymbol{b}(t_i)=(\widetilde{f}_1(t_i),\widetilde{f}_2(t_i),\ldots,\widetilde{f}_m(t_i))$. By the construction of $T_{\Gamma(\widetilde{q},\widetilde{\alpha})}\Delta_s$, in order to show (4.1), it is sufficient to show that the rank of the following matrix M_2 is equal to ℓs:

$$M_2 = \left(\begin{array}{c|c} E_\ell & B(t_1) \\ E_\ell & B(t_2) \\ \vdots & \vdots \\ E_\ell & B(t_s) \end{array} \right)_{t=z}.$$

There exists an $\ell s \times \ell s$ regular matrix Q_1 such that

$$Q_1 M_2 = \left(\begin{array}{c|c} E_\ell & B(t_1) \\ 0 & B(t_2) - B(t_1) \\ \vdots & \vdots \\ 0 & B(t_s) - B(t_1) \end{array} \right)_{t=z}.$$

There exists an $(\ell + m\ell) \times (\ell + m\ell)$ regular matrix Q_2 such that

$$Q_1 M_2 Q_2 = \left(\begin{array}{c|c} E_\ell & 0 \\ 0 & B(t_2) - B(t_1) \\ \vdots & \vdots \\ 0 & B(t_s) - B(t_1) \end{array} \right)_{t=z}$$

$$= \left(\begin{array}{c|cccc} E_\ell & & 0 & & \\ \hline 0 & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} & & & 0 \\ & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} & & \\ & 0 & & \ddots & \\ & & & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} & & & 0 \\ & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} & & \\ & 0 & & \ddots & \\ & & & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} \end{array} \right) \begin{array}{l} \\ \left.\right\} \ell \text{ rows} \\ , \\ \left.\right\} \ell \text{ rows} \end{array}$$

where $\overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_i)} = (\widetilde{f}_1(t_i) - \widetilde{f}_1(t_1), \widetilde{f}_2(t_i) - \widetilde{f}_2(t_1), \ldots, \widetilde{f}_m(t_i) - \widetilde{f}_m(t_1))$ $(2 \leq i \leq s)$ and $t = z$. From $s - 1 \leq s_f - 1$ and the definition of s_f, it follows that

$$\dim \sum_{i=2}^{s} \mathbb{R}\overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_i)} = s - 1,$$

where $t = z$. Thus, by the construction of the matrix $Q_1 M_2 Q_2$ and $s - 1 \leq m$, we have that the rank of the matrix $Q_1 M_2 Q_2$ is equal to ℓs. Hence, the rank of the matrix M_2 must be equal to ℓs. Therefore, we have (4.1). Thus, there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any s $(2 \leq s \leq s_f)$, the mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to the submanifold Δ_s.

Moreover, suppose that the mapping F_π satisfies that $|F_\pi^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$. Since $f : N \to \mathbb{R}^m$ is injective, it follows that $|(F_\pi \circ f)^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$. Hence, it follows that for any positive integer s with $s \geq s_f + 1$, we have $(F_\pi \circ f)^{(s)}(N^{(s)}) \bigcap \Delta_s = \emptyset$. Namely, for any positive integer s with $s \geq s_f + 1$, the mapping $(F_\pi \circ f)^{(s)}$ is transverse to Δ_s. Thus, $F_\pi \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings. □

5. Applications of Theorems 1 and 2.

In Subsection 5.1 (resp., Subsection 5.2), applications of Theorem 1 (resp., Theorem 2) are stated and proved. In Subsection 5.2, applications obtained by combining Theorems 1 and 2 are also given.

5.1. Applications of Theorem 1.

Set

$$\Sigma^k = \left\{ j^1 g(0) \in J^1(n, \ell) \mid \text{corank } Jg(0) = k \right\},$$

where corank $Jg(0) = \min\{n, \ell\} -$ rank $Jg(0)$ and $k = 1, 2, \ldots, \min\{n, \ell\}$. Then, Σ^k is an $\mathcal{A}^1$-invariant submanifold of $J^1(n, \ell)$. Set

$$\Sigma^k(N, \mathbb{R}^\ell) = \bigcup_{\lambda \in \Lambda} \Phi_\lambda^{-1} \left(\varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times \Sigma^k \right),$$

where the mappings Φ_λ and φ_λ are as defined in Section 2. Then, the set $\Sigma^k(N, \mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the fiber Σ^k such that

$$\begin{aligned} \operatorname{codim} \Sigma^k(N, \mathbb{R}^\ell) &= \dim J^1(N, \mathbb{R}^\ell) - \dim \Sigma^k(N, \mathbb{R}^\ell) \\ &= (n - v + k)(\ell - v + k), \end{aligned}$$

where $v = \min\{n, \ell\}$. (For details on Σ^k and $\Sigma^k(N, \mathbb{R}^\ell)$, see for example [**3**], pp. 60–61).

As applications of Theorem 1, we have the following Proposition 1, Corollaries 1, 2, 3 and 4.

Proposition 1. *Let N be a manifold of dimension n. Let f be an immersion of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Then, there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to the submanifold $\Sigma^k(N, \mathbb{R}^\ell)$ for any positive integer k satisfying $1 \leq k \leq v$. Especially, in the case of $\ell \geq 2$, we have $k_0 + 1 \leq v$ and it follows that the mapping $j^1(F_\pi \circ f)$ satisfies that $j^1(F_\pi \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ for any positive integer k satisfying $k_0 + 1 \leq k \leq v$, where k_0 is the maximum integer satisfying $(n - v + k_0)(\ell - v + k_0) \leq n$ $(v = \min\{n, \ell\})$.*

Proof. By Theorem 1, for any positive integer k satisfying $1 \leq k \leq v$, there exists a subset $\widetilde{\Sigma}_k$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in$

$\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \widetilde{\Sigma}_k$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$. Set $\Sigma = \bigcup_{k=1}^{v} \widetilde{\Sigma}_k$. Then, it is clearly seen that Σ is a subset of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero. Hence, it follows that there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to the submanifold $\Sigma^k(N, \mathbb{R}^\ell)$ for any positive integer k satisfying $1 \le k \le v$.

Now, we will consider the case of $\ell \ge 2$. Firstly, we will show that $k_0 + 1 \le v$ in the case. Suppose that $v \le k_0$. Then, by $(n - v + k_0)(\ell - v + k_0) \le n$, we have $n\ell \le n$. This contradicts the assumption $\ell \ge 2$.

Secondly, we will show that in the case of $\ell \ge 2$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ satisfies that $j^1(F_\pi \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ for any positive integer k satisfying $k_0 + 1 \le k \le v$. Suppose that there exist a positive integer k ($k_0 + 1 \le k \le v$) and a point $q \in N$ such that $j^1(F_\pi \circ f)(q) \in \Sigma^k(N, \mathbb{R}^\ell)$. Since the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$ at the point q, the following holds:

$$d(j^1(F_\pi \circ f))_q(T_q N) + T_{j^1(F_\pi \circ f)(q)} \Sigma^k(N, \mathbb{R}^\ell) = T_{j^1(F_\pi \circ f)(q)} J^1(N, \mathbb{R}^\ell).$$

Hence, we have

$$\begin{aligned} & \dim d(j^1(F_\pi \circ f))_q(T_q N) \\ \ge\ & \dim T_{j^1(F_\pi \circ f)(q)} J^1(N, \mathbb{R}^\ell) - \dim T_{j^1(F_\pi \circ f)(q)} \Sigma^k(N, \mathbb{R}^\ell) \\ =\ & \operatorname{codim} T_{j^1(F_\pi \circ f)(q)} \Sigma^k(N, \mathbb{R}^\ell). \end{aligned}$$

Thus, we get $n \ge (n - v + k)(\ell - v + k)$. Since the given integer k_0 is the maximum integer satisfying $n \ge (n - v + k_0)(\ell - v + k_0)$, it follows that $k \le k_0$. This contradicts the assumption $k_0 + 1 \le k$. □

REMARK 2.

1. In Proposition 1, by $(n - v + k_0)(\ell - v + k_0) \le n$, it is clearly seen that $k_0 \ge 0$.
2. In Proposition 1, in the case of $\ell = 1$, we have $k_0 + 1 > v$. Indeed, in the case, by $v = 1$, we get $(n - 1 + k_0)k_0 \le n$. Hence, we have $k_0 = 1$.

A mapping $g : N \to \mathbb{R}$ is called a *Morse function* if all of the singularities of the mapping g are nondegenerate (for details on Morse functions, see for example, [**3**], p. 63). In the case of $(n, \ell) = (n, 1)$, we have the following.

COROLLARY 1. *Let N be a manifold of dimension n. Let f be an immersion of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}$ be a mapping. Then, there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}$ is a Morse function.*

PROOF. By Proposition 1, there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R})$ is transverse to the submanifold $\Sigma^1(N, \mathbb{R})$. Hence, if $q \in N$ is a singular point of the mapping $F_\pi \circ f$, then the point q is nondegenerate. □

For a given mapping $g : N \to \mathbb{R}^{2n-1}$ $(n \geq 2)$, a singular point $q \in N$ is called a *singular point of Whitney umbrella* if there exist two germs of diffeomorphisms $H : (\mathbb{R}^{2n-1}, g(q)) \to (\mathbb{R}^{2n-1}, 0)$ and $h : (N, q) \to (\mathbb{R}^n, 0)$ such that $H \circ g \circ h^{-1}(x_1, x_2, \ldots, x_n) = (x_1^2, x_1x_2, \ldots, x_1x_n, x_2, \ldots, x_n)$, where $(x_1, x_2, \ldots, x_n)$ is a local coordinate around the point $h(q) = 0 \in \mathbb{R}^n$. In the case of $(n, \ell) = (n, 2n-1)$ $(n \geq 2)$, we have the following.

Corollary 2. *Let N be a manifold of dimension n $(n \geq 2)$. Let f be an immersion of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^{2n-1}$ be a mapping. Then, there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma$, any singular point of the mapping $F_\pi \circ f : N \to \mathbb{R}^{2n-1}$ is a singular point of Whitney umbrella.*

Proof. By, for example, [**3**], p. 179, we see that a point $q \in N$ is a singular point of Whitney umbrella of the mapping $F_\pi \circ f$ if $j^1(F_\pi \circ f)(q) \in \Sigma^1(N, \mathbb{R}^{2n-1})$ and the mapping $j^1(F_\pi \circ f)$ is transverse to the submanifold $\Sigma^1(N, \mathbb{R}^{2n-1})$ at q. Set $\ell = 2n-1$ and $v = n$ in Proposition 1. Then, it is clearly seen that we have $k_0 = 1$ in Proposition 1. Hence, there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^{2n-1}$ is transverse to $\Sigma^k(N, \mathbb{R}^{2n-1})$ for any positive integer k satisfying $1 \leq k \leq n$, and the mapping satisfies that $j^1(F_\pi \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^{2n-1}) = \emptyset$ for any positive integer k satisfying $2 \leq k \leq n$. Thus, if a point $q \in N$ is a singular point of the mapping $F_\pi \circ f$, then it follows that $j^1(F_\pi \circ f)(q) \in \Sigma^1(N, \mathbb{R}^{2n-1})$ and $j^1(F_\pi \circ f)$ is transverse to $\Sigma^1(N, \mathbb{R}^{2n-1})$ at q. □

In the case of $\ell \geq 2n$, the immersion property of a given mapping $f : N \to U$ is preserved by composing generic linearly perturbed mappings as follows:

Corollary 3. *Let N be a manifold of dimension n. Let f be an immersion of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping $(\ell \geq 2n)$. Then, there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion.*

Proof. It is clearly seen that the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion if and only if $j^1(F_\pi \circ f)(N) \bigcap \bigcup_{k=1}^n \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$. Set $v = n$ and $\ell \geq 2n$ in Proposition 1. Then, it is clearly seen that $k_0 \leq 0$. By Remark 2, we get $k_0 = 0$. Hence, there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ satisfies that $j^1(F_\pi \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ for any positive integer k $(1 \leq k \leq n)$. □

A mapping $g : N \to \mathbb{R}^\ell$ *has corank at most k singular points* if

$$\sup \{\text{corank } dg_q \mid q \in N\} \leq k,$$

where corank $dg_q = \min\{n, \ell\} - \text{rank } dg_q$. By Proposition 1, we have the following corollary.

Corollary 4. *Let N be a manifold of dimension n. Let f be an immersion of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Let k_0 be the maximum* *integer satisfying* $(n-v+k_0)(\ell-v+k_0) \leq n$ $(v = \min\{n, \ell\})$. *Then, there exists a subset* Σ *of* $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ *with Lebesgue measure zero such that for any* $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, *the mapping* $F_\pi \circ f : N \to \mathbb{R}^\ell$ *has corank at most* k_0 *singular points.*

5.2. Applications of Theorem 2.

Proposition 2. *Let* N *be a manifold of dimension* n. *Let* f *be an injection of* N *into an open subset* U *of* $\mathbb{R}^m$. *Let* $F : U \to \mathbb{R}^\ell$ *be a mapping. If* $(s_f - 1)\ell > ns_f$, *then there exists a subset* Σ *of* $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ *with Lebesgue measure zero such that for any* $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \to \mathbb{R}^\ell$ *is a mapping with normal crossings satisfying* $(F_\pi \circ f)^{(s_f)}(N^{(s_f)}) \bigcap \Delta_{s_f} = \emptyset$.

Proof. By Theorem 2, there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any s $(2 \leq s \leq s_f)$, the mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to the submanifold Δ_s. Hence, in order to show Proposition 2, it is sufficient to show that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $(F_\pi \circ f)^{(s_f)}$ satisfies that $(F_\pi \circ f)^{(s_f)}(N^{(s_f)}) \bigcap \Delta_{s_f} = \emptyset$.

Suppose that there exists an element $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ such that there exists a point $q \in N^{(s_f)}$ satisfying $(F_\pi \circ f)^{(s_f)}(q) \in \Delta_{s_f}$. Since $(F_\pi \circ f)^{(s_f)}$ is transverse to Δ_{s_f}, we have the following:

$$d((F_\pi \circ f)^{(s_f)})_q(T_q N^{(s_f)}) + T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f} = T_{(F_\pi \circ f)^{(s_f)}(q)} (\mathbb{R}^\ell)^{s_f}.$$

Hence, we have

$$\begin{aligned} & \dim d((F_\pi \circ f)^{(s_f)})_q(T_q N^{(s_f)}) \\ \geq\ & \dim T_{(F_\pi \circ f)^{(s_f)}(q)} (\mathbb{R}^\ell)^{s_f} - \dim T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f} \\ =\ & \operatorname{codim} T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f}. \end{aligned}$$

Thus, we get $ns_f \geq (s_f - 1)\ell$. This contradicts the assumption $(s_f - 1)\ell > ns_f$. □

In the case of $\ell > 2n$, the injection property of a given mapping $f : N \to U$ is preserved by composing generic linearly perturbed mappings as follows:

Corollary 5. *Let* N *be a manifold of dimension* n. *Let* f *be an injection of* N *into an open subset* U *of* $\mathbb{R}^m$. *Let* $F : U \to \mathbb{R}^\ell$ *be a mapping. If* $\ell > 2n$, *then there exists a subset* Σ *of* $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ *with Lebesgue measure zero such that for any* $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, *the mapping* $F_\pi \circ f : N \to \mathbb{R}^\ell$ *is injective.*

Proof. Since $s_f \geq 2$ and $\ell > 2n$, it is easily seen that the dimension pair (n, ℓ) satisfies the assumption $(s_f - 1)\ell > ns_f$ of Proposition 2. Indeed, from $\ell > 2n$, it follows that $(s_f - 1)\ell > 2n(s_f - 1)$. By $s_f \geq 2$, we get $2n(s_f - 1) \geq ns_f$.

Hence, by Proposition 2, there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $(F_\pi \circ f)^{(2)} : N^{(2)} \to (\mathbb{R}^\ell)^2$ is transverse to Δ_2. In order to show Corollary 5, it is sufficient to show that the mapping $(F_\pi \circ f)^{(2)}$ satisfies that $(F_\pi \circ f)^{(2)}(N^{(2)}) \bigcap \Delta_2 = \emptyset$.

Suppose that there exists a point $q \in N^{(2)}$ such that $(F_\pi \circ f)^{(2)}(q) \in \Delta_2$. Then, we have the following:

$$d((F_\pi \circ f)^{(2)})_q(T_q N^{(2)}) + T_{(F_\pi \circ f)^{(2)}(q)}\Delta_2 = T_{(F_\pi \circ f)^{(2)}(q)}(\mathbb{R}^\ell)^2.$$

Hence, we have

$$\begin{aligned} &\dim d((F_\pi \circ f)^{(2)})_q(T_q N^{(2)}) \\ \geq\ &\dim T_{(F_\pi \circ f)^{(2)}(q)}(\mathbb{R}^\ell)^2 - \dim T_{(F_\pi \circ f)^{(2)}(q)}\Delta_2 \\ =\ &\operatorname{codim} T_{(F_\pi \circ f)^{(2)}(q)}\Delta_2. \end{aligned}$$

Thus, we get $2n \geq \ell$. This contradicts the assumption $\ell > 2n$. □

By combining Corollaries 3 and 5, we have the following.

COROLLARY 6. *Let N be a manifold of dimension n. Let f be an injective immersion of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $\ell > 2n$, then there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an injective immersion.*

In Corollary 6, suppose that the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is proper. Then, an injective immersion $F_\pi \circ f$ is necessarily an embedding (see [**3**], p. 11). Thus, we get the following.

COROLLARY 7. *Let N be a compact manifold of dimension n. Let f be an embedding of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $\ell > 2n$, then there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an embedding.*

6. Further applications.

6.1. Introduction of generalized distance-squared mappings.

Let $p_i = (p_{i1}, p_{i2}, \ldots, p_{im})$ $(1 \leq i \leq \ell)$ (resp., $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$) be points of $\mathbb{R}^m$ (resp., an $\ell \times m$ matrix with all entries being non-zero real numbers). Set $p = (p_1, p_2, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell$. Let $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$ be the mapping defined by

$$G_{(p,A)}(x) = \left(\sum_{j=1}^{m} a_{1j}(x_j - p_{1j})^2, \sum_{j=1}^{m} a_{2j}(x_j - p_{2j})^2, \ldots, \sum_{j=1}^{m} a_{\ell j}(x_j - p_{\ell j})^2 \right),$$

where $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. The mapping $G_{(p,A)}$ is called a *generalized distance-squared mapping*, and the ℓ-tuple of points $p = (p_1, p_2, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell$ is called the *central point* of the generalized distance-squared mapping $G_{(p,A)}$. A *distance-squared mapping* D_p (resp., *Lorentzian distance-squared mapping* L_p) is the mapping $G_{(p,A)}$ satisfying that each entry of A is equal to 1 (resp., $a_{i1} = -1$ and $a_{ij} = 1$ $(j \neq 1)$).

In [**5**] (resp., [**6**]), a classification result of distance-squared mappings (resp., Lorentzian distance-squared mappings) is given.

In [**9**], a classification result of generalized distance-squared mappings of the plane into the plane is given. If the rank of A is equal to two, then a generalized distance-squared mapping having a generic central point is a mapping of which any singular point is a fold point except one cusp point. The singular set is a rectangular hyperbola. If the rank of A is equal to one, then a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the normal form of fold singularity $(x_1, x_2) \mapsto (x_1, x_2^2)$.

In [**7**], a classification result of generalized distance-squared mappings of $\mathbb{R}^{m+1}$ into $\mathbb{R}^{2m+1}$ is given. If the rank of A is equal to $m+1$, then a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the normal form of Whitney umbrella $(x_1, x_2, \ldots, x_{m+1}) \mapsto (x_1^2, x_1x_2, \ldots, x_1x_{m+1}, x_2, \ldots, x_{m+1})$. If the rank of A is strictly smaller than $m+1$, then a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the inclusion $(x_1, x_2, \ldots, x_{m+1}) \mapsto (x_1, x_2, \ldots, x_{m+1}, 0, \ldots, 0)$.

Namely, in [**5**], [**6**], [**7**] and [**9**], the properties of generic generalized distance-squared mappings are investigated. Hence, it is natural to investigate the properties of compositions with generic generalized distance-squared mappings.

We have another original motivation. Height functions and distance-squared functions have been investigated in detail so far, and they are useful tools in the applications of singularity theory to differential geometry (for instance, see [**2**]). A mapping in which each component is a height function is nothing but a projection. Projections as well as height functions or distance-squared functions have been investigated so far. In [**10**], compositions of generic projections and embeddings are investigated.

On the other hand, a mapping in which each component is a distance-squared function is a distance-squared mapping. In addition, the notion of a generalized distance-squared mapping is an extension of that of a distance-squared mapping. Therefore, it is natural to investigate compositions with generic generalized distance-squared mappings as well as projections.

6.2. Applications of Theorem 1 to $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$.

Proposition 3. *Let N be a manifold of dimension n. Let $f : N \to \mathbb{R}^m$ be an immersion. Let $A = (a_{ij})_{1\le i\le \ell, 1\le j\le m}$ be an $\ell \times m$ matrix with all entries being non-zero real numbers. If X is an $\mathcal{A}^1$-invariant submanifold of $J^1(n, \ell)$, then there exists a subset Σ of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $j^1(G_{(p,A)} \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to the submanifold $X(N, \mathbb{R}^\ell)$.*

Proof. Let $H : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be a diffeomorphism of the target for deleting constant terms. The composition $H \circ G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$ is given as follows:

$$H \circ G_{(p,A)}(x) = \left(\sum_{j=1}^m a_{1j}x_j^2 - 2\sum_{j=1}^m a_{1j}p_{1j}x_j, \sum_{j=1}^m a_{2j}x_j^2 - 2\sum_{j=1}^m a_{2j}p_{2j}x_j, \right.$$
$$\left. \ldots, \sum_{j=1}^m a_{\ell j}x_j^2 - 2\sum_{j=1}^m a_{\ell j}p_{\ell j}x_j \right),$$

where $x = (x_1, x_2, \ldots, x_m)$.

Let $\psi : (\mathbb{R}^m)^\ell \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the mapping defined by

$$\psi(p_{11}, p_{12}, \dots, p_{\ell m}) = -2(a_{11}p_{11}, a_{12}p_{12}, \dots, a_{\ell m}p_{\ell m}).$$

Remark that we have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$. Since $a_{ij} \neq 0$ for any i, j ($1 \leq i \leq \ell$, $1 \leq j \leq m$), it is clearly seen that ψ is a C^∞ diffeomorphism.

Set $F_i(x) = \sum_{j=1}^m a_{ij}x_j^2$ ($1 \leq i \leq \ell$) and $F = (F_1, F_2, \dots, F_\ell)$. By Theorem 1, there exists a subset Σ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $X(N, \mathbb{R}^\ell)$. Since $\psi^{-1} : \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \to (\mathbb{R}^m)^\ell$ is a C^∞ mapping, $\psi^{-1}(\Sigma)$ is a subset of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero. For any $p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma)$, we have $\psi(p) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$. Hence, for any $p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma)$, the mapping $j^1(H \circ G_{(p,A)} \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $X(N, \mathbb{R}^\ell)$. Then, since $H : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is a diffeomorphism, the mapping $j^1(G_{(p,A)} \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $X(N, \mathbb{R}^\ell)$. □

Remark 3. As applications of Proposition 3, regarding generalized distance-squared mappings, we get analogies of Proposition 1, Corollaries 1, 2, 3 and 4.

6.3. Applications of Theorem 2 to $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$.

By Theorem 2, we get the following proposition, which can be proved by the same argument as in the proof of Proposition 3, and we omit the proof.

Proposition 4. *Let N be a manifold of dimension n. Let $f : N \to \mathbb{R}^m$ be an injection. Let $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ be an $\ell \times m$ matrix with all entries being non-zero real numbers. Then, there exists a subset Σ of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, and for any s ($2 \leq s \leq s_f$), the mapping $(G_{(p,A)} \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to the submanifold Δ_s. Moreover, if the mapping $G_{(p,A)}$ satisfies that $|G_{(p,A)}^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$, then $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings.*

Remark 4. As applications of Proposition 4, regarding generalized distance-squared mappings, we get analogies of Proposition 2, Corollaries 5, 6 and 7.

As the special case of the classification result of distance squared mappings (resp., Lorentzian distance-squared mappings) in [**5**] (resp., [**6**]), we have Lemma 2.

Lemma 2 ([**5**], [**6**]). *We have the following.*

1. *For any $p \in \mathbb{R}$, the mappings $D_p : \mathbb{R} \to \mathbb{R}$ and $L_p : \mathbb{R} \to \mathbb{R}$ are $\mathcal{A}$-equivalent to $x \mapsto x^2$.*
2. *For $m \geq 2$, there exists a subset Σ_D (resp., Σ_L) of $(\mathbb{R}^m)^m$ with Lebesgue measure zero such that for any $p \in (\mathbb{R}^m)^m - \Sigma_D$ (resp., $p \in (\mathbb{R}^m)^m - \Sigma_L$), the mapping $D_p : \mathbb{R}^m \to \mathbb{R}^m$ (resp., $L_p : \mathbb{R}^m \to \mathbb{R}^m$) is $\mathcal{A}$-equivalent to the normal form of definite fold mappings $(x_1, x_2, \dots, x_m) \mapsto (x_1, x_2, \dots, x_{m-1}, x_m^2)$.*
3. *In the case of $1 \leq m < \ell$, there exists a subset Σ_D (resp., Σ_L) of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma_D$ (resp., $p \in (\mathbb{R}^m)^\ell - \Sigma_L$),*

the mapping $D_p : \mathbb{R}^m \to \mathbb{R}^\ell$ *(resp.,* $L_p : \mathbb{R}^m \to \mathbb{R}^\ell$*) is* $\mathcal{A}$*-equivalent to the inclusion* $(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_m, 0, \ldots, 0)$.

PROPOSITION 5. *Let* N *be a manifold of dimension* n*. Let* $f : N \to \mathbb{R}^m$ *be an injection. Then, the following holds*:

1. *For* $m \geq 1$*, there exists a subset* Σ_D *(resp.,* Σ_L*) of* $(\mathbb{R}^m)^m$ *with Lebesgue measure zero such that for any* $p \in (\mathbb{R}^m)^m - \Sigma_D$ *(resp.,* $p \in (\mathbb{R}^m)^m - \Sigma_L$*),* $D_p \circ f : N \to \mathbb{R}^m$ *(resp.,* $L_p \circ f : N \to \mathbb{R}^m$*) is a mapping with normal crossings.*
2. *In the case of* $1 \leq m < \ell$*, there exists a subset* Σ_D *(resp.,* Σ_L*) of* $(\mathbb{R}^m)^\ell$ *with Lebesgue measure zero such that for any* $p \in (\mathbb{R}^m)^\ell - \Sigma_D$ *(resp.,* $p \in (\mathbb{R}^m)^\ell - \Sigma_L$*), the mapping* $D_p \circ f : N \to \mathbb{R}^\ell$ *(resp.,* $L_p \circ f : N \to \mathbb{R}^\ell$*) is an injection.*

PROOF. The proof for distance-squared mappings is the same as that for Lorentzian distance-squared mappings. Hence, it is sufficient to give the proof for distance-squared mappings.

Firstly, we will show the assertion 1. From Lemma 2, there exists a subset Σ_1 of $(\mathbb{R}^m)^m$ with Lebesgue measure zero such that for any $p \in (\mathbb{R}^m)^m - \Sigma_1$, the mapping $D_p : \mathbb{R}^m \to \mathbb{R}^m$ satisfies that $|D_p^{-1}(y)| \leq 2$ for any $y \in \mathbb{R}^m$. On the other hand, from Proposition 4, there exists a subset Σ_2 of $(\mathbb{R}^m)^m$ with Lebesgue measure zero such that for any $p \in (\mathbb{R}^m)^m - \Sigma_2$, if D_p satisfies that $|D_p^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^m$, then $D_p \circ f : N \to \mathbb{R}^m$ is a mapping with normal crossings. Set $\Sigma_D = \Sigma_1 \cup \Sigma_2$. It is clearly seen that Σ_D is a subset of $(\mathbb{R}^m)^m$ with Lebesgue measure zero. Then, for any $p \in (\mathbb{R}^m)^m - \Sigma_D$, $D_p \circ f : N \to \mathbb{R}^m$ is a mapping with normal crossings.

In the case of $m < \ell$, since from Lemma 2, there exists a subset Σ_D of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma_D$, the mapping $D_p : \mathbb{R}^m \to \mathbb{R}^\ell$ is $\mathcal{A}$-equivalent to the inclusion, the assertion 2 holds. □

By combining Proposition 5 and the analogy of Corollary 3 in Remark 3, we have the following.

COROLLARY 8. *Let* N *be a manifold of dimension* n*. Let* $f : N \to \mathbb{R}^m$ *be an injective immersion* $(2n \leq m)$*. Then, there exists a subset* Σ_D *(resp.,* Σ_L*) of* $(\mathbb{R}^m)^m$ *with Lebesgue measure zero such that for any* $p \in (\mathbb{R}^m)^m - \Sigma_D$ *(resp.,* $p \in (\mathbb{R}^m)^m - \Sigma_L$*), the mapping* $D_p \circ f : N \to \mathbb{R}^m$ *(resp.,* $L_p \circ f : N \to \mathbb{R}^m$*) is an immersion with normal crossings.*

In Corollary 8, if $m = 2n$ and the mapping $D_p \circ f : N \to \mathbb{R}^{2n}$ (resp., $L_p \circ f : N \to \mathbb{R}^{2n}$) is proper, then the immersion with normal crossings $D_p \circ f : N \to \mathbb{R}^{2n}$ (resp., $L_p \circ f : N \to \mathbb{R}^{2n}$) is necessarily stable (see [**3**], p. 86). Thus, we get the following.

COROLLARY 9. *Let* N *be a compact manifold of dimension* n*. Let* $f : N \to \mathbb{R}^{2n}$ *be an embedding. Then, there exists a subset* Σ_D *(resp.,* Σ_L*) of* $(\mathbb{R}^{2n})^{2n}$ *with Lebesgue measure zero such that for any* $p \in (\mathbb{R}^{2n})^{2n} - \Sigma_D$ *(resp.,* $p \in (\mathbb{R}^{2n})^{2n} - \Sigma_L$*), the mapping* $D_p \circ f : N \to \mathbb{R}^{2n}$ *(resp.,* $L_p \circ f : N \to \mathbb{R}^{2n}$*) is stable.*

Remark that the dimension of the target space in Corollary 9 is smaller than that in Corollary 7.

7. Appendix.

In this section, the main theorems in [**4**] and [**10**] are stated. For this, we prepare some notions.

Let N and P be manifolds. Let ${}_sJ^r(N,P)$ be the space consisting of elements $(j^rg(q_1), j^rg(q_2), \ldots, j^rg(q_s)) \in J^r(N,P)^s$ satisfying $(q_1, q_2, \ldots, q_s) \in N^{(s)}$. Since $N^{(s)}$ is an open submanifold of N^s, the space ${}_sJ^r(N,P)$ is also an open submanifold of $J^r(N,P)^s$. For a given mapping $g : N \to P$, the mapping ${}_sj^rg : N^{(s)} \to {}_sJ^r(N,P)$ is defined by $(q_1, q_2, \ldots, q_s) \mapsto (j^rg(q_1), j^rg(q_2), \ldots, j^rg(q_s))$.

Let W be a submanifold of ${}_sJ^r(N,P)$. A mapping $g : N \to P$ will be said to be *transverse with respect to* W if ${}_sj^rg : N^{(s)} \to {}_sJ^r(N,P)$ is transverse to W.

Following Mather ([**10**]), we can partition P^s as follows. Given any partition Π of $\{1, 2, \ldots, s\}$, let P^Π denote the set of s-tuples $(y_1, y_2, \ldots, y_s) \in P^s$ such that $y_i = y_j$ if and only if the two positive integers i and j are in the same member of the partition Π.

Let Diff N denote the group of diffeomorphisms of N. We have the natural action of Diff $N \times$ Diff P on ${}_sJ^r(N,P)$ such that for a mapping $g : N \to P$, the equality $(h,H) \cdot {}_sj^rg(q) = {}_sj^r(H \circ g \circ h^{-1})(q')$ holds, where $q = (q_1, q_2, \ldots, q_s)$ and $q' = (h(q_1), h(q_2), \ldots, h(q_s))$. A subset W of ${}_sJ^r(N,P)$ is said to be *invariant* if it is invariant under this action.

We recall the following identification (7.1) from [**10**]. For $q = (q_1, q_2, \ldots, q_s) \in N^{(s)}$, let $g : U \to P$ be a mapping defined in a neighborhood U of $\{q_1, q_2, \ldots, q_s\}$ in N, and let $z = {}_sj^rg(q)$, $q' = (g(q_1), g(q_2), \ldots, g(q_s))$. Let ${}_sJ^r(N,P)_q$ and ${}_sJ^r(N,P)_{q,q'}$ denote the fibers of ${}_sJ^r(N,P)$ over q and over (q,q') respectively. Let $J^r(N)_q$ denote the $\mathbb{R}$-algebra of r-jets at q of functions on N. Namely,

$$J^r(N)_q = {}_sJ^r(N,\mathbb{R})_q.$$

Set $g^*TP = \bigcup_{\widetilde{q}\in U} T_{g(\widetilde{q})}P$, where TP is the tangent bundle of P. Let $J^r(g^*TP)_q$ denote the $J^r(N)_q$-module of r-jets at q of sections of the bundle g^*TP. Let $\mathfrak{m}_q$ be the ideal in $J^r(N)_q$ consisting of jets of functions which vanish at q. Namely,

$$\mathfrak{m}_q = \{{}_sj^rh(q) \in {}_sJ^r(N,\mathbb{R})_q \mid h(q_1) = h(q_2) = \cdots = h(q_s) = 0\}.$$

Let $\mathfrak{m}_qJ^r(g^*TP)_q$ be the set consisting of finite sums of products of an element of $\mathfrak{m}_q$ and an element of $J^r(g^*TP)_q$. Namely, we set

$$\mathfrak{m}_qJ^r(g^*TP)_q = J^r(g^*TP)_q \cap \{{}_sj^r\xi(q) \in {}_sJ^r(N,TP)_q \mid \xi(q_1) = \xi(q_2) = \cdots = \xi(q_s) = 0\}.$$

Then, it is easily seen that we have the following canonical identification of $\mathbb{R}$-vector spaces:

$$T({}_sJ^r(N,P)_{q,q'})_z = \mathfrak{m}_qJ^r(g^*TP)_q. \tag{7.1}$$

Let W be a non-empty submanifold of ${}_sJ^r(N,P)$. Choose $q = (q_1, q_2, \ldots, q_s) \in N^{(s)}$

and $g : N \to P$, and set $z = {}_sj^rg(q)$ and $q' = (g(q_1), g(q_2), \ldots, g(q_s))$. Suppose that the choice is made so that $z \in W$. Set $W_{q,q'} = \widetilde{\pi}^{-1}(q, q')$, where $\widetilde{\pi} : W \to N^{(s)} \times P^s$ is defined by $\widetilde{\pi}({}_sj^r\widetilde{g}(\widetilde{q})) = (\widetilde{q}, (\widetilde{g}(\widetilde{q}_1), \widetilde{g}(\widetilde{q}_2), \ldots, \widetilde{g}(\widetilde{q}_s)))$ and $\widetilde{q} = (\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_s) \in N^{(s)}$.

Then, under the identification (7.1), the tangent space $T(W_{q,q'})_z$ can be identified with a vector subspace of $\mathfrak{m}_qJ^r(g^*TP)_q$. We denote this vector subspace by $E(g, q, W)$.

Definition 3. The submanifold W is said to be *modular* if conditions (α) and (β) below are satisfied.

(α) The set W is an invariant submanifold of ${}_sJ^r(N, P)$, and lies over P^Π for some partition Π of $\{1, 2, \ldots, s\}$.

(β) For any $q \in N^{(s)}$ and any mapping $g : N \to P$ such that ${}_sj^rg(q) \in W$, the subspace $E(g, q, W)$ is a $J^r(N)_q$-submodule.

Now, suppose that $P = \mathbb{R}^\ell$. The main theorem in [**10**] is the following.

Theorem 3 ([**10**]). *Let N be a manifold of dimension n. Let f be an embedding of N into $\mathbb{R}^m$. If W is a modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$ and $m > \ell$, then there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

Then, the main theorem in [**4**] is the following.

Theorem 4 ([**4**]). *Let N be a manifold of dimension n. Let f be an embedding of N into an open subset U of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If W is a modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$, then there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

The assertion (6) in Section 1, Corollary 7 in Section 5 and Corollary 9 in Section 6 of the present paper are obtained as corollaries of Theorems 1 and 2 in this paper. On the other hand, they are also corollaries of Theorem 4.

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