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Derived equivalence of Ito–Miura–Okawa–Ueda Calabi–Yau 3-folds

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Abstract. We prove derived equivalence of Calabi–Yau threefolds constructed by Ito–Miura–Okawa–Ueda as an example of non-birational Calabi–Yau varieties whose difference in the Grothendieck ring of varieties is annihilated by the affine line.

In a recent paper [**IMOU**] there was constructed a pair of Calabi–Yau threefolds X and Y such that their classes $[X]$ and $[Y]$ in the Grothendieck group of varieties are different, but

$$([X] - [Y])[\mathbb{A}^1] = 0.$$

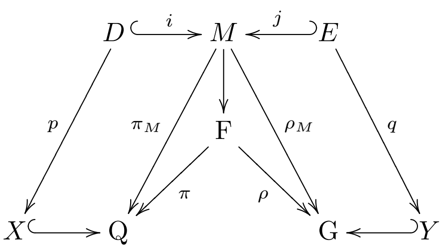
The goal of this short note is to show that these threefolds are derived equivalent

$$\mathbf{D}(X) \cong \mathbf{D}(Y).$$

In course of proof we will construct an explicit equivalence of the categories.

We denote by $\boldsymbol{k}$ the base field. All the functors between triangulated categories are implicitly derived.

As explained in [**IMOU**] the threefolds X and Y are related by the following diagram



Here

- F is the flag variety of the simple algebraic group of type $\boldsymbol{G}_2$,
- Q and G are the Grassmannians of this group:

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  - Q is a 5-dimensional quadric in $\mathbb{P}(V)$, where V is the 7-dimensional fundamental representation, and
  - $\mathrm{G} = \mathrm{Gr}(2, V) \cap \mathbb{P}(W)$, where $W \subset \bigwedge^2 V$ is the 14-dimensional adjoint representation (this intersection is not dimensionally transverse!),

- $\pi\colon \mathrm{F} \to \mathrm{Q}$ and $\rho\colon \mathrm{F} \to \mathrm{G}$ are Zariski locally trivial $\mathbb{P}^1$-fibrations,
- M is a smooth half-anticanonical divisor in F,
- $\pi_M := \pi|_M\colon M \to \mathrm{Q}$ is the blowup with center in the Calabi–Yau threefold X,
- $\rho_M := \rho|_M\colon M \to \mathrm{G}$ is the blowup with center in the Calabi–Yau threefold Y,
- D and E are the exceptional divisors of the blowups,
- $p := \pi|_D\colon D \to X$ and $q := \rho|_E\colon E \to Y$ are the contractions.

We denote by h and H the hyperplane classes of Q and G, as well as their pullbacks to F and M. Then h and H form a basis of Pic(F) in which the canonical classes can be expressed as follows:

$$K_{\mathrm{Q}} = -5h, \qquad K_{\mathrm{G}} = -3H, \qquad K_{\mathrm{F}} = -2H - 2h, \qquad K_M = -H - h. \tag{1}$$

The classes h and H are relative hyperplane classes for the $\mathbb{P}^1$-fibrations $\rho\colon \mathrm{F} \to \mathrm{G}$ and $\pi\colon \mathrm{F} \to \mathrm{Q}$ respectively. We define rank 2 vector bundles $\mathscr{K}$ and $\mathscr{U}$ on Q and G respectively by

$$\pi_*\mathscr{O}_F(H) \cong \mathscr{K}^\vee, \qquad \rho_*\mathscr{O}_F(h) \cong \mathscr{U}^\vee. \tag{2}$$

We also denote the pullbacks of $\mathscr{K}$ and $\mathscr{U}$ to F and M by the same symbols. Then

$$\mathbb{P}_{\mathrm{Q}}(\mathscr{K}) \cong \mathrm{F} \cong \mathbb{P}_{\mathrm{G}}(\mathscr{U}).$$

It follows from (2) that $X \subset \mathrm{Q}$ is the zero locus of a section of the vector bundle $\mathscr{K}^\vee(h)$ on Q and $Y \subset \mathrm{G}$ is the zero locus of a section of the vector bundle $\mathscr{U}^\vee(H)$ on G.

Since H and h are relative hyperplane classes for $\mathrm{F} = \mathbb{P}_{\mathrm{Q}}(\mathscr{K})$ and $\mathrm{F} = \mathbb{P}_{\mathrm{G}}(\mathscr{U})$ respectively, we have on F exact sequences

$$0 \to \omega_{\mathrm{F}/\mathrm{Q}} \to \mathscr{K}^\vee(-H) \to \mathscr{O}_{\mathrm{F}} \to 0, \qquad 0 \to \omega_{\mathrm{F}/\mathrm{G}} \to \mathscr{U}^\vee(-h) \to \mathscr{O}_{\mathrm{F}} \to 0.$$

By (1) we have $\omega_{\mathrm{F}/\mathrm{Q}} \cong \mathscr{O}_{\mathrm{F}}(3h - 2H)$ and $\omega_{\mathrm{F}/\mathrm{G}} \cong \mathscr{O}_{\mathrm{F}}(H - 2h)$. Taking the determinants of the above sequences and dualizing, we deduce

$$\det(\mathscr{K}) \cong \mathscr{O}_{\mathrm{Q}}(-3h), \qquad \det(\mathscr{U}) \cong \mathscr{O}_{\mathrm{G}}(-H). \tag{3}$$

Furthermore, twisting the sequences by $\mathscr{O}_{\mathrm{F}}(H)$ and $\mathscr{O}_{\mathrm{F}}(h)$ respectively, we obtain

$$0 \to \mathscr{O}_{\mathrm{F}}(3h - H) \to \mathscr{K}^\vee \to \mathscr{O}_{\mathrm{F}}(H) \to 0, \tag{4}$$

and

$$0 \to \mathscr{O}_{\mathrm{F}}(H-h) \to \mathscr{U}^\vee \to \mathscr{O}_{\mathrm{F}}(h) \to 0. \tag{5}$$

Derived categories of both Q and G are known to be generated by exceptional collections. In fact, for our purposes the most convenient collections are

$$\mathbf{D}(\mathrm{Q}) = \langle \mathscr{O}_{\mathrm{Q}}(-3h), \mathscr{O}_{\mathrm{Q}}(-2h), \mathscr{O}_{\mathrm{Q}}(-h), \mathscr{S}, \mathscr{O}_{\mathrm{Q}}, \mathscr{O}_{\mathrm{Q}}(h) \rangle, \tag{6}$$

where $\mathscr{S}$ is the spinor vector bundle of rank 4, see [**Kap**], and

$$\mathbf{D}(\mathrm{G}) = \langle \mathscr{O}_{\mathrm{G}}(-H), \mathscr{U}, \mathscr{O}_{\mathrm{G}}, \mathscr{U}^\vee, \mathscr{O}_{\mathrm{G}}(H), \mathscr{U}^\vee(H) \rangle. \tag{7}$$

This collection is obtained from the collection of [**Kuz**, Section 6.4] by a twist (note that $\mathscr{U} \cong \mathscr{U}^\vee(-H)$ by (3)). In fact, for the argument below one even does not need to know that this exceptional collection is full; on a contrary, one can use the argument to prove its fullness, see Remark 6.

Using two blowup representations of M and the corresponding semiorthogonal decompositions

$$\langle \pi_M^*(\mathbf{D}(\mathrm{Q})), i_* p^*(\mathbf{D}(X)) \rangle = \mathbf{D}(M) = \langle \rho_M^*(\mathbf{D}(\mathrm{G})), j_* q^*(\mathbf{D}(Y)) \rangle \tag{8}$$

together with the above exceptional collections, we see that $\mathbf{D}(X)$ and $\mathbf{D}(Y)$ are the complements in $\mathbf{D}(M)$ of exceptional collections of length 6, so one can guess they are equivalent. Below we show that this is the case by constructing a sequence of mutations transforming one exceptional collection to the other.

We start with some cohomology computations:

Lemma 1. (i) *Line bundles* $\mathscr{O}_{\mathrm{F}}(th-H)$ *and* $\mathscr{O}_{\mathrm{F}}(tH-h)$ *are acyclic for all* $t \in \mathbb{Z}$.

(ii) *Line bundles* $\mathscr{O}_{\mathrm{F}}(-2H)$ *and* $\mathscr{O}_{\mathrm{F}}(2h-2H)$ *are acyclic and*

$$H^\bullet(\mathrm{F}, \mathscr{O}_{\mathrm{F}}(3h-2H)) = \boldsymbol{k}[-1].$$

(iii) *Vector bundles* $\mathscr{U}(-2H)$, $\mathscr{U}(-H)$, $\mathscr{U}(h-H)$, *and* $\mathscr{U} \otimes \mathscr{U}(-H)$ *on* F *are acyclic and*

$$H^\bullet(\mathrm{F}, \mathscr{U}(h)) = \boldsymbol{k}, \qquad H^\bullet(\mathrm{F}, \mathscr{U} \otimes \mathscr{U}(h)) \cong \boldsymbol{k}[-1].$$

Proof. Part (i) is easy since $\pi_* \mathscr{O}_{\mathrm{F}}(-H) = 0$ and $\rho_* \mathscr{O}_{\mathrm{F}}(-h) = 0$. For part (ii) we note that

$$\pi_* \mathscr{O}_{\mathrm{F}}(-2H) \cong (\det \mathscr{K})[-1] \cong \mathscr{O}_{\mathrm{Q}}(-3h)[-1], \tag{9}$$

so acyclicity of $\mathscr{O}_{\mathrm{F}}(-2H)$ and $\mathscr{O}_{\mathrm{F}}(2h-2H)$ and the formula for the cohomology of $\mathscr{O}_{\mathrm{F}}(3h-2H)$ follow. For part (iii) we push forward the bundles $\mathscr{U}(-2H)$, $\mathscr{U}(-H)$, $\mathscr{U}(h-H)$, and $\mathscr{U} \otimes \mathscr{U}(-H)$ to G and applying (2) we obtain

$$\mathscr{U}(-2H),\ \mathscr{U}(-H),\ \mathscr{U} \otimes \mathscr{U}^\vee(-H),\ \mathscr{U} \otimes \mathscr{U}(-H).$$

Their acyclicity follows from orthogonality of $\mathscr{U}^\vee(H)$ to the collection $(\mathscr{O}_{\mathrm{G}}(-H), \mathscr{U}, \mathscr{O}_{\mathrm{Q}}, \mathscr{U}^\vee)$ in view of the exceptional collection (7). Analogously, pushing forward $\mathscr{U}(h)$ to G we obtain $\mathscr{U} \otimes \mathscr{U}^\vee$, and its cohomology is $\boldsymbol{k}$ since $\mathscr{U}$ is exceptional. Finally, using (5) we see that $\mathscr{U} \otimes \mathscr{U}(h)$ has a filtration with factors $\mathscr{O}_{\mathrm{F}}(-h)$, $\mathscr{O}_{\mathrm{F}}(h - H)$, and $\mathscr{O}_{\mathrm{F}}(3h - 2H)$. The first two are acyclic by part (i) and the last one has cohomology $\boldsymbol{k}[-1]$ by part (ii). It follows that the cohomology of $\mathscr{U} \otimes \mathscr{U}(h)$ is also $\boldsymbol{k}[-1]$. □

COROLLARY 2. *The following line and vector bundles are acyclic on* M:

$$\mathscr{O}_M(h - H),\ \mathscr{O}_M(3h - H),\ \mathscr{U}(h - H).$$

Moreover,

$$H^\bullet(M, \mathscr{U}(h)) = \boldsymbol{k}, \qquad H^\bullet(M, \mathscr{U} \otimes \mathscr{U}(h)) = \boldsymbol{k}[-1].$$

PROOF. Since $M \subset \mathrm{F}$ is a divisor with class $h + H$ we have a resolution

$$0 \to \mathscr{O}_{\mathrm{F}}(-h - H) \to \mathscr{O}_{\mathrm{F}} \to \mathscr{O}_M \to 0.$$

Tensoring it with the required bundles and using Lemma 1 we obtain the required results. □

PROPOSITION 3. *We have an exact sequence on* F *and* M:

$$0 \to \mathscr{U} \to \mathscr{S}' \to \mathscr{U}^\vee(-h) \to 0, \tag{10}$$

where $\mathscr{S}'$ *is* (*the pullback to* F *or* M *of*) *a rank* 4 *vector bundle on* Q.

Later we will identify the bundle $\mathscr{S}'$ constructed as extension (10) with the spinor bundle $\mathscr{S}$ on Q.

PROOF. We will construct this exact sequence on F, and then restrict it to M. First, note that by Lemma 1 we have $\operatorname{Ext}^\bullet(\mathscr{U}^\vee(-h), \mathscr{U}) \cong H^\bullet(\mathrm{F}, \mathscr{U} \otimes \mathscr{U}(h)) \cong \boldsymbol{k}[-1]$, hence there is a canonical extension of $\mathscr{U}^\vee(-h)$ by $\mathscr{U}$. We denote by $\mathscr{S}'$ the extension, so that we have an exact sequence (10). Obviously, $\mathscr{S}'$ is locally free of rank 4. We have to check that it is a pullback from Q.

Using exact sequences

$$0 \to \mathscr{O}_{\mathrm{F}}(-h) \to \mathscr{U} \to \mathscr{O}_{\mathrm{F}}(h - H) \to 0 \quad \text{and} \quad 0 \to \mathscr{O}_{\mathrm{F}}(H - 2h) \to \mathscr{U}^\vee(-h) \to \mathscr{O}_{\mathrm{F}} \to 0$$

(obtained from (5) by the dualization and a twist) and the cohomology computations of Lemma 1, we see that extension (10) is induced by a class in

$$\operatorname{Ext}^1(\mathscr{O}_{\mathrm{F}}(H - 2h), \mathscr{O}_{\mathrm{F}}(h - H)) \cong H^\bullet(\mathrm{F}, \mathscr{O}_{\mathrm{F}}(3h - 2H)) = \boldsymbol{k}[-1].$$

By (4) the corresponding extension is $\mathscr{K}^\vee(-2h)$. It follows that the sheaf $\mathscr{S}'$ has a 3-step filtration with factors being $\mathscr{O}_{\mathrm{F}}(-h)$, $\mathscr{K}^\vee(-h)$, and $\mathscr{O}_{\mathrm{F}}$. All these sheaves are pullbacks from Q, and since the subcategory $\pi^*(\mathbf{D}(\mathrm{Q})) \subset \mathbf{D}(\mathrm{F})$ is triangulated (because the functor π^* is fully faithful), it follows that $\mathscr{S}'$ is also a pullback from Q. □

Now we are ready to explain the mutations. We start with a semiorthogonal decomposition

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-H), \mathscr{U}, \mathscr{O}_M, \mathscr{U}^\vee, \mathscr{O}_M(H), \mathscr{U}^\vee(H), \Phi_0(\mathbf{D}(Y))\rangle, \tag{11}$$

$$\Phi_0 = j_* \circ q^* \colon \mathbf{D}(Y) \to \mathbf{D}(M), \tag{12}$$

obtained by plugging (7) into the right hand side of (8). Now we apply a sequence of mutations, modifying the functor Φ_0.

First, we mutate $\Phi_0(\mathbf{D}(Y))$ two steps to the left:

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-H), \mathscr{U}, \mathscr{O}_M, \mathscr{U}^\vee, \Phi_1(\mathbf{D}(Y)), \mathscr{O}_M(H), \mathscr{U}^\vee(H)\rangle, \tag{13}$$

$$\Phi_1 = \mathbf{L}_{\langle \mathscr{O}_M(H), \mathscr{U}^\vee(H)\rangle} \circ \Phi_0. \tag{14}$$

Here $\mathbf{L}$ denotes the left mutation functor.

Next, we mutate the last two terms to the far left (these objects got twisted by $K_M = -h - H$):

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-h), \mathscr{U}^\vee(-h), \mathscr{O}_M(-H), \mathscr{U}, \mathscr{O}_M, \mathscr{U}^\vee, \Phi_1(\mathbf{D}(Y))\rangle.$$

Next, we mutate $\mathscr{O}_M(-h)$ and $\mathscr{U}^\vee(-h)$ one step to the right. As

$$\operatorname{Ext}^\bullet(\mathscr{U}^\vee(-h), \mathscr{O}_M(-H)) \cong H^\bullet(M, \mathscr{U}(h-H)) = 0,$$

and

$$\operatorname{Ext}^\bullet(\mathscr{O}_M(-h), \mathscr{O}_M(-H)) \cong H^\bullet(M, \mathscr{O}_M(h-H)) = 0$$

by Corollary 2, we obtain

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-H), \mathscr{O}_M(-h), \mathscr{U}^\vee(-h), \mathscr{U}, \mathscr{O}_M, \mathscr{U}^\vee, \Phi_1(\mathbf{D}(Y))\rangle.$$

Next, we mutate $\mathscr{U}$ one step to the left. As

$$\operatorname{Ext}^\bullet(\mathscr{U}^\vee(-h), \mathscr{U}) \cong H^\bullet(\mathscr{U} \otimes \mathscr{U}(h)) \cong \boldsymbol{k}[-1]$$

by Corollary 2, the resulting mutation is an extension, which in view of (10) gives $\mathscr{S}'$. Thus, we obtain

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-H), \mathscr{O}_M(-h), \mathscr{S}', \mathscr{U}^\vee(-h), \mathscr{O}_M, \mathscr{U}^\vee, \Phi_1(\mathbf{D}(Y))\rangle.$$

Next, we mutate $\mathscr{O}_M(-H)$ to the far right (this object got twisted by $-K_M = h+H$):

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-h), \mathscr{S}', \mathscr{U}^\vee(-h), \mathscr{O}_M, \mathscr{U}^\vee, \Phi_1(\mathbf{D}(Y)), \mathscr{O}_M(h)\rangle.$$

Next, we mutate $\Phi_1(\mathbf{D}(Y))$ one step to the right:

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-h), \mathscr{S}', \mathscr{U}^\vee(-h), \mathscr{O}_M, \mathscr{U}^\vee, \mathscr{O}_M(h), \Phi_2(\mathbf{D}(Y))\rangle,$$

$$\Phi_2 = \mathbf{R}_{\mathscr{O}_M(h)} \circ \Phi_1. \tag{15}$$

Here $\mathbf{R}$ denotes the right mutation functor.

Next, we mutate simultaneously $\mathscr{U}^\vee(-h)$ and $\mathscr{U}^\vee$ one step to the right. As $\operatorname{Ext}^\bullet(\mathscr{U}^\vee(-h), \mathscr{O}_M) \cong \operatorname{Ext}^\bullet(\mathscr{U}^\vee, \mathscr{O}_M(h)) = H^\bullet(M, \mathscr{U}(h)) = \boldsymbol{k}$ by Corollary 2, the resulting mutation is the cone of a morphism, which in view of (5) and its twist by $\mathscr{O}_M(-h)$ gives $\mathscr{O}_M(H-2h)$ and $\mathscr{O}_M(H-h)$ respectively. Thus we obtain

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-h), \mathscr{S}', \mathscr{O}_M, \mathscr{O}_M(H-2h), \mathscr{O}_M(h), \mathscr{O}_M(H-h), \Phi_2(\mathbf{D}(Y))\rangle.$$

Next, we mutate $\mathscr{O}_M(h)$ one step to the left. As

$$\operatorname{Ext}^\bullet(\mathscr{O}_M(H-2h), \mathscr{O}_M(h)) \cong H^\bullet(M, \mathscr{O}_M(3h-H)) = 0$$

by Corollary 2, we obtain

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-h), \mathscr{S}', \mathscr{O}_M, \mathscr{O}_M(h), \mathscr{O}_M(H-2h), \mathscr{O}_M(H-h), \Phi_2(\mathbf{D}(Y))\rangle.$$

Next, we mutate $\Phi_2(\mathbf{D}(Y))$ two steps to the left:

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-h), \mathscr{S}', \mathscr{O}_M, \mathscr{O}_M(h), \Phi_3(\mathbf{D}(Y)), \mathscr{O}_M(H-2h), \mathscr{O}_M(H-h)\rangle,$$
$$\Phi_3 = \mathbf{L}_{\langle \mathscr{O}_M(H-2h), \mathscr{O}_M(H-h)\rangle} \circ \Phi_2. \tag{16}$$

Finally, we mutate $\mathscr{O}_M(H-2h)$ and $\mathscr{O}_M(H-h)$ to the far left:

$$\mathbf{D}(M) = \langle \mathscr{O}_M(-3h), \mathscr{O}_M(-2h), \mathscr{O}_M(-h), \mathscr{S}', \mathscr{O}_M, \mathscr{O}_M(h), \Phi_3(\mathbf{D}(Y))\rangle. \tag{17}$$

Now we finished with mutations, and it remains to check that the resulting semiorthogonal decomposition provides an equivalence of categories. To do this, we first observe the following

Lemma 4. *The bundle $\mathscr{S}'$ is isomorphic to the spinor bundle $\mathscr{S}$ on* Q.

Proof. The first six objects in (17) are pullbacks from Q by π_M. Since π_M^* is fully faithful, the corresponding objects on Q are also semiorthogonal. In particular, the bundle $\mathscr{S}'$ on Q is right orthogonal to $\mathscr{O}_{\mathrm{Q}}$ and $\mathscr{O}_{\mathrm{Q}}(h)$ and left orthogonal to $\mathscr{O}_{\mathrm{Q}}(-3h)$, $\mathscr{O}_{\mathrm{Q}}(-2h)$, and $\mathscr{O}_{\mathrm{Q}}(-h)$. By (6) the intersection of these orthogonals is generated by the spinor bundle $\mathscr{S}$. Therefore, $\mathscr{S}'$ is a multiple of the spinor bundle $\mathscr{S}$. Since the ranks of both $\mathscr{S}'$ and $\mathscr{S}$ are 4, the multiplicity is 1, so $\mathscr{S}' \cong \mathscr{S}$. □

Thus the first six objects of (17) generate $\pi_M^*(\mathbf{D}(\mathrm{Q}))$. Comparing (17) with (6) and (8), we conclude that the last component $\Phi_3(\mathbf{D}(Y))$ coincides with $i_* p^*(\mathbf{D}(X))$. Altogether, this proves the following

Theorem 5. *The functor*

$$\Phi_3 = \mathbf{L}_{\langle \mathscr{O}(H-2h), \mathscr{O}(H-h)\rangle} \circ \mathbf{R}_{\mathscr{O}(h)} \circ \mathbf{L}_{\langle \mathscr{O}(H), \mathscr{U}^\vee(H)\rangle} \circ j_* \circ q^* \colon \mathbf{D}(Y) \to \mathbf{D}(M)$$

is an equivalence of $\mathbf{D}(Y)$ *onto the triangulated subcategory of* $\mathbf{D}(M)$ *equivalent to* $\mathbf{D}(X)$ *via the embedding* $i_* \circ p^* \colon \mathbf{D}(X) \to \mathbf{D}(M)$*. In particular, the functor*

$$\Psi = p_* \circ i^! \circ \mathbf{L}_{\langle \mathscr{O}(H-2h),\mathscr{O}(H-h)\rangle} \circ \mathbf{R}_{\mathscr{O}(h)} \circ \mathbf{L}_{\langle \mathscr{O}(H),\mathscr{U}^\vee(H)\rangle} \circ j_* \circ q^* \colon \mathbf{D}(Y) \to \mathbf{D}(X)$$

is an equivalence of categories.

Remark 6. Let us sketch how the arguments above can be also used to prove fullness of (7). Denote by $\mathscr{C}$ the orthogonal to the collection (7) in $\mathbf{D}(\mathrm{G})$. Then we still have a semiorthogonal decomposition (11), with $\Phi_0(\mathbf{D}(Y))$ replaced by $\langle \mathscr{C}, \Phi_0(\mathbf{D}(Y))\rangle$. We can perform the same sequence of mutations, keeping the subcategory $\mathscr{C}$ together with $\mathbf{D}(Y)$. For instance, in (13) we write $\langle \mathbf{L}_{\langle \mathscr{O}_M(H),\mathscr{U}^\vee(H)\rangle}(\mathscr{C}), \Phi_1(\mathbf{D}(Y))\rangle$ instead of just $\Phi_1(\mathbf{D}(Y))$ and so on. In the end, we arrive at (17) with $\Phi_3(\mathbf{D}(Y))$ replaced by $\langle \mathscr{C}', \Phi_3(\mathbf{D}(Y))\rangle$ with $\mathscr{C}'$ equivalent to $\mathscr{C}$. Comparing it with (6) and (8), we deduce that $\mathbf{D}(X)$ has a semiorthogonal decomposition with two components equivalent to $\mathscr{C}$ and $\mathbf{D}(Y)$. But X is a Calabi–Yau variety, hence its derived category has no nontrivial semiorthogonal decompositions by [**Bri**]. Therefore $\mathscr{C} = 0$ and so exceptional collection (7) is full.

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