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Spectrum for compact operators on Banach spaces

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Abstract. For a two-sided sequence of compact linear operators acting on a Banach space, we consider the notion of spectrum defined in terms of the existence of exponential dichotomies under homotheties of the dynamics. This can be seen as a natural generalization of the spectrum of a matrix—the set of its eigenvalues. We give a characterization of all possible spectra and explicit examples of sequences for which the spectrum takes a form not occurring in finite-dimensional spaces. We also consider the case of a one-sided sequence of compact linear operators.

1. Introduction.

We study a notion of spectrum for a nonautonomous dynamics inspired on a corresponding notion introduced by Sacker and Sell in [**9**] in a finite-dimensional space. This can be seen as a generalization of the spectrum of a matrix (the set of its eigenvalues). More precisely, let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of invertible $d\times d$ matrices and consider the dynamics

$$x_{m+1}=A_m x_m, \quad m\in\mathbb{Z}.$$

For a constant sequence $A_m=A$, a number $a\in\mathbb{R}$ is of the form $a=-\log|\mu|$ for some eigenvalue μ of A if and only if the constant sequence $(e^{-a}A)_{m\in\mathbb{Z}}$ admits an exponential dichotomy. For an arbitrary sequence of matrices, an appropriate notion of spectrum was introduction by Aulbach and Siegmund in [**1**] (following work in [**11**] in the case of continuous time) by declaring that a is in the spectrum if the sequence $(e^{-a}A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy. For related work we refer the reader to [**2**], [**3**], [**8**], [**10**], [**12**].

We consider a corresponding notion in an infinite-dimensional setting. Namely, given a sequence $(A_m)_{m\in\mathbb{Z}}$ of compact linear operators acting on a Banach space, we define its *spectrum* as the set Σ of all $a\in\mathbb{R}$ such that the sequence $(e^{-a}A_m)_{m\in\mathbb{Z}}$ does not admit an exponential dichotomy. We emphasize that in the notion of an exponential dichotomy we only assume the dynamics to be invertible along the unstable direction.

Our main result describes all possible types of spectra and how they relate to certain natural invariant subspaces (see Theorem 3). In particular, we show that each trajectory

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has lower and upper Lyapunov exponents inside the same connected component of the spectrum. We note that this mimics the behavior in the multiplicative ergodic theorem in the particular case of Lyapunov regular trajectories, which is the typical behavior for example in all mechanical systems on a compact energy hypersurface.

Moreover, we provide examples of sequences for which the spectrum takes a form that does not occur in finite-dimensional spaces. More precisely, we provide explicit examples of sequences of compact linear operators acting on the Hilbert space of L^2 functions on the circle with the induced Lebesgue measure for which the spectrum is of the form

$$\Sigma = \bigcup_{j=1}^\infty [a_j, b_j] \quad \text{or} \quad \Sigma = \bigcup_{j=1}^\infty [a_j, b_j] \cup (-\infty, b_\infty],$$

for some numbers

$$b_1 \ge a_1 > b_2 \ge a_2 > b_3 \ge a_3 > \cdots,$$

respectively, with $\lim_{n\to+\infty} a_n = -\infty$ or $\lim_{n\to+\infty} a_n = b_\infty > -\infty$.

Finally, we consider briefly the case of a one-sided sequence of compact linear operators and we describe corresponding notions and results.

2. Spectrum on the line.

2.1. Exponential dichotomies.

Let $(A_n)_{n\in\mathbb{Z}}$ be a sequence of compact linear operators acting on a Banach space $X = (X, \|\cdot\|)$. For each $m \ge n$, let

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n, & m > n, \\ \mathrm{Id}, & m = n. \end{cases} \tag{1}$$

We say that $(A_n)_{n\in\mathbb{Z}}$ admits an *exponential dichotomy* if:

1. there exist projections $P_n \colon X \to X$ for $n \in \mathbb{Z}$ satisfying

$$A_n P_n = P_{n+1} A_n \tag{2}$$

   for $n \in \mathbb{Z}$ such that each map

$$A_n|\operatorname{Ker} P_n \colon \operatorname{Ker} P_n \to \operatorname{Ker} P_{n+1} \tag{3}$$

   is invertible;

2. there exist constants $D, \lambda > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)} \quad \text{for } m \ge n \tag{4}$$

   and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(n-m)} \quad \text{for } m \le n, \tag{5}$$

where $Q_n = \mathrm{Id} - P_n$ and

$$\mathcal{A}(m,n) = (\mathcal{A}(n,m)|\operatorname{Ker} P_m)^{-1} \colon \operatorname{Ker} P_n \to \operatorname{Ker} P_m \tag{6}$$

for $m < n$.

The sets $\operatorname{Im} P_n$ and $\operatorname{Im} Q_n$ are called, respectively, the *stable* and *unstable spaces* of the exponential dichotomy. The following result shows in particular that they are uniquely determined.

Proposition 1. *For each $n \in \mathbb{Z}$,*

$$\operatorname{Im} P_n = \left\{ v \in X : \sup_{m \ge n} \|\mathcal{A}(m,n)v\| < +\infty \right\}$$

and $\operatorname{Im} Q_n$ consists of all $v \in X$ for which there exists a sequence $(x_m)_{m \le n} \subset X$ such that $x_n = v$, $x_m = A_{m-1}x_{m-1}$ for $m \le n$ and $\sup_{m \le n}\|x_m\| < +\infty$. Moreover, $\dim \operatorname{Im} Q_n < +\infty$ for $n \in \mathbb{Z}$.

Proof. By (4) we have

$$\sup_{m \ge n} \|\mathcal{A}(m,n)v\| < +\infty \tag{7}$$

for $v \in \operatorname{Im} P_n$. Conversely, if (7) holds for some $v \in X$, then it follows from (4) that

$$\sup_{m \ge n} \|\mathcal{A}(m,n)Q_n v\| < +\infty. \tag{8}$$

By (5), for $m \ge n$ we have

$$\|Q_n v\| \le De^{-\lambda(m-n)} \|\mathcal{A}(m,n)Q_n v\|,$$

that is,

$$\frac{1}{D} e^{\lambda(m-n)} \|Q_n v\| \le \|\mathcal{A}(m,n)Q_n v\|.$$

Whenever $Q_n v \neq 0$, we obtain

$$\sup_{m \ge n} \|\mathcal{A}(m,n)Q_n v\| = +\infty,$$

which contradicts to (8). Hence, $Q_n v = 0$ and so $v \in \operatorname{Im} P_n$.

Now take $v \in \operatorname{Im} Q_n$ and consider the sequence $x_m = \mathcal{A}(m,n)v$ for $m \le n$. Clearly, $x_n = v$ and $x_m = A_{m-1}x_{m-1}$ for $m \le n$. Moreover, it follows from (5) that $\sup_{m \le n}\|x_m\| < +\infty$. Conversely, one can show that there is no $v \in X \setminus \operatorname{Im} Q_n$ for which there exists a sequence $(x_m)_{m \le n} \subset X$ with the properties in the proposition. Indeed, it follows from (2) and (4) that

$$\|P_n v\| = \|\mathcal{A}(n,m)P_m x_m\| \le De^{-\lambda(n-m)} \|x_m\|$$

for $m \le n$. Hence, if $P_n v \ne 0$, then $\sup_{m\le n}\|x_m\| = +\infty$.

It remains to show that the unstable spaces are finite-dimensional. Let

$$B_n = \{v \in \operatorname{Im} Q_n : \|v\| \le 1\}$$

and take $n \in \mathbb{N}$ such that $e^{\lambda n} > D$. We claim that $B_n \subset \mathcal{A}(n,0)B_0$. For each $v \in B_n$, there exists $x \in \operatorname{Im} Q_0$ such that $v = \mathcal{A}(n,0)x$. If $\|x\| > 1$, then it follows from (5) that

$$1 < \frac{1}{D} e^{\lambda n}\|x\| \le \|\mathcal{A}(n,0)x\| = \|v\|,$$

which contradicts to the assumption that $\|v\| \le 1$. Hence $\|x\| \le 1$ and so $B_n \subset \mathcal{A}(n,0)B_0$. Since B_0 is bounded and $\mathcal{A}(n,0)$ is compact, the set $\mathcal{A}(n,0)B_0$ is relatively compact and so B_n is compact. This shows that $\operatorname{Im} Q_n$ is finite-dimensional. On the other hand, it follows from (2) that the dimensions $\dim \operatorname{Im} Q_m$ are independent of $m \in \mathbb{Z}$. This completes the proof of the proposition. $\square$

2.2. Spectrum.

The *spectrum* of a sequence $(A_n)_{n\in\mathbb{Z}}$ of compact linear operators is the set Σ of all numbers $a \in \mathbb{R}$ such that the sequence $(e^{-a}A_n)_{n\in\mathbb{Z}}$ does not admit an exponential dichotomy. For each $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, let

$$S_a(n) = \left\{ v \in X : \sup_{m\ge n} \left(e^{-a(m-n)}\|\mathcal{A}(m,n)v\|\right) < +\infty \right\}$$

and let $U_a(n)$ be the set of all vectors $v \in X$ for which there exists a sequence $(x_m)_{m\le n} \subset X$ such that $x_n = v$, $x_m = A_{m-1}x_{m-1}$ for $m \le n$ and

$$\sup_{m\le n} \left(e^{-a(m-n)}\|x_m\|\right) < +\infty.$$

We note that if $a < b$, then

$$S_a(n) \subset S_b(n) \quad \text{and} \quad U_b(n) \subset U_a(n)$$

for $n \in \mathbb{Z}$. By Proposition 1, if $a \in \mathbb{R} \setminus \Sigma$, then

$$X = S_a(n) \oplus U_a(n) \quad \text{for } n \in \mathbb{Z}$$

and the projections P_n and Q_n associated to the sequence $(e^{-a}A_n)_{n\in\mathbb{Z}}$ satisfy

$$\operatorname{Im} P_n = S_a(n) \quad \text{and} \quad \operatorname{Im} Q_n = U_a(n).$$

Moreover, by (2), for each $a \in \mathbb{R} \setminus \Sigma$, the numbers $\dim S_a(n)$ and $\dim U_a(n)$ are independent of n. We shall simply denote them by $\dim S_a$ and $\dim U_a$.

Proposition 2. *The following statements hold*:

1. *The set Σ is closed. Moreover, for each $a \in \mathbb{R} \setminus \Sigma$ we have $S_a(n) = S_b(n)$ and $U_a(n) = U_b(n)$ for all $n \in \mathbb{Z}$ and all b in some open neighborhood of a.*

2. *Take* $a_1, a_2 \in \mathbb{R} \setminus \Sigma$ *with* $a_1 < a_2$. *Then* $[a_1, a_2] \cap \Sigma \neq \emptyset$ *if and only if* $\dim U_{a_1} > \dim U_{a_2}$.

3. *For each* $c \notin \Sigma$, *the set* $\Sigma \cap [c, +\infty)$ *consists of finitely many closed intervals.*

PROOF. Given $a \in \mathbb{R} \setminus \Sigma$, there exist projections P_n for $n \in \mathbb{Z}$ satisfying (2) and a constant $\lambda > 0$ such that

$$\|e^{-a(m-n)} \mathcal{A}(m,n) P_n\| \leq D e^{-\lambda(m-n)}$$

for $m \geq n$ and

$$\|e^{-a(m-n)} \mathcal{A}(m,n) Q_n\| \leq D e^{-\lambda(n-m)}$$

for $m \leq n$. Therefore, for each $b \in \mathbb{R}$,

$$\|e^{-b(m-n)} \mathcal{A}(m,n) P_n\| \leq D e^{-(\lambda-a+b)(m-n)}$$

for $m \geq n$ and

$$\|e^{-b(m-n)} \mathcal{A}(m,n) Q_n\| \leq D e^{-(\lambda+a-b)(n-m)}$$

for $m \leq n$. Therefore, $b \in \mathbb{R} \setminus \Sigma$ whenever $|a - b| < \lambda$ and it follows from Proposition 1 that $S_b(n) = S_a(n)$ and $U_b(n) = U_a(n)$ for $n \in \mathbb{Z}$.

For statement 2, assume that $[a_1, a_2] \cap \Sigma \neq \emptyset$. If $\dim U_{a_1} = \dim U_{a_2}$, then $U_{a_1}(n) = U_{a_2}(n)$ and $S_{a_1}(n) = S_{a_2}(n)$ for $n \in \mathbb{Z}$. By Proposition 1, there exist projections P_n for $n \in \mathbb{Z}$ and constants $\lambda_1, \lambda_2 > 0$ such that for $i = 1, 2$ we have

$$\|e^{-a_i(m-n)} \mathcal{A}(m,n) P_n\| \leq D_i e^{-\lambda_i(m-n)} \quad \text{for } m \geq n \tag{9}$$

and

$$\|e^{-a_i(m-n)} \mathcal{A}(m,n) Q_n\| \leq D_i e^{-\lambda_i(n-m)} \quad \text{for } m \leq n. \tag{10}$$

For each $a \in [a_1, a_2]$, by (9),

$$\|e^{-a(m-n)} \mathcal{A}(m,n) P_n\| \leq D_1 e^{-\lambda_1(m-n)} \quad \text{for } m \geq n$$

and similarly, by (10),

$$\|e^{-a(m-n)} \mathcal{A}(m,n) Q_n\| \leq D_2 e^{-\lambda_2(n-m)} \quad \text{for } m \leq n.$$

Taking $\lambda = \min\{\lambda_1, \lambda_2\}$ and $D = \max\{D_1, D_2\}$, we conclude that $[a_1, a_2] \subset \mathbb{R} \setminus \Sigma$, which contradicts to the initial assumption. Therefore, $\dim U_{a_1} = \dim U_{a_2}$. For the converse, assume that $\dim U_{a_1} > \dim U_{a_2}$ and let

$$b = \inf \left\{a \in \mathbb{R} \setminus \Sigma : \dim U_a = \dim U_{a_2}\right\}.$$

Since $\dim U_{a_1} > \dim U_{a_2}$, it follows from statement 1 that $a_1 < b < a_2$. We claim that

$b \in \Sigma$. Otherwise, either $\dim U_b = \dim U_{a_2}$ or $\dim U_b \neq \dim U_{a_2}$. In the first case, by statement 1 there exists $\varepsilon > 0$ such that $\dim U_{b'} = \dim U_{a_2}$ and $b' \in \mathbb{R} \setminus \Sigma$ for $b' \in (b-\varepsilon, b]$. This contradicts to the definition of b. In the second case, by statement 1 there exists $\varepsilon > 0$ such that $\dim U_{b'} \neq \dim U_{a_2}$ and $b' \in \mathbb{R} \setminus \Sigma$ for $b' \in [b, b+\varepsilon)$. Again this contradicts to the definition of b. Hence, $b \in \Sigma$ and $[a_1, a_2] \cap \Sigma \neq \emptyset$.

For statement 3, we proceed by contradiction. Write $\dim U_c = d$ and assume that $\Sigma \cap [c, +\infty)$ contains at least $d+2$ disjoint closed intervals $I_i = [\alpha_i, \beta_i]$, for $i = 1, \ldots, d+2$, where

$$\alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \cdots < \alpha_{d+2} \leq \beta_{d+2} \leq +\infty.$$

For $i = 1, \ldots, d+1$, take $c_i \in (\beta_i, \alpha_{i+1})$. It follows from statement 2 that

$$d > \dim U_{c_1} > \dim U_{c_2} > \cdots > \dim U_{c_{d+1}},$$

which is impossible. This completes the proof of the proposition. $\square$

3. Structure of the spectrum.

Our main result describes all possible forms of the spectrum for a sequence of compact linear operators acting on a Banach space.

Theorem 3. *Let $(A_n)_{n \in \mathbb{Z}}$ be a sequence of compact linear operators for which the spectrum is neither $\emptyset$ nor $\mathbb{R}$.*

1. *One of the following alternatives holds:*
   (a) $\Sigma = I_1 \cup \bigcup_{n=2}^k [a_n, b_n]$, *where $I_1 = [a_1, b_1]$ or $I_1 = [a_1, +\infty)$, for some numbers*
   $$b_1 \geq a_1 > b_2 \geq a_2 > \cdots > b_k \geq a_k; \tag{11}$$
   (b) $\Sigma = I_1 \cup \bigcup_{n=2}^{k-1} [a_n, b_n] \cup (-\infty, b_k]$, *where $I_1 = [a_1, b_1]$ or $I_1 = [a_1, +\infty)$, for some numbers a_n and b_n as in* (11);
   (c) $\Sigma = I_1 \cup \bigcup_{n=2}^{\infty} [a_n, b_n]$, *where $I_1 = [a_1, b_1]$ or $I_1 = [a_1, +\infty)$, for some numbers*
   $$b_1 \geq a_1 > b_2 \geq a_2 > b_3 \geq a_3 > \cdots \tag{12}$$
   and with $\lim_{n \to +\infty} a_n = -\infty$;
   (d) $\Sigma = I_1 \cup \bigcup_{n=2}^{\infty} [a_n, b_n] \cup (-\infty, b_\infty]$, *where $I_1 = [a_1, b_1]$ or $I_1 = [a_1, +\infty)$, for some numbers a_n and b_n as in* (12) *and with $b_\infty := \lim_{n \to +\infty} a_n > -\infty$.*
2. *Let $(c_k)_k$ be a finite or infinite sequence of numbers such that $c_k \in (b_{k+1}, a_k)$ for each k. For each $n \in \mathbb{Z}$ and $v \in S_{c_k}(n) \cap U_{c_{k+1}}(n) \setminus \{0\}$, we have*
   $$\left[\liminf_{m \to +\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\|, \limsup_{m \to +\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\|\right] \subset [a_{k+1}, b_{k+1}].$$
   Moreover, there exists a sequence $(x_m)_{m \leq n} \subset X$ such that $x_n = v$, $x_m = A_{m-1} x_{m-1}$ for $m \leq n$ and

$$\left[\liminf_{m\to-\infty}\frac{1}{m}\log\|x_m\|,\limsup_{m\to-\infty}\frac{1}{m}\log\|x_m\|\right]\subset[a_{k+1},b_{k+1}].$$

PROOF. We first show that Σ has one of the forms in alternatives (a)–(d). Assume first that Σ is not given by alternatives (a)–(b) and take $c_1\notin\Sigma$. By statement 3 of Proposition 2, the set $\Sigma\cap(c_1,+\infty)$ consists of finitely many disjoint closed intervals $I_1,\ldots,I_k$. We note that $\Sigma\cap(-\infty,c_1)\neq\emptyset$, since otherwise we would have $\Sigma=I_1\cup\cdots\cup I_k$, which contradicts to our assumption. Now we observe that there exists $c_2<c_1$ such that $c_2\notin\Sigma$ and $(c_2,c_1)\cap\Sigma\neq\emptyset$. Indeed, otherwise we would have $(-\infty,c_1)\cap\Sigma=(-\infty,a]$ for some $a<c_1$ and thus,

$$\Sigma=(-\infty,a]\cup I_1\cdots\cup I_k,$$

which again contradicts to our assumption. Proceeding inductively, we obtain a decreasing sequence $(c_n)_{n\in\mathbb{N}}$ such that $c_n\notin\Sigma$ and $(c_{n+1},c_n)\cap\Sigma\neq\emptyset$ for each $n\in\mathbb{N}$. Now there are two possibilities: either $\lim_{n\to+\infty}c_n=-\infty$ or $\lim_{n\to+\infty}c_n=b_\infty$ for some $b_\infty\in\mathbb{R}$. In the first case, it follows from statement 2 of Proposition 2 that Σ is given by alternative (c). In the second case, it follows from statement 3 of Proposition 2 that

$$(a_\infty,+\infty)\cap\Sigma=I_1\cup\bigcup_{n=2}^{\infty}[a_n,b_n],$$

where $I_1=[a_1,b_1]$ or $I_1=[a_1,+\infty)$, for some sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ as in (12) with $b_\infty=\lim_{n\to+\infty}a_n$. Again by statement 3 of Proposition 2, we have $(-\infty,b_\infty]\subset\Sigma$ and so Σ is given by alternative (d). This concludes the proof of statement 1.

For statement 2, we first note that it follows easily from statement 2 of Proposition 2 that the subspaces

$$E_k(n)=S_{c_k}(n)\cap U_{c_{k+1}}(n)$$

are independent of the choice of numbers c_k. Since $c_k\notin\Sigma$, the sequence $(e^{-c_k}A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy and so there exist projections P_n for $n\in\mathbb{Z}$ satisfying (2)–(3) and constants $\lambda,D>0$ such that

$$\|\mathcal{A}(m,n)P_n\|\leq De^{(c_k-\lambda)(m-n)}\quad\text{for } m\geq n \tag{13}$$

and

$$\|\mathcal{A}(m,n)Q_n\|\leq De^{-(\lambda+c_k)(n-m)}\quad\text{for } m\leq n,$$

where $Q_n=\mathrm{Id}-P_n$. By Proposition 1, we have $\operatorname{Im}P_n=S_{c_k}(n)$ for $n\in\mathbb{Z}$. Hence, $v\in E_i(n)$ belongs to $\operatorname{Im}P_n$ and it follows from (13) that

$$\limsup_{m\to+\infty}\frac{1}{m}\log\|\mathcal{A}(m,n)v\|\leq c_k-\lambda<c_k.$$

Letting $c_k\searrow b_{k+1}$, we obtain

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\| \le b_{k+1}.$$

Similarly, since $c_{k+1} \notin \Sigma$, the sequence $(e^{-c_{k+1}}A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy and so there exist projections P'_n for $n \in \mathbb{Z}$ satisfying (2)–(3) and constants $\mu, D > 0$ such that

$$\|\mathcal{A}(m,n)P'_n\| \le De^{(c_{k+1}-\mu)(m-n)} \quad \text{for } m \ge n$$

and

$$\|\mathcal{A}(m,n)Q'_n\| \le De^{-(\mu+c_{k+1})(n-m)} \quad \text{for } m \le n, \tag{14}$$

where $Q'_n = \mathrm{Id} - P'_n$. By Proposition 1, we have $\operatorname{Im} Q'_n = U_{c_{k+1}}(n)$ for $n \in \mathbb{Z}$. Hence, $v \in E_i(n)$ belongs to $\operatorname{Im} Q'_n$ and it follows from (14) that

$$\|v\| \le De^{-(\mu+c_{k+1})(m-n)}\|\mathcal{A}(m,n)v\| \quad \text{for } m \ge n.$$

Thus,

$$\liminf_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\| \ge \mu + c_{k+1} > c_{k+1}$$

and letting $c_{k+1} \nearrow a_{k+1}$,

$$\liminf_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\| \ge a_{k+1}.$$

The last statement in the theorem can be proved similarly. □

4. Examples.

Now we provide explicit examples of sequences $(A_n)_{n\in\mathbb{Z}}$ for which the spectrum Σ is given by the last two alternatives in Theorem 3 (the other alternatives already occur in finite-dimensional spaces).

Let $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and consider the space $X = L^2(S^1)$ with respect to the induced Lebesgue measure on S^1. We recall that X is a Banach space when equipped with the norm induced by the scalar product

$$\langle x, y\rangle = \int_{S^1} x(z)\overline{y(z)}\, dz,$$

already identifying functions that are equal Lebesgue-almost everywhere. Moreover, for each $K \in L^2(S^1 \times S^1)$, the operator $A\colon X \to X$ defined by

$$(Af)(z) = \int_{S^1} K(z,w)f(w)\, dw \tag{15}$$

is compact (see [**4**]). For example, one can take $K(z,w) = k(z/w)$, with $k \in X$. We also consider the orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ of X, where $e_n(z) = z^n$, for $z \in S^1$

and $n \in \mathbb{Z}$. Each function $k \in X$ can be written in the form $k = \sum_{n\in\mathbb{Z}} k_n e_n$ for some numbers $k_n \in \mathbb{R}$. Moreover, one can easily verify that $Ae_n = 2\pi k_n e_n$ for each $n \in \mathbb{Z}$, taking $K(z,w) = k(z/w)$.

EXAMPLE 1. Take numbers a_n and b_n as in (12) with $\lim_{n\to+\infty} a_n = -\infty$. We consider the linear operators $A, B\colon X \to X$ such that

$$Ae'_i = e^{b_i} e'_i \quad \text{and} \quad Be'_i = e^{a_i} e'_i$$

for $i \geq 0$, writing $e'_0 = e_0$, $e'_{2n-1} = e_n$ and $e'_{2n} = e_{-n}$ for $n \in \mathbb{N}$. Both A and B are of the form in (15) taking, respectively,

$$k(z) = \frac{1}{2\pi} e^{b_0} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \left(e^{b_{2n-1}} z^n + e^{b_{2n}} \frac{1}{z^n} \right)$$

and

$$k(z) = \frac{1}{2\pi} e^{a_0} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \left(e^{a_{2n-1}} z^n + e^{a_{2n}} \frac{1}{z^n} \right).$$

Now we consider the sequence of compact linear operators

$$A_n = \begin{cases} A, & n \geq 0, \\ B, & n < 0. \end{cases}$$

For each $a \in \mathbb{R}$ and $i \geq 0$, we have

$$e^{-a(m-n)} \mathcal{A}(m,n) e'_i = C_i(m,n) e'_i,$$

where

$$C_i(m,n) = \begin{cases} e^{(b_i - a)(m-n)}, & m, n \geq 0, \\ e^{(b_i - a)m - (a_i - a)n}, & m \geq 0, n < 0, \\ e^{(a_i - a)(m-n)}, & m, n < 0. \end{cases} \tag{16}$$

Take $a > b_1$. For each $x \in X$, we have

$$\|\mathcal{A}(m,n)x\|^2 = \sum_{i=0}^{\infty} C_i(m,n)^2 |\langle x, e'_i \rangle|^2.$$

Since $a_j \leq b_j \leq b_1$, we have $C_i(m,n) \leq e^{(b_1 - a)(m-n)}$ and thus,

$$\|\mathcal{A}(m,n)x\|^2 \leq e^{2(b_1 - a)(m-n)} \sum_{i=0}^{\infty} |\langle x, e'_i \rangle|^2 = e^{2(b_1 - a)(m-n)} \|x\|^2.$$

This shows that

$$\|\mathcal{A}(m,n)x\| \le e^{-(a-b_1)(m-n)}\|x\| \quad \text{for } m \ge n$$

and so $(e^{-a}A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy with projections $P_n = \mathrm{Id}$.

Now take $a \in (b_j, a_{j-1})$ for some $j \ge 2$. Let P_n be the projection given by $P_n e_i' = 0$ for $i < j-1$ and $P_n e_i' = e_i'$ for $i \ge j-1$. For each $x \in X$ and $m \ge n$, we have

$$\|\mathcal{A}(m,n)P_n x\|^2 = \sum_{i=j-1}^{\infty} C_i(m,n)^2 |\langle x, e_i'\rangle|^2 \le e^{2(b_j-a)(m-n)}\|x\|^2$$

and thus,

$$\|\mathcal{A}(m,n)P_n x\| \le e^{-(a-b_j)(m-n)}\|x\|.$$

Similarly, for each $x \in X$ and $m \le n$, we have

$$\|\mathcal{A}(m,n)Q_n x\|^2 = \sum_{i=0}^{j-2} C_i(m,n)^2 |\langle x, e_i'\rangle|^2 \le e^{2(a_{j-1}-a)(m-n)}\|x\|^2$$

and thus,

$$\|\mathcal{A}(m,n)Q_n x\| \le e^{-(a_{j-1}-a)(n-m)}\|x\|.$$

This shows that the sequence $(e^{-a}A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy. Therefore, $\Sigma \subset \bigcup_{j=1}^{\infty}[a_j, b_j]$.

In order to show that $\Sigma = \bigcup_{j=1}^{\infty}[a_j, b_j]$, assume that $[a_j, b_j] \setminus \Sigma$ is nonempty for some $j \in \mathbb{N}$ and take $a \in [a_j, b_j] \setminus \Sigma$. Then the sequence $(e^{-a}A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy say with projections P_n. Let Y be the subspace of X generated by e_i' and let B_n be the restriction of A_n to Y. Clearly, the sequence $(e^{-a}B_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy, say with projections P_n'. Moreover, either $P_n' = \mathrm{Id}$ or $P_n' = 0$, but both alternatives are impossible in view of (16).

Example 2. Now take numbers a_n and b_n as in (12) with $\lim_{n\to+\infty} a_n = b_\infty > -\infty$. We consider the sequence of compact linear operators $(A_n)_{n\in\mathbb{Z}}$ such that

$$A_n e_0' = \begin{cases} e^{b_\infty}, & n \ge 0, \\ e^{-n^2}, & n < 0 \end{cases} \quad \text{and} \quad A_n e_i' = \begin{cases} e^{b_i}/(1+i2^{-|n|}), & n \ge 0, \\ e^{a_i}/(1+i2^{-|n|}), & n < 0, \end{cases} \quad i \ge 1.$$

For each $a \in \mathbb{R}$ and $i \ge 0$, we have

$$e^{-a(m-n)}\mathcal{A}(m,n)e_i' = C_i(m,n)e_i',$$

where

$$C_0(m,n) = \begin{cases} e^{(b_\infty-a)(m-n)}, & m,n \ge 0, \\ e^{(b_\infty-a)m+an-n^2}, & m \ge 0, n < 0, \\ e^{-am+an-n^2+m^2}, & m,n < 0 \end{cases}$$

and

$$C_i(m,n) = \begin{cases} e^{(b_i-a)(m-n)}/\prod_{k=n}^m(1+i2^{-|k|}), & m,n \geq 0, \\ e^{(b_i-a)m-(a_i-a)n}/\prod_{k=n}^m(1+i2^{-|k|}), & m \geq 0, n < 0, \\ e^{(a_i-a)(m-n)}/\prod_{k=n}^m(1+i2^{-|k|}), & m,n < 0 \end{cases}$$

for $i \geq 1$. Notice that $1+j2^{-|n|} \geq 1$. One can proceed as in the previous example to show that for $a > b_1$, the sequence $(e^{-a}A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy with projections $P_n = \mathrm{Id}$. Now take $a \in (b_j, a_{j-1})$. For $i \geq j$ and $m \geq n$, we have $C_i(m,n) \leq e^{(b_j-a)(m-n)}$. Similarly, for $1 \leq i < j$ and $m \leq n$, we have

$$C_i(m,n) \leq e^{(a_j-a)(m-n)} \max_{1\leq i<j} \prod_{p=-\infty}^{\infty}(1+i2^{-|p|}).$$

Proceeding as in the previous example, one can show that the sequence $(e^{-a}A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy, with projections P_n given by $P_n e_i' = 0$ for $1 \leq i < j$ and $P_n e_i' = e_i'$ otherwise. By statement 3 of Proposition 2, we also have $(-\infty, b_\infty] \subset \Sigma$. Hence,

$$\Sigma \subset \bigcup_{j=1}^{\infty}[a_j, b_j] \cup (-\infty, b_\infty].$$

Moreover, one can proceed as in the previous example to show that in fact

$$\Sigma = \bigcup_{j=1}^{\infty}[a_j, b_j] \cup (-\infty, b_\infty].$$

5. Spectrum on the half-line.

5.1. Preliminaries.

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of compact linear operators acting on a Banach space X. For each $m \geq n$ we define $\mathcal{A}(m,n)$ as in (1). We say that $(A_n)_{n\in\mathbb{N}}$ admits an *exponential dichotomy* if there exist projections P_n for $n \in \mathbb{N}$ satisfying (2)–(3) and there exist constants $D, \lambda > 0$ such that (4)–(6) hold with $m, n \in \mathbb{N}$.

The following result can be obtained repeating the first part of the proof of Proposition 1. It shows that the images of the projections P_m are uniquely determined.

Proposition 4. *For each $n \in \mathbb{N}$, we have*

$$\operatorname{Im} P_n = \left\{ v \in X : \sup_{m\geq n} \|\mathcal{A}(m,n)v\| < +\infty \right\}.$$

On the other hand, the images of the projections Q_n are not uniquely determined, in contrast to what happens in the line.

Proposition 5. *Assume that the sequence $(A_m)_{m\in\mathbb{N}}$ admits an exponential di-*

chotomy with respect to projections P_m*. Moreover, let* P'_m*, for* $m \in \mathbb{N}$*, be projections such that*

$$P'_m \mathcal{A}(m,n) = \mathcal{A}(m,n) P'_n \quad \text{for } m,n \in \mathbb{N}$$

and

$$A_n| \operatorname{Ker} P'_n \colon \operatorname{Ker} P'_n \to \operatorname{Ker} P'_{n+1}$$

is invertible. Then $(A_m)_{m\in\mathbb{N}}$ *admits an exponential dichotomy with respect to the projections* P'_m *if and only if* $\operatorname{Im} P_n = \operatorname{Im} P'_n$ *for* $n \in \mathbb{N}$*.*

Proof. If $(A_m)_{m\in\mathbb{N}}$ admits an exponential dichotomy with respect to the projections P'_m, then it follows from Proposition 4 that

$$\operatorname{Im} P'_n = \left\{ v \in X : \sup_{m \ge n} \|\mathcal{A}(m,n)v\| < +\infty \right\} = \operatorname{Im} P_n.$$

Now assume that $\operatorname{Im} P_n = \operatorname{Im} P'_n$. Then

$$P_n P'_n = P'_n \quad \text{and} \quad P'_n P_n = P_n.$$

In particular,

$$P_n - P'_n = P_n(P_n - P'_n) = (P_n - P'_n) Q_n$$

and it follows from (4) and (5) that

$$\begin{aligned}
\|\mathcal{A}(n,1)(P_1 - P'_1)v\| &= \|\mathcal{A}(n,1)P_1(P_1 - P'_1)v\| \\
&\le De^{-\lambda(n-1)} \|(P_1 - P'_1)v\| \\
&= e^{\lambda} De^{-\lambda n} \|(P_1 - P'_1)Q_1 v\| \\
&\le e^{\lambda} De^{-\lambda n} \|P_1 - P'_1\| \cdot \|Q_1 v\| \\
&= e^{\lambda} De^{-\lambda n} \|P_1 - P'_1\| \cdot \|\mathcal{A}(1,m)\mathcal{A}(m,1)Q_1 v\| \\
&= e^{\lambda} De^{-\lambda n} \|P_1 - P'_1\| \cdot \|\mathcal{A}(1,m)Q_m\mathcal{A}(m,1) v\| \\
&\le e^{2\lambda} D^2 e^{-\lambda n - \lambda m} \|P_1 - P'_1\| \cdot \|\mathcal{A}(m,1) v\|
\end{aligned}$$

for $m,n \in \mathbb{N}$ and $v \in X$. Since

$$\begin{aligned}
\mathcal{A}(n,m)(P_m - P'_m) &= \mathcal{A}(n,m)(P_m - P'_m)Q_m \\
&= (P_n - P'_n)\mathcal{A}(n,m)Q_m \\
&= (P_n - P'_n)\mathcal{A}(n,1)Q_1\mathcal{A}(1,m)Q_m \\
&= \mathcal{A}(n,1)(P_1 - P'_1)Q_1\mathcal{A}(1,m)Q_m \\
&= \mathcal{A}(n,1)(P_1 - P'_1)\mathcal{A}(1,m)Q_m,
\end{aligned}$$

we obtain

$$\begin{aligned}\|\mathcal{A}(n,m)P'_m v\| &\le \|\mathcal{A}(n,m)P_m v\| + \|\mathcal{A}(n,m)(P_m - P'_m)v\| \\ &= \|\mathcal{A}(n,m)P_m v\| + \|\mathcal{A}(n,1)(P_1 - P'_1)\mathcal{A}(1,m)Q_m v\| \\ &\le De^{-\lambda(n-m)}\|v\| + e^{2\lambda}D^3 e^{-\lambda(n-m)}\|P_1 - P'_1\| \cdot \|v\| \\ &= D'e^{-\lambda(n-m)}\|v\|\end{aligned}$$

for $n \ge m$ and some constant $D' > 0$. Similarly, letting $Q'_m = \mathrm{Id} - P'_m$ we obtain

$$\begin{aligned}\|\mathcal{A}(n,m)Q'_m v\| &\le \|\mathcal{A}(n,m)Q_m v\| + \|\mathcal{A}(n,m)(P_m - P'_m)v\| \\ &= \|\mathcal{A}(n,m)Q_m v\| + \|\mathcal{A}(n,1)(P_1 - P'_1)\mathcal{A}(1,m)Q_m v\| \\ &\le De^{-\lambda(m-n)}\|v\| + e^{2\lambda}D^3 e^{-\lambda(m-n)}\|P_1 - P'_1\| \cdot \|v\| \\ &= D'e^{-\lambda(m-n)}\|v\|\end{aligned}$$

for $n \le m$. This shows that the sequence $(A_m)_{m\in\mathbb{N}}$ admits an exponential dichotomy with respect to the projections P'_m. ☐

The following proposition is a crucial step of this part.

PROPOSITION 6. *Assume that the sequence $(A_m)_{m\in\mathbb{N}}$ admits an exponential dichotomy with respect to projections P_m and P'_m. Then*

$$\dim \operatorname{Im} Q_m = \dim \operatorname{Im} Q'_m < +\infty, \quad m \in \mathbb{N}.$$

PROOF. By Proposition 5, we have

$$X = \operatorname{Im} P_n \oplus \operatorname{Im} Q_n = \operatorname{Im} P_n \oplus \operatorname{Im} Q'_n$$

and all complements of $\operatorname{Im} P_n$ have the same dimension. ☐

In view this proposition we will denote by d_a the unstable dimension of the sequence $(e^{-a}A_m)_{m\in\mathbb{N}}$ when it admits an exponential dichotomy.

5.2. Spectrum.

Given a sequence $(A_m)_{m\in\mathbb{N}}$ of compact linear operators, its *spectrum* is the set Σ of all numbers $a \in \mathbb{R}$ such that the sequence $(e^{-a}A_m)_{m\in\mathbb{N}}$ does not admit an exponential dichotomy.

For each $a \in \mathbb{R}$ and $n \in \mathbb{N}$, let

$$S_a(n) = \left\{ v \in X : \sup_{m \ge n} \left(e^{-a(m-n)}\|\mathcal{A}(m,n)v\| \right) < +\infty \right\}.$$

Clearly, $S_a(n)$ is a linear space and if $a < b$, then $S_a(n) \subset S_b(n)$.

The following result can be obtained repeating the proof of Proposition 2 with $\dim U_{a_i}$ replaced by d_{a_i}.

PROPOSITION 7. *The following statements hold*:

1. *The set Σ is closed. Moreover, for each $a \in \mathbb{R} \setminus \Sigma$ we have $S_a(n) = S_b(n)$ for all $n \in \mathbb{N}$ and all b in some open neighborhood of a.*

2. *Take $a_1, a_2 \in \mathbb{R} \setminus \Sigma$ with $a_1 < a_2$. Then $[a_1, a_2] \cap \Sigma \neq \emptyset$ if and only if $d_{a_1} > d_{a_2}$.*

3. *For each $c \notin \Sigma$, the set $\Sigma \cap [c, +\infty)$ consists of finitely many closed intervals.*

The following is our main theorem for the half-line. It describes all possible forms of the spectrum for a sequence of compact linear operators.

Theorem 8. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of compact linear operators for which the spectrum is neither $\emptyset$ nor $\mathbb{R}$.*

1. *Property 1 of Theorem 3 holds.*

2. *Let $(c_k)_k$ be a finite or infinite sequence of numbers such that $c_k \in (b_{k+1}, a_k)$ for each k. For each $n \in \mathbb{N}$ and $v \in S_{c_k}(n)$, we have*

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\| \le b_{k+1}.$$

Proof. The proof of statement 1 is the same as in the proof of Theorem 3. For the second statement, since $c_k \notin \Sigma$, the sequence $(e^{-c_k}A_n)_{n\in\mathbb{N}}$ admits an exponential dichotomy and so there exist projections P_n for $n \in \mathbb{N}$ satisfying (2)–(3) and constants $\lambda, D > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{(c_k-\lambda)(m-n)} \quad \text{for } m \ge n \tag{17}$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-(\lambda+c_k)(n-m)} \quad \text{for } m \le n,$$

where $Q_n = \mathrm{Id} - P_n$. By Proposition 7, the space $S_{c_k}(n)$ is independent of the choice of c_k and $\operatorname{Im} P_n = S_{c_k}(n)$ for $n \in \mathbb{N}$. Hence, for $v \in S_{c_k}(n)$, it follows from (17) that

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\| \le c_k - \lambda < c_k$$

and letting $c_k \searrow b_{k+1}$,

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\| \le b_{k+1}.$$

This completes the proof of the theorem. □

6. The case of continuous time.

In this section we consider briefly the case of continuous time and we describe corresponding results to those in the former sections.

Let X be a Banach space. We recall that a family $T(t,s)$ for $t, s \in \mathbb{R}$ with $t \ge s$ of bounded linear operators acting on X is called an *evolution family* if:

1. $T(t,t) = \mathrm{Id}$ for $t \in \mathbb{R}$;
2. $T(t,s) = T(t,r)T(r,s)$ for $t \geq r \geq s$.

We say that $T(t,s)$ admits an *exponential dichotomy* if:

1. there exist projections $P(t)\colon X \to X$ for $t \in \mathbb{R}$ satisfying
$$P(t)T(t,s) = T(t,s)P(s)$$
for $t \geq s$ such that each map
$$T(t,s)|\operatorname{Ker} P(s)\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$$
is invertible;
2. there exist constants $D, \lambda > 0$ such that
$$\|T(t,s)P(s)\| \leq De^{-\lambda(t-s)} \quad \text{for } t \geq s$$
and
$$\|T(t,s)Q(s)\| \leq De^{-\lambda(s-t)} \quad \text{for } t \leq s,$$
where $Q(s) = \mathrm{Id} - P(s)$ and
$$T(t,s) = (T(s,t)|\operatorname{Ker} P(t))^{-1}\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$$
for $t < s$.

The *spectrum* of an evolution family $T(t,s)$ is the set Σ of all numbers $a \in \mathbb{R}$ such that the evolution family
$$T_a(t,s) = e^{-a(t-s)}T(t,s)$$
does not admit an exponential dichotomy.

One can easily adapt the arguments in the proof of Theorem 3 to establish the following result.

THEOREM 9. *Let $T(t,s)$ be an evolution family for which the spectrum is neither $\emptyset$ nor $\mathbb{R}$. If there exists $\rho \geq 0$ such that $T(t,s)$ is compact whenever $t > s + \rho$, then property* 1 *of Theorem* 3 *holds.*

The compactness assumption holds in particular for the evolution families associated to some linear delay equations with $\rho > 0$ (see [**5**]) and some linear parabolic partial differential equations with $\rho = 0$ (see [**6**]). It is also straightforward to establish a version of Theorem 9 for evolution families on the half-line.

Now we consider the particular class of evolution families with bounded growth and we show how one can use *directly* the results for discrete time to obtain corresponding results for this class. More precisely, we obtain the particular case of Theorem 9 for this class with a very simple proof applying Theorem 3, instead of having the need to adapt all arguments in its proof.

Theorem 10. *Let $T(t,s)$ be an evolution family with the property that there exist $K, c > 0$ such that*

$$\|T(t,s)\| \le Ke^{c(t-s)} \quad \text{for } t \ge s. \tag{18}$$

If there exists $\rho \ge 0$ such that $T(t,s)$ is compact whenever $t > s+\rho$, then one of the following alternatives hold:

1. $\Sigma = \emptyset$;
2. $\Sigma = \bigcup_{n=1}^{k}[a_n, b_n]$, *for some numbers a_n and b_n as in* (11);
3. $\Sigma = \bigcup_{n=1}^{k-1}[a_n, b_n] \cup (-\infty, b_k]$, *for some numbers a_n and b_n as in* (11);
4. $\Sigma = \bigcup_{n=1}^{\infty}[a_n, b_n]$, *for some numbers a_n and b_n as in* (12) *and with* $\lim_{n\to+\infty} a_n = -\infty$;
5. $\Sigma = \bigcup_{n=1}^{\infty}[a_n, b_n] \cup (-\infty, b_\infty]$, *for some numbers a_n and b_n as in* (12) *and with* $b_\infty := \lim_{n\to+\infty} a_n > -\infty$.

Proof. We first recall that for an evolution family satisfying (18) the following properties are equivalent (see [**7**, Theorem 1.3]):

1. $T(t,s)$ admits an exponential dichotomy;
2. taking $\alpha > \rho$, the sequence $(A_n)_{n\in\mathbb{Z}}$ with

$$A_n = T(\alpha n, \alpha(n-1)), \quad \text{for } n \in \mathbb{Z},$$

admits an exponential dichotomy.

Hence, Σ coincides with the spectrum of the sequence $(A_n)_{n\in\mathbb{Z}}$. Moreover, in view of the compactness assumption for $T(t,s)$ and since $\alpha > \rho$, each operator A_n is compact. The desired conclusion follows now readily from Theorem 3 since it follows from (18) that $\Sigma \subset (-\infty, c]$. $\square$

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