J. Math. Soc. Japan
Vol. 72, No. 3 (2020) pp. 731–764
doi: 10.2969/jmsj/75117511

Classifying τ-tilting modules over the Auslander algebra of $K[x]/(x^n)$

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(Received May 4, 2016)
(Revised Oct. 27, 2018)

Abstract. We build a bijection between the set sτ-tilt Λ of isomorphism classes of basic support τ-tilting modules over the Auslander algebra Λ of $K[x]/(x^n)$ and the symmetric group $\mathfrak{S}_{n+1}$, which is an anti-isomorphism of partially ordered sets with respect to the generation order on sτ-tilt Λ and the left order on $\mathfrak{S}_{n+1}$. This restricts to the bijection between the set tilt Λ of isomorphism classes of basic tilting Λ-modules and the symmetric group $\mathfrak{S}_n$ due to Brüstle, Hille, Ringel and Röhrle. Regarding the preprojective algebra Γ of Dynkin type A_n as a factor algebra of Λ, we show that the tensor functor $-\otimes_\Lambda \Gamma$ induces a bijection between sτ-tilt $\Lambda \to$ sτ-tilt Γ. This recover Mizuno's anti-isomorphism $\mathfrak{S}_{n+1} \to$ sτ-tilt Γ of posets for type A_n.

1. Introduction.

Tilting theory has been central in the representation theory of finite dimensional algebras since the early seventies [**BGP**], [**AuPR**], [**B**], [**BrB**], [**HaR**]. In this theory, tilting modules play a central role. So it is important to classify tilting modules for a given algebra. There are many algebraists working on this topic which makes the theory fruitful. For more details about classical tilting modules we refer to [**AsSS**], [**AnHK**].

Recently Adachi, Iyama and Reiten [**AIR**] introduced τ-tilting theory to generalize the classical tilting theory from viewpoint of mutations. This is very close to the silting theory (e.g. [**AiI**], [**DF**], [**HKM**], [**KV**]) and the cluster tilting theory (e.g. [**BMRRT**], [**IY**], [**KR**]). The central notion of τ-tilting theory is support τ-tilting modules, and therefore it is important to classify support τ-tilting modules for a given algebra. Recently some authors worked on this topic, e.g. Adachi [**A1**] classified τ-rigid modules for Nakayama algebras, Adachi [**A2**] and Zhang [**Z1**] studied τ-rigid modules for algebras with radical square zero, and Mizuno [**M**] classified support τ-tilting modules for preprojective algebras of Dynkin type. In this context, it is basic to consider algebras with only finitely many support τ-tilting modules, called τ-tilting finite algebras and studied by Demonet, Iyama and Jasso [**DIJ**]. For more details of τ-tilting theory, we refer to [**AAC**], [**AIR**], [**AnMV**], [**DIRRT**], [**HuZ**], [**J**], [**IJY**], [**IRRT**], [**W**], [**Zh**] and so on.

2010 *Mathematics Subject Classification.* Primary 16G10; Secondary 16E10.

Key Words and Phrases. Auslander algebra, tilting module, τ-tilting module, symmetric group, preprojective algebra.

The first author was supported by JSPS Grant-in-Aid for Scientific Research (B) 24340004, (C) 23540045 and (S) 15H05738. The second author was supported by NSFC (Nos.11571164, 11671174) and Jiangsu Government Scholarship for Overseas Studies (JS-2014-352).

In this paper we focus on classifying tilting modules and support τ-tilting modules over a class of Auslander algebras. Recall that an algebra Λ is called an Auslander algebra if the global dimension of Λ is less than or equal to 2 and the dominant dimension of Λ is greater than or equal to 2. It is showed by Auslander that there is a one-to-one correspondence between Auslander algebras and algebras of finite representation type.

In the rest, let Λ be the Auslander algebra of the algebra $K[x]/(x^n)$. Then Λ is presented by the quiver

$$1 \underset{b_2}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_3}{\overset{a_2}{\rightleftarrows}} 3 \underset{b_4}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-2}}{\rightleftarrows}} n-1 \underset{b_n}{\overset{a_{n-1}}{\rightleftarrows}} n$$

with relations $a_1b_2 = 0$ and $a_ib_{i+1} = b_ia_{i-1}$ for any $2 \leq i \leq n-1$. All modules in this paper are right modules. Denote by $\operatorname{tilt}\Lambda$ the set of isomorphism classes of basic tilting Λ-modules. We show that each tilting Λ-module is isomorphic to a product of maximal ideals $I_1, \dots, I_{n-1}$ of Λ. Moreover, we show a strong relationship between basic tilting Λ-modules and the symmetric group $\mathfrak{S}_n$.

For $w, w' \in \mathfrak{S}_n$ and $1 \leq i \leq n$, we denote the product $w'w \in \mathfrak{S}_n$ by $(w'w)(i) := w'(w(i))$. Denote by $s_i \in \mathfrak{S}_n$ the transposition $(i, i+1)$ for $1 \leq i \leq n-1$. The *length* of $w \in \mathfrak{S}_n$ is defined by $l(w) := \#\{(i,j) \mid 1 \leq i < j \leq n, w(i) > w(j)\}$ and an expression $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ of $w \in \mathfrak{S}_n$ is called a *reduced expression* if $l = l(w)$. For elements $w, w' \in \mathfrak{S}_n$, if $l(w') = l(w) + l(w'w^{-1})$ then we write $w \leq w'$. This gives a partial order on $\mathfrak{S}_n$ called the *left order*. The Hasse quiver of $\mathfrak{S}_n$ has vertices w corresponding to each element $w \in \mathfrak{S}_n$, and has arrows $w \to s_iw$ if $l(w) > l(s_iw)$ and $w \leftarrow s_iw$ if $l(w) < l(s_iw)$ for $w \in \mathfrak{S}_n$ and $1 \leq i \leq n-1$. Now we are in a position to state our first main result.

Theorem 1.1 (Theorems 3.9, 3.18). *Let Λ be the Auslander algebra of $K[x]/(x^n)$, and $\langle I_1, \dots, I_{n-1}\rangle$ the ideal semigroup of Λ generated by the maximal ideals $I_1, \dots, I_{n-1}$.*

(1) *The set* $\operatorname{tilt}\Lambda$ *is given by $\langle I_1, \dots, I_{n-1}\rangle$.*

(2) *There exists a well-defined bijection $I : \mathfrak{S}_n \cong \langle I_1, \dots, I_{n-1}\rangle$, which maps w to $I(w) = I_{i_1}\cdots I_{i_l}$ where $w = s_{i_1}\cdots s_{i_l}$ is an arbitrary reduced expression.*

(3) *Consequently there exists a bijection $I : \mathfrak{S}_n \cong \operatorname{tilt}\Lambda$. In particular $\#\operatorname{tilt}\Lambda = n!$.*

(4) *The map I in* (3) *is an anti-isomorphism of posets* (*partially orderd set*).

Theorem 1.1(3) has been shown in [**BHRR**] by using a combinatorial method. Our method in this paper is rather homological, and we shall modify the method in [**IR**], [**BIRS**], [**M**] for preprojective algebras to the Auslander algebra of $K[x]/(x^n)$ by using basic properties of Auslander algebras in Section 2.

Denote by $\mathrm{s}\tau\text{-}\operatorname{tilt}\Lambda$ the set of isomorphism classes of basic support τ-tilting Λ-modules, and by $\mu_i(T)$ the mutation of T with respect to the i-th indecomposable direct summand of T. The set $\mathrm{s}\tau\text{-}\operatorname{tilt}\Lambda$ forms a poset with respect to the generation order (Definition 2.13). We show the following main result of this paper in Section 4, where the map $I : \mathfrak{S}_{n+1} \cong \mathrm{s}\tau\text{-}\operatorname{tilt}\Lambda$ is an extension of the map I in Theorem 1.1.

THEOREM 1.2 (Theorems 4.8, 4.10, 4.12). *Let Λ be the Auslander algebra of $K[x]/(x^n)$.*

(1) $\mathrm{s}\tau\text{-tilt}\,\Lambda$ *is a disjoint union of* $\mu_{i+1}\mu_{i+2}\cdots\mu_n(\mathrm{tilt}\,\Lambda)$ *for* $0 \le i \le n$.

(2) *There exists a bijection* $I : \mathfrak{S}_{n+1} \cong \mathrm{s}\tau\text{-tilt}\,\Lambda$ *which maps* w *to* $I(w) = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_l}(\Lambda)$, *where* $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ *is an arbitrary expression. In particular, we have* $\#\mathrm{s}\tau\text{-tilt}\,\Lambda = (n+1)!$.

(3) *The map* I *in* (2) *is an anti-isomorphism of posets.*

Now let Γ be the preprojective algebra of Dynkin type A_n. Then there exists a natural surjection $\Lambda \to \Gamma$, and we get a tensor functor $- \otimes_\Lambda \Gamma : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$. By using this we get a bijection between $\mathrm{s}\tau\text{-tilt}\,\Lambda$ and $\mathrm{s}\tau\text{-tilt}\,\Gamma$. More precisely, we have:

THEOREM 1.3 (Theorem 5.3). *Let Λ and Γ be as above. Then*

(1) *The map* $- \otimes_\Lambda \Gamma : \mathrm{s}\tau\text{-tilt}\,\Lambda \to \mathrm{s}\tau\text{-tilt}\,\Gamma$ *given by* $U \mapsto U \otimes_\Lambda \Gamma$ *is bijective.*

(2) *The map in* (1) *is an isomorphism of posets.*

As a corollary of Theorems 1.2 and 1.3, we recover Mizuno's anti-isomorphism $\mathfrak{S}_{n+1} \to \mathrm{s}\tau\text{-tilt}\,\Gamma$ [**M**, Theorems 2.21 and 2.30] since it is the composition of $- \otimes_\Lambda \Gamma$ in Theorem 1.3 and I in Theorem 1.2.

COROLLARY 1.4 (Corollary 5.5). *Let Λ and Γ be as above. There are isomorphisms between the following posets*:

(1) *The poset* $\mathrm{s}\tau\text{-tilt}\,\Lambda$ *with the generation order.*

(2) *The poset* $\mathrm{s}\tau\text{-tilt}\,\Gamma$ *with the generation order.*

(3) *The symmetric group* $\mathfrak{S}_{n+1}$ *with the opposite of the left order.*

(4) *The poset* $\mathrm{s}\tau\text{-tilt}\,(\Lambda^{\mathrm{op}})$ *with the opposite of the generation order.*

(5) *The poset* $\mathrm{s}\tau\text{-tilt}\,(\Gamma^{\mathrm{op}})$ *with the opposite of the generation order.*

(6) *The symmetric group* $\mathfrak{S}_{n+1}$ *with the right order.*

The paper is organized as follows: In Section 2, we recall some preliminaries on Auslander algebras, tilting modules and support τ-tilting modules. In Section 3, we focus on the tilting modules over the Auslander algebra of $K[x]/(x^n)$ and we prove Theorem 1.1. In Section 4, we use Theorem 1.1 and some other facts of tilting modules to prove Theorem 1.2. Finally, in Section 5, we apply Theorem 1.2 and Theorem 1.3 to preprojective algebras of Dynkin type A_n and get Mizuno's bijection for preprojective algebras of Dynkin type A_n.

Throughout this paper, we denote by K an arbitrary field, and we consider basic finite dimensional K-algebras. By a module, we mean a finitely generated right module. For an algebra A, we denote by $\mathrm{mod}\,A$ the category of finitely generated right A-modules. For an A-module M, we denote by $\mathrm{add}\,M$ the full subcategory of $\mathrm{mod}\,A$ whose objects are direct summands of M^n for some $n > 0$. The composition of homomorphisms $f : X \to Y$ and $g : Y \to Z$ is denoted by $gf : X \to Z$. Thus $\mathrm{Hom}_\Lambda(X, Y)$ is an $\mathrm{End}_\Lambda(Y)^{\mathrm{op}}$-module and an $\mathrm{End}_\Lambda(X)$-module.

For more recent results on τ-tilting theory of Auslander algebras, we refer to [**IZ**] and [**Z2**].

Acknowledgements. Theorem 1.1 was obtained in the Master thesis of Yusuke Tsujioka [**T**], who was a student of the first author in Graduate school of Mathematics in Nagoya University. The authors thank him for allowing them to include his results in this paper. Other parts of this paper were done when the second author visited Nagoya University in the year 2015. The second author would like to thank Laurent Demonet, Takahide Adachi, Yuta Kimura, Yuya Mizuno and Yingying Zhang for useful discussion and kind help. He also wants to thank the first author for hospitality during his stay in Nagoya. Both of the authors would like to thank the referees for useful suggestions to improve this paper.

2. Preliminaries.

In this section we recall some basic properties of Auslander algebras, tilting modules and support τ-tilting modules. We begin with the definition of Auslander algebras.

For an algebra Λ and a Λ-module M, denote by $\mathrm{gl.dim}\,\Lambda$ the global dimension of Λ, and by $\mathrm{proj.dim}\,M$ (resp. $\mathrm{inj.dim}\,M$) the projective dimension (resp. injective dimension) of M. We recall the following definition.

Definition 2.1. An algebra Λ is called an *Auslander algebra* if $\mathrm{gl.dim}\,\Lambda \leq 2$ and $E_i(\Lambda)$ is projective for $i = 0, 1$, where $E_i(\Lambda)$ is the $(i+1)$-th term in a minimal injective resolution of Λ.

Recall that an algebra R is called *representation-finite* if $\mathrm{mod}R$ admits an additive generator M, that is, $\mathrm{mod}R = \mathrm{add}M$. The following classical result in [**AuRS**] shows the relationship between representation-finite algebras and Auslander algebras.

Theorem 2.2. (1) *For an additive generator M of the category $\mathrm{mod}R$ over a representation-finite algebra R, the algebra $\mathrm{End}_R(M)$ is an Auslander algebra.*

(2) *For an Auslander algebra Λ and an additive generator Q of the category of projective-injective Λ-module, the algebra $\mathrm{End}_\Lambda(Q)$ is representation-finite.*

(3) *The correspondences in* (1) *and* (2) *induce mutually inverse bijections between Morita equivalence classes of representation-finite algebras and Morita equivalence classes of Auslander algebras.*

We call $\Lambda = \mathrm{End}_R(M)$ in Theorem 2.2(1) an *Auslander algebra* of R. In this case, for $X \in \mathrm{mod}R$ we denote

$$P_X = \mathrm{Hom}_R(M, X),\ P^X = \mathrm{Hom}_R(X, M),\ S_X = P_X/\operatorname{rad} P_X \text{ and } S^X = P^X/\operatorname{rad} P^X.$$

Here $P_- = \mathrm{Hom}_R(M,-)$ is an equivalence between $\mathrm{add}\, M$ and $\mathrm{add}\, \Lambda$, and $P^- = \mathrm{Hom}_R(-,M)$ is a duality between $\mathrm{add}\, M$ and $\mathrm{add}\, \Lambda^{\mathrm{op}}$. The following statement [**AuRS**] shows the relationship between almost split sequences of R and projective resolutions of simple Λ-modules.

PROPOSITION 2.3. *Let Λ be an Auslander algebra of R and let X be an indecomposable R-module. Then we have*

(1) $\mathrm{proj.dim}(S_X)_\Lambda \leq 1$ *if and only if X is a projective R-module. Then* $0 \to P_{\mathrm{rad}\, X} \to P_X \to S_X \to 0$ *is a minimal projective resolution of S_X.*

(2) $\mathrm{proj.dim}(S_X)_\Lambda = 2$ *if and only if X is a nonprojective R-module. Then the almost split sequence* $0 \to \tau X \to E \to X \to 0$ *gives a minimal projective resolution* $0 \to P_{\tau X} \to P_E \to P_X \to S_X \to 0$ *of S_X.*

(3) $\mathrm{proj.dim}\,_\Lambda(S^X) \leq 1$ *if and only if X is an injective R-module. Then* $0 \to P^{X/\mathrm{soc} X} \to P^X \to S^X \to 0$ *is a minimal projective resolution of S^X.*

(4) $\mathrm{proj.dim}\,_\Lambda(S^X) = 2$ *if and only if X is a noninjective R-module. Then the almost split sequence* $0 \to X \to E \to \tau^{-1}X \to 0$ *gives a minimal projective resolution* $0 \to P^{\tau^{-1}X} \to P^E \to P^X \to S^X \to 0$ *of S^X.*

Denote by $(-)^* = \mathrm{Hom}_\Lambda(-,\Lambda)$. We also need the following lemma.

LEMMA 2.4. *Let Λ be an Auslander algebra of R and let X be an indecomposable nonprojective R-module. Then we have*

(1) $\mathrm{Ext}^2_\Lambda(S_X,\Lambda) \cong S^{\tau X}$, *and* $\mathrm{Ext}^i_\Lambda(S_X,\Lambda) = 0$ *if* $i \neq 2$.

(2) $\mathrm{Ext}^i_\Lambda(S_X,Y) \cong \mathrm{Tor}^\Lambda_{2-i}(Y,S^{\tau X})$ *for* $Y \in \mathrm{mod}\,\Lambda$ *and* $i \in \mathbb{Z}$.

PROOF. We only prove (2) since the statement (1) follows from (2) immediately. By Proposition 2.3, there exist projective resolutions

$$0 \to P_{\tau X} \to P_E \to P_X \to S_X \to 0, \tag{2.1}$$

$$0 \to P^X \to P^E \to P^{\tau X} \to S^{\tau X} \to 0 \tag{2.2}$$

of S_X and $S^{\tau X}$, respectively. Applying $\mathrm{Hom}_\Lambda(-,Y)$ to (2.1), we obtain a complex

$$0 \to \mathrm{Hom}_\Lambda(P_X,Y) \to \mathrm{Hom}_\Lambda(P_E,Y) \to \mathrm{Hom}_\Lambda(P_{\tau X},Y) \to 0 \tag{2.3}$$

whose homologies are $\mathrm{Ext}^i_\Lambda(S_X,Y)$. Similarly, applying $Y \otimes_\Lambda -$ to (2.2), we obtain a complex

$$0 \to Y \otimes_\Lambda P^X \to Y \otimes_\Lambda P^E \to Y \otimes_\Lambda P^{\tau X} \to 0 \tag{2.4}$$

whose homologies are $\mathrm{Tor}^\Lambda_{2-i}(Y,S^{\tau X})$. Because $\mathrm{Hom}_\Lambda(P_-,Y) \cong Y \otimes_\Lambda {P_-}^* \cong Y \otimes_\Lambda P^-$ holds, (2.3) and (2.4) are isomorphic. Thus we obtain the desired isomorphism. $\square$

The following lemma is useful.

Lemma 2.5. *Let Λ be an Auslander algebra and $Y \in \operatorname{mod}\Lambda$. Then any composition factor of $\operatorname{Ext}^2_\Lambda(Y,\Lambda)$ has projective dimension 2.*

Proof. Without loss of generality, we can assume that Y is simple since any composition factor of $\operatorname{Ext}^2_\Lambda(Y,\Lambda)$ is a composition factor of $\operatorname{Ext}^2_\Lambda(S,\Lambda)$ for some simple Λ-module S. If $\operatorname{proj.dim} Y \le 1$, then the assertion is clear since the zero module has no composition factor. If $\operatorname{proj.dim} Y = 2$, then Proposition 2.3(2) shows that $Y = S_X$ for some indecomposable nonprojective R-module X. Thus $\operatorname{Ext}^2_\Lambda(Y,\Lambda) = S^{\tau X}$ holds by Lemma 2.4(2), and the assertion follows from Proposition 2.3(4). $\square$

We also need the following general result on algebras of global dimension 2.

Lemma 2.6. *Let Λ be an algebra with $\operatorname{gl.dim}\Lambda \le 2$ and $Y \in \operatorname{mod}\Lambda$. Then Y^{**} is a projective Λ-module.*

Proof. Let $Q_1 \to Q_0 \to Y \to 0$ be a projective presentation of Y. Applying $(-)^*$, we obtain an exact sequence $0 \to Y^* \to Q_0^* \to Q_1^*$. Hence Y^* is a projective Λ^{op}-module, since Q_0^* and Q_1^* are projective Λ^{op}-modules and $\operatorname{gl.dim}\Lambda \le 2$. Thus Y^{**} is a projective Λ-module. $\square$

By the lemma above we obtain the following.

Lemma 2.7. *Let Λ be an Auslander algebra, and let Y be a Λ-module with $\operatorname{proj.dim} Y \le 1$. Then the evaluation map $\varphi_Y : Y \to Y^{**}$ is injective, and the projective dimension of any composition factor of Y^{**}/Y is 2.*

Proof. By [**AuB**], we get an exact sequence $0 \to \operatorname{Ext}^1_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} Y,\Lambda) \to Y \to Y^{**} \to \operatorname{Ext}^2_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} Y,\Lambda) \to 0$. Then the latter assertion holds by Lemma 2.5. We prove the former one in two steps.

(1) We show that the projective dimension of any composition factor of $\operatorname{Tr} Y$ is 2.

It suffices to show that $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(P,\operatorname{Tr} Y) = 0$ holds for the projective cover P of any simple Λ^{op}-module S with $\operatorname{proj.dim} S \le 1$. By Proposition 2.3(3), $P = P^I$ for some injective R-module I. On one hand, take a minimal projective resolution of Y:

$$0 \to P_{X_1} \xrightarrow{P_f} P_{X_0} \to Y \to 0. \tag{2.5}$$

Since M is a generator, then we get an R-module monomorphism $f : X_1 \to X_0$. Applying $\operatorname{Hom}_R(-,I)$, one has an epimorphism

$$\operatorname{Hom}_R(X_0,I) \to \operatorname{Hom}_R(X_1,I). \tag{2.6}$$

On the other hand, applying the functor $(-)^*$ to (2.5), we get an exact sequence $P^{X_0} \to P^{X_1} \to \operatorname{Tr} Y \to 0$. Then applying the functor $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(P^I,-)$, one obtains an exact sequence

$$\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(P^I,P^{X_0}) \to \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(P^I,P^{X_1}) \to \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(P^I,\operatorname{Tr} Y) \to 0.$$

This can be rewritten as $\mathrm{Hom}_R(X_0, I) \to \mathrm{Hom}_R(X_1, I) \to \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^I, \mathrm{Tr}\, Y) \to 0$. Thus we obtain $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^I, \mathrm{Tr}\, Y) = 0$ by (2.6).

(2) Now we prove the assertion. By (1) and Proposition 2.3(4), any composition factor of $\mathrm{Tr}\, Y$ has the form S^X for some indecomposable noninjective R-module X. By the dual of Lemma 2.4(1), we have $\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(S^X, \Lambda) = 0$. Thus $\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, Y, \Lambda) = 0$. □

In the rest of this section, Λ is an arbitrary algebra. In the following we recall some basic properties of tilting modules. We begin with the definition of tilting modules.

Definition 2.8. We call $T \in \mathrm{mod}\Lambda$ a *tilting module* if T satisfies the following conditions:

(T1) $\mathrm{proj.dim}\, T \leq 1$.

(T2) $\mathrm{Ext}^1_\Lambda(T, T) = 0$.

(T3) There exists a short exact sequence $0 \to \Lambda \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathrm{add}T$.

The condition (T3) is equivalent to

(T3′) The number of non-isomorphic direct summands of T is equal to that of Λ.

Now let us recall some general properties of tilting modules [**HaU**].

Lemma 2.9. *Let T be a tilting Λ-module, and let $0 \to Q_1 \to Q_0 \to T \to 0$ be a minimal projective resolution of T. Then we have the following*:

(1) $(\mathrm{add}Q_1) \cap (\mathrm{add}Q_0) = 0$ *and* $\mathrm{add}(Q_0 \oplus Q_1) = \mathrm{add}\Lambda$ *hold.*

(2) *For a simple Λ-module S, precisely one of* $\mathrm{Hom}_\Lambda(T, S) = 0$ *and* $\mathrm{Ext}^1_\Lambda(T, S) = 0$ *holds.*

(3) *For a simple Λ^{op}-module S, precisely one of $T \otimes_\Lambda S = 0$ and $\mathrm{Tor}_1^\Lambda(T, S) = 0$ holds.*

We also have the following properties for the tensor products of tilting modules.

Proposition 2.10. *Let T be a tilting Λ-module with $\Gamma = \mathrm{End}_\Lambda(T)$.*

(1) *Let U be a tilting Γ-module. If $\mathrm{Tor}_i^\Gamma(U, T) = 0$ for any $i > 0$ and* $\mathrm{proj.dim}(U \otimes_\Gamma T) \leq 1$, *then $U \otimes_\Gamma T$ is a tilting Λ-module with* $\mathrm{End}_\Lambda(U \otimes_\Gamma T) \cong \mathrm{End}_\Gamma(U)$.

(2) *Let V be a tilting Λ-module. If* $\mathrm{Ext}^i_\Lambda(T, V) = 0$ *for any $i > 0$ and* $\mathrm{proj.dim}\, \mathrm{Hom}_\Lambda(T, V)_\Gamma \leq 1$, *then* $\mathrm{Hom}_\Lambda(T, V)$ *is a tilting Γ-module with* $\mathrm{End}_\Gamma(\mathrm{Hom}_\Lambda(T, V)) \cong \mathrm{End}_\Lambda(V)$.

Proof. (1) Since $- \otimes^{\mathbf{L}}_\Gamma T : \mathrm{D^b}(\mathrm{mod}\Gamma) \to \mathrm{D^b}(\mathrm{mod}\Lambda)$ is a triangle equivalence, $U \otimes^{\mathbf{L}}_\Gamma T$ is a tilting complex of Λ. Since $\mathrm{Tor}_i^\Gamma(U, T) = 0$ for any $i > 0$ by our assumption, $U \otimes_\Gamma T \cong U \otimes^{\mathbf{L}}_\Gamma T$ holds. Since $\mathrm{proj.dim}(U \otimes_\Gamma T) \leq 1$, the assertion holds. One can show (2) similarly. □

Denote by τ the AR-translation and denote by $|N|$ the number of non-isomorphic indecomposable direct summands of N for a Λ-module N. In the following we recall some basic properties of τ-tilting theory. Firstly, we need the following definition in [**AIR**].

Definition 2.11. (1) We call $N \in \operatorname{mod}\Lambda$ *τ-rigid* if $\operatorname{Hom}_\Lambda(N, \tau N) = 0$.

(2) We call $N \in \operatorname{mod}\Lambda$ *τ-tilting* if N is τ-rigid and $|N| = |\Lambda|$.

(3) We call $N \in \operatorname{mod}\Lambda$ *support τ-tilting* if there exists a basic idempotent e of Λ such that N is a τ-tilting $(\Lambda/(e))$-module. In this case, we call $(N, e\Lambda)$ a *support τ-tilting pair.*

It is clear that every tilting Λ-module is a τ-tilting Λ-module, and hence a support τ-tilting module. Moreover, it is showed in [**AIR**] that tilting Λ-modules are exactly faithful support τ-tilting modules. Clearly any support τ-tilting pair $(N, e\Lambda)$ satisfies $|N| + |e\Lambda| = |\Lambda|$.

For a torsion class $\mathcal{T}$ in $\operatorname{mod}\Lambda$, we denote by $P(\mathcal{T})$ the direct sum of one copy of each of the indecomposable Ext-projective objects in $\mathcal{T}$ up to isomorphism. The following properties of τ-rigid modules are important.

Definition-Proposition 2.12 ([**AIR**, Theorem 2.10]). *Let Λ be an algebra and let U be a τ-rigid module. Then $T = P({}^\perp\tau U)$ is a τ-tilting Λ-module, where ${}^\perp\tau U$ consists of Λ-modules X satisfying* $\operatorname{Hom}_\Lambda(X, \tau U) = 0$. *We call T the* Bongartz completion *of U.*

Recall that $\operatorname{s\tau-tilt}\Lambda$ is the set of isomorphism classes of basic support τ-tilting Λ-modules. For a Λ-module X, we define a full subcategory of $\operatorname{mod}\Lambda$ by

$$\operatorname{Fac} X = \{Y \in \operatorname{mod}\Lambda \mid \text{There exists an epimorphism } X^n \to Y \text{ for some } n \geq 0\}.$$

Now we define the partial order on $\operatorname{s\tau-tilt}\Lambda$ as follows:

Definition 2.13. For basic support τ-tilting Λ-modules T, U, we write $T \leq U$ if $\operatorname{Fac} T \subseteq \operatorname{Fac} U$. Then the relation $\leq$ gives a partial order on the set $\operatorname{s\tau-tilt}\Lambda$ by [**AIR**, Theorem 2.7]. We call this partial order a *generation order.*

Clearly Λ is a unique maximal element and 0 is a unique minimal element in $\operatorname{s\tau-tilt}\Lambda$. We now recall the Hasse quiver of general posets.

Definition 2.14. The *Hasse quiver* $\mathrm{H}(P)$ of a poset $(P, \leq)$ is defined as follows:

(1) The vertices are the elements of the poset P.

(2) For $X, Y \in P$, there is an arrow $X \to Y$ if and only if $X > Y$ and there is no $Z \in P$ satisfying $X > Z > Y$.

The following observation is clear.

Lemma 2.15. *Two partial orders on a finite set are the same if and only if their Hasse quivers are the same.*

Now it is time to recall the mutations of support τ-tilting modules from [**AIR**].

DEFINITION 2.16. Let $T, T' \in \mathrm{s}\tau\text{-tilt}\,\Lambda$. We call T' a *mutation of* T if T and T' have the same indecomposable direct summands except one. Precisely speaking, one of the following three cases occurs, where (T, P) and (T', P') are the support τ-tilting pairs.

(1) $T = V \oplus X$ and $T' = V \oplus X'$ with $X \ncong X'$ indecomposable;

(2) $T = T' \oplus X$ and $P' = P \oplus Q'$ with X and Q' indecomposable;

(3) $T' = T \oplus X'$ and $P = P' \oplus Q$ with X' and Q indecomposable.

We call T' a mutation of T *at* X in cases (1) and (2), and *at* Q in case (3). It is uniquely determined by T and the indecomposable direct summand X or Q of T or P respectively.

We call T' a *left mutation* (resp. *right mutation*) of T if $\mathrm{Fac}\,T' \subsetneq \mathrm{Fac}\,T$ (resp. $\mathrm{Fac}\,T' \supsetneq \mathrm{Fac}\,T$).

In the following we give a method of calculating left mutations of support τ-tilting modules due to Adachi, Iyama and Reiten [**AIR**].

THEOREM 2.17 ([**AIR**, Theorem 2.30], [**Zh**, Theorem 1.2]). *Let $T = V \oplus X$ be a basic τ-tilting Λ-module which is the Bongartz completion of V, where X is indecomposable. Let $X \xrightarrow{f} V' \xrightarrow{g} Y \to 0$ be an exact sequence, where f is a minimal left $(\mathrm{add}\,V)$-approximation. Then Y is either indecomposable or zero, and $V \oplus Y$ is a left mutation of T at X in both cases.*

Now let us recall the relationship between mutations and the Hasse quiver, which is given in [**HaU**], [**RS**] for $\mathrm{tilt}\,\Lambda$ and in [**AIR**] for $\mathrm{s}\tau\text{-tilt}\,\Lambda$.

THEOREM 2.18. *Let $T, U \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ (resp. $\mathrm{tilt}\,\Lambda$). The following are equivalent.*

(1) *T is a left mutation of U.*

(2) *U is a right mutation of T.*

(3) *$U > T$ and there is no $V \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ (resp. $\mathrm{tilt}\,\Lambda$) such that $U > V > T$.*

(4) *There is an arrow from U to T in $\mathrm{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ (resp. $\mathrm{H}(\mathrm{tilt}\,\Lambda)$).*

The following result [**AIR**, Corollary 2.38] gives a method of judging an algebra to be τ-tilting finite.

PROPOSITION 2.19. *If $\mathrm{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ admits a finite connected component C, then $\mathrm{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda) = \mathrm{C}$.*

3. Tilting modules over the Auslander algebra of $K[x]/(x^n)$.

Throughout this section, let $R = K[x]/(x^n)$ be a factor algebra of the polynomial ring $K[x]$ with $n \geq 1$, and Λ the Auslander algebra of R. Then the AR-quiver of R is

$$K \rightleftarrows K[x]/(x^2) \rightleftarrows K[x]/(x^3) \rightleftarrows \cdots \rightleftarrows K[x]/(x^{n-1}) \rightleftarrows K[x]/(x^n),$$

and the Auslander algebra Λ is presented by the quiver

$$1 \underset{b_2}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_3}{\overset{a_2}{\rightleftarrows}} 3 \underset{b_4}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-2}}{\rightleftarrows}} n-1 \underset{b_n}{\overset{a_{n-1}}{\rightleftarrows}} n$$

with relations $a_1b_2 = 0$ and $a_ib_{i+1} = b_ia_{i-1}$ for any $2 \le i \le n-1$. In this section, we classify all tilting Λ-modules.

Denote by $\{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents of Λ and denote by $P_i = e_i\Lambda$ (resp. $P^i = \Lambda e_i$) the indecomposable projective Λ-module (resp. Λ^{op}-module). It is easy to see that $P_1, P_2, \dots, P_n$ have the following composition series (see $n = 4$ for example).

$$\left[\, P_1 \mid P_2 \mid P_3 \mid P_4 \,\right] = \left[\begin{array}{c|c|c|c} \begin{smallmatrix} 1&&&\\ &2&&\\ &&3&\\ &&&4 \end{smallmatrix} & \begin{smallmatrix} &2&&\\ 1&&3&\\ &2&&4\\ &&3&\\ &&&4 \end{smallmatrix} & \begin{smallmatrix} &&3&\\ &2&&4\\ 1&&3&\\ &2&&4\\ &&3&\\ &&&4 \end{smallmatrix} & \begin{smallmatrix} &&&4\\ &&3&\\ &2&&4\\ 1&&3&\\ &2&&4\\ &&3&\\ &&&4 \end{smallmatrix} \end{array}\right].$$

For $1 \le i \le n$, we denote by I_i the two-sided ideal generated by $1 - e_i$. This is a maximal left ideal and also a maximal right ideal since there are no loops at the vertex i. Thus we have direct sum decompositions

$$I_i = P_1 \oplus \cdots \oplus (\operatorname{rad} P_i) \oplus \cdots \oplus P_n = P^1 \oplus \cdots \oplus (\operatorname{rad} P^i) \oplus \cdots \oplus P^n.$$

Furthermore, for $1 \le i \le n$, we define a (Λ, Λ)-bimodule by $S_i = \Lambda/I_i$. Clearly we have the following.

Proposition 3.1. *Let Λ be the Auslander algebra of $K[x]/(x^n)$. Then one gets the following.*

(1) *As a Λ-module $S_i \cong P_i/\operatorname{rad} P_i$ is simple. As a Λ^{op}-module $S_i \cong P^i/\operatorname{rad} P^i$ is simple.*

(2) *There exists an isomorphism $P_n \cong DP^n$ of Λ-modules. Thus P_n is a projective-injective Λ-module.*

(3) *For $1 \le i \le n-1$, there exist minimal projective resolutions of Λ-modules*

$$0 \to P_i \to P_{i-1} \oplus P_{i+1} \to P_i \to S_i \to 0 \quad \textit{and} \quad 0 \to P_i \to P_{i-1} \oplus P_{i+1} \to \operatorname{rad} P_i \to 0.$$

(4) *There exist minimal projective resolutions of Λ-modules*

$$0 \to P_{n-1} \to P_n \to S_n \to 0 \quad \textit{and} \quad 0 \to P_{n-1} \to \operatorname{rad} P_n \to 0.$$

Proof. (1) is clear. (3) and (4) are immediate from Proposition 2.3 and the AR-quiver of R above.

(2) Since R is a symmetric K-algebra, we have an isomorphism $\operatorname{Hom}_R(-, R) \cong D\operatorname{Hom}_R(R, -)$ of functors. This gives the desired isomorphism. $\square$

We need the following properties of tilting Λ-modules.

Lemma 3.2. *Let X be a Λ-module. For $1 \le i \le n-1$, there exist isomorphisms $\mathrm{Ext}^2_\Lambda(S_i, X) \cong X \otimes_\Lambda S_i$ and $\mathrm{Ext}^1_\Lambda(S_i, X) \cong \mathrm{Tor}^1_\Lambda(X, S_i)$. If X is tilting, then precisely one of them is zero.*

Proof. Since each indecomposable nonprojective R-module is τ-stable, we have $\mathrm{Ext}^j_\Lambda(S_i, X) \cong \mathrm{Tor}^\Lambda_{2-j}(X, S_i)$ for $j = 1, 2$ by Lemma 2.4(2). The latter statement follows from Proposition 2.9(3). □

Now we are in a position to show the following proposition.

Proposition 3.3. *For $1 \le i \le n-1$, I_i is a tilting Λ-module and a tilting Λ^{op}-module.*

Proof. We only prove the case of a Λ-module since the case of a Λ^{op}-module is similar. By definition, we have $I_i = (\bigoplus_{j\neq i} P_j) \oplus \mathrm{rad}\, P_i$.

(T1) By Proposition 3.1(3), we have $\mathrm{proj.dim}\,\mathrm{rad}\, P_i \le 1$. Thus $\mathrm{proj.dim}\, I_i \le 1$.

(T2) It suffices to show that $\mathrm{Ext}^1_\Lambda(\mathrm{rad}\, P_i, I_i) = 0$. Since there exists an exact sequence $0 \to \mathrm{rad}\, P_i \to P_i \to S_i \to 0$, we have $\mathrm{Ext}^2_\Lambda(S_i, I_i) \cong \mathrm{Ext}^1_\Lambda(\mathrm{rad}\, P_i, I_i)$. By Lemma 3.2, we have $\mathrm{Ext}^2_\Lambda(S_i, I_i) \cong I_i \otimes_\Lambda S_i$. On the other hand, we have $P_j \otimes_\Lambda S_i = e_j\Lambda \otimes_\Lambda S_i = e_j S_i = 0$ for any $j \neq i$. By Proposition 3.1(3), there exists an exact sequence $0 = (P_{i-1} \oplus P_{i+1}) \otimes_\Lambda S_i \to (\mathrm{rad}\, P_i) \otimes_\Lambda S_i \to 0$. Thus we have $(\mathrm{rad}\, P_i) \otimes_\Lambda S_i = 0$ and $I_i \otimes_\Lambda S_i = 0$.

(T3) By Proposition 3.1(3), there exists an exact sequence $0 \to \Lambda \to (\bigoplus_{j\neq i} P_j) \oplus P_{i-1} \oplus P_{i+1} \to \mathrm{rad}\, P_i \to 0$. The middle and right terms of this sequence are contained in $\mathrm{add} I_i$. □

Notice that I_n is not a tilting Λ-module. In fact $I_n = (\bigoplus_{i=1}^{n-1} P_i) \oplus (\mathrm{rad}\, P_n)$ and $\mathrm{rad}\, P_n \cong P_{n-1}$ hold by Proposition 3.1(4), and hence $|I_n| = n-1$. This is not possible for tilting Λ-modules.

To show that any multiplication of ideals $I_1, \ldots, I_{n-1}$ is a tilting Λ-module, we now prepare the following.

Proposition 3.4. (1) *For $1 \le i \le n$, we have $\mathrm{Hom}_\Lambda(I_i, S_i) = 0$.*

(2) *For $1 \le i \le n-1$, the left multiplication $\Lambda \to \mathrm{End}_\Lambda(I_i)$ and the right multiplication $\Lambda^{\mathrm{op}} \to \mathrm{End}_{\Lambda^{\mathrm{op}}}(I_i)$ are isomorphisms.*

Proof. (1) For $j \neq i$, we have $\mathrm{Hom}_\Lambda(P_j, S_i) = 0$. Further, by Proposition 3.1(3) and (4), one gets $\mathrm{Hom}_\Lambda(\mathrm{rad}\, P_i, S_i) = 0$. Thus we have $\mathrm{Hom}_\Lambda(I_i, S_i) = 0$.

(2) Applying $\mathrm{Hom}_\Lambda(-, \Lambda)$ to a short exact sequence

$$0 \to I_i \to \Lambda \to S_i \to 0 \tag{3.1}$$

yields a long exact sequence $0 \to \mathrm{Hom}_\Lambda(S_i, \Lambda) \to \mathrm{Hom}_\Lambda(\Lambda, \Lambda) \to \mathrm{Hom}_\Lambda(I_i, \Lambda) \to \mathrm{Ext}^1_\Lambda(S_i, \Lambda) \to 0$. Then by Lemma 2.4, we have $\mathrm{Hom}_\Lambda(S_i, \Lambda) = \mathrm{Ext}^1_\Lambda(S_i, \Lambda) = 0$, and hence $\mathrm{Hom}_\Lambda(I_i, \Lambda) \cong \mathrm{Hom}_\Lambda(\Lambda, \Lambda) \cong \Lambda$. On the other hand, applying $\mathrm{Hom}_\Lambda(I_i, -)$ to the short exact sequence (3.1), one gets an exact sequence $0 \to \mathrm{Hom}_\Lambda(I_i, I_i) \to \mathrm{Hom}_\Lambda(I_i, \Lambda) \to \mathrm{Hom}_\Lambda(I_i, S_i)$. Using (1), we have $\mathrm{End}_\Lambda(I_i) \cong \mathrm{Hom}_\Lambda(I_i, \Lambda) \cong \Lambda$. □

From the argument above, we have the following proposition on the multiplication of tilting Λ-modules.

Proposition 3.5. *Let T be a tilting Λ-module and $1 \le i \le n-1$. Then we have the following.*

(1) *If $TI_i \neq T$, then $TI_i \cong T \otimes_\Lambda I_i = T \otimes^{\mathbf{L}}_\Lambda I_i$.*

(2) *TI_i is a tilting Λ-module, and $\mathrm{End}_\Lambda(TI_i) \cong \mathrm{End}_\Lambda(T)$.*

Proof. (1) Since $TI_i \neq T$, then $T \otimes_\Lambda S_i \cong T/TI_i \neq 0$, and we have $\mathrm{Tor}_1^\Lambda(T, S_i) = 0$ by Proposition 2.9(3). Applying $T \otimes_\Lambda -$ to the short exact sequence $0 \to I_i \to \Lambda \to S_i \to 0$, one gets an exact sequence $0 = \mathrm{Tor}_1^\Lambda(T, S_i) \to T \otimes_\Lambda I_i \to T \otimes_\Lambda \Lambda \cong T$. Thus the natural map $T \otimes_\Lambda I_i \to T$ is injective and has the image TI_i. Thus we obtain $T \otimes_\Lambda I_i \cong TI_i$. Moreover, we have $\mathrm{Tor}_j^\Lambda(T, I_i) \cong \mathrm{Tor}_{j+1}^\Lambda(T, S_i) = 0$ for $j \ge 1$ since proj.dim $T \le 1$. Thus $T \otimes_\Lambda I_i = T \otimes^{\mathbf{L}}_\Lambda I_i$.

(2) If $TI_i = T$, then the assertion is clear. Now assume that $TI_i \neq T$. Since we have $\mathrm{End}_\Lambda(I_i) \cong \Lambda$ by Proposition 3.4, $T \otimes_\Lambda I_i \cong TI_i$ is a tilting module with $\mathrm{End}_\Lambda(T) \cong \mathrm{End}_\Lambda(TI_i)$ by (1) and Proposition 2.10(1). $\square$

Denote by $\langle I_1, \dots, I_{n-1} \rangle$ the set of ideals of Λ given by products of $I_1, \dots, I_{n-1}$, where the empty product Λ is also contained in this set. Now we can state the following result.

Theorem 3.6. *Any ideal T in $\langle I_1, \dots, I_{n-1} \rangle$ is a basic tilting Λ-module and a basic tilting Λ^{op}-module. The left multiplication $\Lambda \to \mathrm{End}_\Lambda(T)$ and the right multiplication $\Lambda^{\mathrm{op}} \to \mathrm{End}_{\Lambda^{\mathrm{op}}}(T)$ are isomorphisms.*

Proof. We only prove the case of a Λ-module since the case of a Λ^{op}-module is similar.

By Proposition 3.3, each of $I_1, \dots, I_{n-1}$ is a tilting Λ-module such that the left multiplication $\Lambda \to \mathrm{End}_\Lambda(I_i)$ is an isomorphism. Assume that $T = I_{i_1} I_{i_2} \cdots I_{i_{k-1}}$ is a tilting Λ-module such that the left multiplication $\Lambda \to \mathrm{End}_\Lambda(T)$ is an isomorphism. Then, according to Proposition 3.5(2), we obtain that TI_{i_k} is a tilting Λ-module such that the left multiplication $\Lambda \to \mathrm{End}_\Lambda(TI_{i_k})$ is an isomorphism. In particular, TI_{i_k} is basic. Thus we get the assertion inductively. $\square$

By Theorem 3.6, any element in $\langle I_1, \dots, I_{n-1} \rangle$ is a basic tilting Λ-module. In the following we show the converse, that is, all basic tilting Λ-modules are in $\langle I_1, \dots, I_{n-1} \rangle$. For this aim, we start with the following.

Proposition 3.7. *Let T be a tilting Λ-module, and $1 \le i \le n-1$. Then we have the following*:

(1) $\mathrm{Hom}_\Lambda(S_i, T) = 0$.

(2) proj.dim $\mathrm{Hom}_\Lambda(I_i, T) \le 1$.

(3) *There exist natural inclusions $T \subseteq \mathrm{Hom}_\Lambda(I_i, T) \subseteq T^{**} = \mathrm{Hom}_\Lambda(I_i, T)^{**}$.*

(4) $\mathrm{Hom}_\Lambda(I_i,T)/T \cong \mathrm{Ext}^1_\Lambda(S_i,T)$. *If* $T \subsetneq \mathrm{Hom}_\Lambda(I_i,T)$, *then* $\mathrm{Hom}_\Lambda(I_i,T)I_i = T$.

(5) $\mathrm{Hom}_\Lambda(I_i,T)$ *is a tilting* Λ*-module, and* $\mathrm{End}_\Lambda(\mathrm{Hom}_\Lambda(I_i,T)) \cong \mathrm{End}_\Lambda(T)$ *holds.*

(6) *If* T *is not a projective* Λ*-module, then there exists* $1 \le i \le n-1$ *such that* $T \subsetneq \mathrm{Hom}_\Lambda(I_i,T)$.

Proof. We firstly note by Lemma 2.6 that T^{**} is a projective Λ-module. By Lemma 2.4, we have $\mathrm{Ext}^j_\Lambda(S_i,\Lambda) = 0 = \mathrm{Ext}^j_\Lambda(S_i,T^{**})$ for $j \neq 2$. These facts will be used freely in this proof.

(1) By Lemma 2.7, we have an exact sequence

$$0 \to T \xrightarrow{\varphi_T} T^{**} \to T^{**}/T \to 0. \tag{3.2}$$

Applying the functor $\mathrm{Hom}_\Lambda(S_i,-)$, one gets $\mathrm{Hom}_\Lambda(S_i,T) = 0$.

(2) Applying $\mathrm{Hom}_\Lambda(-,T^{**})$ to the short exact sequence $0 \to I_i \to \Lambda \to S_i \to 0$, we have an exact sequence $0 = \mathrm{Hom}_\Lambda(S_i,T^{**}) \to \mathrm{Hom}_\Lambda(\Lambda,T^{**}) \to \mathrm{Hom}_\Lambda(I_i,T^{**}) \to \mathrm{Ext}^1_\Lambda(S_i,T^{**}) = 0$. Thus $\mathrm{Hom}_\Lambda(I_i,T^{**}) \cong T^{**}$ is a projective Λ-module. Then applying the functor $\mathrm{Hom}_\Lambda(I_i,-)$ to the sequence (3.2), one gets that $\mathrm{Hom}_\Lambda(I_i,T)$ is a submodule of the projective Λ-module $\mathrm{Hom}_\Lambda(I_i,T^{**})$. Since $\mathrm{gl.dim}\,\Lambda \le 2$, any submodule of a projective module has projective dimension at most 1.

(3) Applying $\mathrm{Hom}_\Lambda(-,T)$ to the exact sequence $0 \to I_i \to \Lambda \to S_i \to 0$ of (Λ,Λ)-bimodules, we obtain an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(\Lambda,T) \to \mathrm{Hom}_\Lambda(I_i,T) \to \mathrm{Ext}^1_\Lambda(S_i,T) \to 0 \to \mathrm{Ext}^1_\Lambda(I_i,T) \to \mathrm{Ext}^2_\Lambda(S_i,T) \to 0 \tag{3.3}$$

of Λ-modules by (1). Since the Λ^{op}-module S_i is annihilated by I_i, the Λ-module $\mathrm{Ext}^1_\Lambda(S_i,T)$ is annihilated by I_i and hence isomorphic to S_i^m for some $m \ge 0$. Hence (3.3) gives an exact sequence $0 \to T \to \mathrm{Hom}_\Lambda(I_i,T) \to S_i^m \to 0$. Applying $(-)^* = \mathrm{Hom}_\Lambda(-,\Lambda)$, we obtain an exact sequence $0 = (S_i^m)^* \to \mathrm{Hom}_\Lambda(I_i,T)^* \to T^* \to \mathrm{Ext}^1_\Lambda(S_i^m,\Lambda) = 0$. In particular, we have $T^{**} \cong \mathrm{Hom}_\Lambda(I_i,T)^{**}$ and the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "\varphi_T"] & \mathrm{Hom}_\Lambda(I_i,T) \arrow[r] \arrow[d, "\varphi_{\mathrm{Hom}_\Lambda(I_i,T)}"] & S_i^m \arrow[r] & 0. \\
 & T^{**} \arrow[r, equal] & \mathrm{Hom}_\Lambda(I_i,T)^{**} & &
\end{tikzcd}$$

By (2) and Lemma 2.7, $\varphi_{\mathrm{Hom}_\Lambda(I_i,T)}$ is a monomorphism and hence (3) follows.

(4) The former assertion is immediate from the exact sequence (3.3). Since $\mathrm{Ext}^1_\Lambda(S_i,T) \cong S_i^m$ is annihilated by I_i, we have $TI_i \subseteq \mathrm{Hom}_\Lambda(I_i,T)I_i \subseteq T$. For the latter assertion, notice that $\mathrm{Tor}_1^\Lambda(T,S_i) \cong \mathrm{Ext}^1_\Lambda(S_i,T) \neq 0$ by Lemma 3.2. Since $T/TI_i \cong T \otimes_\Lambda S_i = 0$ holds by Lemma 2.9(3), we obtain $\mathrm{Hom}_\Lambda(I_i,T)I_i = T$.

(5) If $T = \mathrm{Hom}_\Lambda(I_i,T)$, then it is obvious. Assume that $T \neq \mathrm{Hom}_\Lambda(I_i,T)$. By (2) and Propositions 3.4(2) and 2.10(2), it suffices to prove that $\mathrm{Ext}^j_\Lambda(I_i,T) = 0$ for any $j > 0$. We only have to consider the case $j = 1$ since $\mathrm{proj.dim}\, I_i \le 1$. We have

$\mathrm{Ext}^1_\Lambda(S_i,T)\neq 0$ by (4), and hence $\mathrm{Ext}^1_\Lambda(I_i,T)\cong \mathrm{Ext}^2_\Lambda(S_i,T)=0$ holds by Lemma 3.2. Thus (5) follows.

(6) By our assumption and Lemma 2.6, $T\neq T^{**}$ holds. By Lemma 2.7 and Proposition 3.1, we can take a simple submodule S_i of T^{**}/T for some $1\leq i\leq n-1$. Applying $\mathrm{Hom}_\Lambda(S_i,-)$ to the exact sequence (3.2), we get an exact sequence $0=\mathrm{Hom}_\Lambda(S_i,T^{**})\to \mathrm{Hom}_\Lambda(S_i,T^{**}/T)\to \mathrm{Ext}^1_\Lambda(S_i,T)$. Thus $\mathrm{Ext}^1_\Lambda(S_i,T)\neq 0$ by our choice of S_i. Thus $\mathrm{Hom}_\Lambda(I_i,T)/T\cong \mathrm{Ext}^1_\Lambda(S_i,T)\neq 0$ holds by (4), and we have $T\subsetneq \mathrm{Hom}_\Lambda(I_i,T)$. □

Lemma 3.8. *Let $T\in\langle I_1,\dots,I_{n-1}\rangle$, and let $f_T:T\to\Lambda$ be a natural inclusion. Then in the following commutative diagram, φ_Λ and f_T^{**} are isomorphisms.*

$$\begin{tikzcd} T \arrow[r, "\varphi_T"] \arrow[d, "f_T"] & T^{**} \arrow[d, "f_T^{**}"] \\ \Lambda \arrow[r, "\varphi_\Lambda"] & \Lambda^{**}. \end{tikzcd}$$

Proof. Since Λ is projective, it is clear that φ_Λ is an isomorphism.

Any composition factor of the Λ-module Λ/T has a form S_i for some $1\leq i\leq n-1$. By Lemma 2.4, we have $\mathrm{Ext}^j_\Lambda(\Lambda/T,\Lambda)=0$ for $j\neq 2$. Applying $(-)^*=\mathrm{Hom}_\Lambda(-,\Lambda)$ to the exact sequence $0\to T\xrightarrow{f_T}\Lambda\to\Lambda/T\to 0$, we have an exact sequence $0=(\Lambda/T)^*\to\Lambda^*\xrightarrow{f_T^*}T^*\to\mathrm{Ext}^1_\Lambda(\Lambda/T,\Lambda)=0$. Thus f_T^* is an isomorphism and hence f_T^{**} is an isomorphism. □

Now we are in a position to state our first main result in this section.

Theorem 3.9. *Let Λ be the Auslander algebra of $K[x]/(x^n)$. Then*

(1) *For any tilting Λ-module T, there exists $U\in\langle I_1,\dots,I_{n-1}\rangle$ such that* $\mathrm{add}T=\mathrm{add}U$.

(2) *If two elements T and U in $\langle I_1,\dots,I_{n-1}\rangle$ are isomorphic as Λ-modules, then $T=U$.*

(3) *The set* $\mathrm{tilt}\Lambda$ *is given by $\langle I_1,\dots,I_{n-1}\rangle$.*

(4) *The statements* (1), (2) *and* (3) *hold also for Λ^{op}-modules.*

Proof. (1) By Proposition 3.7(3), (4), (5) and (6), there exists a finite sequence of tilting Λ-modules

$$T=T_0\subsetneq T_1\subsetneq\cdots\subsetneq T_m=T^{**}$$

and $i_1,\dots,i_m\in\{1,\dots,n-1\}$ such that $T_{k+1}=\mathrm{Hom}_\Lambda(I_{i_{k+1}},T_k)$ and $T_k=T_{k+1}I_{i_{k+1}}$ for any $0\leq k\leq m-1$. In particular, we have $T=T_1I_{i_1}=T_2I_{i_2}I_{i_1}=\cdots=T_mI_{i_m}\cdots I_{i_1}$. Because T^{**} is a projective tilting Λ-module by Lemma 2.6, we have $\mathrm{add}T_m=\mathrm{add}\Lambda$. Thus $\mathrm{add}T=\mathrm{add}U$ holds for $U:=I_{i_m}\cdots I_{i_1}\in\langle I_1,\dots,I_{n-1}\rangle$.

(2) For $T,U\in\langle I_1,\dots,I_{n-1}\rangle$, assume that there exists a Λ-module isomorphism $g:T\cong U$.

By Lemma 3.8, there exists a commutative diagram

$$\begin{tikzcd} & T \arrow[r, "g", "\sim"'] \arrow[dl, "f_T"'] \arrow[d, "\varphi_T"] & U \arrow[d, "\varphi_T"] \arrow[dr, "f_U"] & \\ \Lambda & T^{**} \arrow[l, "e_T"] \arrow[r, "\sim", "g^{**}"'] & U^{**} \arrow[r, "e_U"'] & \Lambda \end{tikzcd}$$

where $e_T := \varphi_\Lambda^{-1} f_T^{**}$ and $e_U := \varphi_\Lambda^{-1} f_U^{**}$ are isomorphisms. Putting $h = e_U g^{**} e_T^{-1} : \Lambda \to \Lambda$, we have a commutative diagram

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "f_T"] & U \arrow[d, "f_U"] \\ \Lambda \arrow[r, "\sim", "h"'] & \Lambda. \end{tikzcd}$$

Since h is given by the left multiplication of an invertible element $x \in \Lambda$, so is g. Since T is an ideal of Λ, we have $U = xT = T$.

(3) This is a consequence of (1), (2) and Theorem 3.6.

(4) One can prove it similarly to (1), (2) and (3). $\square$

The mutations of tilting Λ-modules are described by the following result. Notice that we use the structure of Λ^{op}-modules when we consider mutations of Λ-modules.

PROPOSITION 3.10. *Let $T \in \langle I_1, \ldots, I_{n-1} \rangle$.*

(1) *For each $1 \leq i \leq n-1$, precisely one of the following statements* (a) *and* (b) *holds.*

   (a) *$I_i T \neq T$ and $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T) = T$ hold, and $I_i T = I_i \otimes_\Lambda T$ is a left mutation of T at $e_i T$.*

   (b) *$I_i T = T$ and $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T) \neq T$ hold, and $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T)$ is a right mutation of T at $e_i T$.*

(2) *All mutations of T in* $\mathrm{tilt}\,\Lambda$ *are of the form* (1). *In particular, T has precisely $n-1$ mutations in* $\mathrm{tilt}\,\Lambda$.

(3) *The corresponding statements to* (1) *and* (2) *hold for Λ^{op}-modules.*

PROOF. (1) Applying Proposition 3.5(2) and Proposition 3.7(5) to the tilting Λ^{op}-module T, we have that $I_i T$ and $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T)$ are tilting Λ^{op}-modules with $\mathrm{End}_{\Lambda^{\mathrm{op}}}(I_i T) \cong \mathrm{End}_{\Lambda^{\mathrm{op}}}(T) \cong \mathrm{End}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T))$. Since $\mathrm{End}_{\Lambda^{\mathrm{op}}}(T) \cong \Lambda^{\mathrm{op}}$ holds by Theorem 3.6, we have that $I_i T$ and $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T)$ are tilting Λ-modules. Further we know that

$$I_i T = \bigoplus_{j=1}^{n} e_j I_i T \quad \text{and} \quad \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T) = \bigoplus_{j=1}^{n} \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i e_j, T).$$

Since $e_j I_i = e_j \Lambda$ and $I_i e_j = \Lambda e_j$ hold for any $j \neq i$, the indecomposable direct summands of $I_i T$ (resp. $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T)$) coincide with those of T except one. By Theorem 2.18, $I_i T$ (resp. $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T)$) is either isomorphic to T or a mutation of T. We have

$$I_iT \cong T \iff S_i \otimes_\Lambda T = 0,$$
$$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T) \cong T \iff \mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(S_i, T) = 0$$

by Proposition 3.7. Thus precisely one of these conditions holds by Lemma 3.2.

It remains to decide whether the mutation is left or right. We only have to show $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T) \geq T \geq I_iT$. Taking an epimorphism $\Lambda^m \to I_i$ of Λ-modules, we have an epimorphism $T^m \to I_iT$. Thus, we have $T \geq I_iT$. If $U := \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T) \supsetneq T$, then we have $I_iU = T$ by Proposition 3.7. Thus we have $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T) = U \geq T$.

(2) Any basic tilting Λ-module has precisely n indecomposable direct summands. Since P_n is injective by Proposition 3.1, it is a direct summand of any tilting Λ-module. Therefore the number of mutations of T in $\mathrm{tilt}\,\Lambda$ is at most $n-1$, while we have at least $n-1$ mutations in $\mathrm{tilt}\,\Lambda$ by (1).

(3) One can prove it similarly to (1) and (2). $\square$

Immediately we have the following description of the Hasse quiver of tilting Λ-modules.

Corollary 3.11. *The Hasse quiver of* $\mathrm{tilt}\,\Lambda$ *has the set* $\langle I_1, \ldots, I_{n-1}\rangle$ *of vertices. All arrows starting or ending at* $T \in \langle I_1, \ldots, I_{n-1}\rangle$ *are given by*

$$\mu_i(T) := \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_i, T) \longrightarrow T \quad \text{if} \quad T = I_iT,$$
$$T \longrightarrow \mu_i(T) := I_iT \quad \text{if} \quad T \neq I_iT$$

for each $1 \leq i \leq n-1$, *where* $\mu_i(T)$ *is the mutation of* T *at the direct summand* e_iT (*Definition* 2.16). *Thus the number of arrows starting or ending at* T *is precisely* $n-1$.

We have shown that the set $\mathrm{tilt}\,\Lambda$ is given by $\langle I_1, \ldots, I_{n-1}\rangle$. In the following we give an explicit description of this set. Let us start with the following elementary observation.

Proposition 3.12. *Let* A *be a basic finite dimensional algebra,* $\{e_1, \ldots, e_n\}$ *a complete set of orthogonal primitive idempotents of* A, *and* $S_1, \ldots, S_n$ *the corresponding simple* A-*modules. For a subset* J *of* $\{1, \ldots, n\}$, *we put*

$$e_J = \sum_{i \in J} e_i \quad \text{and} \quad I_J = A(1-e_J)A.$$

Then for any $X \in \mathrm{mod}\,A$, *we have that* XI_J *is the minimum amongst submodules* Y *of* X *satisfying the following condition*:

($\sharp$) *Any composition factor of* X/Y *has the form* S_i *for some* $i \in J$.

Proof. Since $\mathrm{Hom}_A((1-e_J)A, X) \cong X(1-e_J)$, we have

$$XI_J = X(1-e_J)A = \sum_{f \in \mathrm{Hom}_A((1-e_J)A, X)} \mathrm{Im}\, f.$$

The condition ($\sharp$) holds if and only if $\mathrm{Hom}_A((1-e_J)A, X/Y) = 0$ holds if and only if $\mathrm{Im}\, f \subseteq Y$ holds for any $f \in \mathrm{Hom}_A((1-e_J)A, X)$ if and only if $XI_J \subseteq Y$. $\square$

We have the following relations for the multiplication of ideals $I_1, \dots, I_{n-1}$.

PROPOSITION 3.13. *Let I_i be the maximal ideal of Λ as above. Then the following relations hold for any $1 \le i, j \le n-1$.*

(1) $I_i^2 = I_i$.

(2) *If* $|i-j| \ge 2$, *then* $I_i I_j = I_j I_i$.

(3) *If* $|i-j| = 1$, *then* $I_i I_j I_i = I_j I_i I_j$.

PROOF. (1) The assertion is clear from $I_i = \Lambda(1-e_i)\Lambda$.

(2)(3) For $1 \le i \ne j \le n-1$, put $I_{i,j} = \Lambda(1-e_i-e_j)\Lambda$. Removing all vertices except i and j from the quiver with relations of Λ, we have the quiver with relations of $\Lambda/I_{i,j}$. In particular, if $|i-j| \ge 2$, then $\Lambda/I_{i,j} \cong K \times K$. If $|i-j| = 1$, then $\Lambda/I_{i,j}$ is given by the quiver $i \underset{b}{\overset{a}{\rightleftarrows}} j$ with relations $ab = 0 = ba$ and hence $\Lambda/I_{i,j} = \left[\begin{smallmatrix} i \\ & j\end{smallmatrix} \middle| \begin{smallmatrix} & j \\ i & \end{smallmatrix}\right]$.

We prove (2). By Proposition 3.12, $I_i I_j \supseteq I_{i,j}$. Since $\Lambda/I_{i,j} \cong K \times K$, we have $I_i I_j / I_{i,j} = 0$. Hence $I_i I_j = I_{i,j}$ holds, and similarly we have $I_j I_i = I_{i,j}$. Thus $I_i I_j = I_{i,j} = I_j I_i$.

We prove (3). By Proposition 3.12, $I_i I_j I_i \supseteq I_{i,j}$. Since $\Lambda/I_{i,j} = \left[\begin{smallmatrix} i \\ & j\end{smallmatrix} \middle| \begin{smallmatrix} & j \\ i & \end{smallmatrix}\right]$, we have $I_i I_j I_i / I_{i,j} = 0$. Hence $I_i I_j I_i = I_{i,j}$ holds, and similarly we have $I_j I_i I_j = I_{i,j}$. Thus $I_i I_j I_i = I_{i,j} = I_j I_i I_j$. $\square$

Now we recall some well-known properties of the symmetric groups. We consider the action of $\mathfrak{S}_n$ on $\mathbb{R}^n$ given by permuting the standard basis $e_1, \dots, e_n$. Then $\mathfrak{S}_n$ acts on the subspace

$$V := \left\{ x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = 0 \right\},$$

which has a basis $\alpha_i := e_i - e_{i+1}$ with $1 \le i \le n-1$. Clearly the action of $\mathfrak{S}_n$ on V is faithful, and we have an injective homomorphism $\mathfrak{S}_n \to GL(V)$ called the *geometric representation*.

Let s_i be the transposition $(i, i+1) \in \mathfrak{S}_n$. The following elementary fact plays an important role in the proof of our main theorem.

PROPOSITION 3.14. *Let $\mathfrak{S}_n$ be the symmetric group of degree n and $\mathfrak{S}_n \ni w$. Then we have the following*:

(1) [**BjB**, Theorem 3.3.1] *Any expression $s_{i_1} s_{i_2} \cdots s_{i_l}$ of w can be transformed into a reduced expression of w by applying the following operations* (a), (b), (c) *repeatedly.*

   (a) *Remove $s_i s_i$ in the expression.*

   (b) *Replace $s_i s_j$ with $|i-j| \ge 2$ by $s_j s_i$ in the expression.*

   (c) *Replace $s_i s_j s_i$ with $|i-j| = 1$ by $s_j s_i s_j$ in the expression.*

(2) [**BjB**, Theorem 3.3.1] *Every two reduced expressions of w can be transformed into each other by applying the operations* (b) *and* (c) *repeatedly.*

(3) *If* $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ *is a reduced expression, then* $s_{i_1} \cdots s_{i_k}(\alpha_{i_{k+1}})$ *is a positive root for any* $0 \le k \le l-1$.

We also need the following proposition.

Proposition 3.15. *There exists a well-defined surjective map* $\mathfrak{S}_n \to \langle I_1, \ldots, I_{n-1} \rangle$ *which maps* w *to* $I(w) = I_{i_1} \cdots I_{i_l}$, *where* $w = s_{i_1} \cdots s_{i_l}$ *is an arbitrary reduced expression.*

Proof. First, we show that the map is well-defined. Take two reduced expressions $w = s_{i_1} \cdots s_{i_l} = s_{j_1} \cdots s_{j_l}$ of w. These two expressions are transformed into each other by the operation (b) and (c) in Proposition 3.14. Then by Proposition 3.13, we obtain $I_{i_1} \cdots I_{i_l} = I_{j_1} \cdots I_{j_l}$.

Next we show that the map is surjective. For any $I \in \langle I_1, \ldots, I_{n-1} \rangle$, we take a minimal number l such that $I = I_{i_1} \cdots I_{i_l}$ holds for some $i_1, \ldots, i_l \in \{1, \ldots, n-1\}$. Now we put $w := s_{i_1} \cdots s_{i_l}$. This expression is transformed into a reduced expression of w by applying (a), (b) and (c) in Proposition 3.14. Since l is minimal, then (a) would not happen. Therefore $w = s_{i_1} \cdots s_{i_l}$ is a reduced expression and we have $I = I(w)$. $\square$

Since $I(w)$ is a tilting Λ-module with $\mathrm{End}_\Lambda(I(w)) \cong \Lambda$ for any $w \in \mathfrak{S}_n$ by Proposition 3.15, we have an autoequivalence

$$- \otimes^{\mathbf{L}}_\Lambda I(w) : \mathrm{D^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda)$$

whose quasi-inverse is given by $\mathbf{R}\mathrm{Hom}_\Lambda(I(w), -)$. We define a thick subcategory $\mathcal{T}$ of $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$ by

$$\mathcal{T} := \{X \in \mathrm{D^b}(\mathrm{mod}\,\Lambda) \mid \forall i \in \mathbb{Z}\ H^i(X)e_n = 0\}.$$

The Grothendieck group $K_0(\mathcal{T})$ is a free abelian group with basis $[S_1], \ldots, [S_{n-1}]$. We identify V with $\mathbb{R} \otimes_{\mathbb{Z}} K_0(\mathcal{T})$ by $\alpha_i = [S_i]$ for any $1 \le i \le n-1$.

Lemma 3.16. (1) *We have an induced autoequivalence* $- \otimes^{\mathbf{L}}_\Lambda I(w) : \mathcal{T} \to \mathcal{T}$.

(2) *We have* $[- \otimes^{\mathbf{L}}_\Lambda I_i] = s_i$ *in* $GL(V)$ *for any* $1 \le i \le n-1$.

Proof. (1) We have a triangle $I(w) \to \Lambda \to \Lambda/I(w) \to I(w)[1]$ in $\mathrm{D}(\mathrm{mod}\,\Lambda^{\mathrm{op}} \otimes_K \Lambda)$. Applying $X \otimes^{\mathbf{L}}_\Lambda -$ for $X \in \mathcal{T}$, we have a triangle

$$X \otimes^{\mathbf{L}}_\Lambda I(w) \to X \to X \otimes^{\mathbf{L}}_\Lambda (\Lambda/I(w)) \to X \otimes^{\mathbf{L}}_\Lambda I(w)[1] \tag{3.4}$$

in $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$. Since both X and $X \otimes^{\mathbf{L}}_\Lambda (\Lambda/I(w))$ belong to $\mathcal{T}$, so is $X \otimes^{\mathbf{L}}_\Lambda I(w)$. Thus $\mathcal{T} \otimes^{\mathbf{L}}_\Lambda I(w) \subseteq \mathcal{T}$ holds. Similarly one can show $\mathbf{R}\mathrm{Hom}_\Lambda(I(w), \mathcal{T}) \subseteq \mathcal{T}$. Therefore the assertion follows.

(2) For $X \in \mathrm{D^b}(\mathrm{mod}\,\Lambda)$ and $Y \in \mathrm{D^b}(\mathrm{mod}\,\Lambda^{\mathrm{op}})$, let $\chi(X, Y) := \sum_{k \in \mathbb{Z}} (-1)^k \dim_K H^k(X \otimes^{\mathbf{L}}_\Lambda Y)$. Then

$$\chi(S_j, S_i) = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ 0 & |i-j| \geq 2 \end{cases}$$

holds for any $1 \leq j \leq n-1$. We have $[S_j \otimes^{\mathbf{L}}_{\Lambda} I_i] = [S_j] - [S_j \otimes^{\mathbf{L}}_{\Lambda} S_i] = [S_j] - \chi(S_j, S_i)[S_i]$ by applying (3.4) to $X = S_j$ and $w = s_i$. Thus the assertion follows easily. $\square$

We have the following key observations.

PROPOSITION 3.17. *Let $w \in \mathfrak{S}_n$ and $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ a reduced expression.*

(1) *We have $[- \otimes^{\mathbf{L}}_{\Lambda} I(w)] = w^{-1}$ in $GL(V)$.*

(2) *We have $I_{i_l} \supsetneq I_{i_{l-1}} I_{i_l} \supsetneq \cdots \supsetneq I_{i_1} \cdots I_{i_l}$ and $I(w) = I_{i_1} \otimes^{\mathbf{L}}_{\Lambda} \cdots \otimes^{\mathbf{L}}_{\Lambda} I_{i_l}$.*

(3) *Let $1 \leq j \leq n-1$. Then $l(s_j w) > l(w)$ if and only if $I(s_j w) < I(w)$.*

PROOF. The assertion (2) implies (1) since Lemma 3.16(2) implies $[- \otimes^{\mathbf{L}}_{\Lambda} I(w)] = [- \otimes^{\mathbf{L}}_{\Lambda} I_{i_l}] \circ \cdots \circ [- \otimes^{\mathbf{L}}_{\Lambda} I_{i_2}] \circ [- \otimes^{\mathbf{L}}_{\Lambda} I_{i_1}] = s_{i_l} \cdots s_{i_2} s_{i_1} = w^{-1}$.

We prove (2) inductively. This is clear for $l = 1$. For $u := s_{i_2} \cdots s_{i_l}$, we assume $I_{i_l} \supsetneq I_{i_{l-1}} I_{i_l} \supsetneq \cdots \supsetneq I_{i_2} \cdots I_{i_l}$ and $I(u) = I_{i_2} \otimes^{\mathbf{L}}_{\Lambda} \cdots \otimes^{\mathbf{L}}_{\Lambda} I_{i_l}$. Then $[S_{i_1} \otimes^{\mathbf{L}}_{\Lambda} I(u)] = u^{-1}(\alpha_{i_1}) = s_{i_l} \cdots s_{i_2}(\alpha_{i_1})$ is a positive root by Proposition 3.14(3). Hence $S_{i_1} \otimes_{\Lambda} I(u) \neq 0$ holds, and we have $I(u) \supsetneq I_{i_1} I(u) = I(w)$. Thus $I_{i_1} \otimes^{\mathbf{L}}_{\Lambda} I(u) = I(w)$ holds by Proposition 3.5(1), and the assertion follows.

(3) It suffices to show that $l(s_j w) > l(w)$ implies that $I(s_j w) < I(w)$ by replacing $s_j w$ with w if necessary. By (2) we have $I(w) \supsetneq I(s_j w) = I_j I(w)$. Then by Proposition 3.10(1)(a), we have $I(s_i w) < I(w)$. $\square$

Now we have the following main result in this section.

THEOREM 3.18. (1) *There exists a well-defined bijection $\mathfrak{S}_n \cong \langle I_1, \ldots, I_{n-1} \rangle$ which maps w to $I(w) = I_{i_1} \cdots I_{i_l}$, where $w = s_{i_1} \cdots s_{i_l}$ is an arbitrary reduced expression.*

(2) *Consequently, there exists a bijection $I : \mathfrak{S}_n \cong \operatorname{tilt} \Lambda$. In particular $\# \operatorname{tilt} \Lambda = n!$.*

(3) *The bijection I in (2) is an anti-isomorphism of posets with respect to the left order on $\mathfrak{S}_n$ and the generation order on $\operatorname{tilt} \Lambda$.*

PROOF. (1) By Proposition 3.15, I is a well-defined surjective map. Now we show that the map is injective. If $I(w) = I(w')$, then $[- \otimes^{\mathbf{L}}_{\Lambda} I(w)] = [- \otimes^{\mathbf{L}}_{\Lambda} I(w')]$ in $GL(V)$. By Proposition 3.17(1), the images of w and w' in $GL(V)$ are the same. Since $\mathfrak{S}_n \to GL(V)$ is injective, we have $w = w'$.

(2) This is immediate from (1) and Theorem 3.9(3).

(3) In the Hasse quiver of the opposite of left order on $\mathfrak{S}_n$, arrows ending at $w \in \mathfrak{S}_n$ are given by $w \to s_i w$ with $1 \leq i \leq n-1$ satisfying $l(s_i w) > l(w)$. By Proposition 3.17(3), the Hasse quiver of $\operatorname{tilt} \Lambda$ coincides with the opposite of the Hasse quiver of $\mathfrak{S}_n$. Thus I is an anti-isomorphism by Lemma 2.15. $\square$

Immediately we have the following corollary.

Corollary 3.19. *For any expression* $w = s_{i_1}s_{i_2}\cdots s_{i_l} \in \mathfrak{S}_n$, $I(w) = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_l}(\Lambda)$ *holds, where* μ_i *is defined in Corollary* 3.11.

Proof. It suffices to show that, if $l(s_iw) = l(w)+1$, then $I(s_iw) = \mu_i(I(w))$ holds. Since $I(s_iw) \not\supseteq I(w)$ holds by Proposition 3.16(2), the assertion follows from Theorem 3.10(1)(a). $\square$

To compare with the Hasse quiver of tilting Λ-modules, we give the Hasse quiver of the left order on the symmetric group $\mathfrak{S}_n$ for $n = 2, 3$.

Example 3.20. We describe the Hasse quiver of the left order on $\mathfrak{S}_2$ and $\mathfrak{S}_3$.

(1) The Hasse quiver of the left order on $\mathfrak{S}_2$ is the opposite of the following quiver:

$$\mathrm{id} = [12] \longrightarrow [21] = s_1.$$

(2) The Hasse quiver of the left order on $\mathfrak{S}_3$ is the opposite of the following quiver:

$$\begin{tikzcd}[column sep=small]
 & \mathrm{id} = [123] \arrow[dl] \arrow[dr] & \\
s_1 = [213] \arrow[d] & & [132] = s_2 \arrow[d] \\
s_2s_1 = [312] \arrow[dr] & & [231] = s_1s_2 \arrow[dl] \\
 & s_1s_2s_1 = [321] = s_2s_1s_2 &
\end{tikzcd}$$

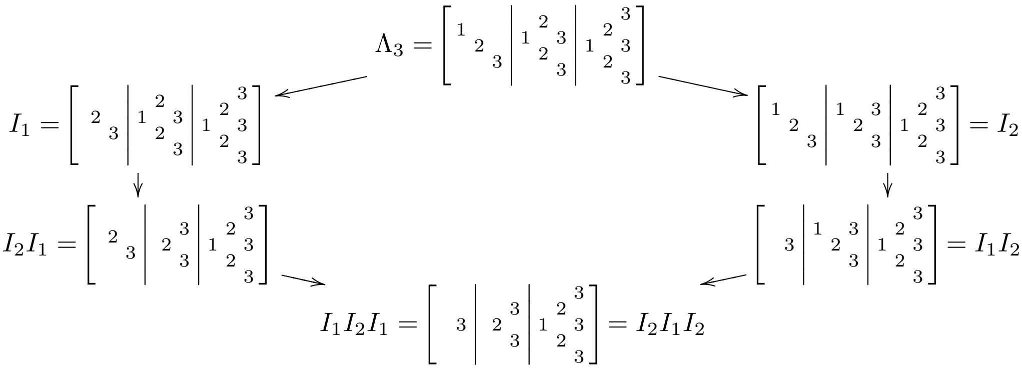
By Corollary 3.11, we can describe the Hasse quiver of tilting modules over the Auslander algebra Λ of $K[x]/(x^n)$ for $n = 2, 3$.

Example 3.21. Denote by Λ_i the Auslander algebra of $K[x]/(x^i)$ for $i = 2, 3$. Then we have

(1) The Hasse quiver $\mathrm{H}(\mathrm{tilt}\,\Lambda_2)$ is the following:

$$\Lambda_2 = \left[\begin{smallmatrix}1\\ & 2\end{smallmatrix}\,\middle|\,\begin{smallmatrix} & 2\\ 1 & \\ & 2\end{smallmatrix}\right] \longrightarrow I_1 = \left[\begin{smallmatrix}2\end{smallmatrix}\,\middle|\,\begin{smallmatrix} & 2\\ 1 & \\ & 2\end{smallmatrix}\right].$$

(2) The Hasse quiver $\mathrm{H}(\mathrm{tilt}\,\Lambda_3)$ is the following:



4. Support τ-tilting modules over the Auslander algebra of $K[x]/(x^n)$.

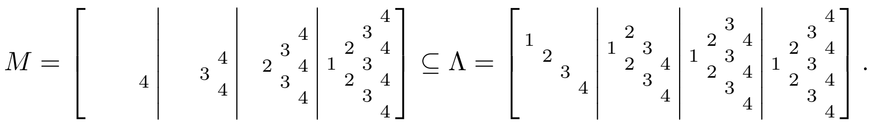
Throughout this section, Λ is the Auslander algebra of $K[x]/(x^n)$. In this section, we firstly construct a bijection from the symmetric group $\mathfrak{S}_{n+1}$ to the set $\mathrm{s}\tau\text{-tilt}\Lambda$ of isomorphism classes of basic support τ-tilting Λ-modules, and then we show that this is an anti-isomorphism of posets. Recall that Λ is presented by the quiver

$$1 \underset{b_2}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_3}{\overset{a_2}{\rightleftarrows}} 3 \underset{b_4}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-2}}{\rightleftarrows}} n-1 \underset{b_n}{\overset{a_{n-1}}{\rightleftarrows}} n$$

with relations $a_1 b_2 = 0$ and $a_i b_{i+1} = b_i a_{i-1}$ for any $2 \leq i \leq n-1$. Let M be the ideal of Λ generated by e_n, and $\overline{\Lambda} := \Lambda/M$. Then we have $M = \bigoplus_{i=1}^{n} M_i$, where $M_i = e_i M$. We often use the functor

$$\overline{(\ )} := - \otimes_\Lambda \overline{\Lambda} : \mathrm{mod}\Lambda \to \mathrm{mod}\overline{\Lambda}.$$

For example, Λ and M in the case $n = 4$ are the following.



We start with some facts on $\mathfrak{S}_{n+1}$. We denote by s_i the transposition $(i, i+1)$ in $\mathfrak{S}_{n+1}$ for $1 \leq i \leq n$.

Lemma 4.1. (1) *$\mathfrak{S}_{n+1} = \bigsqcup_{i=0}^{n} s_{i+1} \cdots s_n \mathfrak{S}_n$, where $s_{i+1} \cdots s_n \mathfrak{S}_n = \mathfrak{S}_n$ for $i = n$.*

(2) *Let $v \in \mathfrak{S}_n$, $1 \leq i \leq n$ and $w = s_{i+1} \cdots s_n v \in \mathfrak{S}_{n+1}$.*

(a) *If $j \leq i-1$, then $s_j w = s_{i+1} \cdots s_n s_j v$.*

(b) *If $j \geq i+2$, then $s_j w = s_{i+1} \cdots s_n s_{j-1} v$.*

Proof. (1) An element $w \in \mathfrak{S}_{n+1}$ belongs to $s_{i+1} \cdots s_n \mathfrak{S}_n$ if and only if $w(n+1) = i+1$ holds. Thus the assertion follows.

(2) (a) is clear. (b) follows from $s_j w = s_{i+1} \cdots s_{j-2} s_j s_{j-1} s_j \cdots s_n v = s_{i+1} \cdots s_{j-1} s_j s_{j-1} s_{j+1} \cdots s_n v = s_{i+1} \cdots s_n s_{j-1} v$. $\square$

By Lemma 4.1, elements in $\mathfrak{S}_{n+1}$ are obtained from elements in $\mathfrak{S}_n$ by multiplying $s_{i+1}\cdots s_n$. Similarly, we will construct support τ-tilting Λ-modules from tilting Λ-modules by applying successive mutations.

In the rest, for $T \in \langle I_1, \ldots, I_{n-1}\rangle$, we consider a direct sum decomposition

$$T = \bigoplus_{i=1}^{n} T_i \quad \text{for} \quad T_i := e_i T.$$

We need the following observations on these direct summands.

LEMMA 4.2. *Let $T \in \langle I_1, \ldots, I_{n-1}\rangle$. For any $1 \le i \le n$, we have*

(1) $\operatorname{soc} T_i \cong S_n$.

(2) *$\overline{T_i}$ is either zero or indecomposable with a simple socle S_{n-i}.*

(3) *$\overline{T_i}$ has no composition factors isomorphic to S_n. In particular* $\operatorname{Hom}_\Lambda(\overline{T_i}, T) = 0$.

(4) *Let $V \in \langle I_1, \ldots, I_{n-1}\rangle$. If $\overline{T_i} \cong \overline{V_i}$, then $T_i \cong V_i$.*

PROOF. (1) Since $M \subseteq T \subseteq \Lambda$, then $M_i \subseteq T_i \subseteq P_i$ and hence $S_n = \operatorname{soc} M_i \subseteq \operatorname{soc} T_i \subseteq \operatorname{soc} P_i = S_n$.

(2) is clear. (3) is immediate from (1).

To prove (4), it suffices to show that T_i can be recovered from $\overline{T_i}$. If $\overline{T_i} = 0$, then $T_i = M_i$. Thus we can assume $\overline{T_i} \neq 0$. Then $\overline{P_i}$ is an injective hull of $\overline{T_i}$ as a $\overline{\Lambda}$-module, and the natural epimorphism $\pi : P_i \to \overline{P_i}$ is a projective cover of $\overline{P_i}$ as a Λ-module. Since $T_i = \pi^{-1}(\overline{T_i})$ holds, the assertion follows. □

The following results on minimal left approximations are also needed to construct support τ-tilting Λ-modules.

LEMMA 4.3. *Let $T \in \langle I_1, \ldots, I_{n-1}\rangle$.*

(1) *The minimal left $\operatorname{add}(\bigoplus_{j=1}^{i-1} T_j)$-approximation of T_i is given by $f_i : T_i \to T_{i-1}$, which is the left multiplication of the arrow $a_{i-1} : i-1 \to i$ in the quiver of Λ. In this case, $f_i(M_i) = M_{i-1}$.*

(2) *The minimal left $\operatorname{add}(\bigoplus_{j=i+1}^{n} T_j)$-approximation of T_i is given by $g_i : T_i \to T_{i+1}$, which is the left multiplication of the arrow $b_{i+1} : i+1 \to i$ in the quiver of Λ. This is a monomorphism.*

PROOF. (1) Since the left multiplication gives an isomorphism $\Lambda \cong \operatorname{End}_\Lambda(T)$, we have an equivalence $\operatorname{Hom}_\Lambda(T, -) : \operatorname{add} T \cong \operatorname{add} \Lambda$. The minimal left $\operatorname{add}(\bigoplus_{j=1}^{i-1} e_j\Lambda)$-approximation of $e_i\Lambda$ is $e_i\Lambda \to e_{i-1}\Lambda$, which is given by the left multiplication of a_{i-1}. Thus the former assertion follows. The latter assertion follows from $f_i(M_i) = a_{i-1}M_i = M_{i-1}$.

(2) One can prove the first assertion similarly to (1). Since the left multiplication of b_{i+1} gives a monomorphism $P_i \to P_{i+1}$, its restriction g_i is also a monomorphism. □

Let $T \in \langle I_1, \ldots, I_{n-1} \rangle$ be a tilting Λ-module. For $0 \le i \le n$, we define

$$\mu_{[i+1,n]}(T) := \mu_{i+1}\mu_{i+2}\cdots\mu_n(T) \in \mathrm{s}\tau\text{-tilt}\,\Lambda$$

as the successive mutation at the direct summands $T_n, T_{n-1}, \ldots, T_{i+1}$ (Definition 2.16), where $\mu_{[i+1,n]}(T) := T$ for $i = n$. The following result plays a crucial role.

Proposition 4.4. *Let $T \in \langle I_1, \ldots, I_{n-1} \rangle$. For $0 \le i \le n$, we have*

(1) $\mu_{[i+1,n]}(T) = \bigoplus_{j=1}^{i} T_j \oplus \bigoplus_{j=i}^{n-1} \overline{T_j}$.

(2) $T > \mu_n(T) > \mu_{[n-1,n]}(T) > \cdots > \mu_{[1,n]}(T)$.

(3) *Let $i \le j \le n-1$. Then $\overline{T_j} = 0$ if and only if S_{n-j} is not a composition factor of $\mu_{[i+1,n]}(T)$.*

(4) *$(\mu_{[i+1,n]}(T), P)$ is a support τ-tilting pair for $P := \bigoplus_{i \le j \le n-1, \overline{T_j}=0} P_{n-j}$.*

Proof. (1) We prove the assertion by descending induction on i. It is clear for $i = n$.

Now we assume that $\mu_{[i+1,n]}(T)$ is $\bigoplus_{j=1}^{i} T_j \oplus \bigoplus_{j=i}^{n-1} \overline{T_j}$. In the following we calculate $\mu_{[i,n]}(T)$ by applying Theorem 2.17.

Firstly, we show that $T_i \notin \mathrm{Fac}\,(\bigoplus_{j=1}^{i-1} T_j \oplus \bigoplus_{j=i}^{n-1} \overline{T_j})$. By Lemma 4.2(3), we have $\mathrm{Hom}_\Lambda(\overline{T_j}, T_i) = 0$. Thus we only have to show $T_i \notin \mathrm{Fac}\,(\bigoplus_{j=1}^{i-1} T_j)$. This is clear since $TM = M$ holds.

Next, by Lemma 4.3(1) and the fact that the natural epimorphism $\pi_i : T_i \to \overline{T_i}$ is a left $(\mathrm{mod}\,\overline{\Lambda})$-approximation of T_i, a left $\mathrm{add}(\bigoplus_{j=1}^{i-1} T_j \oplus \bigoplus_{j=i}^{n-1} \overline{T_j})$-approximation of T_i is given by $f := \binom{f_i}{\pi_i} : T_i \to T_{i-1} \oplus \overline{T_i}$.

Finally, we have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M_i \arrow[r] \arrow[d, equal] & T_i \arrow[r, "\pi_i"] \arrow[d, "f_i"] & \overline{T_i} \arrow[r] \arrow[d] & 0 \\
 & M_i \arrow[r] & T_{i-1} \arrow[r] & \mathrm{Coker}\, f \arrow[r] & 0,
\end{tikzcd}$$

we have $\mathrm{Coker}\, f = T_{i-1}/f_i(M_i) = \overline{T_{i-1}}$ by Lemma 4.3(1). This is indecomposable by Lemma 4.2(2), and we have $\mu_{[i,n]}(T) = \bigoplus_{j=1}^{i-1} T_j \oplus \bigoplus_{j=i-1}^{n-1} \overline{T_j}$ by Theorem 2.17. Thus the assertion follows.

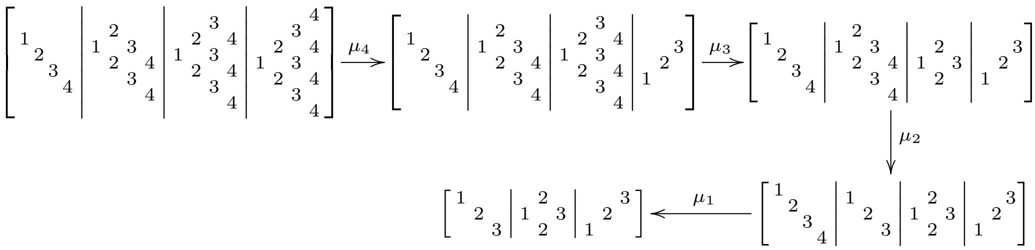
(2) By the proof of (1) we get $\mu_{[i,n]}(T)$ is a left mutation of $\mu_{[i+1,n]}(T)$, and hence the assertion holds.

(3) Notice that the Λ-module $\overline{P_j}$ has the socle S_{n-j}. Since $\overline{T_j}$ is a submodule of $\overline{P_j}$, the "if" part follows. Conversely, assume $\overline{T_j} = 0$. Since $\overline{T}$ is a two-sided ideal of the selfinjective K-algebra $\overline{\Lambda}$, our assumption $\overline{T_j} = 0$ implies that the $\overline{\Lambda}$-module $\overline{T}$ does not have S_{n-j} as a composition factor. Since M_k with $1 \le k \le j$ does not have S_{n-j} as a composition factor, so does $\mu_{[i+1,n]}(T)$.

(4) This is immediate from (3). □

Now we give an example of calculation given in Proposition 4.4.

Example 4.5. Let Λ be the Auslander algebra of $K[x]/(x^4)$. Taking the trivial tilting module Λ, then $\mu_4(\Lambda)$, $\mu_3\mu_4(\Lambda)$, $\mu_2\mu_3\mu_4(\Lambda)$ and $\mu_1\mu_2\mu_3\mu_4(\Lambda)$ are given as follows.



For $0 \leq i \leq n$, we denote by $\mu_{[i+1,n]}(\operatorname{tilt}\Lambda)$ the set of isomorphism classes of support τ-tilting Λ-modules consisting of $\mu_{[i+1,n]}(T)$ for any $T \in \operatorname{tilt}\Lambda$. Then we have the following lemma.

Lemma 4.6. (1) *For any $0 \leq i \leq n$, there is a bijection $\operatorname{tilt}\Lambda \to \mu_{[i+1,n]}(\operatorname{tilt}\Lambda)$, which maps T to $\mu_{[i+1,n]}(T)$.*

(2) *We have $\mu_{[i+1,n]}(\operatorname{tilt}\Lambda) \cap \mu_{[j+1,n]}(\operatorname{tilt}\Lambda) = \emptyset$ for any $0 \leq i \neq j \leq n$.*

Proof. (1) This is clear since each $\mu_j : \mathrm{s}\tau\text{-tilt}\,\Lambda \to \mathrm{s}\tau\text{-tilt}\,\Lambda$ is a bijection.

(2) By Proposition 4.4 and Lemma 4.2(1) and (3), the first i direct summands of $\mu_{[i+1,n]}(T)$ have a composition factor S_n, and the other summands do not have a composition factor S_n. Thus the assertion follows. □

Let $U = \mu_{[i+1,n]}(T) \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ with $T \in \langle I_1, \ldots, I_{n-1}\rangle$ and $0 \leq i \leq n$, given in Proposition 4.4(1). For each $1 \leq k \leq n$, we define $\mu_k(U)$ by

$$\mu_k(U) = \begin{cases} \text{the mutation of } U \text{ at } T_k & \text{if } 1 \leq k \leq i, \\ \text{the mutation of } U \text{ at } \overline{T_{k-1}} & \text{if } i+1 \leq k \leq n \text{ and } \overline{T_{k-1}} \neq 0, \\ \text{the mutation of } U \text{ at } P_{n-k+1} & \text{if } i+1 \leq k \leq n \text{ and } \overline{T_{k-1}} = 0, \end{cases} \tag{4.1}$$

where the third case is well-defined by Proposition 4.4(4). We have the following relations of mutation in $\mathrm{s}\tau\text{-tilt}\,\Lambda$ corresponding to Lemma 4.1(2).

Proposition 4.7. *Let $T \in \langle I_1, \ldots, I_{n-1}\rangle$, $0 \leq i \leq n$ and $U := \mu_{[i+1,n]}(T)$.*

(1) *For any $1 \leq k \leq i-1$, we have $\mu_k(U) = \mu_{[i+1,n]}(\mu_k(T))$. Moreover, $T > \mu_k(T)$ if and only if $U > \mu_k(U)$.*

(2) *For any $i+2 \leq k \leq n$, we have $\mu_k(U) = \mu_{[i+1,n]}(\mu_{k-1}(T))$. Moreover, $T > \mu_{k-1}(T)$ if and only if $U > \mu_k(U)$.*

(3) *We have*

$$\mu_k\mu_{[i+1,n]}(T) = \begin{cases} \mu_{[i+1,n]}\mu_k(T) & k \le i-1, \\ \mu_{[i,n]}(T) & k = i, \\ \mu_{[i+2,n]}(T) & k = i+1, \\ \mu_{[i+1,n]}\mu_{k-1}(T) & k \ge i+2. \end{cases}$$

PROOF. By Proposition 4.4, we have $U = \bigoplus_{j=1}^{i} T_j \oplus \bigoplus_{j=i}^{n-1} \overline{T_j}$.

(1) Let $V := \mu_k(T) = \bigoplus_{j=1}^{k-1} T_j \oplus V_k \oplus \bigoplus_{j=k+1}^{n} T_j$. Then V is a tilting Λ-module with $V_k \ncong T_k$, and applying Proposition 4.4 to V, we have $\mu_{[i+1,n]}(V) = \bigoplus_{j=1}^{k-1} T_j \oplus V_k \oplus \bigoplus_{j=k+1}^{i} T_j \oplus \bigoplus_{j=i}^{n-1} \overline{T_j}$. Since U and $\mu_{[i+1,n]}(V)$ have the same indecomposable direct summands except the k-th one, we have $\mu_k(U) = \mu_{[i+1,n]}(V)$ as desired.

To prove the latter one, it suffices to show that $T > \mu_k(T)$ implies $U > \mu_k(U)$. The condition $T > \mu_k(T)$ is equivalent to $T_k \notin \mathrm{Fac}\,(T/T_k)$. Since U/U_k belongs to $\mathrm{Fac}\,(T/T_k)$ by the explicit form in Proposition 4.4, we have $U_k = T_k \notin \mathrm{Fac}\,(U/U_k)$. Therefore $U > \mu_k(U)$.

(2) Let $V := \mu_{k-1}(T) = \bigoplus_{j=1}^{k-2} T_j \oplus V_{k-1} \oplus \bigoplus_{j=k}^{n} T_j$. Then V is a tilting Λ-module with $V_{k-1} \ncong T_{k-1}$, and applying Proposition 4.4 to V, we have $\mu_{[i+1,n]}(V) = \bigoplus_{j=1}^{i} T_j \oplus \bigoplus_{j=i}^{k-2} \overline{T_j} \oplus \overline{V_{k-1}} \oplus \bigoplus_{j=k}^{n-1} \overline{T_j}$. Since $\overline{V_{k-1}} \ncong \overline{T_{k-1}}$ holds by Lemma 4.2(4), U and $\mu_{[i+1,n]}(V)$ have the same indecomposable direct summands except the k-th one. Thus we have $\mu_k(U) = \mu_{[i+1,n]}(V)$ as desired.

To show the latter one, it suffices to show that $T < \mu_{k-1}(T)$ implies $U < \mu_k(U)$. The condition $T < \mu_{k-1}(T)$ is equivalent to $T_{k-1} \in \mathrm{Fac}\,(T/T_{k-1})$. Since $\overline{T}/\overline{T_{k-1}}$ belongs to $\mathrm{Fac}\,(U/U_k)$ by the explicit form in Proposition 4.4, we have $U_k = \overline{T_{k-1}} \in \mathrm{Fac}\,(\overline{T}/\overline{T_{k-1}}) \subseteq \mathrm{Fac}\,(U/U_k)$. Therefore $U < \mu_k(U)$.

(3) Immediate from (1) and (2). $\square$

Immediately we have the following complete classification of support τ-tilting Λ-modules and indecomposable τ-rigid Λ-modules.

THEOREM 4.8. (1) *We have* $\mathrm{s}\tau\text{-tilt}\,\Lambda = \bigsqcup_{i=0}^{n} \mu_{[i+1,n]}(\mathrm{tilt}\,\Lambda)$. *In particular,* $\#\mathrm{s}\tau\text{-tilt}\,\Lambda = (n+1)!$, *and the mutation* μ_k *for each* $1 \le k \le n$ *is well-defined on* $\mathrm{s}\tau\text{-tilt}\,\Lambda$ *by* (4.1).

(2) *Any support τ-tilting Λ-module has a form $T_1 \oplus \cdots \oplus T_i \oplus \overline{T_i} \oplus \cdots \oplus \overline{T_{n-1}}$ for some $0 \le i \le n$ and $T \in \langle I_1, \ldots, I_{n-1} \rangle$ with $T_j := e_j T$ for $1 \le j \le n$. Moreover such i and T are uniquely determined.*

(3) *Any indecomposable τ-rigid module has a form $T_i = e_i T$ or $\overline{T_i}$ for some $T \in \langle I_1, \ldots, I_{n-1} \rangle$ and $1 \le i \le n$.*

(4) *The statements* (1) *and* (2) *hold for Λ^{op}-modules.*

PROOF. (1) By Lemma 4.6, $\bigcup_{i=0}^{n} \mu_{[i+1,n]}(\mathrm{tilt}\,\Lambda)$ is a disjoint union and contains precisely $(n+1)!$ elements. By Proposition 4.7(3), $\bigsqcup_{i=0}^{n} \mu_{[i+1,n]}(\mathrm{tilt}\,\Lambda)$ is closed under mutation. This is a finite connected component of $\mathrm{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$. By Proposition 2.19, we have $\mathrm{s}\tau\text{-tilt}\,\Lambda = \bigsqcup_{i=0}^{n} \mu_{[i+1,n]}(\mathrm{tilt}\,\Lambda)$.

(2) is clear by (1) and Proposition 4.4.

(3) is a straight result of (2) and Lemma 2.12. □

The following lemma is also needed.

LEMMA 4.9. *Let $U \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ and $1 \le j, k \le n$.*

(1) $\mu_j\mu_j(U) = U$.

(2) *If $|j-k| \ge 2$, then $\mu_j\mu_k(U) = \mu_k\mu_j(U)$.*

(3) *If $|j-k| = 1$, then $\mu_j\mu_k\mu_j(U) = \mu_k\mu_j\mu_k(U)$.*

PROOF. (1) is clear from the definition of mutation.

By Theorem 4.8(1), we can assume that $U = \mu_{[i+1,n]}(T)$ for some $0 \le i \le n$ and $T \in \langle I_1, \dots, I_{n-1}\rangle$. In the following we use Proposition 4.7(3) and Proposition 3.13 frequently.

(2) Without loss of generality, we assume $k < j$. We divide the proof into seven cases.

(a) If $k < j \le i-1$, then $\mu_j\mu_k(U) = \mu_j\mu_k\mu_{[i+1,n]}(T) = \mu_j\mu_{[i+1,n]}\mu_k(T) = \mu_{[i+1,n]}\mu_j\mu_k(T) = \mu_{[i+1,n]}\mu_k\mu_j(T) = \mu_k\mu_j\mu_{[i+1,n]}(T) = \mu_k\mu_j(U)$.

(b) If $i+2 \le k < j$, then the proof is very similar to (a).

(c) If $k \le i-1 < i+2 \le j$, then $\mu_j\mu_k(U) = \mu_j\mu_k\mu_{[i+1,n]}(T) = \mu_j\mu_{[i+1,n]}\mu_k(T) = \mu_{[i+1,n]}\mu_{j-1}\mu_k(T) = \mu_{[i+1,n]}\mu_k\mu_{j-1}(T) = \mu_k\mu_{[i+1,n]}\mu_{j-1}(T) = \mu_k\mu_j\mu_{[i+1,n]}(T) = \mu_k\mu_j(U)$.

(d) The case $k = i < i+2 \le j$, then $\mu_j\mu_k(U) = \mu_j\mu_k\mu_{[i+1,n]}(T) = \mu_j\mu_{[i,n]}(T) = \mu_{[i,n]}\mu_{j-1}(T) = \mu_k\mu_{[i+1,n]}\mu_{j-1}(T) = \mu_k\mu_j\mu_{[i+1,n]}(T) = \mu_k\mu_j(U)$.

(e) If $k \le i-2 < i = j$, then the proof is very similar to (d).

(f) If $k \le i-1 < i+1 = j$, then $\mu_j\mu_k(U) = \mu_j\mu_k\mu_{[i+1,n]}(T) = \mu_{i+1}\mu_{[i+1,n]}\mu_k(T) = \mu_{[i+2,n]}\mu_k(T) = \mu_k\mu_{[i+2,n]}(T) = \mu_k\mu_j\mu_{[i+1,n]}(T) = \mu_k\mu_j(U)$.

(g) If $k = i+1 < i+3 \le j$, then the proof is very similar to (d).

(3) Without loss of generality, we assume $k = j+1$. We also divide the proof into five cases.

(a) If $j \le i-2$, then $\mu_j\mu_k\mu_j(U) = \mu_j\mu_k\mu_j\mu_{[i+1,n]}(T) = \mu_{[i+1,n]}\mu_j\mu_k\mu_j(T) = \mu_{[i+1,n]}\mu_k\mu_j\mu_k(T) = \mu_k\mu_j\mu_k\mu_{[i+1,n]}(T) = \mu_k\mu_j\mu_k(U)$.

(b) If $j \ge i+2$, then the proof is very similar to (a).

(c) If $j = i-1$, then $\mu_{i-1}\mu_i\mu_{i-1}(U) = \mu_{i-1}\mu_i\mu_{i-1}\mu_{[i+1,n]}(T) = \mu_{i-1}\mu_i\mu_{[i+1,n]}\mu_{i-1}(T) = \mu_{[i-1,n]}\mu_{i-1}(T) = \mu_i\mu_{[i-1,n]}(T) = \mu_i\mu_{i-1}\mu_i\mu_{[i+1,n]}(T) = \mu_i\mu_{i-1}\mu_i(U)$.

(d) If $j = i$ or $j = i+1$, then the proof is very similar to (c). □

Now we are in a position to state one of the main results of this section.

THEOREM 4.10. *Let Λ be the Auslander algebra of $K[x]/(x^n)$. Then*

(1) *There exists a bijection $I : \mathfrak{S}_{n+1} \cong \mathrm{s}\tau\text{-tilt}\,\Lambda$ which maps w to $I(w) = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_l}(\Lambda)$, where $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ is an arbitrary (not necessarily reduced) expression.*

(2) *The statement (1) holds for Λ^{op}-modules.*

Proof. (1) Proposition 4.9 and the same argument as in the proof of Theorem 3.18 show that the map I is well-defined. By Theorem 4.8, we have $\#\mathrm{s}\tau\text{-tilt}\,\Lambda = (n+1)! = \#\mathfrak{S}_{n+1}$. Thus we only have to show I is surjective.

By Theorem 4.8, any $U \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ is written as $\mu_{[i+1,n]}(T)$ for some $T \in \mathrm{tilt}\,\Lambda$ and $0 \leq i \leq n$. By Corollary 3.19, there exists $w \in \mathfrak{S}_n$ such that $T = I(w)$. Then we have $I(s_{i+1}\cdots s_n w) = \mu_{[i+1,n]}(T) = U$. Thus the assertion follows.

(2) We only need to replace Λ-modules with Λ^{op}-modules in the proof. □

Our second goal in this section is to show that the map I in Theorem 4.10 is an anti-isomorphism of posets. For this aim, we need the following result.

Proposition 4.11. *For $w \in \mathfrak{S}_{n+1}$ and $1 \leq j \leq n$, $l(s_j w) > l(w)$ if and only if $I(s_j w) < I(w)$.*

Proof. It suffices to show that $l(s_j w) > l(w)$ implies that $I(s_j w) < I(w)$ by replacing $s_j w$ with w if necessary. Write $w = s_{i+1}\cdots s_n v$ with $0 \leq i \leq n$ and $v \in \mathfrak{S}_n$. Then $l(w) = n - i + l(v)$ and $l(s_j w) = n - i + l(v) + 1$ hold by our assumption. We prove the assertion by comparing i with j.

(a) Assume $j \leq i-1$. By Proposition 4.7(3), we have $I(s_j w) = \mu_j \mu_{[i+1,n]}(I(v)) = \mu_{[i+1,n]}\mu_j(I(v)) = \mu_{[i+1,n]}(I(s_j v))$. Since $s_j w = s_{i+1}\cdots s_n s_j v$ holds, we have $n - i + l(v) + 1 = l(s_j w) \leq n - i + l(s_j v)$ and hence $l(v)+1 = l(s_j v)$. Then by Theorem 3.18 one has $I(s_j v) < I(v)$, which implies by Proposition 4.7(1) that $I(s_j w) = \mu_{[i+1,n]}(I(s_j v)) < \mu_{[i+1,n]}(I(v)) = I(w)$.

(b) Assume $j \geq i+2$. We have $I(s_j w) = \mu_j \mu_{[i+1,n]}(I(v)) = \mu_{[i+1,n]}\mu_{j-1}(I(v)) = \mu_{[i+1,n]}(I(s_{j-1}v))$ by Proposition 4.7(3). Since $s_j w = s_{i+1}\cdots s_n s_{j-1} v$ holds by Lemma 4.1(2), we have $n - i + l(v) + 1 = l(s_j w) \leq n - i + l(s_{j-1}v)$ and hence $l(v) + 1 = l(s_{j-1}v)$. Then by Theorem 3.18 one has $I(s_{j-1}v) < I(v)$, which implies by Proposition 4.7(2) that $I(s_j w) = \mu_{[i+1,n]}(I(s_{j-1}v)) < \mu_{[i+1,n]}(I(v)) = I(w)$.

(c) Assume $j = i$. By Proposition 4.7(3), we have $I(s_j w) = \mu_i \mu_{[i+1,n]}(I(v)) = \mu_{[i,n]}(I(v)) < \mu_{[i+1,n]}(I(v)) = I(w)$ by Proposition 4.4(2).

(d) The case $j = i+1$ does not occur. In fact $s_j w = s_{i+2}\cdots s_n v$ implies $l(s_j w) = l(w) - 1$, a contradiction. □

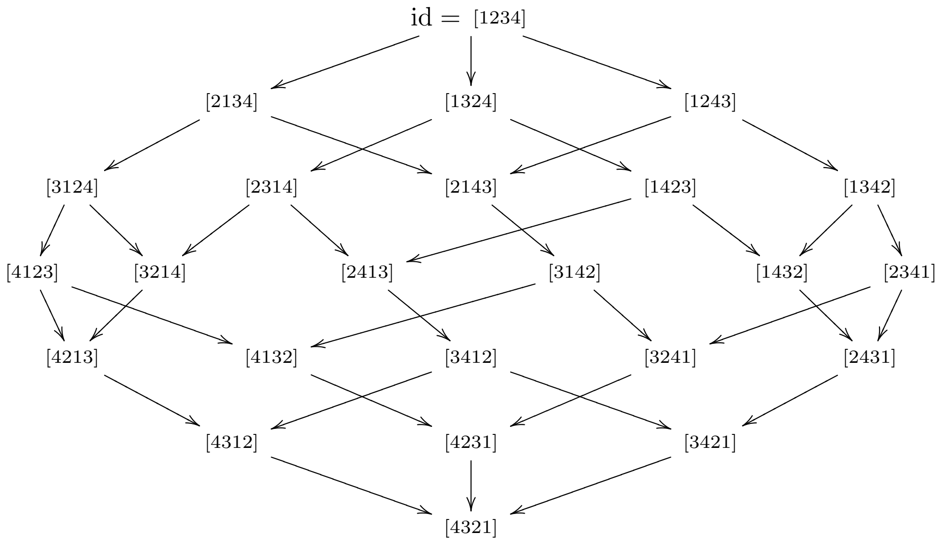
Now we are ready to show the main result on the anti-isomorphisms of posets.

Theorem 4.12. *Let Λ and I be as in Theorem 4.10. Then $I : \mathfrak{S}_{n+1} \to \mathrm{s}\tau\text{-tilt}\,\Lambda$ is an anti-isomorphism of posets with respect to the left order on $\mathfrak{S}_{n+1}$ and the generation order on $\mathrm{s}\tau\text{-tilt}\,\Lambda$, that is, $w_1 \leq w_2$ in $\mathfrak{S}_{n+1}$ if and only if $I(w_1) \geq I(w_2)$ in $\mathrm{s}\tau\text{-tilt}\,\Lambda$.*

Proof. The proof is very similar to the proof of Theorem 3.18(3), we need to use Proposition 4.11 instead of Proposition 3.17(3). □

To compare with the Hasse quiver of support τ-tilting Λ-modules, we give the Hasse quiver of the left order on the symmetric group $\mathfrak{S}_n$ for $n = 4$.

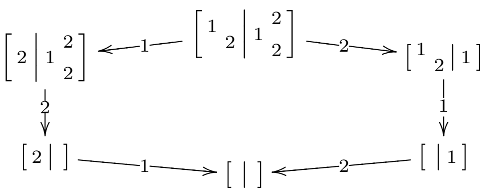
Example 4.13. The Hasse quiver of the left order on $\mathfrak{S}_4$ is the *opposite* of the following quiver:



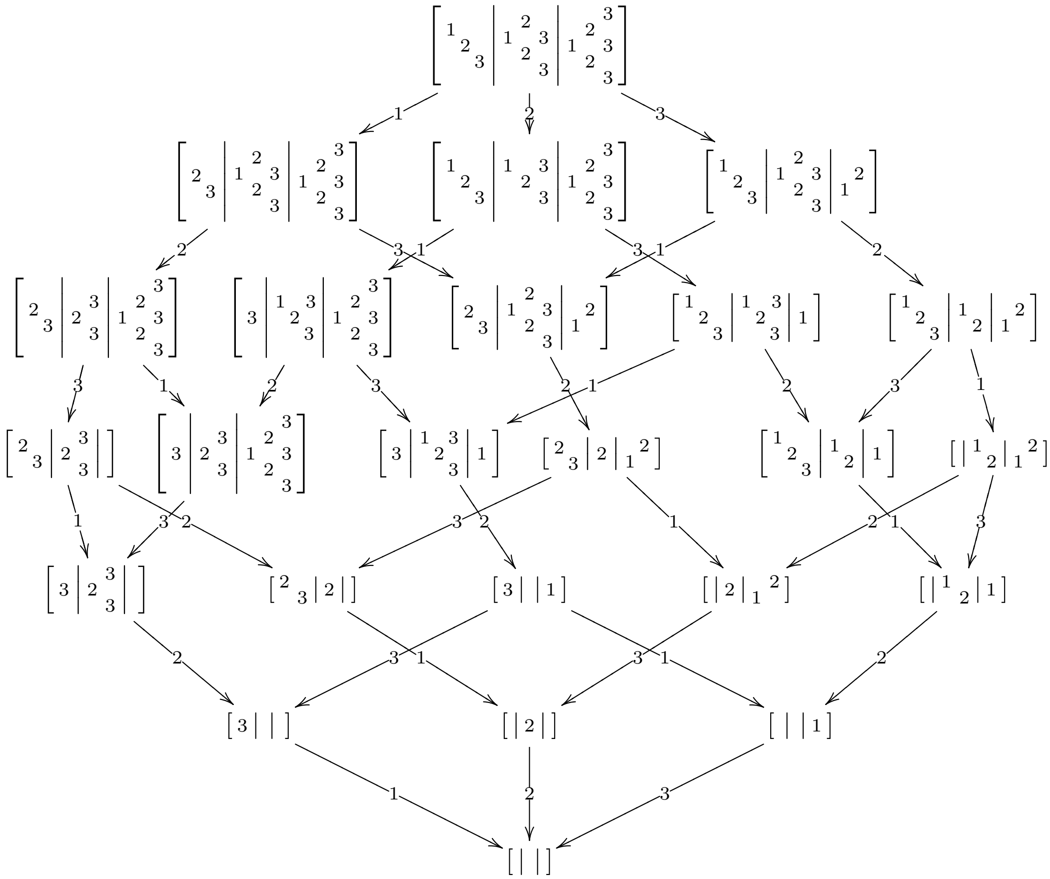
By Theorem 4.14, we give the Hasse quiver of support τ-tilting modules of the Auslander algebra of $K[x]/(x^n)$ for $n = 2, 3$.

Example 4.14. Denote by Λ_i the Auslander algebra of $K[x]/(x^i)$ for $i = 2, 3$. Then

(1) The Hasse quiver $\mathrm{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda_2)$ is of the following form, where $\xrightarrow{i}$ shows μ_i.



(2) The Hasse quiver $\mathrm{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda_3)$ is of the following form, where $\xrightarrow{i}$ shows μ_i.



5. Connection with preprojective algebras of type A_n.

Let Λ be the Auslander algebra of $K[x]/(x^n)$ and Γ be the preprojective algebra of Dynkin type A_n. Thus Γ is presented by the quiver

$$1 \underset{b_2}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_3}{\overset{a_2}{\rightleftarrows}} 3 \underset{b_4}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-2}}{\rightleftarrows}} n-1 \underset{b_n}{\overset{a_{n-1}}{\rightleftarrows}} n$$

with relations $a_1b_2 = 0$, $b_na_{n-1} = 0$ and $a_ib_{i+1} = b_ia_{i-1}$ for any $2 \le i \le n-1$. Thus we have $\Gamma = \Lambda/L$ for the ideal L of Λ generated by b_na_{n-1}. Then we have $L = \bigoplus_{i=1}^n L_i$ for $L_i := e_iL$. For example, Λ and L in the case $n = 4$ is the following.

$$L = \left[\ \ \middle|\ \begin{smallmatrix} 4 \end{smallmatrix}\ \middle|\ \begin{smallmatrix} & 4 \\ 3 & \\ & 4 \end{smallmatrix}\ \middle|\ \begin{smallmatrix} & & 4 \\ & 3 & \\ 2 & & 4 \\ & 3 & \\ & & 4 \end{smallmatrix}\right] \subseteq \Lambda = \left[\begin{smallmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{smallmatrix}\ \middle|\ \begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \\ & & & 4 \end{smallmatrix}\ \middle|\ \begin{smallmatrix} & & 3 & \\ & 2 & & 4 \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \\ & & & 4 \end{smallmatrix}\ \middle|\ \begin{smallmatrix} & & & 4 \\ & & 3 & \\ & 2 & & 4 \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \\ & & & 4 \end{smallmatrix}\right].$$

Our aim in this section is to apply Theorems 4.10 and 4.12 to Γ and prove that the tensor functor

$$-\otimes_\Lambda \Gamma : \mathrm{mod}\Lambda \to \mathrm{mod}\Gamma$$

induces a bijection from sτ-tiltΛ to sτ-tiltΓ. In particular, we can get Mizuno's bijection from the symmetric group $\mathfrak{S}_{n+1}$ to sτ-tiltΓ.

Let us start with the following general properties of support τ-tilting modules over an algebra A and its factor algebra B.

Proposition 5.1 ([**DIRRT**]). *Let A be an algebra and let B be a factor algebra of A.*

(1) *If T is a τ-rigid A-module, then $T \otimes_A B$ is a τ-rigid B-module.*

(2) *If T is a support τ-tilting A-module, then $T \otimes_A B$ is a support τ-tilting B-module. Thus we have a map $-\otimes_A B$: sτ-tilt$A \to$ sτ-tilt$B$, which preserves the generation order.*

(3) *The map in* (2) *is surjective if A is τ-rigid finite.*

Note that $T \otimes_A B$ is not necessarily basic even if T is basic τ-rigid.

Recall that M and L are the ideals defined at the beginning of Sections 4 and 5 respectively, and $M_i = e_i M$ and $L_i = e_i L$ for $1 \le i \le n$. We need the following facts.

Lemma 5.2. *Let $T \in \langle I_1, \dots, I_{n-1}\rangle$ and $T_i := e_i T$ for $1 \le i \le n$. For any $1 \le i \le n$, we have*

(1) *$LM = L = ML$ and $T_i L = L_i$.*

(2) *T_i/L_i is indecomposable with a simple socle S_{n-i+1}.*

(3) *Let $V \in \langle I_1, \dots, I_{n-1}\rangle$. If $T_i/L_i \cong V_i/L_i$, then $T_i \cong V_i$.*

Proof. (1) This is clear. (2) Since $M_i \subseteq T_i \subseteq P_i$, we have $L_i = M_i L \subseteq T_i L \subseteq P_i L = L_i$. The socle of $T_i/L_i \subseteq P_i/L_i$ is S_{n-i+1}. (3) One can prove in a similar method with Lemma 4.2(4). □

Now we can state our main result of this section.

Theorem 5.3. *Let Λ be the Auslander algebra of $K[x]/(x^n)$ and Γ the preprojective algebra of Dynkin type A_n.*

(1) *The map $-\otimes_\Lambda \Gamma$: sτ-tilt$\Lambda \to$ sτ-tiltΓ given by $U \mapsto U \otimes_\Lambda \Gamma$ is bijective.*

(2) *The map in* (1) *is an isomorphism of posets.*

(3) *If X is an indecomposable τ-rigid Λ-module, then $X \otimes_\Lambda \Gamma$ is an indecomposable τ-rigid Γ-module.*

Proof. (1) For any $U \in$ sτ-tiltΛ, there exists $T \in \langle I_1, \dots, I_{n-1}\rangle$ and $0 \le i \le n$ such that

$$U = \mu_{[i+1,n]}(T) = T_1 \oplus \cdots \oplus T_i \oplus \overline{T_i} \oplus \cdots \oplus \overline{T_{n-1}}$$

by Theorem 4.8. In this case, we have

$$U \otimes_\Lambda \Gamma = \begin{cases} (T_1/L_1) \oplus \cdots \oplus (T_i/L_i) \oplus \overline{T_i} \oplus \cdots \oplus \overline{T_{n-1}} & \text{if } i \geq 1, \\ 0 \oplus \overline{T_1} \oplus \cdots \oplus \overline{T_{n-1}} & \text{if } i = 0. \end{cases}$$

For any $1 \leq j \leq n$, $\overline{T_j}$ does not have S_n as a composition factor, and T_j/L_j has S_n as a composition factor. Therefore the integer i can be recovered from U as the number of indecomposable direct summands of U which have S_n as a composition factor. Moreover, by Lemmas 5.2(2) and 4.2(2), the socle of the j-th direct summand of $U \otimes_\Lambda \Gamma$ is S_{n-j+1} if $1 \leq j \leq i$, and either 0 or S_{n-j+1} if $i+1 \leq j \leq n$.

Now assume that another $U' \in \text{s}\tau\text{-tilt}\,\Lambda$ satisfies $U \otimes_\Lambda \Gamma \cong U' \otimes_\Lambda \Gamma$, and take $T' \in \langle I_1, \ldots, I_{n-1} \rangle$ and $1 \leq i' \leq n$ such that $U' = \mu_{[i'+1,n]}(T')$. By the argument above, we have $i = i'$. By looking at the socle of each indecomposable direct summand, we have $T_j/L_j \cong T'_j/L_j$ for any $1 \leq j \leq i$ and $\overline{T_j} \cong \overline{T'_j}$ for any $i \leq j \leq n-1$. They imply $T_j \cong T'_j$ for any $1 \leq j \leq n-1$ by Lemmas 5.2(3) and 4.2(4). Since $T_n = P_n = T'_n$, we have $T \cong T'$ and hence $U = \mu_{[i+1,n]}(T) \cong \mu_{[i+1,n]}(T') = U'$.

(3) By Theorem 4.8(3), X has a form T_i or $\overline{T_i}$ for some $T \in \langle I_1, \ldots, I_{n-1} \rangle$ and $1 \leq i \leq n$. Since $T_i \otimes_\Lambda \Gamma = T_i/L_i$ and $\overline{T_i} \otimes_\Lambda \Gamma = \overline{T_i}$ are indecomposable by Lemmas 5.2(2) and 4.2(2), the assertion follows.

(2) The map $- \otimes_\Lambda \Gamma$ preserves mutations. In fact, if $U = \mu_i(T)$ for $T, U \in \text{s}\tau\text{-tilt}\,\Lambda$, then $U \otimes_\Lambda \Gamma$ and $T \otimes_\Lambda \Gamma$ have the same indecomposable direct summands except the i-th summand by (3) and the injectivity of $- \otimes_\Lambda \Gamma : \text{s}\tau\text{-tilt}\,\Lambda \to \text{s}\tau\text{-tilt}\,\Gamma$. Therefore we have $U \otimes_\Lambda \Gamma = \mu_i(T \otimes_\Lambda \Gamma)$. Moreover, $- \otimes_\Lambda \Gamma$ preserves the generation order clearly.

In particular, $- \otimes_\Lambda \Gamma$ gives an isomorphism $\mathrm{H}(\text{s}\tau\text{-tilt}\,\Lambda) \to \mathrm{H}(\text{s}\tau\text{-tilt}\,\Gamma)$ of Hasse quivers by Theorem 2.18. Thus $- \otimes_\Lambda \Gamma : \text{s}\tau\text{-tilt}\,\Lambda \to \text{s}\tau\text{-tilt}\,\Gamma$ is an isomorphism of posets by Lemma 2.15. □

Remark 5.4. Theorem 5.3 gives another proof of Mizuno's result [**M**, Theorem 2.21].

As a corollary, we get the following.

Corollary 5.5. *Let Λ be the Auslander algebra of $K[x]/(x^n)$ and Γ a preprojective algebra of Dynkin type A_n. There are isomorphisms between the following posets*:

(1) *The poset* $\text{s}\tau\text{-tilt}\,\Lambda$ *with the generation order.*

(2) *The poset* $\text{s}\tau\text{-tilt}\,\Gamma$ *with the generation order.*

(3) *The symmetric group $\mathfrak{S}_{n+1}$ with the opposite of the left order.*

(4) *The poset* $\text{s}\tau\text{-tilt}\,(\Lambda^{\mathrm{op}})$ *with the opposite of the generation order.*

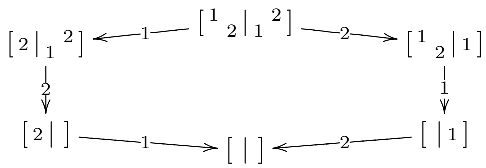
(5) *The poset* $\text{s}\tau\text{-tilt}\,(\Gamma^{\mathrm{op}})$ *with the opposite of the generation order.*

(6) *The symmetric group $\mathfrak{S}_{n+1}$ with the right order.*

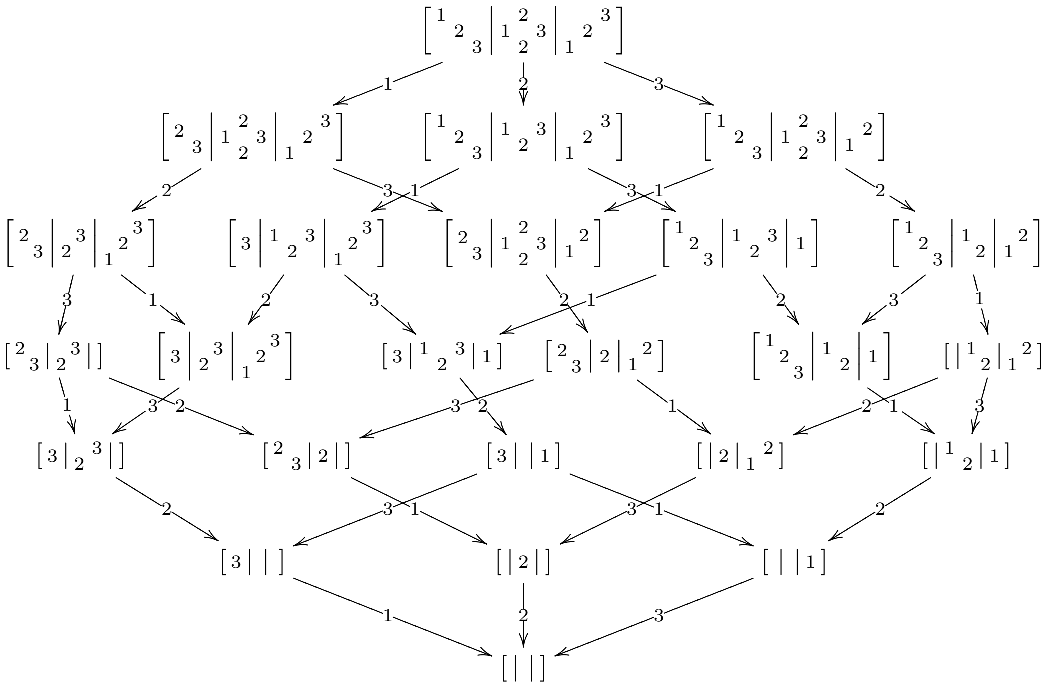
Proof. The isomorphism from (1) to (2) given by $- \otimes_\Lambda \Gamma$ is showed in Theorem 5.3. The isomorphism from (3) to (1) given by I is showed in Theorem 4.12. The isomorphism between (1) and (4) (resp. (2) and (5)) is given in [**AIR**]. □

EXAMPLE 5.6. Denote by Γ_n the preprojective algebra of type A_n. Then

(1) The Hasse quiver $\mathrm{H}(\mathrm{s}\tau\text{-tilt}\,\Gamma_2)$ is of the following form, where $\xrightarrow{i}$ shows μ_i.



(2) The Hasse quiver $\mathrm{H}(\mathrm{s}\tau\text{-tilt}\,\Gamma_3)$ is of the following form, where $\xrightarrow{i}$ shows μ_i.



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