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Common reducing subspaces of several weighted shifts with operator weights

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Abstract. We characterize common reducing subspaces of several weighted shifts with operator weights. As applications, we study the common reducing subspaces of the multiplication operators by powers of coordinate functions on Hilbert spaces of holomorphic functions in several variables. The identification of reducing subspaces also leads to structure theorems for the commutants of von Neumann algebras generated by these multiplication operators. This general approach applies to weighted Hardy spaces, weighted Bergman spaces, Drury–Arveson spaces and Dirichlet spaces of the unit ball or polydisk uniformly.

1. Introduction.

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Let $\Omega \subset B(H)$ be a set of operators. A closed subspace X is an invariant subspace of Ω, if for every $T \in \Omega$, T maps X into X. The space X is a reducing subspace of Ω, if X is invariant under both T and T^* for every $T \in \Omega$. The space X is a minimal invariant (or reducing) subspace of Ω if the only invariant (or reducing) subspaces contained in X are X and $\{0\}$. The set Ω is irreducible if the only reducing subspaces of Ω are $\{0\}$ and the whole space H.

The Beurling invariant subspace theorem [**3**] for the unweighted unilateral shifts of multiplicity one, and its extension to higher multiplicity [**9**] (called the Beurling–Lax–Halmos invariant subspace theorem), are two of the fundamental results in modern operator theory. Despite the substantial advances [**2**] and [**7**], the structure of invariant subspaces of the Bergman shift is still an active research area. In fact this problem is as difficult as the invariant subspace problem (of whether every bounded linear operator on a separable Hilbert space of dimension greater than one has a nontrivial invariant subspace); see for example [**8**]. There also have been extensions of the Beurling invariant subspace theorem on the Hardy space of the polydisk [**10**], [**11**] and [**22**].

On the other hand, there is a nice description of reducing subspaces of powers of weighted shifts with scalar weights [**21**]. This paper and its predecessor [**23**], where reducing subspaces of some analytic Toeplitz operators on the Bergman space of the unit disk were studied, have also been inspirational in the last fifteen years for establishing

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structure of reducing subspaces of Toeplitz operators with Blaschke product symbols on the Bergman space of the unit disk; see a recent monograph [**6**] and extensive references therein. The structure of the reducing subspace lattice for unweighted unilateral shifts was described in [**9**] and [**16**]. The reducing subspaces of some analytic Toeplitz operators on the Hardy space of the unit disk were studied as early as in [**16**] and [**1**].

Recently, the reducing subspaces of some analytic Toeplitz operators on the Bergman space of the bidisk and polydisk were characterized in [**15**], [**18**], [**14**], and [**19**]. In [**5**], the author recovered the results from [**21**] and some results from [**15**] by studying the reducing subspaces of weighted shifts with operator weights.

In this paper, we characterize the common reducing subspaces of several commuting weighted shifts with operator weights as wandering invariant subspaces of the shifts with additional structures. As applications, we study the common reducing subspaces of multiplication operators by powers of coordinate functions on Hilbert spaces of holomorphic functions in several variables.

The identification of reducing subspaces also leads to structure theorems for the commutants of von Neumann algebras generated by these multiplication operators. This general approach applies to weighted Hardy spaces, weighted Bergman spaces, Drury–Arveson spaces, and Dirichlet spaces of the unit ball or polydisk uniformly. Below we give three sample results that are contained in Theorem 8.1, Theorem 5.4, and Theorem 9.5 respectively.

Let $\mathbb{C}$ denote the set of complex numbers. Let $\mathbb{B}^d$ be the unit ball of $\mathbb{C}^d$,

$$\mathbb{B}^d = \left\{ z = (z_1, \dots, z_d) \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 < 1 \right\},$$

and let $\mathbb{S}^d$ be the unit sphere,

$$\mathbb{S}^d = \left\{ z = (z_1, \dots, z_d) \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 = 1 \right\}.$$

The Hardy space $H^2(\mathbb{B}^d)$ is the Hilbert space of holomorphic functions in $\mathbb{B}^d$ such that

$$\|f(z)\|^2 = \sup_{0<r<1} \int_{\mathbb{S}^d} |f(r\zeta)|^2 \, d\sigma(\zeta), \quad f \in H^2(\mathbb{B}^d),$$

where $d\sigma(\zeta)$ is the normalized area measure on $\mathbb{S}^d$. For a multi-index $N = (N_1, \dots, N_d)$, $z^N = z_1^{N_1} \cdots z_d^{N_d}$. Let T_{z^N} be the multiplication operator by z^N, that is

$$T_{z^N} f(z) = z^N f(z), \quad f \in H^2(\mathbb{B}^d).$$

In this paper, for an index set I, $v_i \in H$, Span $\{v_i : i \in I\}$ always means the closed linear span of $\{v_i : i \in I\}$ in H.

THEOREM A. *Let $N = (M, M)$. Let $J_N = \{(\beta_1, \beta_2) : 0 \le \beta_1 < M$ or $0 \le \beta_2 < M\}$. Then any minimal reducing subspace X of T_{z^N} on $H^2(\mathbb{B}^2)$ is of the form* Span $\left\{ f(z_1, z_2)(z_1 z_2)^{kM} : k \ge 0 \right\}$, *where either there exists $\gamma = (\gamma_1, \gamma_2) \in J_N$ such that*

$$f(z) = a z_1^{\gamma_1} z_2^{\gamma_2} + b z_1^{\gamma_2} z_2^{\gamma_1}, \quad a, b \in \mathbb{C},$$

or there exists $0 \le l < M$ *such that*

$$f(z_1, z_2) = a z_1^l z_2^l + b z_1^{l+1} z_2^l + c z_1^l z_2^{l+1}, \quad a, b, c \in \mathbb{C}.$$

Furthermore, any reducing subspace of T_{z^N} *on* $H^2(\mathbb{B}^2)$ *is an orthogonal sum of minimal reducing subspaces.*

The Bergman space $L_a^2(\mathbb{B}^d)$ is the Hilbert space of holomorphic functions in $\mathbb{B}^d$ such that

$$\|f(z)\|^2 = \int_{\mathbb{B}^d} |f(\zeta)|^2 \, dv(\zeta), \quad f \in L_a^2(\mathbb{B}^d),$$

where $dv(\zeta)$ is the normalized volume measure on $\mathbb{B}^d$. Let $\mathbb{D}$ be the open unit disk, and let $\mathbb{D}^d$ be the polydisk. The Bergman space $L_a^2(\mathbb{D}^d)$ is the Hilbert space of holomorphic functions in $\mathbb{D}^d$ such that

$$\|f(z)\|^2 = \int_{\mathbb{D}^d} |f(\zeta)|^2 \, dA(\zeta_1) \cdots dA(\zeta_d), \quad f \in L_a^2(\mathbb{D}^d),$$

where $dA(\zeta_1) \cdots dA(\zeta_d)$ is the normalized product measure on $\mathbb{D}^d$, with $dA(\zeta_1)$ being the normalized area measure of the unit disk $\mathbb{D}$. The following result can also be derived from the discussion of type I weight sequences in [**14**]. The special case $L_a^2(\mathbb{D}^2)$ with $N_1 = N_2$ is contained in Theorem 2.4 [**15**].

Theorem B. *Let* $N = (N_1, \ldots, N_d)$ *be a multi-index such that* $N \ge (1, \ldots, 1)$. *By an abuse of notation, set* $N - 1 = (N_1 - 1, \ldots, N_d - 1)$. *Let*

$$\widehat{N} = \{\beta : 0 \le \beta \le N - 1\}, \text{ and } L = \prod_{i=1}^{d} N_i,$$

where L *is the cardinality of the index set* $\widehat{N}$. *Then*

(i) *For each* $\beta = (\beta_1, \ldots, \beta_d) \in \widehat{N}$, $\mathrm{Span}\{z_1^{\beta_1 + k_1 N_1} \cdots z_d^{\beta_d + k_d N_d} : k = (k_1, \ldots, k_d) \ge 0\}$ *is a common minimal reducing subspace of the tuple* $(T_{z_1^{N_1}}, \ldots, T_{z_d^{N_d}})$ *on* $L_a^2(\mathbb{B}^d)$ *or* $L_a^2(\mathbb{D}^d)$.

(ii) *Those* L *minimal common reducing subspaces are the only minimal common reducing subspaces of the tuple* $(T_{z_1^{N_1}}, \ldots, T_{z_d^{N_d}})$ *on* $L_a^2(\mathbb{B}^d)$ *or* $L_a^2(\mathbb{D}^d)$.

(iii) *There are exactly* $2^L - 1$ *common reducing subspaces of the tuple* $(T_{z_1^{N_1}}, \ldots, T_{z_d^{N_d}})$ *on* $L_a^2(\mathbb{B}^d)$ *or* $L_a^2(\mathbb{D}^d)$.

The Dirichlet space $\mathcal{D}(\mathbb{D}^d)$ on the polydisk $\mathbb{D}^d$ is not as widely studied. Here we define $\mathcal{D}(\mathbb{D}^2)$ and refer to [**13**] for the general case. The Dirichlet space $\mathcal{D}(\mathbb{D}^2)$ is the Hilbert space of holomorphic functions on the bidisk $\mathbb{D}^2$ such that

$$\|f(z_1, z_2)\|_{\mathcal{D}}^2 = \int_{\mathbb{T}^2} |f(\zeta_1, \zeta_2)|^2 \, dm(\zeta_1) dm(\zeta_2) + \int_{\mathbb{D} \times \mathbb{T}} \left| \frac{\partial f(\zeta_1, \zeta_2)}{\partial \zeta_1} \right|^2 dA(\zeta_1) dm(\zeta_2)$$

$$+\int_{\mathbb{T}\times\mathbb{D}}\left|\frac{\partial f(\zeta_1,\zeta_2)}{\partial\zeta_2}\right|^2 dm(\zeta_1)dA(\zeta_2)+\int_{\mathbb{D}^2}\left|\frac{\partial^2 f(\zeta_1,\zeta_2)}{\partial\zeta_2\partial\zeta_1}\right|^2 dA(\zeta_1)dA(\zeta_2),$$

where $dm(\zeta_1)$ is the normalized Lebesgue measure of the unit circle $\mathbb{T}$. The first integral is $\|f(z_1,z_2)\|^2_{H^2(\mathbb{D}^2)}$, which is the norm of $f(z_1,z_2)$ in the Hardy space $H^2(\mathbb{D}^2)$ of the bidisk. Our definition of the norm in $\mathcal{D}(\mathbb{D}^2)$ is equivalent to the norm defined in [**13**], where the Möbius invariance of the fourth integral was studied. Our choice of the norm leads to a reproducing kernel of product form for $\mathcal{D}(\mathbb{D}^d)$ as in (15) below.

Let $N=(N_1,N_2)$ and let $W^*(z_1^{N_1}z_2^{N_2})$ be the von Neumann algebra generated by the analytic Toeplitz operator $T_{z_1^{N_1}z_2^{N_2}}$ on $\mathcal{D}(\mathbb{D}^2)$, and let $v(z_1^{N_1}z_2^{N_2})$ be the commutant of $W^*(z_1^{N_1}z_2^{N_2})$. We have the following structure theorem of $v(z_1^{N_1}z_2^{N_2})$. Let $M_n(\mathbb{C})$ denote the algebra of $n\times n$ matrices.

Theorem C. *The following two statements hold.*

(i) *If $N_1\neq N_2$, then $v(z_1^{N_1}z_2^{N_2})$ on $\mathcal{D}(\mathbb{D}^2)$ is $*$-isomorphic to*

$$\left[\bigoplus_{i=1}^{\infty}\mathbb{C}\right]\bigoplus\left[\bigoplus_{i=1}^{\infty}M_2(\mathbb{C})\right].$$

(ii) *If $N_1=N_2$, then $v(z_1^{N_1}z_2^{N_2})$ on $\mathcal{D}(\mathbb{D}^2)$ is $*$-isomorphic to*

$$\left[\bigoplus_{i=1}^{N_1}\mathbb{C}\right]\bigoplus\left[\bigoplus_{i=1}^{\infty}M_2(\mathbb{C})\right].$$

2. Reducing subspaces of weighted shifts with operator weights.

We first introduce a tuple of d-variable unilateral weighted shifts with operator weights. Here we extend the results for the case $d=1$ by the author [**5**] to the case $d>1$. The classical reference for weighted shifts with scalar weights is [**12**]. Let Z_+ be set of nonnegative integers and

$$Z_+^d=\{\alpha=(\alpha_1,\ldots,\alpha_d):\alpha_i\in Z_+,1\le i\le d\}.$$

We write $\alpha\ge 0$ if $\alpha\in Z_+^d$. More generally, for $\beta=(\beta_1,\ldots,\beta_d)$, $\alpha\ge\beta$ means $\alpha_i\ge\beta_i$ for all $1\le i\le d$. We write $\alpha>\beta$ if $\alpha\ge\beta$ and $\alpha\neq\beta$.

Let $\varepsilon_i=(0,\ldots,1,\ldots,0)$ be the multi-index having 1 at i-th component and 0 elsewhere, and let 0 be the multi-index $(0,0,\ldots,0)$. Let $l^2(Z_+^d)$ be the complex Hilbert space with the standard basis $\{e_\alpha:\alpha\in Z_+^d\}$. Let E be a complex Hilbert space. Let $l_d^2(E)$ denote the tensor product Hilbert space $l^2(Z_+^d)\otimes E$. That is, $l_d^2(E)$ is the E-valued $l^2(Z_+^d)$ space such that

$$l_d^2(E)=\left\{y=\sum_{\alpha\ge 0}y_\alpha e_\alpha : y_\alpha\in E \text{ and } \|y\|^2=\sum_{\alpha\ge 0}\|y_\alpha\|^2<\infty\right\}.$$

We identify E as a subspace of $l_d^2(E)$ by mapping y to ye_0 for $y\in E$. By an abuse of notation, we just write y instead of ye_0 for $y \in E$.

Let $\Phi = \left\{\Phi_{\alpha,i} : \alpha \in Z_+^d, i = 1, \dots, d\right\}$ be a bounded set of invertible operators in $B(E)$ such that

$$\Phi_{\alpha+\varepsilon_i,j}\Phi_{\alpha,i} = \Phi_{\alpha+\varepsilon_j,i}\Phi_{\alpha,j}, \quad \alpha \in Z_+^d, i \neq j, 1 \leq i,j \leq d. \tag{1}$$

Note that we do not assume $\Phi_{\alpha,i}\Phi_{\beta,i} = \Phi_{\beta,i}\Phi_{\alpha,i}$ for $\alpha, \beta \in Z_+^d$.

Definition 2.1. A tuple of d-variable unilateral weighted shifts is a family of d bounded operators on $l_d^2(E)$ with $S_\Phi = (S_1, \dots, S_d)$ defined by

$$S_i\left[ye_\alpha\right] = \left[\Phi_{\alpha,i}y\right]e_{\alpha+\varepsilon_i}, \quad \alpha \in Z_+^d, i = 1, \dots, d, y \in E. \tag{2}$$

Condition (1) on $\Phi_{\alpha,i}$ implies that S_Φ is a tuple of commuting operators, since for $i \neq j, y \in E$,

$$\begin{aligned} S_jS_i\left[ye_\alpha\right] &= S_j\left[\Phi_{\alpha,i}y\right]e_{\alpha+\varepsilon_i} = \left[\Phi_{\alpha+\varepsilon_i,j}\Phi_{\alpha,i}y\right]e_{\alpha+\varepsilon_i+\varepsilon_j}, \text{ and} \\ S_iS_j\left[ye_\alpha\right] &= S_i\left[\Phi_{\alpha,j}y\right]e_{\alpha+\varepsilon_j} = \left[\Phi_{\alpha+\varepsilon_j,i}\Phi_{\alpha,j}y\right]e_{\alpha+\varepsilon_j+\varepsilon_i}. \end{aligned}$$

As in the scalar case, the norm of S_i can be determined by

$$\begin{aligned} \left\|S_i\sum_{\alpha\geq 0}y_\alpha e_\alpha\right\|^2 &= \left\|\sum_{\alpha\geq 0}\left[\Phi_{\alpha,i}y_\alpha\right]e_{\alpha+\varepsilon_i}\right\|^2 \\ &= \sum_{\alpha\geq 0}\left\|\Phi_{\alpha,i}y_\alpha\right\|^2 \leq \sup_{\alpha\geq 0}\left\|\Phi_{\alpha,i}\right\|^2\sum_{\alpha\geq 0}\left\|y_\alpha\right\|^2. \end{aligned} \tag{3}$$

Then S_i is a bounded operator if and only if $\sup_{\alpha\geq 0}\|\Phi_{\alpha,i}\| < \infty$ and $\|S_i\| = \sup_{\alpha\geq 0}\|\Phi_{\alpha,i}\|$. Hence, if Φ is a bounded set in $B(E)$, then S_Φ is a tuple of bounded operators on $l_d^2(E)$. Note also

$$\begin{aligned} &S_i^*\left[ye_\alpha\right] = \left[\Phi_{\alpha-\varepsilon_i,i}^*y\right]e_{\alpha-\varepsilon_i} \quad \text{if } \alpha_i \geq 1, i = 1, \dots, d, \text{ and} \\ &S_i^*\left[ye_\alpha\right] = 0 \quad \text{if } \alpha_i = 0, i = 1, \dots, d, y \in E. \end{aligned}$$

Therefore $\bigcap_{i=1}^d \ker(S_i^*) = E$.

In this section we study the reducing subspace of S_Φ, which is a common reducing subspace of S_i for all $1 \leq i \leq d$. We will often write S instead of S_Φ. Let

$$A_i = \prod_{0\leq k\leq\alpha_i-1}\Phi_{\alpha_1\varepsilon_1+\cdots+\alpha_{i-1}\varepsilon_{i-1}+k\varepsilon_i,i}, \quad W_\alpha = A_dA_{d-1}\cdots A_1,$$
$$\text{where some factors in the product could be missing and } W_0 = I. \tag{4}$$

Then $\Phi_{\alpha,i} = W_{\alpha+\epsilon_i}W_\alpha^{-1}$ and $S^\alpha\left[ye_0\right] = \left[W_\alpha y\right]e_\alpha$ for $y \in E$, where $S^\alpha = S_1^{\alpha_1}\cdots S_d^{\alpha_d}$.

Lemma 2.2. *For a closed subspace E_0 of E, let $V(E_0)$ be defined by*

$$V(E_0) = \operatorname{Span}\left\{S_\Phi^\alpha x : \alpha \geq 0, x \in E_0\right\}. \tag{5}$$

Then $V(E_0)$ is a reducing subspace of S_Φ if and only if E_0 is an invariant subspace of the sequence of operators $\Omega = \left\{W_{\alpha-\epsilon_i}^{-1}\Phi_{\alpha-\epsilon_i,i}^{*}\Phi_{\alpha-\varepsilon_i,i}W_{\alpha-\epsilon_i} : \alpha \geq \varepsilon_i, 1 \leq i \leq d\right\}$. Equivalently, E_0 is an invariant subspace of $\Omega_1 = \{W_\alpha^* W_\alpha : \alpha \geq 0\}$.

PROOF. By the definition, $V(E_0)$ is invariant for S. The space $V(E_0)$ is also invariant for $S^* = (S_1^*, \ldots, S_d^*)$, if and only if $S_i^* S^\alpha x \in V(E_0)$ for any $x \in E_0$, $\alpha \geq 0$, and $1 \leq i \leq d$. For $\alpha = 0$, $S_i^* x = 0$. If $\alpha \geq \varepsilon_i$, then

$$S_i^* S^\alpha x e_0 = S_i^* \left[W_\alpha x e_\alpha\right] = \left[\Phi_{\alpha-\epsilon_i,i}^* W_\alpha x\right] e_{\alpha-\epsilon_i}.$$

By (5), $S_i^* S^\alpha x \in V(E_0)$ if and only if there exists $y \in E_0$ such that

$$S_i^* S^\alpha x e_0 = \left[\Phi_{\alpha-\epsilon_i,i}^* W_\alpha x\right] e_{\alpha-\epsilon_i} = S^{\alpha-\epsilon_i} y = \left[W_{\alpha-\epsilon_i} y\right] e_{\alpha-\epsilon_i}. \tag{6}$$

Since $W_\alpha = \Phi_{\alpha-\epsilon_i,i} W_{\alpha-\epsilon_i}$,

$$W_{\alpha-\epsilon_i}^{-1}\Phi_{\alpha-\epsilon_i,i}^{*}\Phi_{\alpha-\epsilon_i,i}W_{\alpha-\epsilon_i} x = W_{\alpha-\epsilon_i}^{-1}\Phi_{\alpha-\epsilon_i,i}^{*} W_\alpha x = y \in E_0.$$

Therefore E_0 is invariant for Ω.

Note that for $x \in E_0$, since

$$S_\Phi^{*\alpha} S_\Phi^\alpha x e_0 = W_\alpha^* W_\alpha x e_0,$$

$S_\Phi^{*\alpha} S_\Phi^\alpha x e_0 \in V(E_0)$ implies that E_0 is invariant for $W_\alpha^* W_\alpha$. Thus E_0 is invariant for Ω_1.

Assume E_0 is invariant for Ω_1, we now prove E_0 is invariant for Ω. By assumption $W_\alpha^* W_\alpha$ is invertible and positive, so E_0 being invariant for $W_\alpha^* W_\alpha$ implies that

$$W_\alpha^* W_\alpha E_0 = E_0 \text{ and } \left[W_\alpha^* W_\alpha\right]^{-1} E_0 = E_0.$$

Using $\Phi_{\alpha-\varepsilon_i,i} = W_\alpha W_{\alpha-\varepsilon_i}^{-1}$, we have

$$\begin{aligned} W_{\alpha-\epsilon_i}^{-1}\Phi_{\alpha-\epsilon_i,i}^{*}\Phi_{\alpha-\varepsilon_i,i}W_{\alpha-\epsilon_i} &= W_{\alpha-\epsilon_i}^{-1} W_{\alpha-\varepsilon_i}^{*-1} W_\alpha^* W_\alpha W_{\alpha-\varepsilon_i}^{-1} W_{\alpha-\epsilon_i} \\ &= \left[W_{\alpha-\varepsilon_i}^* W_{\alpha-\epsilon_i}\right]^{-1} W_\alpha^* W_\alpha. \end{aligned}$$

Therefore E_0 is invariant for Ω. The proof is complete. □

REMARK 2.3. The space E_0 is also invariant for other operators involving $\Phi_{\alpha,i}$ and W_α by considering the invariance of X for $S^{*\alpha} S^\beta$ for any $\alpha, \beta \geq 0$. The operator $W_{\alpha-\epsilon_i}^{-1}\Phi_{\alpha-\epsilon_i,i}^{*}\Phi_{\alpha-\varepsilon_i,i}W_{\alpha-\epsilon_i} = \Phi_{\alpha-\epsilon_i,i}^{*}\Phi_{\alpha-\varepsilon_i,i}$ under the commuting condition $\left[\Phi_{\alpha-\epsilon_i,i}^{*}\Phi_{\alpha-\varepsilon_i,i}\right] W_{\alpha-\epsilon_i} = W_{\alpha-\epsilon_i}\left[\Phi_{\alpha-\epsilon_i,i}^{*}\Phi_{\alpha-\varepsilon_i,i}\right]$. So $\Omega = \{\Phi_{\alpha-\epsilon_i,i}^{*}\Phi_{\alpha-\varepsilon_i,i} : \alpha \geq \varepsilon_i, 1 \leq i \leq d\}$.

THEOREM 2.4. *A closed subspace X is a (common) reducing subspace of S_Φ if and only if*

$$X = \operatorname{Span}\left\{S_\Phi^\alpha x : \alpha \geq 0, x \in E_0\right\}, \tag{7}$$

where

$$E_0 = \bigcap_{i=1}^{d} X \ominus S_i X = \bigcap_{i=1}^{d} \ker(S_i^* | X) \subseteq E$$

and $E_0 \subseteq E$ is an invariant subspace of the sequence of operators $\Omega_1 = \{W_\alpha^* W_\alpha : \alpha \geq 0\}$. Furthermore, X is a minimal reducing subspace of S_Φ if and only if E_0 is a minimal invariant subspace of Ω_1.

Proof. By Lemma 2.2, we only need to prove that if X is a reducing subspace of S, then X is given by (7) for some $E_0 \subseteq E$. Set $E_0 = \bigcap_{i=1}^{d} X \ominus S_i X$. We first prove that $E_0 \subset E$. Let $f \in X \ominus S_i X$, then

$$\langle S_i^* f, g \rangle = \langle f, S_i g \rangle = 0 \quad \text{for all } g \in X, 1 \leq i \leq d.$$

Since X is also invariant for S_i^*, $S_i^* f \in X$. Hence $S_i^* f = 0$ and $f \in \bigcap_{i=1}^{d} \ker(S_i^*) = E$. This proves that $E_0 \subseteq E$. We claim

$$X = V(E_0) := \operatorname{Span}\left\{ S_\Phi^\alpha x : \alpha \geq 0, x \in E_0 = \bigcap_{i=1}^{d} X \ominus S_i X \right\}.$$

Since $E_0 \subseteq X$, $X \supseteq V(E_0)$. Let $y \in X \ominus V(E_0)$. We need to show that $y = 0$. Write

$$y = \sum_{\alpha \geq 0} y_\alpha e_\alpha, \quad y_\alpha \in E.$$

Since X is invariant for $S^{*\beta}$ for any $\beta \geq 0$, $S^{*\beta} y \in X$. For all $x \in E_0 = \bigcap_{i=1}^{d} X \ominus S_i X$ and $\beta \geq 0$, note that $y \in X \ominus V(E_0)$ implies

$$0 = \left\langle y, S^\beta x \right\rangle = \left\langle S^{*\beta} y, x \right\rangle.$$

That is, $S^{*\beta} y \in X \ominus [\bigcap_{i=1}^{d} X \ominus S_i X]$. Set

$$M_X = X \ominus \left[\bigcap_{i=1}^{d} X \ominus S_i X \right] \quad \text{and} \quad M = \left\{ y = \sum_{\alpha \geq 0, \alpha \neq 0} f_\alpha e_\alpha : f_\alpha \in E \right\}.$$

Then

$$M_X = \operatorname{Span}\{S_i X : 1 \leq i \leq d\} \subseteq M$$

and $M_X^\perp \supseteq M^\perp = l_d^2(E) \ominus M = E$. Note that

$$\begin{aligned} S^{*\beta} y &= \sum_{\alpha \geq 0} S^{*\beta} y_\alpha e_\alpha = \sum_{\alpha \geq \beta} S^{*\beta} y_\alpha e_\alpha \\ &= S^{*\beta} y_\beta e_\beta + \sum_{\alpha \geq \beta, \alpha \neq \beta} S^{*\beta} y_\alpha e_\alpha \end{aligned}$$

and

$$S^{*\beta} y_\beta e_\beta \in E, \qquad \sum_{\alpha \geq \beta, \alpha \neq \beta} S^{*\beta} y_\alpha e_\alpha \in M.$$

Thus $S^{*\beta} y \in M_X$ implies that

$$0 = S^{*\beta} y_\beta e_\beta = \left[W^*_\beta y_\beta\right] e_0.$$

By assumption W_β is invertible, so $y_\beta = 0$ for $\beta \geq 0$. In conclusion $y = 0$. The proof is complete. □

By the above theorem, the lattice of reducing subspaces of S_Φ is completely determined by the lattice of invariant subspaces of Ω_1. This topics has been discussed extensively in literature, and many results are known, in particular when Ω_1 is a set of finite matrices, see the book [**17**].

It is well-known that the weighted shifts with nonzero scalar weights are irreducible.

Corollary 2.5. *The tuple of weighted shifts $S_\Phi = (S_1, \ldots, S_d)$ on $l^2_d(E)$ with operator weights is irreducible if and only if $\Omega_1 = \{W^*_\alpha W_\alpha : \alpha \geq 0\}$ is irreducible.*

A simple but remarkable fact is that $(S_1^{k_1}, \ldots, S_d^{k_d})$ with $k_i \geq 1$ for $i = 1, \ldots, d$, is again a tuple of commuting weighted shifts with operator weights. The above theorem also applies to $(S_1^{k_1}, \ldots, S_d^{k_d})$. This idea will become clear when we apply the above theorem to multiplication operators by powers of coordinator functions.

3. Multiplication operators on weighted Hardy spaces of several variables.

Let $z \in \mathbb{C}^d$ be the multivariable,

$$z = (z_1, \ldots, z_d), \quad \overline{z} = (\overline{z_1}, \ldots, \overline{z_d}).$$

An analytic polynomial $p(z)$ is of the form

$$p(z) = \sum_{|\alpha|=0}^{m} c_\alpha z^\alpha, \quad c_\alpha \in \mathbb{C},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \geq 0$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$. It is known from [**12**] that several weighted shifts with nonzero scalar weights is unitarily equivalently to multiplications by z_i on weighted Hardy spaces with positive scalar weights. In this section, we introduce weighted Hardy spaces of multivariable z with operator weights. We show that multiplications by z_i on those weighted Hardy spaces are the weighted shift S_Φ studied in the last section. First note that for $A \in B(H)$ and $h \in H$,

$$\langle Ah, Ah \rangle = \langle A^*Ah, h \rangle = \left\langle \sqrt{A^*A}h, \sqrt{A^*A}h \right\rangle,$$

and $\sqrt{A^*A} \geq 0$. Thus in the definition of weighted Hardy spaces we will use positive operators. Let $\Delta = \{W_\alpha : \alpha \geq 0\}$ be a bounded set of invertible positive operators in

$B(E)$. The weighted Hardy space $H^2_\Delta(E)$ is defined by

$$H^2_\Delta(E) = \left\{ f(z) = \sum_{\alpha\geq 0} f_\alpha z^\alpha : f_\alpha \in E, \quad \|f(z)\|^2 = \sum_{\alpha\geq 0} \|W_\alpha f_\alpha\|^2 < \infty \right\}. \tag{8}$$

Then the multiplication operator M_{z_i} by z_i on $H^2_\Delta(E)$ for $1 \leq i \leq d$, denoted by $M_z = (M_{z_1}, \ldots, M_{z_d})$, can be identified with the weighted shift S_Φ on $l^2_d(E)$ with

$$\Phi = \left\{ \Phi_{\alpha,i} = W_{\alpha+\varepsilon_i} W_\alpha^{-1} : \alpha \geq 0, 1 \leq i \leq d \right\}.$$

(Note that $W_{\alpha+\varepsilon_i} W_\alpha^{-1}$ is not necessary positive since no commuting condition is imposed on W_α.) More precisely, let U be the linear operator from $l^2_d(E)$ onto $H^2_\Delta(E)$ defined by

$$U[ye_\alpha] = \left[W_\alpha^{-1} y\right] z^\alpha, \quad \alpha \geq 0, y \in E.$$

Then

$$\begin{aligned}\left\| U\left(\sum_{\alpha\geq 0} y_\alpha e_\alpha\right)\right\|^2_{H^2_\Delta(E)} &= \left\| \sum_{\alpha\geq 0} \left[W_\alpha^{-1} y_\alpha\right] z^\alpha \right\|^2_{H^2_\Delta(E)} = \sum_{\alpha\geq 0} \left\| W_\alpha W_\alpha^{-1} y_\alpha \right\|^2_E \\ &= \sum_{\alpha\geq 0} \|y_\alpha\|^2_E = \left\| \sum_{\alpha\geq 0} y_\alpha e_\alpha \right\|^2_{l^2_d(E)}.\end{aligned}$$

Thus U is an onto isometry. Furthermore,

$$\begin{aligned} M_{z_i} U y e_\alpha &= M_{z_i}\left(W_\alpha^{-1} y z^\alpha\right) = W_\alpha^{-1} y z^{\alpha+\varepsilon_i}, \\ U S_i y e_\alpha &= U(\Phi_{\alpha,i} y e_{\alpha+\varepsilon_i}) = W^{-1}_{\alpha+\varepsilon_i} \Phi_{\alpha,i} y z^{\alpha+\varepsilon_i} \\ &= W^{-1}_{\alpha+\varepsilon_i} W_{\alpha+\varepsilon_i} W_\alpha^{-1} y z^{\alpha+\varepsilon_i} = W_\alpha^{-1} y z^{\alpha+\varepsilon_i}. \end{aligned}$$

Therefore

$$M_z U = (M_{z_1} U, \ldots, M_{z_d} U) = U S_\Phi = (US_1, \ldots, US_d).$$

By (3), M_z is a tuple of commuting bounded operators if and only if

$$\|M_{z_i}\| = \sup_{\alpha\geq 0} \|\Phi_{\alpha,i}\| = \sup_{\alpha\geq 0} \left\| W_{\alpha+\varepsilon_i} W_\alpha^{-1} \right\| < \infty, \quad 1 \leq i \leq d, \tag{9}$$

which we shall assume. Note that (1) is automatically satisfied, since by $\Phi_{\alpha,i} = W_{\alpha+\varepsilon_i} W_\alpha^{-1}$, for $i \neq j$,

$$\begin{aligned} \Phi_{\alpha+\varepsilon_i,j} \Phi_{\alpha,i} &= W_{\alpha+\varepsilon_i+\varepsilon_j} W^{-1}_{\alpha+\varepsilon_i} W_{\alpha+\varepsilon_i} W_\alpha^{-1} = W_{\alpha+\varepsilon_i+\varepsilon_j} W_\alpha^{-1}, \\ \Phi_{\alpha+\varepsilon_j,i} \Phi_{\alpha,j} &= W_{\alpha+\varepsilon_j+\varepsilon_i} W^{-1}_{\alpha+\varepsilon_j} W_{\alpha+\varepsilon_j} W_\alpha^{-1} = W_{\alpha+\varepsilon_j+\varepsilon_i} W_\alpha^{-1}. \end{aligned}$$

The reducing subspaces (or minimal reducing subspaces) of M_z and S_Φ are in one to one correspondence. Now Theorem 2.4 can be reformulated as the following theorem which generalizes a similar result [**5**] in one variable case.

THEOREM 3.1. *Any common reducing subspace X of M_z on $H^2_\Delta(E)$ is of the form $H^2_\Delta(E_0)$, where*

$$E_0 = \bigcap_{i=1}^{d} X \ominus M_{z_i} X = \bigcap_{i=1}^{d} \ker(M_{z_i}^* | X) \subseteq E,$$

and $E_0 \subseteq E$ is an invariant subspace of $\Omega = \{W_\alpha : \alpha \geq 0\}$. Furthermore $H^2_\Delta(E_0)$ is a minimal reducing subspace of M_z if and only if E_0 is a minimal invariant subspace of Ω.

PROOF. By Theorem 2.4,

$$\Omega_1 = \{W_\alpha^* W_\alpha : \alpha \geq 0\}.$$

But here we assume W_α is positive, so $W_\alpha^* W_\alpha = W_\alpha^2$. The space E_0 is invariant for W_α^2 if and only if it is invariant for W_α. $\square$

If E is a finite dimensional complex Hilbert space and $E_0 \subseteq E$ is a nontrivial invariant subspace of $\Omega = \{W_\alpha : \alpha \geq 0\}$, then E_0 contains a minimal invariant subspace of Ω. Since W_α is positive, E_0 is in fact a reducing subspace of Ω and it is an orthogonal sum of several minimal reducing subspaces of Ω.

COROLLARY 3.2. *Assume $N = \dim(E) < \infty$. Then any nontrivial reducing subspace of M_z on $H^2_\Delta(E)$ contains a minimal reducing subspace. Furthermore it is a direct sum of at most N minimal reducing subspaces of M_z.*

As mentioned in [**21**], there are operators which possess many reducing subspaces but have no minimal reducing subspaces at all. For example, the operator of multiplication by z on the Lebesgue space $L^2(\mathbb{D}, dA)$, where dA is the area measure on the unit disk $\mathbb{D}$, is one. In view of the above corollary, if $\dim(E) < \infty$, then the only question remaining is how to describe the minimal reducing subspaces of Ω.

4. Hilbert spaces of holomorphic functions of several variables.

Let $\omega = \{\omega_\alpha : \alpha \geq 0\}$ be a set of positive numbers. Let $\mathbb{C}$ denote the set of complex numbers viewed as an one dimensional Hilbert space. Let H^2_ω be the weighted Hardy space as in [**12**]

$$H^2_\omega = \left\{ f(z) = \sum_{\alpha \geq 0} f_\alpha z^\alpha : f_\alpha \in \mathbb{C}, \quad \|f(z)\|^2 = \sum_{\alpha \geq 0} \omega_\alpha |f_\alpha|^2 < \infty \right\}. \tag{10}$$

Considering $E = \mathbb{C}$ and $\Delta = \left\{\sqrt{\omega_\alpha} : \alpha \geq 0\right\}$ in Section 3, we have $H^2_\omega = H^2_\Delta(\mathbb{C})$. By (9), we assume

$$\|M_{z_i}\| = \sup_{\alpha \geq 0} \sqrt{\frac{\omega_{\alpha+\varepsilon_i}}{\omega_\alpha}} < \infty, \quad 1 \leq i \leq d.$$

Let $N = (N_1, \ldots, N_d) \in Z_+^d$ be such that $N \geq (1, \ldots, 1)$. By an abuse of notation, set $N - 1 = (N_1 - 1, \ldots, N_d - 1)$. Let $L = \prod_{i=1}^d N_i$, E be the L-dimensional subspace of H_ω^2 defined by

$$E = \left\{ \sum_{0 \leq \beta \leq N-1} f_\beta z^\beta : f_\beta \in \mathbb{C} \right\},$$

and $\left\{ z^\beta / \sqrt{\omega_\beta} : 0 \leq \beta \leq N - 1 \right\}$ be the standard basis of E. For two multi-indices $k = (k_1, \ldots, k_d)$ and $N = (N_1, \ldots, N_d)$, let kN denote the multi-index $kN = (k_1 N_1, \ldots, k_d N_d)$. Let $\Delta = \{W_k : k \geq 0\}$ be the set of diagonal operators where W_k is the diagonal matrix (with respect to the standard basis of E) defined by

$$W_k \left(\frac{z^\beta}{\sqrt{\omega_\beta}} \right) = \frac{\sqrt{\omega_{\beta+kN}}}{\sqrt{\omega_\beta}} \left(\frac{z^\beta}{\sqrt{\omega_\beta}} \right), \quad 0 \leq \beta \leq N - 1, k \geq 0. \tag{11}$$

Then the tuple $M_z = \left(M_{z_1^{N_1}}, \ldots, M_{z_d^{N_d}} \right)$ on H_ω^2 can be identified with $M_z = (M_{z_1}, \ldots, M_{z_d})$ on $H_\Delta^2(E)$. To see this, we write

$$\sum_{\alpha \geq 0} f_\alpha z^\alpha = \sum_{k \geq 0} \left(\sum_{0 \leq \beta \leq N-1} f_{\beta+kN} z^\beta \right) z^{kN}.$$

Let U be the linear operator from H_ω^2 onto $H_\Delta^2(E)$ defined by

$$U \sum_{\alpha \geq 0} f_\alpha z^\alpha = \sum_{k \geq 0} g_k z^k \quad \text{where} \quad g_k = \sum_{0 \leq \beta \leq N-1} f_{\beta+kN} z^\beta \in E.$$

Since U maps z^{kN} in H_ω^2 into z^k in $H_\Delta^2(E)$, it is easy to see that $UM_{z^N} = M_z U$. We now verify that U is an onto isometry.

$$\begin{aligned}
\left\| U \sum_{\alpha \geq 0} f_\alpha z^\alpha \right\|_{H_\Delta^2(E)}^2 &= \left\| \sum_{k \geq 0} g_k z^k \right\|_{H_\Delta^2(E)}^2 = \sum_{k \geq 0} \|W_k g_k\|_E^2 = \sum_{k \geq 0} \left\| W_k \sum_{0 \leq \beta \leq N-1} f_{\beta+kN} z^\beta \right\|_E^2 \\
&= \sum_{k \geq 0} \sum_{0 \leq \beta \leq N-1} \left\| W_k f_{\beta+kN} z^\beta \right\|_E^2 = \sum_{k \geq 0} \sum_{0 \leq \beta \leq N-1} \left\| \frac{\sqrt{\omega_{\beta+kN}}}{\sqrt{\omega_\beta}} f_{\beta+kN} z^\beta \right\|_E^2 \\
&= \sum_{k \geq 0} \sum_{0 \leq \beta \leq N-1} \omega_\beta \left| \frac{\sqrt{\omega_{\beta+kN}}}{\sqrt{\omega_\beta}} f_{\beta+kN} \right|^2 = \sum_{k \geq 0} \sum_{0 \leq \beta \leq N-1} \omega_{\beta+kN} \left| f_{\beta+kN} \right|^2 \\
&= \left\| \sum_{\alpha \geq 0} f_\alpha z^\alpha \right\|_{H_\omega^2}^2.
\end{aligned}$$

Since Δ consists of diagonal matrices, the following result, which is Lemma 6 in [**5**], is useful.

LEMMA 4.1. (i) *Let Ω be a set of invertible diagonal matrices on $\mathbb{C}^L$ with respect to an orthonormal basis $\{e_1, \dots, e_L\}$. Then any minimal invariant subspace of Ω is one dimensional.*

(ii) *Any invariant subspace of Ω is an orthogonal sum of several one dimensional invariant subspaces of Ω.*

(iii) *Let $v = \sum_{i=1}^{k} v_{n_i} e_{n_i}$, where all v_{n_i} are nonzero. Then* $\mathrm{Span}\,\{v\}$ *is invariant for Ω if and only if each diagonal matrix in Ω restricted to* $\mathrm{Span}\,\{e_{n_1}, \dots, e_{n_k}\}$ *is a constant multiple of the identity matrix.*

Combining Theorem 3.1 and Lemma 4.1, we immediately have the following theorem, which contains Theorem A and Theorem D in [**21**], and Theorem 6 in [**14**] as special cases. This theorem can also be derived from the work of [**14**]. Set

$$\widehat{N} = \{\beta : 0 \le \beta \le N - 1\}.$$

THEOREM 4.2. (i) *A reducing subspace of $(M_{z_1^{N_1}}, \dots, M_{z_d^{N_d}})$ on H^2_ω is a direct sum of at most L (singly generated) minimal reducing subspaces, where L is the cardinality of the index set $\widehat{N}$. That is, $L = \prod_{i=1}^{d} N_i$.*

(ii) *A minimal reducing subspace of $(M_{z_1^{N_1}}, \dots, M_{z_d^{N_d}})$ on H^2_ω is of the form*

$$\mathrm{Span}\left\{p(z) z^{kN} : k \ge 0\right\},$$

where

$$p(z) = \sum_{\gamma \in J} f_\gamma z^\gamma, \quad f_\gamma \in \mathbb{C},\ f_\gamma \ne 0 \text{ for all } \gamma \in J, \tag{12}$$

and $J \subseteq \widehat{N}$ and $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $\gamma, \delta \in J$ and $k \ge 0$.

(iii) *For each $\gamma \in \widehat{N}$,* $\mathrm{Span}\left\{z^\gamma z^{kN} : k \ge 0\right\}$ *is a (singly generated) minimal reducing subspace of $(M_{z_1^{N_1}}, \dots, M_{z_d^{N_d}})$.*

PROOF. Conclusion (i) is clear from (i) and (ii) of Lemma 4.1. To see (ii), we note that W_k as in (11) is a constant multiple of the identity on the $\mathrm{Span}\{z^\gamma : \gamma \in J\}$ if and only if $\omega_{\gamma+kN}/\omega\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $\gamma, \delta \in J$. The proof is complete. $\square$

We say the reducing subspaces as in (iii) are the obvious ones.

DEFINITION 4.3. Let $\kappa(J)$ denote the cardinality of the index set J. We say $p(z)$ as in (12) is of length $\kappa(J)$ and $\mathrm{Span}\left\{p(z) z^{kN} : k \ge 0\right\}$ is a minimal reducing subspace of length $\kappa(J)$.

Thus $\kappa(J) = 1$ for the reducing subspaces as in (iii) of Theorem 4.2. As we will see, in most classical function spaces, $\kappa(J) = 1$, so that there are exactly L minimal reducing subspaces of $(M_{z_1^{N_1}}, \ldots, M_{z_d^{N_d}})$ on H^2_ω and there are exactly $2^L - 1$ reducing subspaces of $(M_{z_1^{N_1}}, \ldots, M_{z_d^{N_d}})$.

Lemma 4.1 can be extended to the set of diagonal operators on the infinite dimensional l^2 space. For convenience, we recall Lemma 8 and Corollary 9 from [**5**]. Let $\mathbb{N}$ be the set of positive integers. In the infinite dimensional case, all subspaces are assumed to be closed.

Lemma 4.4. *Let Ω be a set of injective diagonal operators on l^2 with respect to an orthonormal basis $\{e_n : n \in \mathbb{N}\}$.*

(i) *Let $v = \sum_{i=1}^{\infty} v_{n_i} e_{n_i}$, where all v_{n_i} are nonzero. Then* Span $\{v\}$ *is invariant for Ω if and only if the restriction of each diagonal operator in Ω to* Span $\{e_{n_1}, e_{n_2}, \ldots\}$ *is a constant multiple of the identity operator.*

(ii) *Any minimal invariant subspace of Ω is one dimensional.*

(iii) *Any invariant subspace of Ω is an orthogonal sum of finite or infinite many one dimensional invariant subspaces of Ω.*

The following corollary tells us when the invariant subspaces of Ω are the obvious ones.

Corollary 4.5. (1) *Let Ω be a set of invertible diagonal matrices on $\mathbb{C}^L$ with respect to an orthonormal basis $\{e_1, \ldots, e_L\}$. The following two statements are equivalent.*

(i) *For any $i \neq j$, there is $A \in \Omega$ such that $Ae_i = \lambda_i e_i, Ae_j = \lambda_j e_j$ with $\lambda_i \neq \lambda_j$.*

(ii) *There are exactly L minimal invariant subspaces of Ω, namely,* Span $\{e_i\}$ *for $i = 1, \ldots, L$.*

(2) *Let Ω be a set of injective diagonal operators on l^2 with respect to an orthonormal basis $\{e_n, n \in \mathbb{N}\}$. The following two statements are equivalent.*

(i) *For any $i, j \in \mathbb{N}$ and $i \neq j$, there is $A \in \Omega$ such that $Ae_i = \lambda_i e_i, Ae_j = \lambda_j e_j$ with $\lambda_i \neq \lambda_j$.*

(ii) *The minimal invariant subspaces of Ω are* Span $\{e_i\}$ *for $i \in \mathbb{N}$.*

Statement (i) holds as long as Ω contains a diagonal operator with distinct diagonals.

5. Examples.

Let $\mathbb{B}^d$ be the unit ball of $\mathbb{C}^d$,

$$\mathbb{B}^d = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 < 1 \right\}.$$

Let $w = (w_1, \ldots, w_d) \in \mathbb{B}^d$ and $\langle z, w \rangle$ be the inner product defined by

$$\langle z, w\rangle = \sum_{i=1}^{d} z_i \overline{w_i}.$$

Let $\mathcal{K}_\rho(\mathbb{B}^d)$ (for $\rho > 0$) be the Hilbert space of analytic functions on the ball $\mathbb{B}^d$ with reproducing kernel

$$K(z, w) = \frac{1}{(1 - \langle z, w\rangle)^\rho}.$$

This scale of spaces contains the Bergman space $L_a^2(\mathbb{B}^d)$ ($\rho = d + 1$), the Hardy space $H^2(\mathbb{B}^d)$ ($\rho = d$), and the Drury–Arveson space $H_d^2(\mathbb{B}^d)$ ($\rho = 1$). By the expansion formula,

$$\begin{aligned} K(z, w) &= \frac{1}{(1 - \langle z, w\rangle)^\rho} = \sum_{i=0}^{\infty} \frac{\Gamma(\rho + i)}{i!\Gamma(\rho)} \langle z, w\rangle^i \\ &= \sum_{i=0}^{\infty} \frac{\Gamma(\rho + i)}{i!\Gamma(\rho)} \sum_{|\alpha|=i} \frac{i!}{\alpha!} z^\alpha \overline{w}^\alpha = \sum_{\alpha \geq 0} \frac{\Gamma(\rho + |\alpha|)}{\alpha!\Gamma(\rho)} z^\alpha \overline{w}^\alpha. \end{aligned}$$

Therefore

$$\mathcal{K}_\rho(\mathbb{B}^d) = H_\omega^2 \quad \text{with} \quad \omega = \left\{ \omega_\alpha = \frac{\alpha!\Gamma(\rho)}{\Gamma(\rho + |\alpha|)} : \alpha \geq 0 \right\}. \tag{13}$$

Let $\mathcal{D}(\mathbb{B}^d)$ denote the holomorphic Dirichlet space on $\mathbb{B}^d$ with reproducing kernel

$$K(z, w) = -\frac{1}{\langle z, w\rangle} \ln(1 - \langle z, w\rangle).$$

Note that

$$\begin{aligned} K(z, w) &= -\frac{1}{\langle z, w\rangle} \ln(1 - \langle z, w\rangle) = \sum_{n=0}^{\infty} \frac{1}{n+1} \langle z, w\rangle^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{|\alpha|=n} \frac{n! z^\alpha \overline{w}^\alpha}{\alpha!} = \sum_{\alpha \geq 0} \frac{|\alpha|! z^\alpha \overline{w}^\alpha}{\alpha! \, (|\alpha| + 1)}. \end{aligned}$$

Therefore

$$\mathcal{D}(\mathbb{B}^d) = H_\omega^2 \quad \text{with} \quad \omega = \left\{ \omega_\alpha = \frac{\alpha! \, (|\alpha| + 1)}{|\alpha|!} : \alpha \geq 0 \right\}.$$

Let $\mathbb{D}$ be the unit disk and $\mathbb{D}^d$ be the polydisk. We use $\mathcal{K}_\rho(\mathbb{D}^d)$ (for $\rho > 0$) to denote the Hilbert space of analytic functions on the polydisk $\mathbb{D}^d$ with reproducing kernel

$$K(z, w) = \frac{1}{\prod_{i=1}^{d} (1 - z_i \overline{w_i})^\rho}.$$

This scale of spaces contains the Bergman space $L_a^2(\mathbb{D}^d)$ ($\rho = 2$) and Hardy space $H^2(\mathbb{D}^d)$

($\rho = 1$). When $\rho > 1$, $\mathcal{K}_\rho(\mathbb{D}^d)$ is often called the weighted Bergman space on polydisk. By the expansion formula,

$$K(z,w) = \frac{1}{\prod_{i=1}^d (1 - z_i\overline{w_i})^\rho} = \prod_{i=1}^d \left(\sum_{\alpha_i=0}^\infty \frac{\Gamma(\rho+\alpha_i)}{\alpha_i!\Gamma(\rho)} (z_i\overline{w_i})^{\alpha_i} \right)$$
$$= \sum_{\alpha\geq 0} \prod_{i=1}^d \frac{\Gamma(\rho+\alpha_i)}{\alpha_i!\Gamma(\rho)} z^\alpha \overline{w}^\alpha.$$

Therefore

$$\mathcal{K}_\rho(\mathbb{D}^d) = H^2_\omega \quad \text{with} \quad \omega = \left\{ \omega_\alpha = \prod_{i=1}^d \frac{\alpha_i!\Gamma(\rho)}{\Gamma(\rho+\alpha_i)} : \alpha \geq 0 \right\}. \tag{14}$$

Let $\mathcal{D}(\mathbb{D}^d)$ denote the holomorphic Dirichlet space on $\mathbb{D}^d$ with reproducing kernel

$$K(z,w) = (-1)^d \prod_{i=1}^d \frac{1}{z_i\overline{w_i}} \ln(1 - z_i\overline{w_i}). \tag{15}$$

Note that

$$K(z,w) = (-1)^d \prod_{i=1}^d \frac{1}{z_i\overline{w_i}} \ln(1 - z_i\overline{w_i}) = \prod_{i=1}^d \left(\sum_{\alpha_i=0}^\infty \frac{1}{\alpha_i+1} (z_i\overline{w_i})^{\alpha_i} \right)$$
$$= \sum_{\alpha\geq 0} \frac{1}{\prod_{i=1}^d (\alpha_i+1)} z^\alpha \overline{w}^\alpha.$$

Therefore

$$\mathcal{D}(\mathbb{D}^d) = H^2_\omega \quad \text{with} \quad \omega = \left\{ \omega_\alpha = \prod_{i=1}^d (\alpha_i+1) : \alpha \geq 0 \right\}.$$

Let $N = (N_1, \ldots, N_d) \in Z^d_+$ be such that $N \geq (1, \ldots, 1)$.

LEMMA 5.1. *We use notations as above. Let $\gamma, \delta \geq 0$ be two multi-indices. Then for ω_α as in* (13) *with $d \geq 2$,*

$$\frac{\omega_{\gamma+kN}}{\omega_\gamma} = \frac{\omega_{\delta+kN}}{\omega_\delta} \quad \textit{for all } k \geq 0$$

if and only if $\gamma = \delta$.

PROOF. Note that $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ is the same as

$$\frac{\omega_{\gamma+kN}}{\omega_{\delta+kN}} = \frac{\omega_\gamma}{\omega_\delta} \quad \text{or} \quad \frac{(\gamma+kN)!\Gamma(\rho+|\delta+kN|)}{(\delta+kN)!\Gamma(\rho+|\gamma+kN|)} = \frac{\omega_\gamma}{\omega_\delta}.$$

Note that the limit of

$$\frac{(\gamma+kN)!\Gamma(\rho+|\delta+kN|)}{(\delta+kN)!\Gamma(\rho+|\gamma+kN|)} \approx k_1^{\gamma_1-\delta_1} k_1^{|\delta|-|\gamma|},$$

as $k_1 \to \infty$, is 0 or ∞ unless

$$\gamma_1 + |\delta| = \delta_1 + |\gamma| \,.$$

Similarly

$$\gamma_i + |\delta| = \delta_i + |\gamma| \,, \quad i = 1, \ldots, d.$$

This implies that $\gamma = \delta$ since $d \geq 2$. $\square$

REMARK 5.2. The above lemma also holds when $d = 1$ and $\rho \neq 1$, see the lemma below.

LEMMA 5.3. *We use notations as above. Let $\gamma, \delta \geq 0$ be two multi-indices. Assume $\rho \neq 1$. Then for ω_α as in* (14),

$$\frac{\omega_{\gamma+kN}}{\omega_\gamma} = \frac{\omega_{\delta+kN}}{\omega_\delta} \quad \textit{for all } k \geq 0$$

if and only if $\gamma = \delta$.

PROOF. Note that $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ is the same as

$$\frac{\omega_{\gamma+kN}}{\omega_{\delta+kN}} = \frac{\omega_\gamma}{\omega_\delta} \quad \text{or} \quad \prod_{i=1}^{d} \frac{(\gamma_i+k_iN_i)!\Gamma(\rho)\Gamma(\rho+\delta_i+k_iN_i)}{(\delta_i+k_iN_i)!\Gamma(\rho)\Gamma(\rho+\gamma_i+k_iN_i)} = \frac{\omega_\gamma}{\omega_\delta}.$$

Equivalently, for each $i = 1, \ldots, d$,

$$\frac{(\gamma_i+k_iN_i)!\Gamma(\rho+\delta_i+k_iN_i)}{(\delta_i+k_iN_i)!\Gamma(\rho+\gamma_i+k_iN_i)} = \frac{(\gamma_i)!\Gamma(\rho+\delta_i)}{(\delta_i)!\Gamma(\rho+\gamma_i)}.$$

Taking limit of the above expression as $k_i \to \infty$, we see that both sides of the above expression are equal to one. That is,

$$\frac{(\gamma_i+k_iN_i)!\Gamma(\rho+\delta_i+k_iN_i)}{(\delta_i+k_iN_i)!\Gamma(\rho+\gamma_i+k_iN_i)} = 1 \quad \text{for all } k_i \geq 0.$$

Without loss of generality, assume $\gamma_i = \delta_i + l$ for some $l > 0$. Then

$$\frac{(1+\delta_i+k_iN_i)\cdots(l+\delta_i+k_iN_i)}{(\rho+\delta_i+k_iN_i)\cdots(\rho+l-1+\delta_i+k_iN_i)} = 1 \quad \text{for all } k_i \geq 0,$$

which is impossible unless $\rho = 1$. Hence $\gamma_i = \delta_i$ and $\gamma = \delta$. $\square$

THEOREM 5.4. *The tuple $(M_{z_1^{N_1}}, \ldots, M_{z_d^{N_d}})$ on $\mathcal{K}_\rho(\mathbb{B}^d)$, $\mathcal{K}_\rho(\mathbb{D}^d)$ $(\rho \neq 1)$, $\mathcal{D}(\mathbb{B}^d)$, and $\mathcal{D}(\mathbb{D}^d)$ has only the obvious L (singly generated) minimal reducing subspaces of length one.*

PROOF. The results follow from proceeding lemmas. The Hardy space on the polydisk is the only exception. The proofs for the Dirichlet spaces are the same. □

Theorem 2.4 in [**15**] corresponds to the special case of the above theorem on $\mathcal{K}_\rho(\mathbb{D}^d)$ for $d = 2, \rho > 1$ and $N_1 = N_2$. A description of minimal reducing subspaces of $(M_{z_1^{N_1}}, \dots, M_{z_d^{N_d}})$ on $\mathcal{K}_\rho(\mathbb{D}^d)$ for $\rho = 1$ readily follows from Theorem 4.2 by noting that $\omega_\alpha = 1$ for all $\alpha \geq 0$. See also a related result for the case $\rho = 1$ in Theorem 9.1.

6. Product of weighted shifts.

In this section we demonstrate a more subtle observation that the product of several commuting weighted shifts (with operator weights) is again a weighted shift with operator weights. In the remaining part of the paper except the last section, $k \in Z_+$ is not a multi-index. For $N = (N_1, \dots, N_d)$, $kN = (kN_1, \dots, kN_d)$. Recall $S_\Phi = (S_1, \dots, S_d)$ is defined on $l_d^2(E)$ by

$$S_i\,[ye_\alpha] = [\Phi_{\alpha,i} y]\, e_{\alpha+\varepsilon_i}, \quad \alpha \in Z_+^d, \quad i = 1, \dots, d, y \in E. \tag{16}$$

Let $J_0 = \{\alpha : \alpha \geq 0 \text{ and } \min\{\alpha_i : i = 1, \dots, d\} = 0\}$ and

$$\widehat{E} = \operatorname{Span}\{ye_\alpha : y \in E, \alpha \in J_0\} = \ker\left(\prod_{i=1}^d S_i^*\right).$$

Let $\{g_k\}_{k=0}^\infty$ be the standard basis of l^2 and

$$l^2(\widehat{E}) = \left\{ y = \sum_{k=0}^\infty y_k g_k : y_k \in \widehat{E} \text{ and } \|y\|^2 = \sum_{k=0}^\infty \|y_k\|^2 < \infty \right\}.$$

Again we identify $\widehat{E}$ with the subspace $\{yg_0, y \in \widehat{E}\}$.

PROPOSITION 6.1. *The operator $\prod_{i=1}^d S_i$ is unitarily equivalent to a weighted shift S_Ψ defined on $l^2(\widehat{E})$ with $\Psi = \{\Psi_k : k \geq 0\}$, where $\Psi_k \in B(\widehat{E})$ is defined by*

$$\Psi_k\,(ye_\beta) = \left[\left(\Phi_{\beta+k(1,\dots,1)+\varepsilon_1+\cdots+\varepsilon_{d-1},d} \cdots \Phi_{\beta+k(1,\dots,1)+\varepsilon_1,2}\Phi_{\beta+k(1,\dots,1),1}\right) y\right] e_\beta, \quad y \in E, \beta \in J_0. \tag{17}$$

PROOF. Let U be the isometry from $l_d^2(E)$ into $l^2(\widehat{E})$ defined by

$$Uye_\alpha = \left[ye_{\alpha-k(1,\dots,1)}\right] g_k, \text{ where } k = \min\{\alpha_i : i = 1, \dots, d\}, \alpha \geq 0.$$

Then U is an onto isometry. Note that for $y \in E, \alpha \geq 0$,

$$\begin{aligned} U\left(\prod_{i=1}^d S_i\right) ye_\alpha &= U\left[\left(\Phi_{\alpha+\varepsilon_1+\cdots+\varepsilon_{d-1},d} \cdots \Phi_{\alpha+\varepsilon_1,2}\Phi_{\alpha,1}\right) y\right] e_{\alpha+(1,\dots,1)} \\ &= \left[\left\{\left(\Phi_{\alpha+\varepsilon_1+\cdots+\varepsilon_{d-1},d} \cdots \Phi_{\alpha+\varepsilon_1,2}\Phi_{\alpha,1}\right) y\right\} e_{\alpha+(1,\dots,1)-(k+1)(1,\dots,1)}\right] g_{k+1} \end{aligned}$$

$$= \left[\left\{\left(\Phi_{\alpha+\varepsilon_1+\cdots+\varepsilon_{d-1},d}\cdots\Phi_{\alpha+\varepsilon_1,2}\Phi_{\alpha,1}\right)y\right\}e_{\alpha-k(1,\dots,1)}\right]g_{k+1},$$

since $k = \min\{\alpha_i, i = 1, \dots, d\}$ implies that $k+1 = \min\{\alpha_i + 1 : i = 1, \dots, d\}$. On the other hand, by (17) with $\beta = \alpha - k(1, \dots, 1)$,

$$\begin{aligned} S_\Psi U\left[ye_\alpha\right] &= S_\Psi\left(\left[ye_{\alpha-k(1,\dots,1)}\right]g_k\right) \\ &= \left[\Psi_k\left(ye_{\alpha-k(1,\dots,1)}\right)\right]g_{k+1} \\ &= \left[\left\{\left(\Phi_{\alpha+\varepsilon_1+\cdots+\varepsilon_{d-1},d}\cdots\Phi_{\alpha+\varepsilon_1,2}\Phi_{\alpha,1}\right)y\right\}e_{\alpha-k(1,\dots,1)}\right]g_{k+1}. \end{aligned}$$

Therefore,

$$U\left(\prod_{i=1}^d S_i\right) = S_\Psi U.$$

The proof is complete. □

By Theorem 2.4 with $d = 1$, we have the following result. Let $T := \prod_{i=1}^d S_i$. Set

$$V_k = \Psi_{k-1}\cdots\Psi_1\Psi_0. \tag{18}$$

Corollary 6.2. *A closed subspace X is a reducing subspace of $\prod_{i=1}^d S_i$ if and only if*

$$X = \operatorname{Span}\left\{T^k x : k \geq 0, x \in \widehat{E_0} = X \ominus TX = \ker(T^*|X)\right\}, \tag{19}$$

where $\widehat{E_0} \subseteq \widehat{E}$ is an invariant subspace of the sequence of operators $V = \{V_k^* V_k : k \geq 0\}$ and V_k is defined by (17) and (18). Furthermore, X is a minimal reducing subspace of T if and only if $\widehat{E_0}$ is a minimal invariant subspace of V.

Because $M_z = (M_{z_1}, \dots, M_{z_d})$, we use T_z to denote the multiplication operator by z on $H^2_\Delta(E)$ as in (8). That is,

$$T_z = \prod_{i=1}^d M_{z_i}.$$

Let $\widehat{E} \subset H^2_\Delta(E)$ be the subspace given by

$$\widehat{E} = \left\{f(z) = \sum_{\alpha\in J_0} f_\alpha z^\alpha : f_\alpha \in E, \quad \|f(z)\|^2 = \sum_{\alpha\in J_0}\|W_\alpha f_\alpha\|^2 < \infty\right\} = \ker\left(T_z^*\right).$$

Let $\Psi_k \in B(\widehat{E})$ be given by

$$\begin{aligned} \Psi_k\left(yz^\beta\right) &= \left[\left(\Phi_{\beta+k(1,\dots,1)+\varepsilon_1+\cdots+\varepsilon_{d-1},d}\cdots\Phi_{\beta+k(1,\dots,1)+\varepsilon_1,2}\Phi_{\beta+k(1,\dots,1),1}\right)y\right]z^\beta, \\ &= \left[\left(\begin{matrix} W_{\beta+k(1,\dots,1)+(1,\dots,1)}W^{-1}_{\beta+k(1,\dots,1)+\varepsilon_1+\cdots+\varepsilon_{d-1}}\cdots \\ \cdots W_{\beta+k(1,\dots,1)+\varepsilon_1+\varepsilon_2}W^{-1}_{\beta+k(1,\dots,1)+\varepsilon_1}W_{\alpha+k(1,\dots,1)+\varepsilon_1}W^{-1}_{\alpha+k(1,\dots,1)}\end{matrix}\right)y\right]z^\beta \end{aligned}$$

$$= \left[\left(W_{\beta+(k+1)(1,\ldots,1)}W^{-1}_{\beta+k(1,\ldots,1)}\right) y\right] z^{\beta}, \quad y \in E, \beta \in J_0. \tag{20}$$

Note that Ψ_k is not necessarily positive because no commuting condition is imposed on W_α. Let V_k be defined by

$$V_k\left(yz^{\beta}\right) = [\Psi_{k-1}\cdots\Psi_1\Psi_0]\left(yz^{\beta}\right) = \left[\left(W_{\beta+k(1,\ldots,1)}W_{\beta}^{-1}\right) y\right] z^{\beta}, \quad y \in E, \beta \in J_0. \tag{21}$$

Then the proceeding corollary takes the following form on $H^2_\Delta(E)$.

COROLLARY 6.3. *A closed subspace X of $H^2_\Delta(E)$ is a reducing subspace of T_z if and only if*

$$X = \operatorname{Span}\left\{T_z^k x : k \geq 0, x \in \widehat{E_0} = X \ominus T_z X = \ker(T_z^*|X)\right\}, \tag{22}$$

where $\widehat{E_0} \subseteq \widehat{E}$ is an invariant subspace of the sequence of operators $V = \{V_k^* V_k : k \geq 0\}$ and V_k is given by (21). Furthermore, X is a minimal reducing subspace of T if and only if $\widehat{E_0}$ is a minimal invariant subspace of V.

7. Reducing subspaces of multiplication operators on H^2_ω.

Let $N = (N_1, \ldots, N_d) \in Z_+^d$ and $N \geq (1, \ldots, 1)$. Let T_{z^N} denote the multiplication operator by z^N on H^2_ω as in (10). That is,

$$T_{z^N} = \prod_{i=1}^{d} M_{z_i^{N_i}}.$$

Let

$$J_N = \{\alpha : \alpha \geq 0 \text{ and } \min\{\alpha_i - N_i : i = 1, \ldots, d\} < 0\},$$

and $\widehat{E} \subset H^2_\omega$ be the subspace given by

$$\widehat{E} = \left\{ f(z) = \sum_{\alpha \in J_N} f_\alpha z^\alpha : f_\alpha \in \mathbb{C}, \quad \|f(z)\|^2 = \sum_{\alpha \in J_N} \omega_\alpha |f_\alpha|^2 < \infty \right\} = \ker(T^*_{z^N}).$$

Let

$$\Psi_k\left(\frac{z^\beta}{\sqrt{\omega_\beta}}\right) = \frac{\sqrt{\omega_{\beta+kN}}}{\sqrt{\omega_\beta}}\left(\frac{z^\beta}{\sqrt{\omega_\beta}}\right), \quad \beta \in J_N, k \geq 0. \tag{23}$$

THEOREM 7.1. (i) *If a closed subspace X of H^2_ω is a reducing subspace of T_{z^N}, then*

$$X = \operatorname{Span}\left\{T^k_{z^N} x : k \geq 0, x \in \widehat{E_0}\right\}, \tag{24}$$

where

$$\widehat{E_0} = X \ominus T_{z^N} X = \ker\left(T^*_{z^N}|X\right),$$

and $\widehat{E_0} \subseteq \widehat{E}$ *is an invariant subspace of* $\Psi = \{\Psi_k : k \geq 0\}$ *defined by* (23).

(ii) *Any reducing subspace of* T_{z^N} *on* H^2_ω *is a direct sum of* (*singly generated*) *minimal reducing subspaces.*

(iii) *Any minimal reducing subspace of* T_{z^N} *on* H^2_ω *is of the form* $\operatorname{Span}\{f(z)z^{kN} : k \geq 0\}$, *where*

$$f(z) = \sum_{\gamma \in J} f_\gamma z^\gamma, \quad f_\gamma \neq 0 \text{ for all } \gamma \in J, \tag{25}$$

and $J \subseteq J_N$ *and* $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ *for all* $\gamma, \delta \in J, k \geq 0$.

Proof. Note that in this case Ψ_k in (23) is a diagonal operator (with positive diagonals). Therefore $\widehat{E_0} \subseteq \widehat{E}$ is an invariant subspace of the sequence of operators $\left\{\Psi_k^2 : k \geq 0\right\}$ if and only if $\widehat{E_0}$ is invariant for $\Psi = \{\Psi_k : k \geq 0\}$. So (i) follows from Corollary 6.3. Items (ii) and (iii) follow from Corollary 6.3 and Lemma 4.4. $\square$

Since $\omega_\alpha = 1$ for all $\alpha \geq 0$ on $\mathcal{K}_\rho(\mathbb{D}^d)$ for $\rho = 1$ (the Hardy space of polydisk), any minimal reducing subspace of T_{z^N} on $\mathcal{K}_1(\mathbb{D}^d)$ is described as in (iii) above where J is an arbitrary subset of J_N.

The above theorem formally looks the same as Theorem 4.2. But the condition on ω_α is less restrictive because k is not a multi-index, and the index set J_N is infinite, so in general there are many J such that $\kappa(J) > 1$ as we demonstrate below. However, here we only make a couple of observations and also work out the details for a few clean cases.

Let $P(d)$ denote the permutation group of $\{1, 2, \ldots, d\}$. For $\sigma \in P(d)$ and a multi-index $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d)$,

$$\gamma_\sigma = \left(\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \ldots, \gamma_{\sigma(d)}\right).$$

Then we have the following proposition.

Proposition 7.2. *Assume* $N = (M, M, \ldots, M)$. *Given* $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d) \in J_N$, *let*

$$f(z) = \sum_{\sigma \in P(d)} f_\sigma z^{\gamma_\sigma}, \quad f_\sigma \in \mathbb{C} \text{ for all } \sigma \in P(d). \tag{26}$$

Then $\operatorname{Span}\left\{f(z)z^{kN} : k \geq 0\right\}$ *is a minimal reducing subspace of* T_{z^N} *on* $\mathcal{K}_\rho(\mathbb{B}^d), \mathcal{K}_\rho(\mathbb{D}^d)$ $(\rho \neq 1)$, $\mathcal{D}(\mathbb{B}^d)$ *and* $\mathcal{D}(\mathbb{D}^d)$.

Proof. We just prove for T_{z^N} on $\mathcal{K}_\rho(\mathbb{B}^d)$. Recall that

$$\omega = \left\{\omega_\alpha = \frac{\alpha!\Gamma(\rho)}{\Gamma(\rho + |\alpha|)} : \alpha \geq 0\right\}.$$

Thus for given $\gamma \in J_N$ and for any $\sigma \in P(d)$,

$$
\begin{aligned}
\frac{\omega_{\gamma+kN}}{\omega_\gamma} &= \frac{\prod_{i=1}^d (\gamma_i+kM)!\Gamma(\rho+\sum_{i=1}^d \gamma_i)}{\prod_{i=1}^d \gamma_i!\Gamma(\rho+\sum_{i=1}^d \gamma_i+dkM)},\\
\frac{\omega_{\gamma_\sigma+kN}}{\omega_{\gamma_\sigma}} &= \frac{\prod_{i=1}^d \left(\gamma_{\sigma(i)}+kM\right)!\Gamma(\rho+\sum_{i=1}^d \gamma_{\sigma(i)})}{\prod_{i=1}^d \gamma_{\sigma(i)}!\Gamma(\rho+\sum_{i=1}^d \gamma_{\sigma(i)}+dkM)}.
\end{aligned}
$$

Since $\sum_{i=1}^d \gamma_{\sigma(i)} = \sum_{i=1}^d \gamma_i$ and $\prod_{i=1}^d \gamma_{\sigma(i)}! = \prod_{i=1}^d \gamma_i!$,

$$
\frac{\omega_{\gamma+kN}}{\omega_\gamma} = \frac{\omega_{\gamma_\sigma+kN}}{\omega_{\gamma_\sigma}}.
$$

The result now follows from the proceeding theorem. $\square$

The space Span $\left\{f(z)z^{kN} : k \geq 0\right\}$ above is the closed linear span in different spaces accordingly. In (26), we allow the coefficients f_σ to be zero. Furthermore, γ_{σ_1} could be same as γ_{σ_2} for two different permutations σ_1 and σ_2. The length of $f(z)$ in (26) is $d!$ if γ_i are distinct for $i = 1, \ldots, d$ and all f_σ are not zero. It turns out we can prove the converse of the above proposition if ρ is not a positive integer. The proof of the following lemma is more streamlined by comparing the roots of polynomials as Lemma 7 in [**19**], where reducing subspaces on weighted Bergman spaces on $\mathbb{D}^3$ are discussed.

Lemma 7.3. *Assume $N = (M, M, \ldots, M)$. Let ω be on $\mathcal{K}_\rho(\mathbb{B}^d)$, where ρ is not a positive integer, or ω be on $\mathcal{K}_\rho(\mathbb{D}^d)$ ($\rho \neq 1$). For $\gamma, \delta \in J_N$, $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$ if and only if there exists a permutation $\sigma \in P(d)$ such that $\delta = \gamma_\sigma$.*

Proof. We first prove this lemma on $\mathcal{K}_\rho(\mathbb{B}^d)$. If $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$, then

$$
\frac{\omega_{\gamma+kN}}{\omega_{\gamma+(k+1)N}} = \frac{\omega_{\delta+kN}}{\omega_{\delta+(k+1)N}} \quad \text{for all } k \geq 0. \tag{27}
$$

Equivalently

$$
\frac{\prod_{j=1}^{dM} (\rho+|\gamma|+dkM+j-1)}{\prod_{i=1}^d \prod_{j=1}^M (\gamma_i+kM+j)} = \frac{\prod_{j=1}^{dM} (\rho+|\delta|+dkM+j-1)}{\prod_{i=1}^d \prod_{j=1}^M (\delta_i+kM+j)}.
$$

We define $G(\lambda)$ by replacing k with λ,

$$
\begin{aligned}
&G(\lambda) = p(\lambda) - q(\lambda), \text{ where}\\
&p(\lambda) = \prod_{j=1}^{dM} (\rho+|\gamma|+d\lambda M+j-1) \prod_{i=1}^d \prod_{j=1}^M (\delta_i+\lambda M+j), \text{ and}\\
&q(\lambda) = \prod_{j=1}^{dM} (\rho+|\delta|+d\lambda M+j-1) \prod_{i=1}^d \prod_{j=1}^M (\gamma_i+\lambda M+j).
\end{aligned}
$$

Then $G(\lambda) \equiv 0$ and the roots of the two polynomials $p(\lambda)$ and $q(\lambda)$ are the same. In particular,

$$\text{either} \quad \frac{\rho+|\gamma|+dM-1}{dM} = \frac{\gamma_i+j}{M} \quad \text{for some } i \text{ and } j, \tag{28}$$

$$\frac{\rho+|\delta|+dM-1}{dM} = \frac{\delta_i+j}{M} \quad \text{for some } i \text{ and } j, \tag{29}$$

$$\text{or} \quad \frac{\rho+|\gamma|+dM-1}{dM} = \frac{\rho+|\delta|+dM-1}{dM}. \tag{30}$$

Both sides of (30) are the largest roots (in absolute value) of $p(\lambda)$ and $q(\lambda)$ containing ρ. If ρ is not a positive integer, (28) or (29) can not happen. So (30) implies that

$$\sum_{i=1}^{d} \delta_i = \sum_{i=1}^{d} \gamma_i.$$

Now (27) implies that

$$G_1(\lambda) = \prod_{i=1}^{d}\prod_{j=1}^{M} (\delta_i + \lambda M + j) - \prod_{i=1}^{d}\prod_{j=1}^{M} (\gamma_i + \lambda M + j) \equiv 0. \tag{31}$$

Therefore

$$\left\{ \frac{\delta_i+j}{M} : 1 \le i \le d, 1 \le j \le M \right\} = \left\{ \frac{\gamma_i+j}{M} : 1 \le i \le d, 1 \le j \le M \right\}.$$

This implies that $\delta = \gamma_\sigma$ for some permutation $\sigma \in P(d)$.

We now prove on $\mathcal{K}_\rho(\mathbb{D}^d)$ where $\rho \neq 1$. We will be brief. If $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \ge 0$, then

$$\frac{\prod_{i=1}^{d}\prod_{j=1}^{M} (\rho+\gamma_i+kM+j-1)}{\prod_{i=1}^{d}\prod_{j=1}^{M} (\gamma_i+kM+j)} = \frac{\prod_{i=1}^{d}\prod_{j=1}^{M} (\rho+\delta_i+kM+j-1)}{\prod_{i=1}^{d}\prod_{j=1}^{M} (\delta_i+kM+j)}. \tag{32}$$

Then the roots of the two polynomials are the same, equivalently $F_1 = F_2$ where

$$F_1 = \{\rho+\gamma_i+j-1, \delta_i+j : 1 \le i \le d, 1 \le j \le M\},$$
$$F_2 = \{\rho+\delta_i+j-1, \gamma_i+j : 1 \le i \le d, 1 \le j \le M\}.$$

Without loss of generality, assume $\gamma_d = \max\{\gamma_i : 1 \le i \le d\}$. Let $\delta_l = \max\{\delta_i : 1 \le i \le d\}$. Note that

$$\max F_1 = \max\{\rho+\gamma_d+M-1, \delta_l+M\},$$
$$\max F_2 = \max\{\rho+\delta_l+M-1, \gamma_d+M\}.$$

We claim $\delta_l = \gamma_d$. Assume $\delta_l > \gamma_d$. Then, in the case $\rho > 1$,

$$\max F_2 = \rho+\delta_l+M-1 > \max F_1,$$

which is a contradiction. In the case $0 < \rho < 1$,

$$\max F_1 = \delta_l + M > \max F_2,$$

which is a contradiction. Similarly $\delta_l < \gamma_d$ will also lead to contradictions. Thus $\delta_l = \gamma_d$. By using a permutation, we can assume $\delta_d = \gamma_d$. Now (32) becomes a new equation with d replaced by $d-1$. Continuing this process, we see that $\delta = \gamma_\sigma$ for some $\sigma \in P(d)$. $\square$

The case of ρ being a positive integer is most interesting since $\mathcal{K}_\rho(\mathbb{B}^d)$ contains the Hardy space, the Bergman space, and the Drury–Arveson space. Unfortunately, in this case, if $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$, we can only establish that $|\gamma| = |\delta|$ or $|\gamma| = |\delta| \pm 1$. If $|\gamma| = |\delta|$, then $\delta = \gamma_\sigma$ for some $\sigma \in P(d)$. We will resolve the case $|\gamma| = |\delta| \pm 1$ when $d = 2$. But first we prove a similar lemma on Dirichlet spaces.

Lemma 7.4. *Assume $N = (M, M, \ldots, M)$. Let ω be on $\mathcal{D}(\mathbb{B}^d)$ or $\mathcal{D}(\mathbb{D}^d)$. For $\gamma, \delta \in J_N$, then $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$ if and only if there exists a permutation $\sigma \in P(d)$ such that $\delta = \gamma_\sigma$.*

Proof. When $d = 1$, we need to prove $\gamma = \delta$. We skip this short proof, assume now $d \geq 2$. Since the proof on $\mathcal{D}(\mathbb{B}^d)$ is similar to the attempted (but failed) proof of the previous lemma on $\mathcal{K}_\rho(\mathbb{B}^d)$ for $\rho = 1$, we include the details to demonstrate the subtlety. Recall

$$\omega_\alpha = \frac{\alpha!\,(|\alpha|+1)}{|\alpha|!}, \quad \alpha \geq 0.$$

If $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$, then

$$\frac{\omega_{\gamma+kN}}{\omega_{\gamma+(k+1)N}} = \frac{\omega_{\delta+kN}}{\omega_{\delta+(k+1)N}} \quad \text{for all } k \geq 0.$$

Equivalently,

$$\begin{aligned}
&\frac{(|\gamma| + dkM + 1)\prod_{j=1}^{dM}(|\gamma| + dkM + j)}{(|\gamma| + dkM + dM + 1)\prod_{i=1}^{d}\prod_{j=1}^{M}(\gamma_i + kM + j)} \\
&\qquad = \frac{(|\delta| + dkM + 1)\prod_{j=1}^{dM}(|\delta| + dkM + j)}{(|\delta| + dkM + dM + 1)\prod_{i=1}^{d}\prod_{j=1}^{M}(\delta_i + kM + j)}.
\end{aligned} \tag{33}$$

We define $G(\lambda)$ by replacing k with λ, $G(\lambda) = p(\lambda) - q(\lambda)$, where

$$p(\lambda) = (|\gamma| + d\lambda M + 1)\,(|\delta| + d\lambda M + dM + 1)\prod_{j=1}^{dM}(|\gamma| + d\lambda M + j)\prod_{i=1}^{d}\prod_{j=1}^{M}(\delta_i + \lambda M + j),$$

$$q(\lambda) = (|\delta| + d\lambda M + 1)\,(|\gamma| + d\lambda M + dM + 1)\prod_{j=1}^{dM}(|\delta| + d\lambda M + j)\prod_{i=1}^{d}\prod_{j=1}^{M}(\gamma_i + \lambda M + j).$$

Then $G(\lambda) \equiv 0$ and the roots of the two polynomials $p(\lambda)$ and $q(\lambda)$ are the same. By multiplying all the roots of $p(\lambda)$ and $q(\lambda)$ by $-dM$, we have

$$E_1 \cup F_1 \cup G_1 = E_2 \cup F_2 \cup G_2,$$

where

$$\begin{aligned}
&E_1 = \{|\gamma| + j : 1 \le j \le dM\}, \quad F_1 = \{|\gamma| + 1, |\delta| + dM + 1\}, \\
&G_1 = \{d(\delta_i + j) : 1 \le i \le d, 1 \le j \le M\}, \\
&E_2 = \{|\delta| + j : 1 \le j \le dM\}, \quad F_2 = \{|\delta| + 1, |\gamma| + dM + 1\}, \\
&G_2 = \{d(\gamma_i + j) : 1 \le i \le d, 1 \le j \le M\}.
\end{aligned}$$

We claim $|\gamma| = |\delta|$. Assume to the contrary, $|\gamma| < |\delta|$. Note that $|\delta| + dM$ from E_2 does not belong to $E_1 \cup F_1$, so $|\delta| + dM \in G_1$. That is

$$|\delta| + dM = d(\delta_i + j) \quad \text{for some } i, j. \tag{34}$$

Note that $|\delta| + 1$ belongs to both E_2 and F_2. Since E_1 consists of consecutive integers which can only has at most one $|\delta|+1$ and $|\delta|+1 \notin F_1$, so there is another $|\delta|+1$ in G_1. That is

$$|\delta| + 1 = d(\delta_{i'} + j') \quad \text{for some } i', j'. \tag{35}$$

Equations (34) and (35) can not hold at the same time for $d \ge 2$. Similarly, $|\gamma| > |\delta|$ will also lead to a contradiction. Therefore $|\gamma| = |\delta|$. Now (33) simplifies to

$$\prod_{i=1}^{d} \prod_{j=1}^{M} (\gamma_i + kM + j) = \prod_{i=1}^{d} \prod_{j=1}^{M} (\delta_i + kM + j) \quad \text{for all } k \ge 0.$$

As in (31), this implies that $\delta = \gamma_\sigma$ for some permutation $\sigma \in P(d)$. The proof on $\mathcal{D}(\mathbb{D}^d)$ is similar and much simpler. $\square$

On $\mathcal{K}_\rho(\mathbb{B}^d)$ when ρ is an integer, the sets F_1 and F_2 are absent, so we are unable to conclude $|\gamma| = |\delta|$. A little reflection can establish that $|\gamma| = |\delta|$ or $|\gamma| = |\delta| \pm 1$ as we will do in the case $d = 2$ in next section. The following theorem follows from Theorem 7.1, Proposition 7.2, Lemma 7.3, and Lemma 7.4.

Theorem 7.5. *Assume $N = (M, M, \dots, M)$. Then any minimal reducing subspace X of T_{z^N} on $\mathcal{K}_\rho(\mathbb{B}^d)$ (ρ is not a positive integer), $\mathcal{K}_\rho(\mathbb{D}^d)$ ($\rho \ne 1$), $\mathcal{D}(\mathbb{B}^d)$, and $\mathcal{D}(\mathbb{D}^d)$, is of the form* Span $\left\{f(z)z^{kN} : k \ge 0\right\}$, *where there exists $\gamma \in J_N$ such that*

$$f(z) = \sum_{\sigma \in P(d)} f_\sigma z^{\gamma_\sigma}, \quad f_\sigma \in \mathbb{C}.$$

Thus the length of a minimal reducing subspace of T_{z^N} can be any integer between 1 *and* $d!$.

Note that $f(z)$ is a homogenous polynomial of degree $|\gamma|$. Theorem 1.1 in [**15**] corresponds to the special case of the above theorem on $\mathcal{K}_\rho(\mathbb{D}^d)$ for $d = 2$ and $\rho > 1$.

8. On spaces of holomorphic functions of two variables.

To compare with the previous theorem immediately, we first state the following theorem, then prove the lemma needed for the proof of this theorem.

THEOREM 8.1. *Let $N = (M, M)$. Assume ρ is a positive integer. Then any minimal reducing subspace X of T_{z^N} on $\mathcal{K}_\rho(\mathbb{B}^2)$ is of the form* $\operatorname{Span}\{f(z)z^{kN} : k \geq 0\}$, *where either there exists $\gamma = (\gamma_1, \gamma_2) \in J_N$ such that*

$$f(z) = az_1^{\gamma_1} z_2^{\gamma_2} + bz_1^{\gamma_2} z_2^{\gamma_1}, \quad a, b \in \mathbb{C}, \tag{36}$$

or there exists $0 \leq l < M$ such that

$$f(z) = az_1^{l+1} z_2^{l+1} + bz_1^{l+1} z_2^{l} + cz_1^{l} z_2^{l+1}, \quad a, b, c \in \mathbb{C} \tag{37}$$

on the Drury–Arveson space ($\rho = 1$), or

$$f(z) = az_1^{l} z_2^{l} + bz_1^{l+1} z_2^{l} + cz_1^{l} z_2^{l+1}, \quad a, b, c \in \mathbb{C} \tag{38}$$

on the Hardy space ($\rho = 2$), or

$$f(z) = a_1 z_1^{\rho-1+l} z_2^{l} + a_2 z_1^{l} z_2^{\rho-1+l} + b_1 z_1^{\rho-2+l} z_2^{l} + b_2 z_1^{l} z_2^{\rho-2+l}, \quad a_1, a_2, b_1, b_2 \in \mathbb{C} \tag{39}$$

on the Bergman space ($\rho = 3$) and $\mathcal{K}_\rho(\mathbb{B}^2)$ with $\rho > 3$. Thus the length of a minimal reducing subspace of T_{z^N} can be 1, 2, and 3 on the Drury–Arveson space and the Hardy space, and the length of a minimal reducing subspace of T_{z^N} can be 1, 2, 3, and 4 on the Bergman space ($\rho = 3$) and on $\mathcal{K}_\rho(\mathbb{B}^2)$ with $\rho > 3$.

The above theorem follows from the following lemma.

LEMMA 8.2. *Let $N = (M, M)$. Assume ρ is a positive integer. Let ω be on $\mathcal{K}_\rho(\mathbb{B}^2)$. Let $\gamma = (\gamma_1, \gamma_2)$, $\delta = (\delta_1, \delta_2) \in J_N$ be such that $|\delta| < |\gamma|$. Then $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$ if and only if (modulo permutations)*

$$(\gamma_1, \gamma_2) = (\delta_1, \rho - 1 + \delta_1), \quad (\delta_1, \delta_2) = (\delta_1, \rho - 2 + \delta_1). \tag{40}$$

PROOF. As in the proofs of Lemmas 7.3 and 7.4, $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$ if and only if

$$E_1 \cup F_1 \cup G_1 = E_2 \cup F_2 \cup G_2, \tag{41}$$

where

$$\begin{aligned}
&E_1 = \{\rho - 1 + |\gamma| + j : 1 \leq j \leq 2M\}, \quad F_1 = \{2(\delta_1 + j) : 1 \leq j \leq M\}, \\
&G_1 = \{2(\delta_2 + j) : 1 \leq j \leq M\}, \\
&E_2 = \{\rho - 1 + |\delta| + j : 1 \leq j \leq 2M\}, \quad F_2 = \{2(\gamma_1 + j) : 1 \leq j \leq M\}, \\
&G_2 = \{2(\gamma_2 + j) : 1 \leq j \leq M\}.
\end{aligned}$$

Since E_1 consists of consecutive integers, and F_1 and G_1 consist of consecutive even integers, $|\gamma| = |\delta| + 1$. By using permutations, we may assume $\gamma_1 \leq \gamma_2$ and $\delta_1 \leq \delta_2$. Now (41) becomes

$$E_1' \cup F_1 \cup G_1 = E_2' \cup F_2 \cup G_2, \tag{42}$$

where

$$E_1' = \{\rho - 1 + |\gamma| + 2M\}, \quad E_2' = \{\rho + |\delta|\}.$$

Case (1): $\rho - 1 + |\gamma| + 2M = \max G_1 = 2(\delta_2 + M)$. In this case, $2(\delta_2 + M)$ appears twice in the left side of (42); hence

$$2(\delta_2 + M) = 2(\gamma_1 + M) = 2(\gamma_2 + M).$$

That is, $\delta_2 = \gamma_1 = \gamma_2$. By $|\gamma| = |\delta| + 1, \delta_1 = \delta_2 - 1$. By $\rho - 1 + |\gamma| + 2M = 2(\delta_2 + M)$, $\rho - 1 + 2\gamma_1 = 2\gamma_1$, and $\rho = 1$. This corresponds to (40) (modulo permutations) for $\rho = 1$.

Case (2): $\rho - 1 + |\gamma| + 2M > \max G_1 = 2(\delta_2 + M)$. In this case,

$$\rho - 1 + |\gamma| + 2M = 2(\gamma_2 + M).$$

That is, $\gamma_2 = \rho - 1 + \gamma_1$. It also follows that $\rho - 1 + |\gamma| + 2M = 2(\delta_2 + M) + 2$ since otherwise $\rho - 1 + |\gamma| + 2M - 2 = 2(\gamma_2 + M - 1)$ belongs to the right side of (42) (G_2), but does not belong to the left side of (42). Hence $\delta_2 = \gamma_2 - 1 = \rho - 2 + \gamma_1$. Now $|\gamma| = |\delta| + 1$ yields $\gamma_1 = \delta_1$. Therefore (40) holds.

Case (3): $\rho - 1 + |\gamma| + 2M < \max G_1 = 2(\delta_2 + M)$. Here,

$$2(\delta_2 + M) = \max G_1 = \max G_2 = 2(\gamma_2 + M).$$

Now $|\gamma| = |\delta| + 1$ yields $\gamma_1 = \delta_1 + 1$. Then (42) becomes

$$\{\rho - 1 + |\gamma| + 2M, 2(\delta_1 + 1)\} = \{\rho + |\delta|, 2(\gamma_1 + M)\}.$$

In particular $2(\delta_1 + 1) = \rho + |\delta|$. That is, $\delta_1 = \rho + \delta_2$, which is excluded by our assumption $\delta_1 \leq \delta_2$.

The proof is complete. □

We next discuss reducing subspaces of T_{z^N} where $N = (N_1, N_2, \ldots, N_d)$ and some of N_i are distinct. If $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$, the relationship between γ and δ could be complicated for $d \geq 3$ as shown in [**19**] on $\mathcal{K}_\rho(\mathbb{D}^d)$. In particular, the reducing subspaces of T_{z^N} for $N = (N_1, N_2, N_3)$ with distinct N_i on $\mathcal{K}_\rho(\mathbb{D}^d)$ for $d = 3, \rho > 1$ are completely worked out there. Here we will discuss Dirichlet spaces $\mathcal{D}(\mathbb{B}^d)$ or $\mathcal{D}(\mathbb{D}^d)$ for $d = 2$, and the answers are still relatively compact. Surprisingly the answers are quite different. Let $\mathrm{GCD}(N_1, N_2)$ denote the greatest common factor of N_1 and N_2.

Lemma 8.3. *Assume $N = (N_1, N_2)$. Write*

$$N_1 = N_1' M, \quad N_2 = N_2' M, \quad M = \mathrm{GCD}(N_1, N_2).$$

Let ω be on $\mathcal{D}(\mathbb{D}^2)$. Let $\gamma = (\gamma_1, \gamma_2)$, $\delta = (\delta_1, \delta_2) \in J_N$ and $\gamma \neq \delta$. Then $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$ if and only if there exist positive integers l and m such that $\min\{l, m\} \leq M, l \neq m$ and

$$(\gamma_1, \gamma_2) = (lN_1' - 1, mN_2' - 1), \quad (\delta_1, \delta_2) = (mN_1' - 1, lN_2' - 1). \tag{43}$$

Proof. Let ω be on $\mathcal{D}(\mathbb{D}^2)$. Recall $\omega_\alpha = (\alpha_1 + 1)(\alpha_2 + 1)$. If $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$ and $G(k) = \omega_{\gamma+kN}\omega_\delta - \omega_{\delta+kN}\omega_\gamma = 0$, then with k replaced by λ,

$$\begin{aligned} G(\lambda) &= p(\lambda) - q(\lambda), \text{ where} \\ p(\lambda) &= (\gamma_1 + \lambda N_1 + 1)(\gamma_2 + \lambda N_2 + 1)(\delta_1 + 1)(\delta_2 + 1), \\ q(\lambda) &= (\delta_1 + \lambda N_1 + 1)(\delta_2 + \gamma N_2 + 1)(\gamma_1 + 1)(\gamma_2 + 1), \end{aligned}$$

$G(\lambda) \equiv 0$, and the roots of the two polynomials $p(\lambda)$ and $q(\lambda)$ are the same. Multiplying the roots by $-N_1'N_2'M$, we have

$$\{(\gamma_1 + 1)N_2', (\gamma_2 + 1)N_1'\} = \{(\delta_1 + 1)N_2', (\delta_2 + 1)N_1'\}.$$

Thus

$$(\gamma_1 + 1)N_2' = (\delta_2 + 1)N_1', \quad (\gamma_2 + 1)N_1' = (\delta_1 + 1)N_2'.$$

Since N_1' and N_2' are coprime, there exist integers l and m such that (43) holds. □

The case for $\mathcal{D}(\mathbb{B}^2)$ is more difficult, but the result is simple.

Lemma 8.4. *Assume $N = (N_1, N_2)$ with $N_1 \neq N_2$. Let ω be on $\mathcal{D}(\mathbb{B}^2)$. Let γ, δ be two multi-indices in J_N. If $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$, then $\gamma = \delta$.*

Proof. Let ω be on $\mathcal{D}(\mathbb{B}^2)$. Write

$$N_1 = N_1'M, \quad N_2 = N_2'M, \quad M = \mathrm{GCD}(N_1, N_2).$$

If $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$, then as in (33), $G(\lambda) \equiv 0$ and the roots of the two polynomials $p(\lambda)$ and $q(\lambda)$ are the same, where

$$G(\lambda) = p(\lambda) - q(\lambda),$$

$$p(\lambda) = (|\gamma| + \lambda|N| + 1)(|\delta| + (\lambda + 1)|N| + 1)\prod_{j=1}^{|N|}(|\gamma| + \lambda|N| + j)\prod_{i=1}^{2}\prod_{j=1}^{N_i}(\delta_i + \lambda N_i + j),$$

$$q(\lambda) = (|\delta| + \lambda|N| + 1)(|\gamma| + (\lambda + 1)|N| + 1)\prod_{j=1}^{|N|}(|\delta| + \lambda|N| + j)\prod_{i=1}^{2}\prod_{j=1}^{N_i}(\gamma_i + \lambda N_i + j).$$

By multiplying all the roots of $p(\lambda)$ and $q(\lambda)$ by $-(N_1' + N_2')N_1'N_2'M$, we have

$$E_1 \cup F_1 \cup G_1 = E_2 \cup F_2 \cup G_2,$$

where

$$\begin{aligned}
E_1 &= \{(|\gamma| + j)\, N_1' N_2' : 1 \le j \le |N|\}\,, \\
F_1 &= \{(|\gamma| + 1)\, N_1' N_2', (|\delta| + |N| + 1)\, N_1' N_2'\}\,, \\
G_1 &= \{(\delta_1 + j)\, (N_1' + N_2')\, N_2' : 1 \le j \le N_1\} \cup \{(\delta_2 + j)\, (N_1' + N_2')\, N_1' : 1 \le j \le N_2\}\,, \\
E_2 &= \{(|\delta| + j)\, N_1' N_2' : 1 \le j \le |N|\}\,, \\
F_2 &= \{(|\delta| + 1)\, N_1' N_2', (|\gamma| + |N| + 1)\, N_1' N_2'\}\,, \\
G_2 &= \{(\gamma_1 + j)\, (N_1' + N_2')\, N_2' : 1 \le j \le N_1\} \cup \{(\gamma_2 + j)\, (N_1' + N_2')\, N_1' : 1 \le j \le N_2\}\,.
\end{aligned}$$

We claim $|\gamma| = |\delta|$. Assume to the contrary, $|\gamma| < |\delta|$. Note that $(|\delta| + |N|)\, N_1' N_2' > \max E_1$, so $(|\delta| + |N|)\, N_1' N_2'$ from E_2 does not belong to $E_1 \cup F_1$. Hence $(|\delta|+|N|) N_1' N_2' \in G_1$. That is,

$$\begin{aligned}
\text{either }\ (|\delta| + |N|)\, N_1' N_2' &= (\delta_1 + j)\, (N_1' + N_2')\, N_2' \quad \text{for some } j \\
\text{or }\ (|\delta| + |N|)\, N_1' N_2' &= (\delta_2 + j)\, (N_1' + N_2')\, N_1' \quad \text{for some } j.
\end{aligned}$$

In either case, since N_1' and $N_1' + N_2'$ are coprime and N_2' and $N_1' + N_2'$ are coprime,

$$|\delta| + |N| = a\, (N_1' + N_2') \quad \text{for some integer } a. \tag{44}$$

Note that $(|\delta| + |N| + 1)\, N_1' N_2' > \max E_2$, so $(|\delta| + |N| + 1)\, N_1' N_2'$ from F_1 does not belong to $E_2 \cup F_2$. Thus $(|\delta| + |N| + 1)\, N_1' N_2' \in G_2$. That is,

$$\begin{aligned}
\text{either }\ (|\delta| + |N| + 1)\, N_1' N_2' &= (\gamma_1 + j)\, (N_1' + N_2')\, N_2' \quad \text{for some } j \\
\text{or }\ (|\delta| + |N| + 1)\, N_1' N_2' &= (\gamma_2 + j)\, (N_1' + N_2')\, N_1' \quad \text{for some } j.
\end{aligned}$$

In either case

$$|\delta| + |N| + 1 = b\, (N_1' + N_2') \quad \text{for some integer } b. \tag{45}$$

Equations (44) and (45) can not hold at the same time since $N_1' + N_2' > 1$. Similarly, $|\gamma| > |\delta|$ will also lead to a contradiction. Therefore $|\gamma| = |\delta|$. Now we have $G_1 = G_2$. Thus

$$\begin{aligned}
(\delta_1 + N_1)\, (N_1' + N_2')\, N_2' &= (\gamma_2 + N_2)\, (N_1' + N_2')\, N_1', \\
(\delta_2 + N_2)\, (N_1' + N_2')\, N_1' &= (\gamma_1 + N_1)\, (N_1' + N_2')\, N_2'.
\end{aligned}$$

Since N_1' and N_2' are coprime, there exist integers l and m such that

$$(\gamma_1, \gamma_2) = (l N_1', m N_2')\,, \quad (\delta_1, \delta_2) = (m N_1', l N_2')\,.$$

But $\gamma_1 + \gamma_2 = \delta_1 + \delta_2$ or $l N_1' + m N_2' = m N_1' + l N_2'$ implies that $l = m$. □

The following theorem together with Theorem 7.5 gives a complete description of reducing subspaces of T_{z^N} on $\mathcal{D}(\mathbb{D}^d)$ or $\mathcal{D}(\mathbb{B}^d)$ for $d = 2$. Note $kN = (kN_1, kN_2)$ in the theorem.

THEOREM 8.5. *Assume $N = (N_1, N_2)$ with $N_1 \neq N_2$. Write*

$$N_1 = N_1' M, \quad N_2 = N_2' M, \quad M = \mathrm{GCD}(N_1, N_2).$$

(i) *Any minimal reducing subspace X of T_{z^N} on $\mathcal{D}(\mathbb{D}^2)$ is of the form* $\mathrm{Span}\{f(z)z^{kN} : k \geq 0\}$, *where either $f(z) = z^\gamma$ for some $\gamma = (\gamma_1, \gamma_2) \in J_N$ or*

$$f(z) = az^\gamma + bz^\delta, \quad a, b \in \mathbb{C} \text{ and } ab \neq 0$$

with

$$(\gamma_1, \gamma_2) = (lN_1' - 1, mN_2' - 1), \quad (\delta_1, \delta_2) = (mN_1' - 1, lN_2' - 1),$$

for some positive integers l and m such that $\min\{l, m\} \leq M$ *and $l \neq m$.*

(ii) *Any minimal reducing subspace X of T_{z^N} on $\mathcal{D}(\mathbb{B}^2)$ is of the form* $\mathrm{Span}\{z^\gamma z^{kN} : k \geq 0\}$ *for some $\gamma = (\gamma_1, \gamma_2) \in J_N$.*

Next we characterize reducing subspaces of T_{z^N} on $\mathcal{K}_\rho(\mathbb{D}^d)$ for $d = 2$. The case $\rho > 1$ is treated in Theorem 2.4 ($\rho = 2$) and Theorem 3.2 ($\rho > 1$ and $\rho \neq 2$) in [**18**]. We include a self-contained exposition for completeness. We give a slightly improved and unified proof for both $\rho > 1$ and $\rho < 1$ by extending some ideas from [**18**]. We first state the result, which is presented slightly differently from [**18**], then we prove the lemma. Part (i) of Theorem 8.6 is similar to Part (i) Theorem 8.5. Indeed we will state a unified Theorem 9.5 for a result related to von Neumann algebras on $\mathcal{D}(\mathbb{D}^d)$ and $\mathcal{K}_\rho(\mathbb{D}^d)$ for $d = 2$.

THEOREM 8.6. *Assume $N = (N_1, N_2)$ with $N_1 \neq N_2$. Write*

$$N_1 = N_1' M, \quad N_2 = N_2' M, \quad M = \mathrm{GCD}(N_1, N_2).$$

(i) *Any minimal reducing subspace X of T_{z^N} on $\mathcal{K}_\rho(\mathbb{D}^2)$ for $\rho = 2$ is of the form* $\mathrm{Span}\left\{f(z)z^{kN} : k \geq 0\right\}$, *where either $f(z) = z^\gamma$ for some $\gamma = (\gamma_1, \gamma_2) \in J_N$ or*

$$f(z) = az^\gamma + bz^\delta, \quad a, b \in \mathbb{C} \text{ and } ab \neq 0$$

with

$$(\gamma_1, \gamma_2) = (lN_1' - 1, mN_2' - 1), \quad (\delta_1, \delta_2) = (mN_1' - 1, lN_2' - 1),$$

for some positive integers l and m such that $\min\{l, m\} \leq M$ *and $l \neq m$.*

(ii) *Any minimal reducing subspace X of T_{z^N} on $\mathcal{K}_\rho(\mathbb{D}^2)$ for $\rho \neq 1, 2$ is of the form* $\mathrm{Span}\left\{z^\gamma z^{kN} : k \geq 0\right\}$ *for some $\gamma = (\gamma_1, \gamma_2) \in J_N$.*

LEMMA 8.7. *Assume $N = (N_1, N_2)$ with $N_1 \neq N_2$. Write*

$$N_1 = N_1' M, \quad N_2 = N_2' M, \quad M = \mathrm{GCD}(N_1, N_2).$$

Let ω be on $\mathcal{K}_\rho(\mathbb{D}^2)$. Let γ, δ be two multi-indices in J_N. If $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$, then $\gamma = \delta$ in the case $\rho \neq 1, 2$. In the case $\rho = 1$, $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$. In the case $\rho = 2$, if $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$, then either $\gamma = \delta$ or

$$(\gamma_1, \gamma_2) = (lN_1' - 1, mN_2' - 1), \quad (\delta_1, \delta_2) = (mN_1' - 1, lN_2' - 1), \tag{46}$$

for some positive integers l and m such that $\min\{l, m\} \leq M$ and $l \neq m$.

Proof. As Lemma 5.3,

$$\frac{\omega_{\gamma+kN}}{\omega_{\delta+kN}} \to 1 \text{ as } k \to \infty.$$

Assume $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k \geq 0$, then

$$\frac{\omega_{\gamma+kN}}{\omega_{\delta+kN}} = \frac{\omega_\gamma}{\omega_\delta} = 1 \quad \text{for all } k \geq 0.$$

Assume $\rho \neq 1, 2$. We will prove $\gamma = \delta$. Note that if $\gamma_1 = \delta_1$, then it follows from $\omega_\gamma = \omega_\delta$ that $\delta_2 = \gamma_2$. Thus, by symmetry, we can assume $\gamma_1 < \delta_1$. We claim that $\gamma_2 > \delta_2$. If $\gamma_2 < \delta_2$, then $\omega_{\gamma+kN} = \omega_{\delta+kN}$ for all $k \geq 0$ implies that

$$\begin{aligned}&\prod_{i=1}^{\delta_1-\gamma_1} (\gamma_1 + \lambda N_1 + i) \prod_{j=1}^{\delta_2-\gamma_2} (\gamma_2 + \lambda N_2 + j) \\ &\quad = \prod_{i=1}^{\delta_1-\gamma_1} (\rho + \gamma_1 + \lambda N_1 + i - 1) \prod_{j=1}^{\delta_2-\gamma_2} (\rho + \gamma_2 + \lambda N_2 + j - 1).\end{aligned}$$

Then the roots of the two polynomials are the same. In particular, the sum of the roots of the polynomial on the left side minus the sum of the roots of the polynomial on the right side is zero. That is,

$$-\frac{1}{N_1}(1-\rho)(\delta_1 - \gamma_1) - \frac{1}{N_2}(1-\rho)(\delta_2 - \gamma_2) = 0,$$

which is impossible. So we have $\gamma_1 < \delta_1$ and $\gamma_2 > \delta_2$. Now $\omega_{\gamma+kN} = \omega_{\delta+kN}$ for all $k \geq 0$ implies that

$$\begin{aligned}&\prod_{i=1}^{\delta_1-\gamma_1} (\rho + \gamma_1 + \lambda N_1 + i - 1) \prod_{j=1}^{\gamma_2-\delta_2} (\delta_2 + \lambda N_2 + j) \\ &\quad = \prod_{i=1}^{\delta_1-\gamma_1} (\gamma_1 + \lambda N_1 + i) \prod_{j=1}^{\gamma_2-\delta_2} (\rho + \delta_2 + \lambda N_2 + j - 1).\end{aligned}$$

Then the roots of the two polynomials are the same. As before, by considering the sum of the roots, we have

$$-\frac{1}{N_1}(\rho-1)(\delta_1-\gamma_1)+\frac{1}{N_2}(\rho-1)(\gamma_2-\delta_2)=0$$
$$\text{or } (\delta_1-\gamma_1)N_2'-(\gamma_2-\delta_2)N_1'=0. \tag{47}$$

By multiplying the roots by $-N_1'N_2'M$, we have

$$F_1\cup G_1=F_2\cup G_2,$$

where

$$F_1=\{(\rho+\gamma_1+i-1)\,N_2' : 1\le i\le \delta_1-\gamma_1\},\quad G_1=\{(\delta_2+j)\,N_1' : 1\le j\le \gamma_2-\delta_2\},$$
$$F_2=\{(\gamma_1+i)\,N_2' : 1\le i\le \delta_1-\gamma_1\},\quad G_2=\{(\rho+\delta_2+j-1)\,N_1' : 1\le j\le \gamma_2-\delta_2\}.$$

In the case $0<\rho<1$, note that

$$\max F_1\cup G_1=\max\{(\rho+\delta_1-1)\,N_2',\ \gamma_2N_1'\},$$
$$\max F_2\cup G_2=\max\{\delta_1N_2',\ (\rho+\gamma_2-1)\,N_1'\}.$$

Thus

$$\gamma_2N_1'=\delta_1N_2'. \tag{48}$$

Similarly, $\min F_1\cup G_1=\min F_2\cup G_2$ implies that

$$(\rho+\delta_2)\,N_1'=(\rho+\gamma_1)\,N_2'. \tag{49}$$

Equations (47), (48), and (49) imply that $\rho N_1'=\rho N_2'$, which is a contradiction.

In the case $\rho>1$, $\max F_1\cup G_1=\max F_2\cup G_2$ implies that

$$(\rho+\delta_1-1)\,N_2'=(\rho+\gamma_2-1)\,N_1', \tag{50}$$

and $\min F_1\cup G_1=\min F_2\cup G_2$ implies that

$$(\gamma_1+1)\,N_2'=(\delta_2+1)\,N_1'. \tag{51}$$

Equations (47), (50), and (51) imply that $(\rho-2)\,N_1'=(\rho-2)\,N_2'$, which is a contradiction if $\rho\neq 2$.

In the case $\rho=2$, (50) or (51) imply (46), since N_1' and N_2' are coprime. The proof is complete. □

9. Reducing subspaces and von Neumann algebras.

In this section we will use reducing subspaces of operators to reveal the structures of von Neumann algebras associated with these operators. Let $\mathcal{A}$ be a von Neumann algebra in $B(H)$ and $\mathcal{A}'$ be the commutant of $\mathcal{A}$. A von Neumann algebra is the norm closed linear span of its projections (Proposition 13.3 [**4**]). Note that a projection $P_{H_0}\in\mathcal{A}'$ if and only if H_0 is a reducing subspace of $\mathcal{A}$. Therefore

$$\mathcal{A}' = \text{Span}\{P_{H_0} : H_0 \text{ is any reducing subspace of } \mathcal{A}\}.$$

Thus knowing reducing subspaces of $\mathcal{A}$ will help us to identify $\mathcal{A}'$. For two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \approx \mathcal{B}$ means $\mathcal{A}$ is $*$-isomorphic to $\mathcal{B}$. Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ complex matrices, and let $M_\infty(\mathbb{C})$ denote $B(H)$ for an infinite dimensional separable complex Hilbert space. For example, let $W^*(z_1^{N_1}, \ldots, z_d^{N_d})$ be the von Neumann algebra generated by $M_{z_1^{N_1}}, \ldots, M_{z_d^{N_d}}, M^*_{z_1^{N_1}}, \ldots, M^*_{z_d^{N_d}}$ in $B(H^2_\omega)$ as in (10), and let $v(z_1^{N_1}, \ldots, z_d^{N_d})$ be the commutant of $W^*(z_1^{N_1}, \ldots, z_d^{N_d})$. Theorem 5.4 implies the following result.

Theorem 9.1. *Let ω be on $\mathcal{K}_\rho(\mathbb{B}^d)$, $\mathcal{K}_\rho(\mathbb{D}^d)$ $(\rho \neq 1)$, $\mathcal{D}(\mathbb{B}^d)$, and $\mathcal{D}(\mathbb{D}^d)$. Then $v(z_1^{N_1}, \ldots, z_d^{N_d})$ is abelian. In fact*

$$v(z_1^{N_1}, \ldots, z_d^{N_d}) \approx \bigoplus_{i=1}^{L} \mathbb{C}, \text{ where } L = N_1 \cdots N_d. \tag{52}$$

In the case $\mathcal{K}_\rho(\mathbb{D}^d)$ for $\rho = 1$, $v(z_1^{N_1}, \ldots, z_d^{N_d})$ is not abelian unless $N_1 = \cdots = N_d = 1$. In fact

$$v(z_1^{N_1}, \ldots, z_d^{N_d}) \approx M_L(\mathbb{C}). \tag{53}$$

To prove the above theorem, we first have to study when two reducing subspaces are equivalent in a von Neumann algebra. Again we find it is more convenient to discuss in the general framework of weighted shifts with operator weights. Let $S_\Phi = (S_1, \ldots, S_d)$ be a tuple of weighted shifts on $l^2_d(E)$. Let $\{g_i\}_{i=1}^\infty$ be an orthonormal basis of E. Assume $\Phi = \{\Phi_{\alpha,i} : \alpha \in Z^d_+, i = 1, \ldots, d\}$ is a bounded set of invertible positive diagonal operators (with respect to the basis $\{g_i\}_{i=1}^\infty$ of E) in $B(E)$. Then by Theorem 2.4,

$$V(g_i) = \text{Span}\{S^\alpha_\Phi g_i : \alpha \geq 0\}$$

is a (common) reducing subspace of S_Φ. Recall two projections P_1 and P_2 are equivalent in a von Neumann algebra $\mathcal{A}$ in $B(H)$ if there exists a partial isometry U in $\mathcal{A}$ such that $UU^* = P_1$ and $U^*U = P_2$. We say two subspaces H_1 and H_2 of H are equivalent in $\mathcal{A}$ if P_{H_1} and P_{H_2} are equivalent in $\mathcal{A}$. As before, let $W^*(S_\Phi)$ be the von Neumann algebra generated by S_Φ and S^*_Φ, and let $v(S_\Phi) = \{W^*(S_\Phi)\}'$.

Lemma 9.2. *For $i \neq j$, the following are equivalent.*

(i) *$V(g_i)$ is equivalent to $V(g_j)$ in $v(S_\Phi)$.*

(ii) *Each $\Phi_{\alpha,i}$ in Φ restricted to* Span $\{g_i, g_j\}$ *is a constant multiple of the identity.*

(iii) *Each W_α (as defined in (4)) restricted to* Span $\{g_i, g_j\}$ *is a constant multiple of the identity.*

(iv) *For any $a, b \neq 0$, $V(ag_i + bg_j) =$* Span $\{S^\alpha_\Phi(ag_i + bg_j) : \alpha \geq 0\}$ *is also a reducing subspace of S_Φ.*

PROOF. We prove (i) implies (iii). Without loss of generality, assume $i = 1$ and $j = 2$. A partial isometry U, which maps $V(g_1)$ onto $V(g_2)$, belongs to $v(S_\Phi)$ if and only if $US_\Phi^\alpha = S_\Phi^\alpha U$ and $US_\Phi^{*\alpha} = S_\Phi^{*\alpha} U$ for every $\alpha \geq 0$. Thus $S_\Phi^{*\alpha} U g_1 e_0 = U S_\Phi^{*\alpha} g_1 e_0 = 0$ for $\alpha \neq 0$ implies that $U g_1 e_0 = \lambda g_2 e_0$ for some complex number λ with $|\lambda| = 1$. Now

$$\begin{aligned} US_\Phi^\alpha g_1 e_0 &= U\left[(W_\alpha g_1) e_\alpha\right], \text{ and} \\ S_\Phi^\alpha U g_1 e_0 &= S_\Phi^\alpha\left(\lambda g_2 e_0\right) = \lambda (W_\alpha g_2) e_\alpha. \end{aligned}$$

Since W_α is a positive diagonal operator, $W_\alpha g_1 = \lambda_1 g_1$ and $W_\alpha g_2 = \lambda_2 g_2$. Thus

$$\begin{aligned} \|U\left[(W_\alpha g_1) e_\alpha\right]\| &= \|(W_\alpha g_1) e_\alpha\| = \lambda_1 \\ &= \|\lambda (W_\alpha g_2) e_\alpha\| = \lambda_2. \end{aligned}$$

This proves (iii). The implication (iii) to (i) is also clear by defining the partial isometry U as $U(S_\Phi^\alpha g_1) = S_\Phi^\alpha g_2$ for $\alpha \geq 0$.

The equivalence of (ii) and (iii) follows from (4). The equivalence of (iii) and (iv) follows from Lemma 4.4. $\square$

The above lemma also holds when E is a finite dimensional complex Hilbert space.

PROOF OF THEOREM 9.1. Note that

$$H_\omega^2 = \bigoplus_{\gamma \in J_N} H_\gamma \text{ where } H_\gamma = \operatorname{Span}\left\{z^\gamma z^{kN} : k = (k_1, \ldots, k_d) \geq 0\right\}.$$

By Theorem 4.2 and Lemma 9.2, each H_γ is a minimal reducing subspace of $(M_{z_1^{N_1}}, \ldots, M_{z_d^{N_d}})$. Furthermore, in the case $\rho \neq 1$, by Theorem 5.4, H_γ is not equivalent to H_δ for $\gamma \neq \delta$. By Theorem 50.19 [**4**], we have (52) where L is the cardinality of the index set J_N.

In the case $\mathcal{K}_\rho(\mathbb{D}^d)$ with $\rho = 1$,

$$H_\omega^2 = \bigoplus_{\gamma \in J_N} H_\gamma.$$

Each H_γ is a minimal reducing subspace of $(M_{z_1^{N_1}}, \ldots, M_{z_d^{N_d}})$. Furthermore, H_γ is equivalent to H_δ for $\gamma \neq \delta$ since W_α is the identity operator. Therefore $v(z_1^{N_1}, \ldots, z_d^{N_d})$ is a homogenous von Neumann algebra. By Corollary 50.16 [**4**], we have (53). $\square$

Similarly, Theorem 8.1 and Theorem 7.5 also lead to the structures of various von Neumann algebras. Let $W^*(z_1^M \cdots z_d^M)$ be the von Neumann algebra generated by $T_{z_1^M \cdots z_d^M}$ and $T^*_{z_1^M \cdots z_d^M}$ in $B(H_\omega^2)$ as in (10), and let $v(z_1^M \cdots z_d^M)$ be the commutant of $W^*(z_1^M \cdots z_d^M)$.

THEOREM 9.3. *Let ω be on $\mathcal{K}_\rho(\mathbb{B}^d)$ (ρ is not a positive integer), $\mathcal{K}_\rho(\mathbb{D}^d)$ ($\rho \neq 1$), $\mathcal{D}(\mathbb{B}^d)$, and $\mathcal{D}(\mathbb{D}^d)$. Then $v(z_1^M \cdots z_d^M)$ is $*$-isomorphic to*

$$\left[\bigoplus_{i=1}^{M}\mathbb{C}\right]\bigoplus_{j=2}^{d}\left[\bigoplus_{i=1}^{\infty}M_{d(d-1)\cdots(d-j+2)}(\mathbb{C})\right].$$

PROOF. Write $N=(M,\ldots,M)$ and

$$J_N=\bigcup_{j=1}^{d}J_{N,j},\ \text{where}$$
$$J_{N,j}=\{\gamma=(\gamma_1,\ldots,\gamma_d)\in J_N:\ \text{there are } j \text{ distinct numbers in } \gamma_1,\ldots,\gamma_d\}.$$

For $j=1$,

$$J_{N,1}=\{\gamma=(\gamma_1,\gamma_1,\ldots,\gamma_1):0\le\gamma_1<M\},$$

and the cardinality of $J_{N,1}$ is M. For $j\ge 2$, write

$$J_{N,j}=\bigcup_{\gamma\in J'_{N,j}}J_{N,j,\gamma},$$

where $J'_{N,j}$ is a subset of $J_{N,j}$ and

$$J_{N,j,\gamma}=\{\delta=\gamma_\sigma:\ \sigma\in P(d)\}.$$

Recall $P(d)$ is the permutation group. The cardinality of $J'_{N,j}$ is infinite and the cardinality of $J_{N,j,\gamma}$ is $d(d-1)\cdots(d-j+2)$. Write

$$H^2_\omega=\bigoplus_{j=1}^{d}\left[\bigoplus_{\gamma\in J_{N,j}}H_\gamma\right]=\left[\bigoplus_{\gamma\in J_{N,1}}H_\gamma\right]\bigoplus_{j=2}^{d}\left[\bigoplus_{\gamma\in J'_{N,j}}\left(\bigoplus_{\delta\in J_{N,j,\gamma}}H_\delta\right)\right].$$

Here

$$H_\delta=\operatorname{Span}\left\{z^\delta z^{kN}:k\ge 0\right\}.$$

By Theorem 7.5 and Lemma 9.2, H_δ is equivalent to H_β for $\delta,\beta\in J_{N,j,\gamma}$ and $\left\{P_{H_\beta}:\beta\in J_{N,j,\gamma}\right\}$ is a (maximal) set of mutually equivalent minimal projections. Thus the part $\bigoplus_{\delta\in J_{N,j,\gamma}}H_\delta$ gives rise to $M_{d(d-1)\cdots(d-j+2)}(\mathbb{C})$. Similarly, $\bigoplus_{\gamma\in J_{N,1}}H_\gamma$ gives rise to $\bigoplus_{i=1}^{M}\mathbb{C}$. The proof is complete. □

THEOREM 9.4. *Let ω be on $\mathcal{K}_\rho(\mathbb{B}^2)$ where ρ is a positive integer.*

(i) *Then $v(z_1^Mz_2^M)$ on the Hardy space ($\rho=2$) is $*$-isomorphic to*

$$\left[\bigoplus_{i=1}^{M}\mathbb{C}\right]\bigoplus\left[\bigoplus_{i=1}^{\infty}M_2(\mathbb{C})\right]\bigoplus\left[\bigoplus_{i=1}^{M}M_3(\mathbb{C})\right].$$

(ii) *Then $v(z_1^Mz_2^M)$ on the Drury–Arveson space ($\rho=1$) is $*$-isomorphic to*

$$\left[\bigoplus_{i=1}^{M}\mathbb{C}\right]\bigoplus\left[\bigoplus_{i=1}^{\infty}M_2(\mathbb{C})\right]\bigoplus\left[\bigoplus_{i=1}^{M-1}M_3(\mathbb{C})\right].$$

(iii) *Then $v(z_1^M z_2^M)$ on the Bergman space ($\rho = 3$) and on $\mathcal{K}_\rho(\mathbb{B}^2)$ with $\rho > 3$ is $*$-isomorphic to*

$$\left[\bigoplus_{i=1}^{M}\mathbb{C}\right]\bigoplus\left[\bigoplus_{i=1}^{\infty}M_2(\mathbb{C})\right]\bigoplus\left[\bigoplus_{i=1}^{M}M_4(\mathbb{C})\right].$$

Proof. The proof is similar to the proof of proceeding theorem by using Theorem 8.1 and Lemma 9.2. We give some brief explanations. For (i), (ii) and (iii), (36) with $\gamma_1 = \gamma_2$ gives rise to $[\bigoplus_{i=1}^M \mathbb{C}]$ and (36) with $\gamma_1 \neq \gamma_2$ gives rise to $[\bigoplus_{i=1}^\infty M_2(\mathbb{C})]$. For (i), (38) gives rise to $[\bigoplus_{i=1}^M M_3(\mathbb{C})]$. For (ii), (37) gives rise to $[\bigoplus_{i=1}^{M-1} M_3(\mathbb{C})]$ because $(l+1, l+1) \notin J_N$ for $l = M-1$. So in fact (37) hold for $0 \le l \le M-2$. For (iii), (39) gives rise to $[\bigoplus_{i=1}^M M_4(\mathbb{C})]$. □

Theorems 8.5 and 8.6 yield the following result.

Theorem 9.5. *Assume $N = (N_1, N_2)$ with $N_1 \neq N_2$. Write*

$$N_1 = N_1'M, \quad N_2 = N_2'M, \quad M = \mathrm{GCD}(N_1, N_2).$$

(i) *Then $v(z_1^{N_1} z_2^{N_2})$ on $\mathcal{D}(\mathbb{D}^2)$ or $\mathcal{K}_\rho(\mathbb{D}^2)$ for $\rho = 2$ is $*$-isomorphic to*

$$\left[\bigoplus_{i=1}^{\infty}\mathbb{C}\right]\bigoplus\left[\bigoplus_{i=1}^{\infty}M_2(\mathbb{C})\right].$$

(ii) *Then $v(z_1^{N_1} z_2^{N_2})$ on $\mathcal{D}(\mathbb{B}^2)$ or $\mathcal{K}_\rho(\mathbb{D}^2)$ for $\rho \neq 1, 2$ is $*$-isomorphic to*

$$\bigoplus_{i=1}^{\infty}\mathbb{C}.$$

Note that by Theorem 9.3, in the case $N_1 = N_2 = M$, $v(z_1^{N_1} z_2^{N_2})$ on $\mathcal{D}(\mathbb{D}^2)$ or $\mathcal{D}(\mathbb{B}^2)$ or $\mathcal{K}_\rho(\mathbb{D}^2)$ ($\rho \neq 1$) is $*$-isomorphic to

$$\left[\bigoplus_{i=1}^{M}\mathbb{C}\right]\bigoplus\left[\bigoplus_{i=1}^{\infty}M_2(\mathbb{C})\right].$$

This result is different to results in the case $N_1 \neq N_2$.

10. Final remarks.

Recall that $S_\Phi = (S_1, \ldots, S_d)$ is a tuple of weighted shifts on $l_d^2(E)$ with operator weights $\Phi = \{\Phi_{\alpha,i} : \alpha \in Z_+^d, i = 1, \ldots, d\}$. We have discussed the common reducing subspaces of S_Φ. We have also studied the reducing subspaces of the product $\prod_{i=1}^d S_i$, which can be viewed as a single weighted shift with operator weights. We would remark that the general approach of using weighted shifts with operator weights also allows us to discuss the reducing subspaces of related operators such as S_1 or the common reducing subspaces of $(S_1,\ S_2S_3)$. The operator S_1 on $l^2_d(E)$ can be viewed as a single weighted shift with operator weights on $l^2(\widehat{E})$ for a Hilbert space $\widehat{E}$. The tuple (S_1, S_2S_3) can be viewed as two commuting weighted shifts with operator weights on $l^2_2(\widehat{E})$ for a Hilbert space $\widehat{E}$.

For example, we have the following theorem which is a combination of Theorem 4.2 and Theorem 7.1.

We need to introduce notations. These notations are not only useful for describing this specific result, but is also suggestive of more general results. Assume $d \geq 3$, since we are studying the common reducing subspaces of $(T_{z_1^{N_1}}, T_{z_2^{N_2}z_3^{N_3}})$ on H^2_ω, and let

$$\begin{aligned} k &= (k_1, k_2, k_2, 0, \ldots, 0), \quad N = (N_1, N_2, N_3, 0, \ldots, 0), \quad k, N \in Z^d_+, \\ kN &= (k_1N_1, k_2N_2, k_2N_3, 0, \ldots, 0). \end{aligned}$$

Let $N \geq (1, 1, 1, 0, \ldots, 0)$ be given. Set

$$J_N = \{\alpha \geq 0 : \alpha_1 < N_1 \text{ and } \min\{\alpha_2 - N_2, \alpha_3 - N_3\} < 0\},$$

and let $\widehat{E} \subset H^2_\omega$ be the subspace given by

$$\begin{aligned} \widehat{E} &= \left\{ f(z) = \sum_{\alpha \in J_N} f_\alpha z^\alpha : f_\alpha \in \mathbb{C}, \quad \|f(z)\|^2 = \sum_{\alpha \in J_N} \omega_\alpha |f_\alpha|^2 < \infty \right\} \\ &= \ker\left(T^*_{z_1^{N_1}}\right) \cap \ker\left(T^*_{z_2^{N_2}z_3^{N_3}}\right). \end{aligned}$$

Theorem 10.1. (i) *If a closed subspace X of H^2_ω is a common reducing subspace of $(T_{z_1^{N_1}}, T_{z_2^{N_2}z_3^{N_3}})$, then*

$$X = \operatorname{Span}\left\{ T^{k_1}_{z_1^{N_1}} T^{k_2}_{z_2^{N_2}z_3^{N_3}} x : (k_1, k_2) \geq 0, x \in \widehat{E_0} \right\},$$

where

$$\widehat{E_0} = \left(X \ominus T_{z_1^{N_1}} X\right) \cap \left(X \ominus T_{z_2^{N_2}z_3^{N_3}} X\right) \subseteq \widehat{E},$$

and $\widehat{E_0}$ is an invariant subspace of a set of diagonal operators with positive diagonals.

(ii) *Any common reducing subspace of $(T_{z_1^{N_1}}, T_{z_2^{N_2}z_3^{N_3}})$ on H^2_ω is a direct sum of (singly generated) minimal reducing subspaces.*

(iii) *Any minimal reducing subspace of $(T_{z_1^{N_1}}, T_{z_2^{N_2}z_3^{N_3}})$ on H^2_ω is of the form*

$$\operatorname{Span}\left\{ f(z) z_1^{k_1N_1} z_2^{k_2N_2} z_3^{k_2N_3} : (k_1, k_2) \geq 0 \right\} = \operatorname{Span}\left\{ f(z) z^{kN} : k \geq 0 \right\},$$

where

$$f(z) = \sum_{\gamma \in J} f_\gamma z^\gamma, \quad f_\gamma \in \mathbb{C} \text{ and } f_\gamma \neq 0 \text{ for all } \gamma \in J,$$

and $J \subseteq J_N$ and $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $\gamma, \delta \in J, k \geq 0$. To be clear, we recall the notation:

$$\gamma + kN = (\gamma_1 + k_1 N_1, \gamma_2 + k_2 N_2, \gamma_3 + k_2 N_3, \gamma_4, \dots, \gamma_d)\,.$$

Since ω_α on the polydisk is of a product form, the above idea together with theorems in last section readily yields several interesting results. We state a couple of concrete results on the Dirichlet space $\mathcal{D}(\mathbb{D}^d)$ with brief explanations. We will only state Part (iii). Part (i) and (ii) are similar.

Corollary 10.2. (i) *The minimal reducing subspaces of $T_{z_1^{N_1}}$ on $\mathcal{K}_\rho(\mathbb{D}^d)$ ($\rho \neq 1$) or $\mathcal{D}(\mathbb{D}^d)$ for $d \geq 2$ are the obvious ones. Namely, any minimal reducing subspaces of $T_{z_1^{N_1}}$ is of the form* $\operatorname{Span}\{f(z) z_1^{k_1 N_1} : k_1 \geq 0\}$, *where*

$$f(z) = z_1^{\gamma_1} g(z_2, \dots, z_d)$$

for some $0 \leq \gamma_1 < N_1$, and g is a holomorphic function only depending on $(z_2, \dots, z_d)$ and $g \in \mathcal{K}_\rho(\mathbb{D}^{d-1})$ or $g \in \mathcal{D}(\mathbb{D}^{d-1})$.

(ii) *Let $v(z_1^{N_1})$ be the commutant of the von Neumann algebra generated by $T_{z_1^{N_1}}$ and $T^*_{z_1^{N_1}}$ on $\mathcal{K}_\rho(\mathbb{D}^d)$ ($\rho \neq 1$) or $\mathcal{D}(\mathbb{D}^d)$ for $d \geq 2$. Then*

$$v(z_1^{N_1}) \approx \bigoplus_{i=1}^{N_1} M_\infty(\mathbb{C}).$$

Proof. Note that in this case, $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ is the same as $\omega_{\gamma_1+k_1N_1}/\omega_{\gamma_1} = \omega_{\delta_1+k_1N_1}/\omega_{\delta_1}$. So the result in (i) can be viewed as a special case of Theorem 5.4 with $d = 1$. Thus (ii) follows from (i) and Lemma 9.2. ☐

Corollary 10.3. *Assume $N = (N_1, N_2)$. Write*

$$N_1 = N_1' M, \quad N_2 = N_2' M, \quad M = \operatorname{GCD}(N_1, N_2).$$

(i) *Any minimal reducing subspace of $T_{z_1^{N_1} z_2^{N_2}}$ on $\mathcal{D}(\mathbb{D}^d)$ for $d \geq 3$ is of the form*

$$\operatorname{Span}\left\{ f(z) z_1^{kN_1} z_2^{kN_2} : k \geq 0 \right\},$$

where either

$$f(z) = z_1^{\gamma_1} z_2^{\gamma_2} g_1(z_3, \dots, z_d)$$

for (γ_1, γ_2) such that $\min\{\gamma_1 - N_1, \gamma_2 - N_2\} < 0$ *or*

$$f(z) = z_1^{\gamma_1} z_2^{\gamma_2} g_1(z_3, \dots, z_d) + z_1^{\delta_1} z_2^{\delta_2} g_2(z_3, \dots, z_d)$$

with $g_1, g_2 \in \mathcal{D}(\mathbb{D}^{d-2})$,

$$(\gamma_1, \gamma_2) = (lN_1' - 1, mN_2' - 1)\,, \quad (\delta_1, \delta_2) = (mN_1' - 1, lN_2' - 1)\,, \tag{54}$$

for some positive integers l *and* m *such that* $\min\{l, m\} \leq M$ *and* $l \neq m$.

(ii) *Let* $v(z_1^{N_1} z_2^{N_2})$ *be the commutant of the von Neumann algebra generated by* $T_{z_1^{N_1} z_2^{N_2}}$ *and* $T^*_{z_1^{N_1} z_2^{N_2}}$ *on* $\mathcal{D}(\mathbb{D}^d)$ *for* $d \geq 3$. *Then*

$$v(z_1^{N_1} z_2^{N_2}) \approx \bigoplus_{i=1}^{\infty} M_\infty(\mathbb{C}).$$

PROOF. Note that (i) follows essentially from Theorem 8.5 for $N_1 \neq N_2$ and Theorem 7.5 for $N_1 = N_2$. In the case $N_1 = N_2$, just note that $(\gamma_1, \gamma_2) = (\delta_2, \delta_1)$ in (54). So (ii) follows from (i) and Lemma 9.2 and the fact that

$$\left[\bigoplus_{i=1}^{\infty} M_\infty(\mathbb{C})\right] \bigoplus \left[\bigoplus_{i=1}^{\infty} M_\infty(\mathbb{C})\right] \approx \bigoplus_{i=1}^{\infty} M_\infty(\mathbb{C}).$$

The proof is complete. □

In the previous two results, the length of a singly generated reducing subspace could be infinite if g_1 or g_2 is not an analytic polynomial. The next result shows that on the unit ball, the length is always finite.

THEOREM 10.4. (i) *Any minimal reducing subspace of* $T_{z_1^{N_1}}$ *on* $\mathcal{K}_\rho(\mathbb{B}^d)$ $(\rho \neq 1)$ *or* $\mathcal{D}(\mathbb{B}^d)$ *for* $d \geq 2$ *is of the form* $\mathrm{Span}\{f(z) z_1^{k_1 N_1} : k_1 \geq 0\}$, *where*

$$f(z) = z_1^{\gamma_1} g(z_2, \ldots, z_d)$$

for some $\gamma \geq 0$ *such that* $0 \leq \gamma_1 < N_1$ *and* g *is a homogenous polynomial in* $(z_2, \ldots, z_d)$ *of degree* $|\gamma| - \gamma_1$.

(ii) *Any minimal reducing subspaces of* $T_{z_1^{N_1}}$ *on* $\mathcal{K}_\rho(\mathbb{B}^d)$ *with* $\rho = 1$ *for* $d \geq 2$ *(the Drury–Arveson space) is of the form* $\mathrm{Span}\{f(z) z_1^{k_1 N_1} : k_1 \geq 0\}$, *where either*

$$f(z) = z_1^{\gamma_1} g(z_2, \ldots, z_d)$$

for some $\gamma \geq 0$ *such that* $0 \leq \gamma_1 < N_1$ *and* g *is a homogenous polynomial in* $(z_2, \ldots, z_d)$ *of degree* $|\gamma| - \gamma_1 \neq 0$ *or*

$$f(z) = \sum_{i=0}^{N_1 - 1} a_i z_1^i, \quad a_i \in \mathbb{C}. \tag{55}$$

(iii) *Let* $v(z_1^{N_1})$ *be the commutant of the von Neumann algebra generated by* $T_{z_1^{N_1}}$ *and* $T^*_{z_1^{N_1}}$ *on* $\mathcal{K}_\rho(\mathbb{B}^d)$ $(\rho \neq 1)$ *or* $\mathcal{D}(\mathbb{B}^d)$ *for* $d \geq 2$. *Then*

$$v(z_1^{N_1}) \approx \bigoplus_{i=1}^{N_1} \left[\bigoplus_{n=0}^{\infty} M_{r_{n,d}}(\mathbb{C}) \right], \ \textit{where } r_{n,d} = \binom{n+d-2}{d-2}. \tag{56}$$

In the case $\mathcal{K}_\rho(\mathbb{B}^d)$ *with* $\rho = 1$ *for* $d \geq 2$*,*

$$v(z_1^{N_1}) \approx M_{N_1}(\mathbb{C}) \oplus \bigoplus_{i=1}^{N_1} \left[\bigoplus_{n=1}^{\infty} M_{r_{n,d}}(\mathbb{C}) \right], \ \textit{where } r_{n,d} = \binom{n+d-2}{d-2}. \tag{57}$$

Proof. We first prove (i). We will show that $\omega_{\gamma+kN}/\omega_\gamma = \omega_{\delta+kN}/\omega_\delta$ for all $k = (k_1, 0, \ldots, 0) \geq 0$ if and only if $\gamma_1 = \delta_1$ and $|\gamma| = |\delta|$. Let ω_α be on $\mathcal{K}_\rho(\mathbb{B}^d)$. Then

$$\omega_{\gamma+kN} = \frac{(\gamma_1 + k_1 N_1)! \gamma_2! \cdots \gamma_d! \Gamma(\rho)}{\Gamma(\rho + |\gamma| + k_1 N_1)}.$$

Thus $\omega_{\gamma+kN}/\omega_{\gamma+(k+1)N} = \omega_{\delta+kN}/\omega_{\delta+(k+1)N}$ is equivalent to

$$\frac{\prod_{j=1}^{N_1} (\rho + |\gamma| + k_1 N_1 + j - 1)}{\prod_{j=1}^{N_1} (\gamma_1 + k_1 N_1 + j)} = \frac{\prod_{j=1}^{N_1} (\rho + |\delta| + k_1 N_1 + j - 1)}{\prod_{j=1}^{N_1} (\delta_1 + k_1 N_1 + j)} \quad \text{for all } k \geq 0. \tag{58}$$

If $\rho > 1$, then $\rho + |\delta| + N_1 > \delta_1 + N_1$. This implies that $\gamma_1 = \delta_1$ and $|\gamma| = |\delta|$. If $0 < \rho < 1$, the above equation also implies that $\gamma_1 = \delta_1$ and $|\gamma| = |\delta|$. Therefore

$$az^\gamma + bz^\delta = z_1^{\gamma_1}(a z_2^{\gamma_2} \cdots z_d^{\gamma_d} + b z_2^{\delta_2} \cdots z_d^{\delta_d}).$$

Given γ, let $J_\gamma = \{\delta \geq 0 : \delta_1 = \gamma_1 \text{ and } |\delta| = |\gamma|\}$. Then

$$f(z) = \sum_{\delta \in J_\gamma} f_\delta z^\delta = z_1^{\gamma_1} g(z_2, \ldots, z_d), \quad f_\delta \in \mathbb{C}$$

where g is a homogenous polynomial. The number of monomial terms that can appear in g is the cardinality of the index set J_γ. Recall that a weak composition of an integer n into i parts is a way of writing n as the sum of a sequence of i nonnegative integers such that two sequences that differ in the order of their terms define different weak compositions. The number of weak compositions is

$$\binom{n+i-1}{i-1}.$$

Thus the cardinality $\kappa(J_\gamma)$ of the index set J_γ is the number of weak compositions of $|\gamma| - \gamma_1$ into $d-1$ parts,

$$\kappa(J_\gamma) = \binom{|\gamma| - \gamma_1 + d - 2}{d-2} = r_{|\gamma|-\gamma_1, d}.$$

The length of $f(z)$ could be any integer between 1 and $\kappa(J_\gamma)$.

In the case $\mathcal{K}_\rho(\mathbb{B}^d)$ with $\rho = 1$ for $d \geq 2$, Equation (58) implies that either $\gamma_1 = \delta_1$, $|\gamma| = |\delta|$, and $|\gamma| - \gamma_1 \neq 0$ or $|\gamma| = \gamma_1$ and $|\delta| = \delta_1$. This proves (ii).

Relation (56) follows from (i) and Lemma 9.2 by noting that there are N_1 values for γ_1, and $|\gamma| - \gamma_1$ could be any nonnegative integer. Relation (57) follows from (ii) and Lemma 9.2 by noting that $f(z)$ as in (55) gives rise to $M_{N_1}(\mathbb{C})$ and $n = |\gamma| - \gamma_1 \neq 0$. $\square$

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