J. Math. Soc. Japan
Vol. 55, No. 1, 2003

Residues of Chern classes

To the memory of Katsuo Kawakubo

By Tatsuo SUWA

(Received May 1, 2001)
(Revised Oct. 12, 2001)

Abstract. If we have a finite number of sections of a complex vector bundle E over a manifold M, certain Chern classes of E are localized at the singular set S, i.e., the set of points where the sections fail to be linearly independent. When S is compact, the localizations define the residues at each connected component of S by the Alexander duality. If M itself is compact, the sum of the residues is equal to the Poincaré dual of the corresponding Chern class. This type of theory is also developed for vector bundles over a possibly singular subvariety in a complex manifold. Explicit formulas for the residues at an isolated singular point are also given, which express the residues in terms of Grothendieck residues relative to the subvariety.

Let E be a complex vector bundle of rank r over a manifold M. If we have a section s of E, the top Chern class $c^r(E)$ of E is "localized" at the zero set, or "the singular set", S of s. More presicely, there is a canonical class $c^r(E,s)$ in the relative cohomology $H^{2r}(M, M\backslash S)$ which lifts the class $c^r(E)$ in $H^{2r}(M)$. If S is compact, by the Alexander duality, $c^r(E,s)$ defines the "residue" in the homology of each connected component of S. If M itself is compact, we have the residue formula which says that the sum of the residues is equal to the Poincaré dual of $c^r(E)$.

The residue at an isolated singular point is expressed as a Grothendieck residue when M is a complex manifold of dimension $n = r$ and when E and s are holomorphic, see for example [**Su2**, Theorem 3.1]. Special cases of this include the Poincaré-Hopf index of a holomorphic vector field as a section of the tangent bundle and the multiplicity (or Milnor number) of a holomorphic function with its differential considered as a section of the cotangent bundle. In the global situation, the residue formula leads to the Poincaré-Hopf theorem, in the first case, and to the "multiplicity formula" (or the "Milnor number formula") [**I**], see also [**F**, Example 14.1.5] and [**HL**, VI 3], in the second case.

2000 *Mathematics Subject Classification.* Primary 14C17, 32A27, 57R20; Secondary 14B05, 32S05.

Key Words and Phrases. Chern classes, frames of vector bundles, localization, Grothendieck residues relative to subvarieties.

Partially supported by a grant from the Japan Society for the Promotion of Science.

In this note, we consider the case where we are given an ℓ-tuple $\mathbf{s}$ of sections of the bundle E. In this case, if we denote by S the set of points where the members of $\mathbf{s}$ fail to be linearly independent, there is a canonical localization $c^i(E,\mathbf{s})$ in $H^{2i}(M, M\backslash S)$ of the Chern class $c^i(E)$, for each $i = r-\ell+1,\dots,r$. Again, if S is compact, by the Alexander duality, $c^i(E,s)$ defines the residue in the homology of each connected component of S and, if M is compact, we have the residue formula (Proposition 2.5 below). We give an explicit formula for the residue at an isolated singular point when M is a complex manifold of dimension $n = r-\ell+1$ and E and $\mathbf{s}$ are holomorphic. It is also expressed by a Grothendieck residue (Theorem 5.2 and Section 6).

The definition of residues and the residue formula are readily generalized to the case of vector bundles over singular subvarieties in complex manifolds (Proposition 3.3). We also have a similar expression for the residue at an isolated singular point of a variety as a Grothendieck residue relative to the subvariety (Theorem 5.7 and Section 6).

The above localization theory fits nicely into the framework of the theory of integration on the Čech-de Rham cohomology, which we recall in Section 1. The computation of the residues is also done in this framework. We give, in Section 4, some fundamental material necessary for this purpose.

For an application, we refer to [**IS**], where the multiplicity of a function on a singular variety is defined and the aforementioned multiplicity formula is generalized to the case of functions on singular varieties. The multiplicity at an isolated singular point can be computed using the formulas in this note.

I would like to thank M. Oka, M. Saito and H. Yamada for helpful conversations.

1. Chern classes in the Čech-de Rham cohomology.

We refer to [**BT**], [**L1-2**] and [**Su1**] for the material in this section.

(A) Čech-de Rham cohomology and dualities.

Let M be a (connected) oriented C^∞ manifold of dimension m. For an open set U in M, we denote by $A^q(U)$ the space of complex valued C^∞ q-forms on U. For an open covering $\mathscr{U} = \{U_\alpha\}_\alpha$ of M, we denote by $A^*(\mathscr{U})$ the Čech-de Rham complex associated to the covering $\mathscr{U}$ with differential D and by $H^q(A^*(\mathscr{U}))$ its cohomology [**Su1**, Chapter II, 3]. We have a canonical isomorphism

$$H^q(M,\boldsymbol{C}) \xrightarrow{\sim} H^q(A^*(\mathscr{U})), \tag{1.1}$$

where $H^q(M,\boldsymbol{C})$ denotes the de Rham cohomology of M. We also have the cup product in $A^*(\mathscr{U})$, which induces the cup product in $H^*(A^*(\mathscr{U}))$ compatible, via (1.1), with the usual one in the de Rham cohomology.

If M is compact, taking a "system of honey-comb cells" $\{R_\alpha\}_\alpha$ adapted to $\mathscr{U}$, we may define the integration

$$\int_M : H^m(A^*(\mathscr{U})) \to \boldsymbol{C},$$

which is compatible, via (1.1), with the usual integration on the de Rham cohomology. The composition of the cup product and the integration induces the Poincaré duality

$$P : H^q(M, \boldsymbol{C}) \simeq H^q(A^*(\mathscr{U})) \xrightarrow{\sim} H^{m-q}(A^*(\mathscr{U}))^\vee \simeq H_{m-q}(M, \boldsymbol{C}).$$

Now let S be a closed set in M. Letting $U_0 = M \backslash S$ and U_1 a neighborhood of S in M, we consider the covering $\mathscr{U} = \{U_0, U_1\}$ of M. In this case, an element σ in $A^q(\mathscr{U})$ may be written as $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ with σ_0 and σ_1 q-forms on U_0 and U_1, respectively, and σ_{01} a $(q-1)$-form on $U_{01} = U_0 \cap U_1$. If we set

$$A^q(\mathscr{U}, U_0) = \{\sigma \in A^q(\mathscr{U}) \mid \sigma_0 = 0\},$$

$A^*(\mathscr{U}, U_0)$ forms a subcomplex of $A^*(\mathscr{U})$ and we have a canonical isomorphism

$$H^q(A^*(\mathscr{U}, U_0)) \simeq H^q(M, M \backslash S; \boldsymbol{C}).$$

Suppose S is compact (M may not be). Let R_1 be a compact manifold of dimension m with C^∞ boundary ∂R_1 in U_1, containing S in its interior $\operatorname{Int} R_1$, and set $R_0 = M \backslash \operatorname{Int} R_1$. Then $\{R_0, R_1\}$ is a system of honey-comb cells adapted to $\mathscr{U}$. In this situation, we have the integration on $A^m(\mathscr{U}, U_0)$ defined by

$$\int_M \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}, \tag{1.2}$$

where $R_{01} = R_0 \cap R_1 = -\partial R_1$ (∂R_1 with opposite orientation). This induces the integration

$$\int_M : H^m(A^*(\mathscr{U}, U_0)) \to \boldsymbol{C}.$$

The cup product in $A^*(\mathscr{U})$ induces a pairing $A^q(\mathscr{U}, U_0) \times A^{m-q}(U_1) \to A^m(\mathscr{U}, U_0)$ given by $((0, \sigma_1, \sigma_{01}), \tau_1) \mapsto (0, \sigma_1 \wedge \tau_1, \sigma_{01} \wedge \tau_1)$. This, followed by the integration, gives a pairing

$$A^q(\mathscr{U}, U_0) \times A^{m-q}(U_1) \to \boldsymbol{C}.$$

If we further assume that U_1 is a regular neighborhood of S, this induces the Alexander duality

$$A : H^q(M, M \backslash S; \boldsymbol{C}) \simeq H^q(A^*(\mathscr{U}, U_0)) \xrightarrow{\sim} H^{m-q}(U_1, \boldsymbol{C})^\vee \simeq H_{m-q}(S, \boldsymbol{C}).$$

If M is compact, the following diagram is commutative:

$$\begin{array}{ccc} H^q(M, M\backslash S; \boldsymbol{C}) & \xrightarrow{j^*} & H^q(M, \boldsymbol{C}) \\ \wr \big\downarrow A & & \wr \big\downarrow P \\ H_{m-q}(S, \boldsymbol{C}) & \xrightarrow{i_*} & H_{m-q}(M, \boldsymbol{C}), \end{array} \tag{1.3}$$

where i and j denote, respectively, the inclusions $S \hookrightarrow M$ and $(M, \varnothing) \hookrightarrow (M, M\backslash S)$.

(B) Representatives of Chern classes.

Let M be a C^∞ manifold of dimension m, as above, and let E be a C^∞ complex vector bundle of (complex) rank r on M. For a connection ∇ for E and for $i = 1, \dots, r$, we denote by $c^i(\nabla)$ the i-th Chern form defined by ∇. Recall that it is defined by $c^i(\nabla) = (\sqrt{-1}/(2\pi))^i \sigma_i(\kappa)$, where $\sigma_i(\kappa)$ denotes the i-th symmetric form of the curvature matrix κ of ∇ and is a closed $2i$-form on M. Its class $[c^i(\nabla)]$ in $H^{2i}(M, \boldsymbol{C})$ is the i-th Chern class $c^i(E)$ of E.

If we have $p+1$ connections $\nabla_0, \dots, \nabla_p$ for E there is a $(2i-p)$-form $c^i(\nabla_0, \dots, \nabla_p)$ alternating in the $p+1$ entries and satisfying

$$\sum_{\nu=0}^{p} (-1)^\nu c^i(\nabla_0, \dots, \widehat{\nabla_\nu}, \dots, \nabla_p) + (-1)^p \, dc^i(\nabla_0, \dots, \nabla_p) = 0, \tag{1.4}$$

cf. [**Bo**]. Here we use a different sign convention, see [**Su1**, Chapter II, (7.10)].

Let $\mathscr{U} = \{U_\alpha\}_\alpha$ be an open covering of M. For each α, we choose a connection ∇_α for E on U_α, and for the collection $\nabla_\star = (\nabla_\alpha)_\alpha$, we define the element $c^i(\nabla_\star)$ in $A^{2i}(\mathscr{U})$ by

$$c^i(\nabla_\star)_{\alpha_0 \cdots \alpha_p} = c^i(\nabla_{\alpha_0}, \dots, \nabla_{\alpha_p}).$$

Then we have $Dc^i(\nabla_\star) = 0$ by (1.4). Moreover, it is shown that the class of $c^i(\nabla_\star)$ in $H^{2i}(A^*(\mathscr{U}))$ does not depend on the choice of the collection of connections $\nabla_\star$. The class $[c^i(\nabla_\star)]$ corresponds to $c^i(E)$ in $H^{2i}(M, \boldsymbol{C})$ under the isomorphism (1.1).

2. Localization of Chern classes.

Let E be a C^∞ complex vector bundle of rank r over an oriented C^∞ manifold M of dimension m as in the previous section. Let $\mathbf{s} = (s_1, \dots, s_\ell)$, $1 \le \ell \le r$, be a C^∞ ℓ-frame of E on some open set U, namely, a set of ℓ C^∞ sections of E linearly independent at each point of U. In the sequel, an r-frame is simply called a frame. We say that a connection ∇ for E is trivial with respect to $\mathbf{s}$ (simply, $\mathbf{s}$-trivial), if $\nabla(s_i) = 0$ for $i = 1, \dots, \ell$. Note that if $\nabla_0, \dots, \nabla_p$ are $\mathbf{s}$-trivial connections, we have the vanishing (see, for example, [**Su1**, Chapter II, Proposition 9.1])

$$c^i(\nabla_0, \dots, \nabla_p) = 0 \quad \text{for } i \geq r - \ell + 1. \tag{2.1}$$

Let S be a closed set in M and suppose we have a C^∞ ℓ-frame $\mathbf{s}$ of E on $M \backslash S$. Then, from the above fact, we will see that there is a "localization" $c^i(E, \mathbf{s})$ in $H^{2i}(M, M \backslash S; \boldsymbol{C})$ of the Chern class $c^i(E)$ in $H^{2i}(M, \boldsymbol{C})$, for $i \geq r - \ell + 1$.

Letting $U_0 = M \backslash S$ and U_1 a neighborhood of S, we consider the covering $\mathscr{U} = \{U_0, U_1\}$ of M. Recall the Chern class $c^i(E)$ is represented by the cocycle $c^i(\nabla_\star)$ in $A^{2i}(\mathscr{U})$ given by

$$c^i(\nabla_\star) = (c^i(\nabla_0), c^i(\nabla_1), c^i(\nabla_0, \nabla_1)), \tag{2.2}$$

where ∇_0 and ∇_1 denote connections for E on U_0 and U_1, respectively. If we take as ∇_0 an $\mathbf{s}$-trivial connection, then $c^i(\nabla_0) = 0$ and the cocycle is in $A^{2i}(\mathscr{U}, U_0)$. Thus it defines a class in the relative cohomology $H^{2i}(M, M \backslash S; \boldsymbol{C})$, which we denote by $c^i(E, \mathbf{s})$. It is sent to the class $c^i(E)$ by the canonical homomorphism $j^* : H^{2i}(M, M \backslash S; \boldsymbol{C}) \to H^{2i}(M, \boldsymbol{C})$. It does not depend on the choice of the connection ∇_1 or on the choice of the $\mathbf{s}$-trivial connection ∇_0 [**Su1**, Chapter III, Lemma 3.1]. We call $c^i(E, \mathbf{s})$ the localization of $c^i(E)$ at S with respect to $\mathbf{s}$.

Suppose now that S is a compact set admitting a regular neighborhood and let $(S_\lambda)_\lambda$ be the connected components of S. Then we have the Alexander duality

$$A : H^{2i}(M, M \backslash S; \boldsymbol{C}) \xrightarrow{\sim} H_{m-2i}(S, \boldsymbol{C}) = \bigoplus_\lambda H_{m-2i}(S_\lambda, \boldsymbol{C}).$$

Thus, for each λ, $c^i(E, \mathbf{s})$ defines a class in $H_{m-2i}(S_\lambda, \boldsymbol{C})$, which we call the residue of $\mathbf{s}$ at S_λ with respect to c^i and denote by $\operatorname{Res}_{c^i}(\mathbf{s}, E; S_\lambda)$. It is also called a residue of $c^i(E)$ for brevity.

For each λ, we choose a neighborhood U_λ of S_λ in U_1, so that the U_λ's are mutually disjoint, and let R_λ be an m-dimensional manifold with C^∞ boundary in U_λ containing S_λ in its interior. We set $R_{0\lambda} = -\partial R_\lambda$. Then the residue $\operatorname{Res}_{c^i}(\mathbf{s}, E; S_\lambda)$ is represented by an $(m-2i)$-cycle C in S_λ such that

$$\int_C \tau_1 = \int_{R_\lambda} c^i(\nabla_1) \wedge \tau + \int_{R_{0\lambda}} c^i(\nabla_0, \nabla_1) \wedge \tau \tag{2.3}$$

for every closed $(m-2i)$-form τ on U_λ. If $2i = m$, the residue is a number given by

$$\text{(2.4)} \qquad \mathrm{Res}_{c^i}(\mathbf{s}, E; S_\lambda) = \int_{R_\lambda} c^i(\nabla_1) + \int_{R_{0\lambda}} c^i(\nabla_0, \nabla_1).$$

From the commutativity of (1.3), we have the "residue formula":

PROPOSITION 2.5. *If M is compact, for $i = r - \ell + 1, \dots, r$, we have*

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_{c^i}(\mathbf{s}, E; S_\lambda) = c^i(E) \frown [M] \quad \text{in } H_{m-2i}(M, \boldsymbol{C}),$$

where i_λ denotes the inclusion $S_\lambda \hookrightarrow M$ and the sum is taken over the connected components of S.

REMARKS 2.6. 1. From the fact that $c^i(E, \mathbf{s})$ does not depend on ∇_0, we see that, for $i = r - \ell + 2, \dots, r$, we have $c^i(E, \mathbf{s}) = c^i(E, \mathbf{s}')$, where $\mathbf{s}'$ denotes an $(r - i + 1)$-frame made of $r - i + 1$ arbitrary members of $\mathbf{s}$. Thus the case $i = r - \ell + 1$ will be of essential interest.

2. It is rather a strong hypothesis to assume the existence of an ℓ-frame on $M \backslash S$, unless $m = 2(r - \ell + 1)$. It is more reasonable to assume, taking some triangulation or cellular decomposition of M compatible with S, the existence of an ℓ-frame on the $2(r - \ell + 1)$-skeleton of $M \backslash S$, see [**St**]. We may still define a canonical localization $c^i(E, \mathbf{s})$ in $H^{2i}(M, M \backslash S; \boldsymbol{C})$ and the residue $\mathrm{Res}_{c^i}(\mathbf{s}, E; S_\lambda)$ in $H_{m-2i}(S_\lambda, \boldsymbol{C})$, for $i = r - \ell + 1$, by modifying the above arguments, see [**BLSS**] and [**L3**].

3. We may also define the localization $c^i(E, \mathbf{s})$ via obstruction theory. In this case, $c^i(E, \mathbf{s})$ is in the integral cohomology $H^{2i}(M, M \backslash S; \boldsymbol{Z})$, which shows that the above residue $\mathrm{Res}_{c^i}(\mathbf{s}, E; S_\lambda)$ is in fact in the integral homology (and is an integer, if $2i = m$).

3. Chern classes on singular varieties.

We refer to [**Su1**, Chapter IV, 2, Chapter VI, 4] for details of the material in this section.

Let V be an analytic variety of pure dimension n in a complex manifold W of dimension $n + k$. We denote by $\mathrm{Sing}(V)$ the singular set of V and set $V' = V \backslash \mathrm{Sing}(V)$. First, suppose V is compact and let $\tilde{U}$ be a regular neighborhood of V in W. Let $\mathscr{U} = \{\tilde{U}_\alpha\}_\alpha$ be an open covering of $\tilde{U}$. Taking a system $\{\tilde{R}_\alpha\}_\alpha$ of honey-comb cells adapted to $\mathscr{U}$ such that V is transverse to each $\tilde{R}_{\alpha_0 \cdots \alpha_p} = \tilde{R}_{\alpha_0} \cap \cdots \cap \tilde{R}_{\alpha_p}$, we may define the integration

$$\int_V : H^{2n}(A^*(\mathscr{U})) \to \boldsymbol{C}.$$

We have $H^{2n}(A^*(\mathscr{U})) \simeq H^{2n}(\tilde{U}, \boldsymbol{C})$ and the above integration is compatible with the integration $\int_V : H^{2n}(\tilde{U}, \boldsymbol{C}) \to \boldsymbol{C}$ induced from the integration of $2n$-forms on $\tilde{U}$ over the $2n$-cycle V.

Also the bilinear pairing defined as the composition of the cup product in $A^*(\mathscr{U})$ and the integration over V induces the "Poincaré homomorphism"

$$P : H^q(V, \boldsymbol{C}) \simeq H^q(A^*(\mathscr{U})) \to H^{2n-q}(A^*(\mathscr{U}))^\vee \simeq H_{2n-q}(V, \boldsymbol{C}),$$

which is not an isomorphism in general. Note that in [**Br**] the above homomorphism P, as well as the Alexander homomorphism defined below, is defined by a combinatorial method in homology and cohomology with integral coefficients. The homomorphism P is given by the cap product with the fundamental class $[V]$.

Now suppose V may not be compact. Let S be a compact set in V admitting a regular neighborhood in W such that there is an open set U in V with $S \subset U$ and $U \backslash S \subset V'$. Let $\tilde{U}_1$ be a regular neighborhood of S in W with $\tilde{U}_1 \cap V \subset U$ and $\tilde{U}_0$ a tubular neighborhood of $U \backslash S$ in W with C^∞ projection $\rho : \tilde{U}_0 \to U \backslash S$. We consider the covering $\mathscr{U} = \{\tilde{U}_0, \tilde{U}_1\}$ of $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$, which may be assumed to be a regular neighborhood of U. We also define the subcomplex $A^*(\mathscr{U}, \tilde{U}_0)$ of $A^*(\mathscr{U})$ as in Section 1 (A). Then we see that

$$H^q(A^*(\mathscr{U}, \tilde{U}_0)) \simeq H^q(U, U \backslash S; \boldsymbol{C}).$$

Let $\tilde{R}_1$ be a compact real $2(n+k)$ dimensional manifold with C^∞ boundary in $\tilde{U}_1$ such that S is in its interior and that $\partial \tilde{R}_1$ is transverse to U. We set $R_1 = \tilde{R}_1 \cap U$ and $R_{01} = -\partial R_1$. Then we have the integration on $A^{2n}(\mathscr{U}, \tilde{U}_0)$ given as (1.2), with M replaced by U, for $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^{2n}(\mathscr{U}, \tilde{U}_0)$. This again induces the integration on the cohomology

$$\int_U : H^{2n}(A^*(\mathscr{U}, \tilde{U}_0)) \to \boldsymbol{C}.$$

As in Section 1 (A), we have a bilinear pairing $A^q(\mathscr{U}, \tilde{U}_0) \times A^{2n-q}(\tilde{U}_1) \to \boldsymbol{C}$, which induces the "Alexander homomorphism"

$$A : H^q(U, U \backslash S; \boldsymbol{C}) \simeq H^q(A^*(\mathscr{U}, \tilde{U}_0)) \to H^{2n-q}(\tilde{U}_1, \boldsymbol{C})^\vee \simeq H_{2n-q}(S, \boldsymbol{C}).$$

Note that the above is not an isomorphism in general.

Suppose V is compact and let S be a compact set in V which admits a regular neighborhood in W and contains $\mathrm{Sing}(V)$. Then the following diagram is commutative [**Su1**, Chapter VI, Proposition 4.4]:

$$\begin{array}{ccc} H^q(V, V \backslash S; \boldsymbol{C}) & \xrightarrow{j^*} & H^q(V, \boldsymbol{C}) \\ \Big\downarrow{\scriptstyle A} & & \Big\downarrow{\scriptstyle P} \\ H_{2n-q}(S, \boldsymbol{C}) & \xrightarrow{i_*} & H_{2n-q}(V, \boldsymbol{C}), \end{array} \tag{3.1}$$

where i and j denote, respectively, the inclusions $S \hookrightarrow V$ and $(V, \emptyset) \hookrightarrow (V, V \backslash S)$.

REMARK 3.2. In the above, the assumption that $U \backslash S$ is in the regular part $V' = V \backslash \mathrm{Sing}(V)$ is not necessary. However, with this condition, to define a cochain $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ in $A^q(\mathcal{U})$, we only need to define σ_0 on $U \backslash S$, since there is a C^∞ retraction $\rho : \tilde{U}_0 \to U \backslash S$.

Again, let V be a variety of dimension n in a complex manifold W. First suppose V is compact and let $\tilde{U}$ and $\mathcal{U}$ be as above. For a complex vector bundle E over $\tilde{U}$, we have the i-th Chern class $c^i(E)$ in $H^{2i}(A^*(\mathcal{U})) \simeq H^{2i}(V, \boldsymbol{C})$. The corresponding class in $H^{2i}(V, \boldsymbol{C})$ is denoted by $c^i(E|_V)$. We also have the class $P(c^i(E)) = c^i(E) \frown [V]$ in $H_{2n-2i}(V, \boldsymbol{C})$.

Next, let S be a compact set in V (V may not be compact) admitting a regular neighborhood in W such that there is an open set U in V with $S \subset U$ and $U \backslash S \subset V'$. Let $\tilde{U}_1, \tilde{U}_0$, $\mathcal{U} = \{\tilde{U}_0, \tilde{U}_1\}$ and $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$ be as above. For a complex vector bundle E over $\tilde{U}$, the Chern class $c^i(E)$ is represented by the cocycle $c^i(\nabla_\star)$ in $A^{2i}(\mathcal{U})$ given as (2.2) with ∇_0 and ∇_1 connections for E on $\tilde{U}_0$ and $\tilde{U}_1$, respectively. Note that it is sufficient if ∇_0 is defined only on $U_0 = U \backslash S$ (see Remark 3.2). Suppose that we have a C^∞ ℓ-frame $\mathbf{s} = (s_1, \ldots, s_\ell)$ on U_0 and let ∇_0 be $\mathbf{s}$-trivial. Then we have the vanishing $c^i(\nabla_0) = 0$, for $i \geq r - \ell + 1$, and the above cocycle $c^i(\nabla_\star)$ is in $A^{2i}(\mathcal{U}, \tilde{U}_0)$. Thus it defines a class $c^i(E|_V, \mathbf{s})$ in $H^{2i}(U, U \backslash S; \boldsymbol{C})$. It does not depend on the choices of various connections. We have the Alexander homomorphism

$$A : H^{2i}(U, U \backslash S; \boldsymbol{C}) \to H_{2n-2i}(S, \boldsymbol{C}) = \bigoplus_\lambda H_{2n-2i}(S_\lambda, \boldsymbol{C}),$$

where $(S_\lambda)_\lambda$ are the connected components of S. Thus, for each λ, $c^i(E|_V, \mathbf{s})$ defines a class in $H_{2n-2i}(S_\lambda, \boldsymbol{C})$, which we call the residue of $\mathbf{s}$ at S_λ with respect to c^i and denote by $\mathrm{Res}_{c^i}(\mathbf{s}, E|_V; S_\lambda)$. It is also called a residue of $c^i(E|_V)$.

For each λ, we choose a neighborhood $\tilde{U}_\lambda$ of S_λ in $\tilde{U}_1$, so that the $\tilde{U}_\lambda$'s are mutually disjoint. Let $\tilde{R}_\lambda$ be a real $2(n+k)$-dimensional manifold with C^∞ boundary in $\tilde{U}_\lambda$ containing S_λ in its interior such that the boundary $\partial \tilde{R}_\lambda$ is transverse to V. We set $R_{0\lambda} = -\partial \tilde{R}_\lambda \cap V$. Then the residue $\mathrm{Res}_{c^i}(\mathbf{s}, E|_V; S_\lambda)$ is represented by a $2(n-i)$-cycle C in S_λ satisfying an identity similar to (2.3) with τ an arbitrary closed $2(n-i)$-form on $\tilde{U}_\lambda$. In particular, if $i = n$, the residue is a number given by a formula as (2.4). From the commutativity of (3.1), we have the residue formula:

PROPOSITION 3.3. *Let V be a compact variety of dimension n in a complex manifold W and E a complex vector bundle over a neighborhood of V in W. Let $(S_\lambda)_\lambda$ be a finite number of compact connected sets in V such that each S_λ admits a* *regular neighborhood disjoint one another and that* $\bigcup_\lambda S_\lambda$ *contains* $\mathrm{Sing}(V)$. *Then, for an* ℓ*-frame* $\mathbf{s}$ *of* E *on* $V \backslash \bigcup_\lambda S_\lambda$ *and for* $i \geq r - \ell + 1$,

$$\sum_\lambda (i_\lambda)_* \operatorname{Res}_{c^i}(\mathbf{s}, E|_V; S_\lambda) = c^i(E) \frown [V] \quad \text{in } H_{2n-2i}(V, \boldsymbol{C}).$$

Note that the residues $\operatorname{Res}_{c^i}(\mathbf{s}, E|_V; S_\lambda)$ are in fact in the integral homology and the above formula holds in the integral homology (cf. Remark 2.6.3).

4. Some local analytic geometry and others.

(A) Lemmas.

We denote by $\mathcal{O}_{n+k}$ the ring of germs of holomorphic functions at the origin 0 in $\boldsymbol{C}^{n+k}$ and, for germs $a_1, \dots, a_r$ in $\mathcal{O}_{n+k}$, we denote by $V(a_1, \dots, a_r)$ the germ of variety defined by $a_1, \dots, a_r$. The following is proved similarly as [**LS**, Lemma 3] and [**Su1**, Chapter IV, Lemma 4.4]. Here we give a more detailed proof for later use.

Lemma 4.1. *Let* V *be a germ of variety of dimension* n *at* 0 *in* $\boldsymbol{C}^{n+k}$ *and let* $g_1, \dots, g_N$ *be germs in the ring* $\mathcal{O}_{n+k}$. *Suppose* $V(g_1, \dots, g_N) \cap V = \{0\}$. *Then there exists an* $N \times n$ *matrix* $C = (c_{ij})$ *of complex numbers such that, for germs* $f_j = \sum_{i=1}^N c_{ij} g_i$, $j = 1, \dots, n$, *we have* $V(f_1, \dots, f_n) \cap V = \{0\}$.

Proof. Since $\dim V = n$, it suffices to show the following for $\ell = 1, \dots, n$:

(*) If there exists an $N \times (\ell - 1)$ matrix (c_{ij}) such that $\dim V(f_1, \dots, f_{\ell-1}) \cap V = n - \ell + 1$, for $f_j = \sum_{i=1}^N c_{ij} g_i$, $j = 1, \dots, \ell - 1$, then there exist complex numbers $c_{i\ell}$, $i = 1, \dots, N$, such that $\dim V(f_1, \dots, f_\ell) \cap V = n - \ell$, for $f_\ell = \sum_{i=1}^N c_{i\ell} g_i$.

In the above, when $\ell = 1$, $V(f_1, \dots, f_{\ell-1})$ is understood to be (the germ at 0 of) $\boldsymbol{C}^{n+k}$. To show (*), let $V(f_1, \dots, f_{\ell-1}) \cap V = \bigcup_q V_q$ be the irreducible decomposition. Since $V(g_1, \dots, g_N) \cap V = \{0\}$, for each q, there exist a point x_q in V_q and g_i with $g_i(x_q) \neq 0$. Let H_q denote the hyperplane in $\boldsymbol{C}^N = \{(\xi_1, \dots, \xi_N)\}$ defined by $\sum_{i=1}^N g_i(x_q)\xi_i = 0$. Let $(c_{1\ell}, \dots, c_{N\ell})$ be a point in $\boldsymbol{C}^N \backslash \bigcup_q H_q$ and set $f_\ell = \sum_{i=1}^N c_{i\ell} g_i$. Then, $V_i \not\subset V(f_\ell)$, for each i. We have

$$V(f_1, \dots, f_\ell) \cap V = \bigcup_q (V_q \cap V(f_\ell)).$$

Since each V_q is irreducible and $V_q \not\subset V(f_\ell)$, $\dim V_q \cap V(f_\ell) = \dim V_q - 1$. Therefore, we have (*) and the lemma. □

Note that the above holds when $k = 0$, in which case V is the germ of the total space $\boldsymbol{C}^n$.

Let r and ℓ be integers with $1 \leq \ell \leq r$ and denote by $M(r)$ the space of $r \times r$ matrices with the standard topology homeomorphic with $\boldsymbol{C}^{r^2}$. Let $\mathscr{I}$ be the set of ℓ-tuples of integers $(i_1, \dots, i_\ell)$ with $1 \leq i_1 < \cdots < i_\ell \leq r$. Let $N = \binom{r}{\ell}$. Thus $\mathscr{I}$ contains N elements. We endow $\mathscr{I}$ with the lexicographic order. For a matrix A in $M(r)$, its ℓ-th exterior power $\Lambda^\ell A$ is an $N \times N$ matrix whose entries are given by $\det A_{IJ}$, the $\ell \times \ell$ minor of A consisting of the rows corresponding to $I = (i_1, \dots, i_\ell)$ and of the columns corresponding to $J = (j_1, \dots, j_\ell)$.

LEMMA 4.2. *For a matrix A and a neighborhood of A, there exists a matrix A' in the neighborhood such that the matrix C consisting of the first n columns of $\Lambda^\ell A'$ satisfies the condition of Lemma* 4.1.

PROOF. Recall that, for each $j = 1, \dots, n$, $(c_{1j}, \dots, c_{Nj})$ is determined so that it avoids a finite number of hyperplanes in $\boldsymbol{C}^N$. Suppose the j-th column $(\det A_{IJ})_I$ satisfies the equation $\sum_{i=1}^N \alpha_i \xi_i = 0$ for one of the hyperplanes, with $J = (j_1, \dots, j_\ell)$ the index corresponding to j. Suppose $\alpha_{i_0} \neq 0$ and let I_0 be the index corresponding to i_0. If $\det A_{I_0 J} = 0$, then there is a matrix A' in the given neighborhood of A such that $\det A'_{I_0 J} \neq 0$. We may choose A' so that if some column of $\Lambda^\ell A$ does not satisfy a linear equation, the corresponding column of $\Lambda^\ell A'$ does not satisfy the equation either. So we may assume that $\det A_{I_0 J} \neq 0$ from the beginning. We may write

$$\alpha_{i_0} \cdot \det A_{I_0 J} = \sum_i \det B_i, \quad B_i = \begin{pmatrix} a_{ij_1} & \cdots & a_{ij_\ell} \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix},$$

where the sum is taken over i which is not in I_0. In the above, we arrange the B_i's so that, if $i < i'$, $a_{ij_1}, \dots, a_{ij_\ell}$ do not appear in $B_{i'}$. Let i be the smallest i such that $\det B_i \neq 0$. By changing $a_{ij_1}, \dots, a_{ij_\ell}$ a little, we see that there is a matrix A' in the given neighborhood of A such that the above equation does not hold for A'. We may choose A' so that if some column of $\Lambda^\ell A$ does not satisfy a linear equation, the corresponding column of $\Lambda^\ell A'$ does not satisfy the equation either. Continuing this process, we prove the lemma. □

(B) Grothendieck residues relative to a subvariety.

Let U be a neighborhood of 0 in $\boldsymbol{C}^n$ and let $f_1, \dots, f_n$ be holomorphic functions on U with $V(f_1, \dots, f_n) = \{0\}$. Thus there exists a positive number δ such that $f^{-1}(D_\delta)$ is a compact set in U, where f denotes the map $(f_1, \dots, f_n) : U \to \boldsymbol{C}^n$ and D_δ the closed polydisk of radius δ. For a holomorphic n-from ω on U, the usual Grothendieck residue is defined by

$$\operatorname{Res}_0\left[\begin{matrix}\omega\\ f_1,\dots,f_n\end{matrix}\right]=\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\int_\Gamma\frac{\omega}{f_1\cdots f_n},$$

where Γ denotes the n-cycle in U given by

$$\Gamma=\Gamma_\varepsilon=\{p\in U\mid |f_i(p)|=\varepsilon_i,\ \ i=1,\dots,n\}$$

for ε_i with $0<\varepsilon_i<\delta$. It is oriented so that $d\arg(f_1)\wedge\cdots\wedge d\arg(f_n)\ge 0$ (e.g., [**GH**, Chapter 5]). Note that, for various $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)$, the cycles Γ_ε are homologous to one another and the above integral is well-defined. Moreover, for almost all ε, Γ_ε is a C^∞ manifold. In the sequel, we set $\varepsilon_1=\cdots=\varepsilon_n=\varepsilon$ for simplicity.

Now let $\tilde U$ be a neighborhood of 0 in $\boldsymbol{C}^{n+k}$ and V a subvariety of dimension n in $\tilde U$ which contains 0 as at most an isolated singular point. Also, let $f_1,\dots,f_n$ be holomorphic functions on $\tilde U$ with $V(f_1,\dots,f_n)\cap V=\{0\}$. For a holomorphic n-from ω on $\tilde U$, the Grothendieck residue relative to V is defined by

$$\operatorname{Res}_0\left[\begin{matrix}\omega\\ f_1,\dots,f_n\end{matrix}\right]_V=\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\int_\Gamma\frac{\omega}{f_1\cdots f_n},$$

where Γ is the n-cycle expressed as above with $U=\tilde U\cap V$ ([**LS**], [**Su1**, Chapter IV, 8]).

If V is a complete intersection defined by $h_1=\cdots=h_k=0$ in $\tilde U$, by an iterated use of the projection formula, we see that

$$\operatorname{Res}_0\left[\begin{matrix}\omega\\ f_1,\dots,f_n\end{matrix}\right]_V=\operatorname{Res}_0\left[\begin{matrix}\omega\wedge dh_1\wedge\cdots\wedge dh_k\\ f_1,\dots,f_n,h_1,\dots,h_k\end{matrix}\right].$$

(C) Determinants of matrices of forms.

Let $\Omega=(\omega_{ij})$ be an $r\times r$ matrix with differential forms ω_{ij} in its entries. We define the determinant of Ω, denoted by $\det(\Omega)$ or $|\Omega|$ as usual, by

$$\det\Omega=\sum_{\sigma\in\mathscr{S}_r}\operatorname{sgn}\sigma\cdot\omega_{\sigma(1)1}\cdots\omega_{\sigma(r)r},$$

where $\mathscr{S}_r$ denotes the symmetric group of degree r and the products of forms are exterior products. Note that if the entries ω_{ij} are forms of even degree, possibly except for the ones in a single column, the products in the above are commutative and we may treat $\det\Omega$ in the same way as a usual matrix of numbers. Also, for $n=1,\dots,r$, we define $\sigma_n(\Omega)$ to be the coefficient of t^n in $\det(I+t\Omega)$. Let $\mathscr{A}$ denote the set of n-tuples of integers $(a_1,\dots,a_n)$ with $1\le a_1<\cdots<a_n\le r$. For an element $A=(a_1,\dots,a_n)$ in $\mathscr{A}$, we denote by Ω_A the $n\times n$ matrix whose (i,j)-entry is the (a_i,a_j)-entry of Ω. Then we have

$$\sigma_n(\Omega)=\sum_{A\in\mathscr{A}}\det\Omega_A. \tag{4.3}$$

5. Residue at an isolated singularity.

(A) Residue on a manifold.

Let p_0 be a point in a complex manifold M of dimension n and let E be a holomorphic vector bundle of rank r over a neighborhood U of p_0 in M, with $1 \le n \le r$. Let $\ell = r - n + 1$ and suppose we have ℓ holomorphic sections $s_1, \dots, s_\ell$ of E on U which are linearly independent at each point of $U_0 = U \backslash \{p_0\}$. Thus $\mathbf{s} = (s_1, \dots, s_\ell)$ is a holomorphic ℓ-frame on U_0 and, in this situation, we have the residue $\mathrm{Res}_{c^n}(\mathbf{s}, E; p_0)$ in $H_0(\{p_0\}, \boldsymbol{C}) = \boldsymbol{C}$ (in fact in $\boldsymbol{Z}$, cf. Remarks 2.6). In the following, we compute this residue.

We may assume that E is trivial over U and let $\mathbf{e} = (e_1, \dots, e_r)$ be a holomorphic frame of E on U. We write $s_i = \sum_{j=1}^{r} f_{ij} e_j$, $i = 1, \dots, \ell$, with f_{ij} holomorphic functions on U. Let F be the $\ell \times r$ matrix whose (i, j)-entry is f_{ij}. We set

$$\mathscr{I} = \{(i_1, \dots, i_\ell) \mid 1 \le i_1 < \cdots < i_\ell \le r\}$$

as in Section 4 (A). For an element $I = (i_1, \dots, i_\ell)$ in $\mathscr{I}$, let F_I denote the $\ell \times \ell$ matrix consisting of the columns of F corresponding to I and set $f_I = \det F_I$. If we write $e_I = e_{i_1} \wedge \cdots \wedge e_{i_\ell}$, we have

$$s_1 \wedge \cdots \wedge s_\ell = \sum_{I \in \mathscr{I}} f_I e_I.$$

Noting that the set of common zeros of the f_I's consists only of p_0, we have, from Lemmas 4.1 and 4.2:

Lemma 5.1. *We may choose a holomorphic frame $\mathbf{e} = (e_1, \dots, e_r)$ of E so that there exist n elements $I^{(1)}, \dots, I^{(n)}$ in $\mathscr{I}$ with the property*

$$\{p \in U \mid f_{I^{(1)}}(p) = \cdots = f_{I^{(n)}}(p) = 0\} = \{p_0\}.$$

Note that we may assume that $I^{(1)}, \dots, I^{(n)}$ are the first n elements in $\mathscr{I}$ with the lexicographic order. Let $\mathbf{e} = (e_1, \dots, e_r)$ be a frame of E on U as in Lemma 5.1. Let us write $I^{(\alpha)} = (i_1^{(\alpha)}, \dots, i_\ell^{(\alpha)})$, $\alpha = 1, \dots, n$, and let $F^{(\alpha)}$ be the $r \times r$ matrix obtained by replacing the $i_j^{(\alpha)}$-th row of the $r \times r$ identity matrix by the j-th row of F, $j = 1, \dots, \ell$. Note that $\det F^{(\alpha)} = f_{I^{(\alpha)}}$. Let $\check{F}^{(\alpha)}$ denote the adjoint matrix of $F^{(\alpha)}$ and set

$$\Theta^{(\alpha)} = \check{F}^{(\alpha)} \cdot dF^{(\alpha)},$$

which is an $r \times r$-matrix whose entries are holomorphic 1-forms.

Recall that (Section 4, (C)), for an n-tuple of integers $A = (a_1, \dots, a_n)$ with $1 \le a_1 < \cdots < a_n \le r$, we denote by $\Theta_A^{(\alpha)}$ the $n \times n$ matrix whose (i, j)-entry is the (a_i, a_j)-entry of $\Theta^{(\alpha)}$. For a permutation ρ of degree n, we denote by $\Theta_A(\rho)$

the $n \times n$-matrix whose i-th column is that of $\Theta_A^{(\rho(i))}$ and, for the collection $\Theta = \{\Theta^{(\alpha)}\}_\alpha$, we set

$$\sigma_n(\Theta) = \frac{1}{n!} \sum_{A \in \mathscr{A}} \sum_{\rho \in \mathscr{S}_n} \operatorname{sgn} \rho \cdot \det \Theta_A(\rho).$$

Note that $\sigma_n(\Theta)$ is a holomorphic n-form on U.

Theorem 5.2. *In the above situation, we have*

$$\operatorname{Res}_{c^n}(\mathbf{s}, E; p_0) = \operatorname{Res}_{p_0} \begin{bmatrix} \sigma_n(\Theta) \\ f_{I^{(1)}}, \dots, f_{I^{(n)}} \end{bmatrix}.$$

Proof. This is done similarly as for [**Su2**, Theorem 3.1]. The main differences are:

1) We cannot say that we have the vanishing as (3.5) in [**Su2**] simply because of the triviality of the connections involved. We need to look at this form a little more closely to have this.

2) The computation of the n-form finally to be integrated is much more complicated.

On U_0, we let ∇_0 be an $\mathbf{s}$-trivial connection for E and, on U_1, we let ∇_1 be the connection for E trivial with respect to the frame $\mathbf{e}$. Since $c^n(\nabla_1) = 0$ and $R_{01} = -\partial R_1$, from (2.4) we have

$$\operatorname{Res}_{c^n}(\mathbf{s}, E; p) = -\int_{\partial R_1} c^n(\nabla_0, \nabla_1). \tag{5.3}$$

We consider the covering $\mathscr{U} = \{U^{(\alpha)}\}_{\alpha=1}^n$ of U_0 defined by

$$U^{(\alpha)} = \{p \in U_0 \mid f_{I^{(\alpha)}}(p) \neq 0\},$$

and work on the Čech-de Rham cohomology with respect to $\mathscr{U}$. On $U^{(\alpha)}$, we may replace, in the frame $\mathbf{e}$, $(e_{i_1^{(\alpha)}}, \dots, e_{i_\ell^{(\alpha)}})$ by $(s_1, \dots, s_\ell)$ to obtain a frame $\mathbf{e}^{(\alpha)}$ for E. We denote by $\nabla^{(\alpha)}$ the connection for E on $U^{(\alpha)}$ trivial with respect to the frame $\mathbf{e}^{(\alpha)}$. Then the connection matrix $\theta^{(\alpha)}$ of $\nabla^{(\alpha)}$ with respect to the frame $\mathbf{e}$ is given by

$$\theta^{(\alpha)} = dF^{(\alpha)^{-1}} \cdot F^{(\alpha)} = -F^{(\alpha)^{-1}} \cdot dF^{(\alpha)} = -\frac{1}{f_{I^{(\alpha)}}} \Theta^{(\alpha)}.$$

Let $\mathscr{U}$ be the covering of U_0 as above and define a cochain τ in $A^{2n-2}(\mathscr{U})$ by

$$\tau_{\alpha_1 \cdots \alpha_q} = c^n(\nabla_0, \nabla_1, \nabla^{(\alpha_1)}, \dots, \nabla^{(\alpha_q)}),$$

which is a $(2n - q - 1)$-form on $U^{(\alpha_1 \cdots \alpha_q)} = U^{(\alpha_1)} \cap \cdots \cap U^{(\alpha_q)}$. Since ∇_0 and $\nabla^{(\alpha)}$ are all $\mathbf{s}$-trivial, we have

$$c^n(\nabla_0, \nabla^{(\alpha_1)}, \dots, \nabla^{(\alpha_q)}) = 0 \tag{5.4}$$

for $q \geq 1$. Now we compute $D\tau$. First for $q = 1$, we have, by (5.4),

$$(D\tau)_\alpha = dc^n(\nabla_0, \nabla_1, \nabla^{(\alpha)}) = -c^n(\nabla_1, \nabla^{(\alpha)}) - c^n(\nabla_0, \nabla_1).$$

For $q = 2, \dots, n$, we compute using (5.4) as in [**Su2**], to get

$$(D\tau)_{\alpha_1 \cdots \alpha_q} = -c^n(\nabla_1, \nabla^{(\alpha_1)}, \dots, \nabla^{(\alpha_q)}).$$

We set

$$R_1 = \{p \in U \mid |f_{I^{(1)}}(p)|^2 + \cdots + |f_{I^{(n)}}(p)|^2 \leq n\varepsilon^2\}$$

for a small positive number ε. Denoting by ι the inclusion map $\partial R_1 \hookrightarrow U_0$, we let $\iota^*\mathscr{U}$ be the covering of ∂R_1 by the open sets $\partial R_1 \cap U^{(\alpha)}$. Then, as a system $\{R^{(\alpha)}\}_{\alpha=1}^n$ of honey-comb cells adapted to $\iota^*\mathscr{U}$, we take

$$R^{(\alpha)} = \{p \in \partial R_1 \mid |f_{I^{(\alpha)}}(p)| \geq |f_{I^{(\beta)}}(p)| \text{ for all } \beta\}$$

and, for a q-tuple $(\alpha_1 \cdots \alpha_q)$ with $1 \leq \alpha_1 < \cdots < \alpha_q \leq n$, we set $R^{(\alpha_1 \cdots \alpha_q)} = R^{(\alpha_1)} \cap \cdots \cap R^{(\alpha_q)}$, which is a $(2n-q)$-dimensional manifold with boundary oriented as an intersection of honey-comb cells. Considering the integration

$$\int_{\partial R_1} : A^{2n-1}(\iota^*\mathscr{U}) \to \boldsymbol{C},$$

from the identity $\int_{\partial R_1} D\tau = 0$, we get, as in [**Su2**],

$$\operatorname{Res}_{c^n}(\mathbf{s}, E; p_0) = \sum_{q=1}^{n} \sum_{1 \leq \alpha_1 < \cdots < \alpha_q \leq n} \int_{R^{(\alpha_1 \cdots \alpha_q)}} c^n(\nabla_1, \nabla^{(\alpha_1)}, \dots, \nabla^{(\alpha_q)}).$$

Now we compute the $(2n-q)$-form $c^n(\nabla_1, \nabla^{(\alpha_1)}, \dots, \nabla^{(\alpha_q)})$. For this, let $\tilde{\nabla}$ denote the connection for the bundle $E \times \boldsymbol{R}^q$ over $U^{(\alpha_1 \cdots \alpha_q)} \times \boldsymbol{R}^q$ given by $\tilde{\nabla} = (1 - \sum_{\nu=1}^q t_\nu)\nabla_1 + \sum_{\nu=1}^q t_\nu \nabla^{(\alpha_\nu)}$. Then the connection matrix $\tilde{\theta}$ of $\tilde{\nabla}$ with respect to the frame $\mathbf{e}$ is given by

$$\tilde{\theta} = \left(1 - \sum_{\nu=1}^{q} t_\nu\right)\theta_1 + \sum_{\nu=1}^{q} t_\nu \theta^{(\alpha_\nu)},$$

where θ_1 is the connection matrix of ∇_1 with respect to the frame $\mathbf{e}$ and is equal to zero. The curvature matrix $\tilde{\kappa}$ of $\tilde{\nabla}$ with respect to the frame $\mathbf{e}$ is then given by

$$\tilde{\kappa} = d\tilde{\theta} - \tilde{\theta} \wedge \tilde{\theta} = \sum_{\nu=1}^{q} dt_\nu \wedge \theta^{(\alpha_\nu)} + \sum_{\nu=1}^{q} t_\nu \, d\theta^{(\alpha_\nu)} - \sum_{\nu, \mu=1}^{q} t_\nu t_\mu \theta^{(\alpha_\nu)} \wedge \theta^{(\alpha_\mu)}. \tag{5.5}$$

By definition,

$$c^n(\nabla_1, \nabla^{(\alpha_1)}, \dots, \nabla^{(\alpha_q)}) = \pi_* c^n(\tilde{\nabla}) = \left(\frac{\sqrt{-1}}{2\pi}\right)^n \pi_* \sigma_n(\tilde{\kappa}),$$

where $\sigma_n(\tilde{\kappa})$ denotes the n-th symmetric form of $\tilde{\kappa}$ and π_* the integration along the fibers of the projection $\pi : U^{(\alpha_1 \cdots \alpha_q)} \times \Delta^q \to U^{(\alpha_1 \cdots \alpha_q)}$ with Δ^q the standard q-simplex in $\boldsymbol{R}^q$.

We claim that

$$c^n(\nabla_1, \nabla^{(\alpha_1)}, \dots, \nabla^{(\alpha_q)}) = 0, \quad \text{if } 1 \le q \le n-1.$$

In fact, when we compute $\pi_* \det \tilde{\kappa}_A$ (cf. (4.3)), only the term involving $dt_1 \wedge \cdots \wedge dt_q$ matters. Its coefficient is a holomorphic $(2n-q)$-form on an open set of M, which is zero if $q < n$.

Thus we have

$$\mathrm{Res}_{c^n}(\mathbf{s}, E; p_0) = \int_{R^{(1\cdots n)}} c^n(\nabla_1, \nabla^{(1)}, \dots, \nabla^{(n)}).$$

To compute $c^n(\nabla_1, \nabla^{(1)}, \dots, \nabla^{(n)})$, fix A and let ρ be a permution of degree n. Then, by (5.5), the term in $\det \tilde{\kappa}_A$ involving $dt_1 \wedge \cdots \wedge dt_n$ is given by

$$(-1)^{n(n-1)/2} \sum_{\rho} \operatorname{sgn} \rho \cdot dt_1 \wedge \cdots \wedge dt_n \wedge \det \theta_A(\rho),$$

where $\theta_A(\rho)$ is defined similarly as for $\Theta_A(\rho)$. Therefore we obtain

$$c^n(\nabla_1, \nabla^{(1)}, \dots, \nabla^{(n)}) = \sum_A \sum_{\rho} \operatorname{sgn} \rho \cdot c \cdot \det \theta_A(\rho),$$

where

$$c = (-1)^{n(n-1)/2} \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{\Delta^n} dt_1 \cdots dt_n = (-1)^{n(n-1)/2} \left(\frac{\sqrt{-1}}{2\pi}\right)^n \frac{1}{n!}.$$

Noting that $\det \theta_A(\rho) = (-1)^n (1/(f_{I^{(1)}} \cdots f_{I^{(n)}})) \det \Theta_A(\rho)$ and that the n-cycle Γ appearing in the Grothendieck residue with respect to $f_{I^{(1)}}, \dots, f_{I^{(n)}}$ is given by $\Gamma = (-1)^{n(n-1)/2} R^{(1\cdots n)}$, we obtain the formula. $\square$

(B) Residue on a singular variety.

Let V be a subvariety of dimension n in a complex manifold W of dimension $n+k$, as before. Let p_0 be an isolated singular point in V and let E be a holomorphic vector bundle of rank r, $1 \le n \le r$, over a neighborhood $\tilde{U}$ of p_0 in W. Let $\ell = r - n + 1$ and suppose we have ℓ holomorphic sections $s_1, \dots, s_\ell$ of E on $\tilde{U}$ such that

$$\{p \in \tilde{U} \mid s_1 \wedge \cdots \wedge s_\ell(p) = 0\} \cap V = \{p_0\}.$$

Thus $\mathbf{s} = (s_1, \dots, s_\ell)$ is an ℓ-frame of E on $U_0 = U \backslash \{p_0\}$, $U = \tilde{U} \cap V$, and we have the residue $\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; p_0)$ in $H_0(\{p_0\}, \boldsymbol{C}) = \boldsymbol{C}$ (in fact in $\boldsymbol{Z}$, cf. Remarks 2.6). In the following, we compute this residue. We may assume that E is trivial on $\tilde{U}$ and let $\mathbf{e} = (e_1, \dots, e_r)$ be a holomorphic frame of E on $\tilde{U}$. We write $s_i = \sum_{j=1}^r f_{ij} e_j$, $i = 1, \dots, \ell$, with f_{ij} holomorphic functions on $\tilde{U}$. If we let F and c be as in (A), we have

$$s_1 \wedge \cdots \wedge s_\ell = \sum_{I \in \mathscr{I}} f_I e_I.$$

From Lemmas 4.1 and 4.2, we have:

Lemma 5.6. *We may choose a holomorphic frame $\mathbf{e} = (e_1, \dots, e_r)$ of E so that there exist n elements $I^{(1)}, \dots, I^{(n)}$ in $\mathscr{I}$ with the property*

$$\{p \in \tilde{U} \mid f_{I^{(1)}}(p) = \cdots = f_{I^{(n)}}(p) = 0\} \cap V = \{p_0\}.$$

Note that we may assume that $I^{(1)}, \dots, I^{(n)}$ are the first n elements in $\mathscr{I}$ with the lexicographic order. Once we choose a frame $\mathbf{e} = (e_1, \dots, e_r)$ of E on $\tilde{U}$ as in Lemma 5.6, the rest goes exactly the same way as in (A). The only difference is that, in (A), U is a neighborhood of p_0 in a manifold M, while in this subsection, it is a neighborhood of p_0 in a possibly singular variety V. In both cases, $U_0 = U \backslash \{p_0\}$ is non-singular, where everything is performed. Thus by similar notation as in (A), we have:

Theorem 5.7. *In the above situation,*

$$\mathrm{Res}_{c^n}(\mathbf{s}, E|_V; p_0) = \mathrm{Res}_{p_0} \begin{bmatrix} \sigma_n(\Theta) \\ f_{I^{(1)}}, \dots, f_{I^{(n)}} \end{bmatrix}_V.$$

6. Special cases.

We consider the situations of Section 5. Thus p_0 will be either

(I) a point in a complex manifold M of dimension n, or

(II) an isolated singular point of a subvariety V of dimension n in a complex manifold.

Let U be a neighborhood of p_0 in M or in V as in Section 5. In what follows, in the case (II), we denote $\mathrm{Res}_{c_n}(\mathbf{s}, E|_V; p_0)$ simply by $\mathrm{Res}_{c_n}(\mathbf{s}, E; p_0)$ and omit the suffix V in the residue symbol so that the residues are expressed in the same way in the both cases.

(1) The case $\ell = 1$ and $r = n$, with n arbitrary.

Let $\mathbf{e} = (e_1, \dots, e_n)$ be an arbitrary frame of E in a neighborhood of p_0 and write $s_1 = \sum_{i=1}^n f_i e_i$. Then we have

$$\mathrm{Res}_{c_n}(\mathbf{s}, E; p_0) = \mathrm{Res}_{p_0} \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \dots, f_n \end{bmatrix}.$$

In fact, in Theorem 5.2 or 5.7, we have $f_{I^{(i)}} = f_i$, $i = 1, \dots, n$, and we readily see that $\sigma_n(\Theta) = df_1 \wedge \cdots \wedge df_n$ (see also [**Su2**, Theorem 3.1]).

(2) The case $n = 1$ and $\ell = r$, with r arbitrary.

Let $\mathbf{e} = (e_1, \dots, e_r)$ be an arbitrary frame of E in a neighborhood of p_0 and write $s_i = \sum_{j=1}^r f_{ij} e_j$, $i = 1, \dots, r$. Let $F = (f_{ij})$ and set $f = \det F$. Then we have

$$\operatorname{Res}_{c^1}(\mathbf{s}, E; p_0) = \operatorname{Res}_{p_0}\begin{bmatrix} df \\ f \end{bmatrix}.$$

In fact, in Theorem 5.2 or 5.7, we have $f_{I^{(1)}} = f$ and we easily see that $\sigma_n(\Theta) = df$.

Note that $\operatorname{Res}_{c^1}(\mathbf{s}, E; p_0)$ coincides with the residue $\operatorname{Res}_{c^1}(s, \det E; p_0)$ of the section $s = s_1 \wedge \cdots \wedge s_r$ of the line bundle $\det E = \Lambda^r E$ at p_0.

(3) The case $\ell = 2$ and $r = n + 1$ with n arbitrary.

Let $\mathbf{s} = (s_1, s_2)$ be a 2-frame on $U_0 = U \backslash \{p_0\}$ and $\mathbf{e} = (e_1, \dots, e_{n+1})$ a frame on U or $\tilde{U}$ satisfying the condition of Lemma 5.1 or Lemma 5.6, respectively. We write $s_i = \sum_{j=1}^{n+1} f_{ij} e_j$, $i = 1, 2$, as in Section 5. We may suppose that $I^{(\alpha)} = (1, \alpha + 1)$, $\alpha = 1, \dots, n$, so that

$$f_{I^{(\alpha)}} = \begin{vmatrix} f_{11} & f_{1,\alpha+1} \\ f_{21} & f_{2,\alpha+1} \end{vmatrix}$$

and that

$$\{p \mid f_{I^{(1)}}(p) = \cdots = f_{I^{(n)}}(p) = 0\} = \{p_0\},$$

in the case (I) or

$$\{p \mid f_{I^{(1)}}(p) = \cdots = f_{I^{(n)}}(p) = 0\} \cap V = \{p_0\},$$

in the case (II). In the sequel, we introduce the following notation:

$$\varphi_{ij} = \begin{vmatrix} f_{1i} & f_{1j} \\ f_{2i} & f_{2j} \end{vmatrix}, \quad \theta_{ij} = \begin{vmatrix} f_{1i} & df_{1j} \\ f_{2i} & df_{2j} \end{vmatrix}.$$

Thus $f_{I^{(\alpha)}} = \varphi_{1,\alpha+1}$. With these notation, we claim that

$$\operatorname{Res}_{c^n}(\mathbf{s}, E; p_0) = \operatorname{Res}_{p_0}\begin{bmatrix} \sigma_n(\Theta) \\ f_{I^{(1)}}, \dots, f_{I^{(n)}} \end{bmatrix}$$

with

$$\sigma_n(\Theta) = \frac{1}{n}\Biggl(\sum_{i=2}^{n+1} \theta_{12} \wedge \cdots \wedge \theta_{1,i-1} \wedge df_{I^{(i-1)}} \wedge \theta_{1,i+1} \wedge \cdots \wedge \theta_{1,n+1}$$

$$+ \sum_{2 \le i < j \le n+1} (-1)^{i+j} \theta_{11} \wedge d\varphi_{ij} \wedge \theta_{12} \wedge \cdots \wedge \widehat{\theta_{1i}} \wedge \cdots \wedge \widehat{\theta_{1j}} \wedge \cdots \wedge \theta_{1,n+1}\Biggr).$$

To show this, we compute and see that the rows of the matrix $\Theta^{(\alpha)}$ are zero except for the first and the $(\alpha+1)$-st, where we have $-(\theta_{\alpha+1,1},\dots,\theta_{\alpha+1,n+1})$ and $(\theta_{11},\dots,\theta_{1,n+1})$, respectively. For $i=1,\dots,n+1$, let A_i denote the n-tuple obtained from $(1,\dots,n+1)$ by removing i. Then we have

$$\sigma_n(\Theta)=\frac{1}{n!}\sum_{i=1}^{n+1}\sum_{\rho\in\mathscr{S}_n}\operatorname{sgn}\rho\cdot\det\Theta_{A_i}(\rho).$$

Denoting by $(\Theta_{A_i}^{(\alpha)})_{ab}$ the (a,b) entry of the matrix $\Theta_{A_i}^{(\alpha)}$, we compute

$$\sum_{\rho\in\mathscr{S}_n}\operatorname{sgn}\rho\cdot\det\Theta_{A_i}(\rho)=\sum_{\sigma,\rho\in\mathscr{S}_n}\operatorname{sgn}\sigma\cdot\operatorname{sgn}\rho\cdot(\Theta_{A_i}^{(1)})_{\sigma(1)\rho(1)}\cdots(\Theta_{A_i}^{(n)})_{\sigma(n)\rho(n)}. \tag{6.1}$$

First we compute the right hand side of (6.1) for $i=1$. The rows of $\Theta_{A_1}^{(\alpha)}$ are zero except for the α-th, where we have $(\theta_{12},\dots,\theta_{1,n+1})$. Hence we have

$$\sum_{\rho\in\mathscr{S}_n}\operatorname{sgn}\rho\cdot\det\Theta_{A_1}(\rho)=n!\cdot\theta_{12}\wedge\cdots\wedge\theta_{1,n+1}. \tag{6.2}$$

Next we compute the right hand side of (6.1) for $i=2,\dots,n+1$. For $\alpha=1,\dots,i-2$, the rows of $\Theta_{A_i}^{(\alpha)}$ are zero except for the first and the $(\alpha+1)$-st, where we have $-(\theta_{\alpha+1,1},\dots,\widehat{\theta_{\alpha+1,i}},\dots,\theta_{\alpha+1,n+1})$ and $(\theta_{11},\dots,\widehat{\theta_{1i}},\dots,\theta_{1,n+1})$, respectively. Here "$\hat{\ }$" means that the symbol under it is to be removed. The rows of the matrix $\Theta_{A_i}^{(i-1)}$ are zero except for the first, where we have $-(\theta_{i1},\dots,\widehat{\theta_{ii}},\dots,\theta_{i,n+1})$. For $\alpha=i,\dots,n$, the rows of $\Theta_{A_i}^{(\alpha)}$ are zero except for the first and the α-th, where we have $-(\theta_{\alpha+1,1},\dots,\widehat{\theta_{\alpha+1,i}},\dots,\theta_{\alpha+1,n+1})$ and $(\theta_{11},\dots,\widehat{\theta_{1i}},\dots,\theta_{1,n+1})$, respectively. Thus the terms in (6.1) are zero except for $\sigma=(1,\dots,i-1)$, the cyclic permutation of order $i-1$, whose signature is $(-1)^i$. Then we compute and see that, for $i=2,\dots,n+1$,

$$\begin{aligned}\sum_{\rho\in\mathscr{S}_n}\operatorname{sgn}\rho\cdot\det\Theta_{A_i}(\rho)&=(-1)^{i-1}(n-1)!\\&\quad\cdot\left(\sum_{j=1}^{i-1}\theta_{11}\wedge\cdots\wedge\theta_{1,j-1}\wedge\theta_{ij}\wedge\theta_{1,j+1}\wedge\cdots\wedge\widehat{\theta_{1i}}\wedge\cdots\wedge\theta_{1,n+1}\right.\\&\qquad\left.+\sum_{j=i+1}^{n+1}\theta_{11}\wedge\cdots\wedge\widehat{\theta_{1i}}\wedge\cdots\wedge\theta_{1,j-1}\wedge\theta_{ij}\wedge\theta_{1,j+1}\wedge\cdots\wedge\theta_{1,n+1}\right).\end{aligned}$$

Finally, using (6.2) and the above, we get the formula.

The formula in (2) or (3) can be used to compute the multiplicity of a holomorphic function on a possibly singular curve of arbitrary codimension or on a possibly singular hypersurface of arbitrary dimension, respectively, see [**IS**].

EXAMPLE 6.3. In the situation of (3), let $n = 2$ and suppose that p_0 is a non-singular point. Let (z_1, z_2) be a coordinate system around p_0. If $s_1 = z_1 e_1 - z_2 e_3$ and $s_2 = z_2 e_1 + z_1 e_2$, we have

$$\mathrm{Res}_{c^2}(\mathbf{s}, E; p_0) = \mathrm{Res}_{p_0}\begin{bmatrix} 3 z_1 z_2 \, dz_1 \wedge dz_2 \\ z_1^2, z_2^2 \end{bmatrix} = \mathrm{Res}_{p_0}\begin{bmatrix} 3 \, dz_1 \wedge dz_2 \\ z_1, z_2 \end{bmatrix} = 3.$$

References

[Bo] R. Bott, Lectures on characteristic classes and foliations, Lectures on Algebraic and Differential Topology, Lecture Notes in Math., **279**, Springer-Verlag, New York, Heidelberg, Berlin, 1972, pp. 1–94.

[BT] R. Bott and L. Tu, Differential Forms in Algebraic Topology, Grad. Texts in Math., **82**, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[Br] J.-P. Brasselet, Définition combinatoire des homomorphismes de Poincaré, Alexander et Thom, pour une pseudo-variété, Caractéristique d'Euler-Poincaré, Astérisque **82**–**83**, Société Mathématique de France, 1981, pp. 71–91.

[BLSS] J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, Milnor classes of local complete intersections, Trans. Amer. Math. Soc., **354** (2002), 1351–1371.

[F] W. Fulton, Intersection Theory, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[GH] P. Griffiths and J. Harris, Principles of Algebraic Geometry, John Wiley, New York, Chichester, Brisbane, Toronto, 1978.

[HL] F. R. Harvey and H. B. Lawson, A theory of characteristic currents associated with a singular connection, Astérisque, **213**, Société Mathématique de France, 1993.

[I] B. Iversen, Critical points of an algebraic function, Invent. Math., **12** (1971), 210–224.

[IS] T. Izawa and T. Suwa, Multiplicity of functions on singular varieties, preprint.

[L1] D. Lehmann, Systèmes d'alvéoles et intégration sur le complexe de Čech-de Rham, Publications de l'IRMA, **23**, N^o VI, Université de Lille I, 1991.

[L2] D. Lehmann, Résidus des sous-variétés invariantes d'un feuilletage singulier, Ann. Inst. Fourier, **41** (1991), 211–258.

[L3] D. Lehmann, A Chern-Weil theory for Milnor classes, to appear in Singularities-Sapporo 1998, Adv. Stud. Pure Math., **29**, Math. Soc. Japan.

[LS] D. Lehmann and T. Suwa, Residues of holomorphic vector fields relative to singular invariant subvarieties, J. Differential Geom., **42** (1995), 165–192.

[St] N. Steenrod, The Topology of Fibre Bundles, Princeton Univ. Press, Princeton, 1951.

[Su1] T. Suwa, Indices of Vector Fields and Residues of Singular Holomorphic Foliations, Actualités Math., Hermann, Paris, 1998.

[Su2] T. Suwa, Dual class of a subvariety, Tokyo J. Math., **23** (2000), 51–68.

Tatsuo SUWA
Department of Mathematics
Hokkaido University
Sapporo 060-0810
Japan
E-mail: suwa@math.sci.hokudai.ac.jp