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Minimal Lagrangian submanifolds in adjoint orbits and upper bounds on the first eigenvalue of the Laplacian

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Abstract. Let G be a compact semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $(\,,\,)$ an Ad_G-invariant inner product on $\mathfrak{g}$, and M an adjoint orbit in $\mathfrak{g}$. In this article, if $(M, (\,,\,)_{|M})$ is Kähler with respect to its canonical complex structure, then we give, for a closed minimal Lagrangian submanifold $L \subset M$, upper bounds on the first positive eigenvalue $\lambda_1(L)$ of the Laplacian Δ_L, which acts on $C^\infty(L)$, and lower bounds on the volume of L. In particular, when $(M, (\,,\,)_{|M})$ is Kähler-Einstein, ($\rho = c\omega$, where ρ and ω are Ricci form and Kähler form of $(M, (\,,\,)_{|M})$ with respect to the canonical complex structure respectively, and c is a positive constant,) we prove $\lambda_1(L) \le c$. Combining with a result of Oh [**5**], we can see that L is Hamiltonian stable if and only if $\lambda_1(L) = c$.

1. Introduction.

Let $\boldsymbol{CP}^n$ be the n-dimensional complex projective space and g be the Fubini-Study metric of $\boldsymbol{CP}^n$ with its holomorphic sectional curvature 1. In [**6**], A. Ros gave upper bound of the first positive eigenvalue of the Laplacian and lower bound of the volume of closed CR-minimal submanifolds of $\boldsymbol{CP}^n$. The technique used in that paper is as follows; let $HM(n+1) = \{A \in \mathfrak{gl}(n+1; \boldsymbol{C}) \mid \bar{A} = {}^tA\}$ and define an inner product $(\,,\,)$ on $HM(n+1)$ as $(A, B) = 2\,\mathrm{trace}(AB)$. Then $(\boldsymbol{CP}^n, g)$ is isometrically embedded in $(HM(n+1), (\,,\,))$, and using estimates for the total mean curvature of a closed Riemannian manifold isometrically embedded in a Euclidean space, proved by B.-Y. Chen in [**2**], [**3**], the desired bounds were obtained. In this article, we will apply this technique to closed minimal Lagrangian submanifolds in adjoint orbits.

Let G be a compact semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $(\,,\,)$ an Ad_G-invariant inner product on $\mathfrak{g}$, and M an adjoint orbit in $\mathfrak{g}$. Suppose that the Lie group G acts on M effectively. (In this paper, when we say "adjoint orbit", we suppose that it satisfies this condition.) Then M has canonical complex structure J, and canonical symplectic form F, which is Kähler with respect to J, see [**1**] or section 3 below. We regard that M is isometrically embedded in $(\mathfrak{g}, (\,,\,))$. The

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2-form associated with $(\, , \,)_{|M}$ and J, which is defined by $\omega(X, Y) := (JX, Y)_{|M}$, is not always Kähler. But, when this associated 2-form is equal to α times the canonical symplectic form for a positive constant α, we get the following bounds.

THEOREM 1.1. *Let G be a compact semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $(\, , \,)$ an Ad_G-invariant inner product on $\mathfrak{g}$, and M^{2m} an adjoint orbit in $\mathfrak{g}$ with the associated 2-form being equal to α times the canonical symplectic form for a positive constant α, i.e. $\omega(X, Y) := (JX, Y)_{|M} = \alpha F(X, Y)$. If $L \subset M$ is a closed minimal Lagrangian submanifold, then*

$$\mathrm{Vol}(L) \geq \left(\frac{2m^2}{s} \right)^{m/2} c_m,$$

where s is the scalar curvature of $(\, , \,)_{|M}$ and c_m is the volume of unit m-sphere.

Moreover, if $(\, , \,)_{|M}$ is Kähler-Einstein with respect to the canonical complex structure J, and its Ricci form equals to $c\omega$ for a positive constant c, we have

$$\mathrm{Vol}(L) \geq \left(\frac{m}{c} \right)^{m/2} c_m.$$

THEOREM 1.2. *Suppose that the situation is the same as Theorem 1.1. Then*

$$\lambda_1(L) \leq \frac{s}{2m},$$

where $\lambda_1(L)$ is the first positive eigenvalue of the Laplacian Δ_L, which acts on $C^\infty(L)$. Let $l : L \to \mathfrak{g}$ denote the embedding. Then the equality holds if and only if there is a constant vector d in $\mathfrak{g}$ such that, $l - d$ is an embedding of order 1, namely, for a fixed basis of $\mathfrak{g}$, all of its coordinate functions $l^j - d^j$ are $\lambda_1(L)$-eigenfunctions. (This property is independent of the choice of the basis.) Also the dimension of the space of $\lambda_1(L)$-eigenfunctions is greater than m.

Moreover, if $(\, , \,)_{|M}$ is Kähler-Einstein with respect to the canonical complex structure J, and its Ricci form equals to $c\omega$ for a positive constant c, we have

$$\lambda_1(L) \leq c,$$

and the constant $d \in \mathfrak{g}$ above is equal to 0.

For minimal Lagrangian submanifolds in a Kähler manifold (X, ω), Y.-G. Oh defined a Hamiltonian stability in [**5**] as follows. Let $\iota : L \hookrightarrow X$ be a Lagrangian embedding and V be a normal variation vector along L. Since L is totally real and $2 \dim L = \dim M$, we can regard $\iota^*(V \rfloor \omega)$ as a 1-form on L. When the 1-form $\iota^*(V \rfloor \omega)$ is exact, V is called a Hamiltonian variation vector. A smooth family $\{\iota_t\}$ of embeddings of L into X is called a Hamiltonian deformation, if its derivative is Hamiltonian. Note that Hamiltonian deformations leave Lagrangian submanifolds Lagrangian. We say that a minimal

Lagrangian submanifold is Hamiltonian stable, if, for any Hamiltonian variation V, the second variation along V of the volume functional is non-negative. When (X, ω) is Kähler-Einstein with a positive scalar curvature and its Ricci form satisfies $\rho = c\omega$, Oh [**5**] proved that a compact minimal Lagrangian submanifold L is Hamiltonian stable if and only if $\lambda_1(L) \geq c$.

So we have the following corollary.

Corollary 1.3. *The situation being as in Theorem* 1.1, *suppose that* $(\,,\,)_{|M}$ *is Kähler-Einstein with respect to the canonical complex structure J, and its Ricci form equals to* $c\omega$ *for a positive constant c. Then the following three conditions are equivalent*;

(1) *L is Hamiltonian stable.*
(2) $\lambda_1(L) = c$.
(3) *All of the coordinate functions l^i are $\lambda_1(L)$-eigenfunctions.*

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2. Estimates on volume and λ_1.

Let an m-dimensional Riemannian manifold (X^m, g) be isometrically embedded in a Euclidean space $(\boldsymbol{R}^k, (\,,\,))$, and $Y^n \subset X$ be a closed minimal submanifold. Then we can obtain the upper bound of the first positive eigenvalue $\lambda_1(Y)$ of the Laplacian Δ_Y, which acts on $C^\infty(Y)$, and the lower bound of the volume of Y as follows.

For the embeddings $X \hookrightarrow \boldsymbol{R}^k$, $Y \hookrightarrow X$ and $Y \hookrightarrow \boldsymbol{R}^k$, we denote their second fundamental forms σ, $\bar{\sigma}$ and $\tilde{\sigma}$ respectively. Then, from the definition of the second fundamental forms, we have

$$\tilde{\sigma}_x(A, B) = \bar{\sigma}_x(A, B) + \sigma_x(A, B) \quad \text{for } x \in Y,\ A, B \in T_x Y. \tag{2.1}$$

We can think of (2.1) as the decomposition of $\tilde{\sigma}$ to the component tangent to X, which is $\bar{\sigma}$, and the one normal to X, which is σ. Since $Y \hookrightarrow X$ is minimal, the mean curvature vector $\tilde{H}$ of embedding $Y \hookrightarrow \boldsymbol{R}^k$ is obtained by

$$\tilde{H}_x = \tilde{H}_x^\perp := \frac{1}{n} \sum_{j=1}^{n} \sigma_x(e_j, e_j), \tag{2.2}$$

where $x \in Y$ and $\{e_j\}_{j=1}^n$ is an orthonormal basis of $T_x Y$.

To get the bounds of $\lambda_1(Y)$ and volume of Y, we use next two theorems by B.-Y. Chen ([**2**] and [**3**]).

Theorem 2.1 (Chen [**2**]). *Let M be an m-dimensional closed submanifold of a Euclidean space* $(\boldsymbol{R}^k, (\,,\,))$, *and H be its mean curvature vector. Then we have*

$$\int_M (H,H)^{m/2}\, dv \geq c_m, \tag{2.3}$$

where c_m is the volume of unit m-sphere. The equality holds if and only if M is embedded as the standard m-sphere in an affine $(m+1)$-space.

Theorem 2.2 (Chen [**3**]). *Let $x : (M^m, g) \to (\boldsymbol{R}^k, (\,,\,))$ be an isometric immersion of a closed m-dimensional Riemannian manifold into a Euclidean space, and H be its mean curvature vector. Then we have*

$$\int_M (H,H)^{m/2}\, dv \geq \left(\frac{\lambda_1(M)}{m}\right)^{m/2} \mathrm{Vol}(M), \tag{2.4}$$

where $\lambda_1(M)$ is the first positive eigenvalue of the Laplacian Δ_M. The equality holds if and only if there is a vector c in $\boldsymbol{R}^k$ such that $x - c$ is an embedding of order 1, *namely, its j-th coordinate function $x^j - c^j$ is the first eigenfunction of Δ_M, for each j.*

We apply these theorems to the case $x : (X^m, g) \hookrightarrow (\boldsymbol{R}^k, (\,,\,))$ is an isometric embedding and $Y^n \subset X$ is a closed minimal submanifold.

Corollary 2.3. *Let (X^m, g) be an m-dimensional Riemannian manifold and $Y^n \subset X$ be a closed n-dimensional minimal submanifold. Suppose that there is an isometric embedding $x : (X, g) \hookrightarrow (\boldsymbol{R}^k, (\,,\,))$ of X into the Euclidean space $(\boldsymbol{R}^k, (\,,\,))$. Then we have*

$$\mathrm{Vol}(Y) \geq \frac{c_n}{\max_{y \in Y} (\tilde{H}_y^{\perp}, \tilde{H}_y^{\perp})^{n/2}}, \tag{2.5}$$

where $\tilde{H}_y^{\perp} := (1/n) \sum_{j=1}^{n} \sigma_y(e_j, e_j)$, σ is the second fundamental form of embedding x, $y \in Y$, and $\{e_j\}_{j=1}^{n}$ is an orthonormal basis of $T_y Y$.

Corollary 2.4. *Notation being as in Corollary* 2.3, *we have*

$$\lambda_1(Y) \leq n \left(\frac{\int_Y (\tilde{H}^{\perp}, \tilde{H}^{\perp})^{n/2}\, dv}{\mathrm{Vol}(Y)}\right)^{2/n}. \tag{2.6}$$

The equality holds if and only if there is a constant vector $c \in \boldsymbol{R}^k$ such that the embedding $x_{|Y} - c : Y \to \boldsymbol{R}^k$ is an embedding of order 1.

In section 4, we will apply these corollaries to the case of the adjoint orbits.

3. The adjoint orbits of compact semisimple Lie groups.

In this section, we review the chapter 8 of [**1**].

Let G be a compact semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $(\,,\,)$ an Ad_G-invariant inner product on $\mathfrak{g}$, and M an adjoint orbit in $\mathfrak{g}$. Suppose that the Lie group G acts on M effectively. In this paper, when we say "adjoint orbit", we assume that it satisfies this condition. For $U \in \mathfrak{g}$, the fundamental vector field attached to U, X_U is defined by

$$X_U(w) = [U, w] \quad (w \in M), \tag{3.1}$$

under the identification $\mathfrak{g} \simeq T_w\mathfrak{g} \supset T_wM$, $(w \in M)$. Since G acts on M transitively, any tangent vector in T_wM is written as the value of a fundamental vector field, and we can identify

$$T_wM \simeq \mathrm{Im}(\mathrm{ad}_w : \mathfrak{g} \to \mathfrak{g}) =: M_w \quad (w \in M).$$

Similarly, we have an identification

$$N_wM \simeq \mathrm{Ker}(\mathrm{ad}_w : \mathfrak{g} \to \mathfrak{g}) =: L_w \quad (w \in M),$$

where N_wM is the normal space of $M \subset \mathfrak{g}$ at $w \in M$.

Next, we will define the canonical complex structure J on M. For $w \in M$, let $G_w := \{g \in G \mid \mathrm{Ad}(g)w = w\}$, S_w the connected center of G_w, and $\mathfrak{s}_w$ the Lie algebra of S_w. Note that $w \in \mathfrak{s}_w$. Then M_w is preserved by Ad_{G_w} and ad_{L_w}. Since the restriction of the adjoint action of G_w on M_w to S_w is completely reducible, we have an Ad_{S_w} invariant orthogonal direct sum decomposition

$$M_w = \sum_{j=1}^{m} E_{w,j} \quad (\dim M = 2m), \tag{3.2}$$

where each $E_{w,j}$ is a real 2-dimensional vector space isomorphic, as a S_w-representation space, to the irreducible representation $\Gamma_{a_j} : S_w \to GL(2, \boldsymbol{R})$ defined by

$$\Gamma_{a_j}(\exp(s)) = \begin{pmatrix} \cos a_j(s) & -\sin a_j(s) \\ \sin a_j(s) & \cos a_j(s) \end{pmatrix} \quad (s \in \mathfrak{s}_w). \tag{3.3}$$

Here $a_j \in \mathfrak{s}_w^*$ is the weight of Γ_{a_j} (via $(\,,\,)_{|\mathfrak{s}_w}$, a_j may be viewed as an element of $\mathfrak{s}_w$). We choose each a_j so that $a_j(w) > 0$. Then $E_{w,j}$ is oriented by the basis for which the action of S_w is represented by Γ_{a_j}. The almost complex structure J on TM is defined as

$$J_wX = \frac{1}{a_j(w)}[w, X] \quad (w \in M, \ X \in E_{w,j}). \tag{3.4}$$

This almost complex structure is integrable and G-invariant, see [**1**]. We call J the canonical complex structure of M.

Finally, we define two G-invariant closed 2-forms, canonical symplectic form and G-invariant Ricci form. Let the 2-form F on M be defined by

$$F_w(X, Y) = (w, [U, V]) \quad (w \in M, \ X, Y \in T_w M), \tag{3.5}$$

where $U, V \in \mathfrak{g}$ such that $X = [U, w]$, $Y = [V, w]$. Then it is proved that F is the G-invariant Kähler form of a G-invariant Kähler struture compatible with the canonical complex structure of M, see [**1**]. We refer to F as the canonical symplectic structure of M.

There is unique, up to multiplication by a positive constant, G-invariant volume form Ω on M, which is the volume form of $(\, , \,)_{|M}$. Since the Ricci form of a Kähler metric depends only on the complex structure and the volume form, the Ricci form of any G-invariant Kähler metric on M relative to the canonical complex structure equals to $C\rho$, where ρ is the G-invariant Ricci form determined by J and Ω, C is a positive constant. This G-invariant Ricci form ρ is computed as

$$\rho_w(X, Y) = (\gamma(w), [U, V]) \quad (w \in M, \ X, Y \in T_w M), \tag{3.6}$$

where $U, V \in \mathfrak{g}$ such that $X = [U, w]$, $Y = [V, w]$, $\mathfrak{s}_w \ni \gamma(w) = \sum_{j=1}^{m} [X_j, JX_j]$, $\{X_j, JX_j\}$ is a positively oriented orthonormal basis of $E_{w,j}$. Note that ρ is positive definte, see [**1**].

4. The Proofs of Theorem 1.1 and 1.2.

In the same setting as in the previous section, we define the 2-form ω by $\omega(X, Y) := (JX, Y)$. In general, this 2-form is positive definite and type $(1,1)$ with respect to J but not closed. We have the following lemma.

Lemma 4.1. *For a positive constant α, $\omega = \alpha F$ if and only if $a_j(w) = a_k(w) = \alpha$ for some $w \in M$ and any j, k.*

Proof. For $w \in M$, $X_j \in E_{w,j}$, $X_k \in E_{w,k}$ with $j \neq k$, we have $[(1/(a_j(w)))J_w X_j, w] = X_j$ and thus

$$\begin{aligned} F_w(X_j, X_k) &= \left(w, \left[\frac{1}{a_j(w)} J_w X_j, \frac{1}{a_k(w)} J_w X_k \right] \right) \\ &= \frac{1}{a_j(w) a_k(w)} ([w, J_w X_j], J_w X_k) \\ &= 0, \end{aligned}$$

where the second equality is derived from the Ad_G invariance of inner product

$(,)$ on $\mathfrak{g}$, and the third one is derived from $[w, J_w X_j] \in E_{w,j}$ and $J_w X_k \in E_{w,k}$. Similarly, we have $[-(1/(a_j(w)))X_j, w] = J_w X_j$ and thus

$$F_w(X_j, J_w X_j) = \left(w, \left[\frac{1}{a_j(w)} J_w X_j, -\frac{1}{a_j(w)} X_j\right]\right)$$
$$= \frac{1}{(a_j(w))^2}([w, X_j], J_w X_j)$$
$$= \frac{1}{a_j(w)} \omega(X_j, JX_j).$$

Since $a_j(w)$ is Ad_G-invariant, Lemma 4.1 follows immediately. □

LEMMA 4.2. *Let $M \subset \mathfrak{g}$ be an adjoint orbit with $\omega = \alpha F$, and σ, H be the second fundamental form and the mean curvature vector of embedding $M \subset \mathfrak{g}$ respectively. Then, for each $w \in M$, we have*

$$\sigma_w(X, Y) = p_w([V, [U, w]]) \quad (X, Y \text{ are vector fields on } M), \tag{4.1}$$

$$\sigma_w(JX, JY) = \sigma_w(X, Y), \tag{4.2}$$

$$H_w = \frac{-1}{m\alpha} \gamma(w), \tag{4.3}$$

$$(H, H) = \frac{s}{2m^2} \quad (s \text{ is the scalar curvature of } (,)_{|M}), \tag{4.4}$$

where $U, V \in \mathfrak{g}$ such that $X(w) = [U, w]$, $Y(w) = [V, w]$, and $p_w : \mathfrak{g} \to L_w$ is the orthogonal projection.

PROOF. For the equation (4.1), since σ is tensor, it is sufficient that we prove (4.1) for fundamental vector fields X_U, X_V. But we easily see that $D_{X_U} X_V(w) = [V, [U, w]]$, where D is the Levi-Civita connection of $(\mathfrak{g}, (,))$. So, by the definition of the second fundamental form, we have proved the equation (4.1).

From the equation (4.1), we have

$$\sigma_w(JX, JY) = \sigma_w(JY, JX)$$
$$= p_w\left[-\frac{X(w)}{\alpha}, JY(w)\right]$$
$$= p_w\left[\frac{JY(w)}{\alpha}, X(w)\right]$$
$$= \sigma_w(X, Y).$$

Let $\{e_j, Je_j\}$ be the orthonormal basis of $E_{w,j} \subset M_w$. Then

$$H_w = \frac{1}{2m}\sum_{j=1}^{m}\{\sigma_w(e_j, e_j) + \sigma_w(Je_j, Je_j)\}$$

$$= \frac{1}{m}\sum_{j=1}^{m}\{\sigma_w(e_j, e_j)\}$$

$$= \frac{1}{m}p_w\left(\sum_{j=1}^{m}\left[\frac{Je_j}{\alpha}, e_j\right]\right)$$

$$= \frac{-1}{m\alpha}\gamma(w).$$

The last equality holds since $\gamma(w) \in \mathfrak{s}_w \subset L_w$.

Finally, from the direct computation, we have

$$\frac{s}{2} = \sum_{j=1}^{m} \rho_w(e_j, Je_j)$$

$$= \sum_{j=1}^{m}\left(\gamma(w), \left[\frac{Je_j}{\alpha}, -\frac{e_j}{\alpha}\right]\right)$$

$$= \frac{1}{\alpha^2}(\gamma(w), \gamma(w)). \qquad \square$$

Corollary 4.3. *Let $x : M \hookrightarrow \mathfrak{g}$ be a closed adjoint orbit with $\omega = \alpha F$. Moreover, suppose that (M, ω) is Kähler-Einstein with respect to the canonical complex structure J and that its Ricci form equals to $c\omega$ for a positive constant c. Then x is the embedding of order* 1.

Proof. We apply Theorem 2.2 to the embedding $x : M \hookrightarrow \mathfrak{g}$. Then we have $\lambda_1(M) \leq 2c$, by Lemma 4.2. On the other hand, since the Lie algebra of Killing vector fields on M is non trivial, by Theorem of Matsushima [**4**] (see also Theorem 11.52 of [**1**]), we have $\lambda_1(M) = 2c$. By (3.5), (3.6) and the assumption $\rho = c\omega = \alpha cF$, we have $\gamma(x) = \alpha cx$. So, by (4.3), we have

$$\Delta_M x = -2mH_x$$

$$= -2m\left(\frac{-1}{m\alpha}\alpha cx\right)$$

$$= 2cx. \qquad \square$$

Let $L \subset M$ be a Lagrangian submanifold. Then $\tilde{H}^{\perp}$ of the embeddings $L \subset M \subset \mathfrak{g}$, the definition of $\tilde{H}^{\perp}$ being in Section 2, is equal to H.

PROPOSITION 4.4. *Let $M \subset \mathfrak{g}$ be an adjoint orbit with $\omega = \alpha F$, and H be the mean curvature vector of embedding $M \subset \mathfrak{g}$. For a Lagrangian submanifold $L \subset M$, we have*

$$\tilde{H}_w^{\perp} = H_w \quad (w \in L).$$

PROOF. Let $\{e_1, \dots, e_m\}$ be an orthnormal basis of $T_w L$. Then $\{e_j, J_w e_j\}_{j=1}^m$ is the orthonormal basis of $T_w M$, since L is totally real. So, by the definition of $\tilde{H}^{\perp}$ and (4.2), we have

$$\begin{aligned}\tilde{H}_w^{\perp} &= \frac{1}{m}\sum_{j=1}^{m} \sigma_w(e_j, e_j) \\ &= \frac{1}{2m}\sum_{j=1}^{m}\{\sigma_w(e_j, e_j) + \sigma_w(Je_j, Je_j)\} \\ &= H_w. \end{aligned}$$

□

PROOF OF THEOREM 1.1. Let $(M^{2m}, (\ ,\ )_{M})$ be an adjoint orbit with $\omega = \alpha F$ and $L^m \subset M$ a closed minimal Lagrangian submanifold. Then we have

$$\begin{aligned}\mathrm{Vol}(L) &\geq \frac{c_m}{(\tilde{H}^{\perp}, \tilde{H}^{\perp})^{m/2}} \\ &= \left(\frac{2m^2}{s}\right)^{m/2} c_m,\end{aligned}$$

by Corollary 2.3, Proposition 4.4 and (4.4). □

PROOF OF THEOREM 1.2. Let $(M^{2m}, (\ ,\ )_{|M})$ be an adjoint orbit with $\omega = \alpha F$ and $L^m \subset M$ a closed minimal Lagrangian submanifold. Then we have

$$\begin{aligned}\lambda_1(L) &\leq m(\tilde{H}^{\perp}, \tilde{H}^{\perp}) \\ &= \frac{s}{2m},\end{aligned}$$

by Corollary 2.4, Proposition 4.4 and (4.4).

Moreover, if $(M, (\ ,\ )_{|M})$ is Kähler-Einstein, we have

$$\begin{aligned}\Delta_L l &= -m\tilde{H}_l \quad (\tilde{H}\text{: the mean curvature vector of } L \subset M) \\ &= -m\tilde{H}_l^{\perp} \quad (\text{by } (2.2)) \\ &= -mH_l \quad (\text{by Proposition 4.4}) \\ &= cl \quad (\text{by Corollary 4.3}).\end{aligned}$$

□

5. Example.

In this section, as an example, we investigate the case $G = SU(n)$, $\mathfrak{g} = \mathfrak{su}(n)$, and $(X, Y) = -\mathrm{trace}\, XY$, $X, Y \in \mathfrak{su}(n)$.

Let $w_0 \in \mathfrak{su}(n)$ be

$$w_0 = \begin{pmatrix} i\lambda I_p & 0 \\ 0 & i\mu I_{n-p} \end{pmatrix} \quad (\lambda, \mu \in \boldsymbol{R},\ \ \lambda - \mu > 0,\ \ p\lambda + (n-p)\mu = 0),$$

where $I_p \in \mathfrak{gl}(p, \boldsymbol{R})$, $I_{n-p} \in \mathfrak{gl}(n-p, \boldsymbol{R})$ are the identity matrixes. We consider the orbit $M \subset \mathfrak{su}(n)$ of w_0.

The orbit M is identified with the Grassmann manifold $\mathrm{Gr}_{n,p}(\boldsymbol{C})$ by

$$x \mapsto i(A_{jk})_{j,k=1}^n \in \mathfrak{su}(n) \quad (x \in \mathrm{Gr}_{n,p}(\boldsymbol{C})), \tag{5.1}$$

where, if x is represented by a complex p-dimensional subspace in $\boldsymbol{C}^n$ spanned by orthonormal vectors $(a_{1j}, \dots, a_{nj}) \in \boldsymbol{C}^n$ $(j = 1, \dots, p)$, A_{jk} is defined as

$$A_{jj} = (\lambda - \mu)(|a_{j1}|^2 + \cdots + |a_{jp}|^2) + \mu, \tag{5.2}$$

$$A_{jk} = (\lambda - \mu)(a_{j1}\bar{a}_{k1} + \cdots + a_{jp}\bar{a}_{kp}). \tag{5.3}$$

In this example, geometrical objects at w_0 (tangent space, canonical complex structure, and so on) are

$$M_{w_0} = \left\{ \begin{pmatrix} 0 & A \\ -{}^t\bar{A} & 0 \end{pmatrix} \in \mathfrak{su}(n) \right\}$$

$$\simeq \{X \in \mathfrak{su}(n) \mid \mathrm{ad}_{w_0} X = (\lambda - \mu) J_0 X\},$$

where

$$J_0 = i \begin{pmatrix} I_p & 0 \\ 0 & -I_{n-p} \end{pmatrix},$$

the canonical complex structure at w_0 is the left multiplication by J_0,

$$\mathfrak{s}_{w_0} = \mathrm{span}_{\boldsymbol{R}}\langle w_0 \rangle,$$

and

$$\omega = (\lambda - \mu) F.$$

Since $\dim \mathfrak{s}_{w_0} = 1$ and $\omega = (\lambda - \mu)F$, $(M, (\ ,\ )_{|M})$ is Kähler-Einstein. So, by Corollary 4.3,

$$x \mapsto iA_{jk} \in \mathfrak{su}(n) \quad (x \in \mathrm{Gr}_{n,p}(\boldsymbol{C}))$$

is the embedding of order 1.

LEMMA 5.1. *If the Ricci form of* $(\, , \,)_{|M}$ *equals to* $c\omega$, *then*

$$c = \frac{n}{(\lambda - \mu)^2}.$$

PROOF. Let $X = \begin{pmatrix} 0 & A \\ -{}^t\bar{A} & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & B \\ -{}^t\bar{B} & 0 \end{pmatrix}$. Then

$$\begin{aligned}\omega_{w_0}(X, Y) &= -\operatorname{trace} J_0 XY \\ &= i(\operatorname{trace} A\,{}^t\bar{B} - \operatorname{trace} {}^t\bar{A}B).\end{aligned}$$

On the other hand, it is easily seen that

$$\gamma(w_0) = i\begin{pmatrix} (n-p)I_p & 0 \\ 0 & -pI_{n-p} \end{pmatrix}.$$

So we have

$$\begin{aligned}\rho_{w_0}(X, Y) &= -\frac{1}{(\lambda - \mu)^2}\operatorname{trace}(\gamma(w_0)[J_0 X, J_0 Y]) \\ &= \frac{ni}{(\lambda - \mu)^2}(\operatorname{trace} A\,{}^t\bar{B} - \operatorname{trace} {}^t\bar{A}B).\end{aligned}$$

□

For example, by Lemma 5.1, when we regard the Grassmann manifold $\mathrm{Gr}_{n,p}(\boldsymbol{C}) \simeq M$ as the Hermitian symmetric space $SU(n)/S(U(p) \times U(n-p))$, with the metric induced from the Killing form of $\mathfrak{su}(n)$, we have

$$-2n \operatorname{trace} UV = (X_U, X_V) = -(\lambda - \mu)^2 \operatorname{trace} UV,$$

where $U, V \in T_{[I_n]}SU(n)/S(U(p) \times U(n-p)) \subset \mathfrak{su}(n)$, so $c = 1/2$.

In [**5**], Oh gave some examples of Hamiltonian stable closed minimal Lagrangian submanifolds in Hermitian symmetric spaces.

Let $\sigma : \mathrm{Gr}_{n,p}(\boldsymbol{C}) \to \mathrm{Gr}_{n,p}(\boldsymbol{C})$ be an involutive anti-holomorphic isometry defined as $x \mapsto \bar{x}$, where, for $x \in \mathrm{Gr}_{n,p}(\boldsymbol{C})$ which is represented by a p-dimensional subspace in $\boldsymbol{C}^n$ spanned by orthonormal vectors $(a_{1j}, \dots, a_{nj}) \in \boldsymbol{C}^n$ $(j = 1, \dots, p)$, $\bar{x}$ is represented by the subspace spanned by $\{(\bar{a}_{1j}, \dots, \bar{a}_{nj})\}_{j=1}^p$. Then the fixed point set of σ

$$L = \{x \in \mathrm{Gr}_{n,p}(\boldsymbol{C}) \mid x = \sigma(x)\}$$

is a totally geodesic Lagrangian submanifold, by Proposition 6.1 of [**5**] or Lemma 1.1 of [**7**]. Moreover, in [**7**], it was proved that $\lambda_1(L) = 1/2$, when we regard the Grassmann manifold $\mathrm{Gr}_{n,p}(\boldsymbol{C}) \simeq M$ as the Hermitian symmetric space $SU(n)/S(U(p) \times U(n-p))$ with the metric induced from the Killing form of

$\mathfrak{su}(n)$. So L is the Hamiltonian stable totally geodesic Lagrangian submanifold in M. The element $x \in L$ is represented by a p-dimensional subspace in $\boldsymbol{C}^n$ spanned by orthogonal vectors $(a_{1j}, \dots, a_{nj}) \in \boldsymbol{R}^n \subset \boldsymbol{C}^n$ $(j = a, \dots, p)$. Applying Corollary 1.3 to L, we see that

$$L \ni x \mapsto a_{j1}a_{k1} + \cdots + a_{jp}a_{kp}$$

are the eigenfunctions of $\lambda_1(L)$.

Another example is the Clifford torus $\tilde{L}$ embedded in $\boldsymbol{CP}^n$ defined as

$$\tilde{L} = \{[z_0 : \cdots : z_n] \in \boldsymbol{CP}^n \mid |z_0| = \cdots = |z_n|\}.$$

In particular, if the representative $z = (z_0, \dots, z_n) \in \boldsymbol{C}^{n+1}$ satisfies $|z| = 1$, then the norm of each component z_j is $1/\sqrt{n+1}$. The Clifford torus $\tilde{L}$ is a minimal Lagrangian submanifold in $\boldsymbol{CP}^n$, [**5**]. Moreover, by computing $\lambda_1(\tilde{L})$, Oh proved that, in [**5**], $\tilde{L}$ is Hamiltonian stable. So, by Corollary 1.3,

$$\tilde{L} \ni [z_0 : \cdots : z_n] \mapsto \operatorname{Re} z_j \bar{z}_k \quad (j \neq k)$$

and

$$\tilde{L} \ni [z_0 : \cdots : z_n] \mapsto \operatorname{Im} z_j \bar{z}_k \quad (j \neq k)$$

are the eigenfunctions of $\lambda_1(\tilde{L})$, where $|z_j|^2 = 1/(n+1)$ for $j = 0, \dots, n$.

References

[1] A. L. Besse, Einstein manifolds, Ergeb. Math. Grenzgeb., **3**, Springer-Verlag, 1987.

[2] B.-Y. Chen, On the total curvature of immersed manifolds, I, Amer. J. Math., **93** (1971), pp. 148–162.

[3] B.-Y. Chen, Geometry of submanifolds and its applications, Science University of Tokyo, 1981.

[4] Y. Matsushima, Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne, Nagoya Math J., **11** (1957), pp. 145–150.

[5] Y.-G. Oh, Second variation and stabilities of minimal lagrangian submanifolds in Kähler manifolds, Invent. Math., **101** (1990), pp. 501–519.

[6] A. Ros, Spectral geometry of CR-minimal submanifolds in the complex projective space, Kodai Math. J., **6** (1983), pp. 88–99.

[7] M. Takeuchi, Stability of certain minimal submanifolds of compact Hermitian symmetric spaces, Tohoku Math. J., **36** (1984), pp. 293–314.

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