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Characterization of a differentiable point of the distance function to the cut locus

Dedicated to Professor Takashi Sakai on the occasion of his sixtieth birthday

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Abstract. We give a necessary and sufficient condition for a given point on the unit normal bundle of a closed submanifold N of a 2-dimensional complete Riemannian manifold M to be a differentiable point of the distance function to the cut locus of N.

Let N be a closed submanifold of a complete Riemannian manifold M and $\pi : \mathrm{U}\nu \to N$ denote the unit sphere normal bundle over N. A unit speed geodesic segment $\gamma : [0, a] \to M$ emanating from N is called an *N-segment* if $t = d(N, \gamma(t))$ on $[0, a]$, where $d(N, \cdot)$ denotes the Riemannian distance function from N. In [**8**], two functions ρ and λ_1 on $\mathrm{U}\nu$ are defined by

$$\rho(v) := \sup\{t > 0; \gamma_v|_{[0,t]} \ \textit{is an N-segment}\},$$

which is called the *distance function to the cut locus* of N and

$$\lambda_1(v) := \sup\{t > 0; \gamma_v|_{[0,t]} \ \textit{has no focal point of N}\},$$

where γ_v is the geodesic in M with $\dot{\gamma}_v(0) = v$. The *cut locus* C_N of N is defined by

$$C_N := \{\exp(\rho(v)v); v \in \mathrm{U}\nu, \rho(v) < \infty\},$$

where exp denotes the exponential map on the tangent bundle over M. Each point of the cut locus is called a *cut point* of N. Note that $\gamma_v(\lambda_1(v))$ is the first focal point of N (cf. [**1**] or [**10**]) along γ_v, when $\lambda_1(v)$ is finite. Some properties of these functions were investigated in the paper [**8**]. For example, it was proved that the function ρ on $\mathrm{U}\nu$ is locally Lipschitz where ρ is finite. Therefore, from

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Rademacher's theorem (cf. [**2**] and [**9**]) it follows that the function $\min(\rho, r)$ is differentiable almost everywhere for each $r > 0$, but this theorem does not tell us whether a given point is a differentiable one of ρ or not. It is well-known that ρ is differentiable at v_0 if $\exp(\rho(v_0)v_0)$ is a *normal* cut point, i.e., a cut point q of N is called *normal* if there exist exactly two N-segments through q, which is not a focal point along either of these two N-segments. In this article, we give a necessary and sufficient condition for a given point of $\mathrm{U}\nu$ to be a differentiable point of ρ in the case where the manifold M is 2-dimensional.

MAIN THEOREM. *Let N be a closed smooth (C^∞) submanifold of a complete 2-dimensional smooth Riemannian manifold M and $\mathrm{U}\nu$ the unit sphere normal bundle over N. A point $v \in \mathrm{U}\nu$ with $\rho(v) < \infty$ is a differentiable one of the distance function ρ to the cut locus of N if and only if $\gamma_v(\rho(v))$ is a focal point of N along γ_v or there exist at most two N-segments through $\gamma_v(\rho(v))$.*

REMARK. Under the same assumption in the Main Theorem, the set of all normal cut points is open and dense in each component of C_N, unless the component consists of a single point. This Main Theorem was motivated by Kokkendorff's conjecture ([**13**]), which was in turn a result of experimentation with the software tool "Loki".

We refer some basic tools in Riemannian geometry to [**1**] or [**10**]. From now on let (M, g) denote a complete 2-dimensional smooth Riemannian manifold with Riemannian metric g. We need the detailed structure of the cut locus of N (cf. [**4**], [**5**], [**6**], [**7**], [**11**], [**8**] and [**12**]) to prove our Main Theorem. Notice that we may assume that each connected component of N is 1-dimensional, because if N contains an isolated point q, then the point q and the distance function ρ to the cut locus can be replaced by the distance circle $\{\exp(\varepsilon v) \mid v \in \mathrm{U}\nu, \pi(v) = q\}$ and ρ_ε, where $\rho_\varepsilon(\dot\gamma_w(\varepsilon)) := \rho(w) - \varepsilon$ for each $w \in \mathrm{U}\nu \cap \pi^{-1}(q)$, respectively by taking a sufficiently small positive ε. Therefore we prove the Main Theorem by assuming that *each connected component of N is 1-dimensional*.

From the Gauss-Bonnet theorem and the Rauch comparison theorem, we get

LEMMA 1. *Let $\triangle(p\ q\ r)$ be a geodesic triangle in an open ball $B(p, \delta_0)$ centered at a point p with radius δ_0. If the Gaussian curvature G of M satisfies $-a^2 \le G \le a^2$ on the open ball $B(p, \delta_0)$ for some positive number a and if δ_0 is less than the convexity radius at p, then*

$$(2 - \cosh 2a\delta_0)\angle q \le \pi - \angle p$$

holds, where $\angle p$ and $\angle q$ denote the inner angle of the triangle at the vertices p and q respectively.

The following four lemmas on the cut loci are fundamental.

Lemma 2. *The inequality $\lambda_1 \geq \rho$ holds on* Uv, *and λ_1 is smooth where λ_1 is finite. Furthermore, if $\lambda_1(v_0) = \rho(v_0) < \infty$, then the differential $d\lambda_1$ of λ_1 is zero at v_0.*

For convenience we introduce a smooth Riemannian metric on Uv. The following two lemmas follow from Lemmas 2.4 and 2.5 in [**8**] respectively.

Lemma 3. *Let $w(t)$ be a unit speed smooth curve in* Uv *with $\rho(w(0)) < \infty$. Then there exist positive constants δ and C_1 such that*

$$C_1|t - s| \leq \angle(\dot{\gamma}_{w(t)}(\rho(w(t))), \dot{\gamma}_{w(s)}(\rho(w(s))))$$

holds for any $s, t \in [-\delta, \delta]$ with $\gamma_{w(t)}(\rho(w(t))) = \gamma_{w(s)}(\rho(w(s)))$. Here

$$\angle(\dot{\gamma}_{w(t)}(\rho(w(t))), \dot{\gamma}_{w(s)}(\rho(w(s))))$$

denotes the angle made by the two tangent vectors $\dot{\gamma}_{w(t)}(\rho(w(t)))$ and $\dot{\gamma}_{w(s)}(\rho(w(s)))$.

Lemma 4. *Let $w(t)$ be a unit speed smooth curve in* Uv *with $\rho(w(0)) = \lambda_1(w(0)) < \infty$. Then for any $\varepsilon > 0$ there exists a positive number δ such that for any $t \in [-\delta, \delta]$, $\gamma_{w(0)}[\rho(w(0)) - \varepsilon, \rho(w(0)) + \varepsilon]$ and $\gamma_{w(t)}[\rho(w(t)) - \varepsilon, \rho(w(t)) + \varepsilon]$ have a common point.*

Let p be a cut point of N and δ a positive number less than the injectivity radius at p. Each component of $B(p, \delta) \setminus \bigcup_{\gamma \in \Gamma_p} \gamma[d(N, p) - \delta, d(N, p)]$, where Γ_p denotes the set of all N-segments through p, is called a *sector* at p. It was proved in [**5**] (cf. also [**12**]) that for any cut point p of N and any neighborhood U around p, there exists a neighborhood $V \subset U$ around p such that for any $x, y \in V \cap C_N$, x and y can be joined by a unique rectifiable *Jordan arc*, i.e., an arc homeomorphic to a closed interval, in $V \cap C_N$. This property was proved by making use of a sector. The following lemma is proved in [**12**].

Lemma 5. *Let Σ be a sector at a cut point p of N and $m : [0, 1] \to \{p\} \cup (C_N \cap \Sigma)$ a Jordan arc issuing from $p = m(0)$. Then the curve m bisects the sector Σ at p. Furthermore, let $\{\alpha_n : [0, l_n] \to C_N\}$ denote an infinite sequence of arcs in $C_N \cap \Sigma$ with $\alpha_n(0) \notin m[0, 1]$ such that each α_n is the unit speed minimal arc in C_N from $\alpha_n(0)$ to $m[0, 1]$ and $\lim_{n \to \infty} \alpha_n(0) = p$. Then there exists a sequence $\{\Sigma_n\}$ of sectors Σ_n at the cut point $q_n := \alpha_n(l_n) \in m[0, 1]$, which is the nearest point on $m[0, 1]$ from $\alpha_n(0)$, satisfying the following four properties.*

1. *$q_n \neq p$ for any n and $\lim_{n \to \infty} q_n = p$.*
2. *$\alpha_n(0) \in \Sigma_n$ for any sufficiently large n.*
3. *The sequence of the inner angles of the sectors Σ_n at q_n converges to zero.*
4. *The two N-segments γ_{v_n} and $\gamma_{\tilde{v}_n}$, which determine Σ_n, bound a disk domain $D(\Sigma_n)$ together with the subarc of N cut off by these two N-segments, if n is sufficiently large.*

Let $v(t)$ be a unit speed smooth curve on $\mathrm{U}\nu$ with $\lambda_1(v(0)) = \rho(v(0)) < \infty$. For simplicity, we put

$$\rho(t) := \rho(v(t)), \quad \lambda(t) := \lambda_1(v(t)), \quad p := \exp(\rho(0)v(0)).$$

By Lemma 2, we have

$$(1) \qquad \liminf_{t\to+0} \frac{\rho(t)-\rho(0)}{t} \le \limsup_{t\to+0} \frac{\rho(t)-\rho(0)}{t} \le d\lambda_1(\dot v(0)) = 0$$

and

$$(2) \qquad 0 = d\lambda_1(\dot v(0)) \le \liminf_{t\to-0} \frac{\rho(t)-\rho(0)}{t} \le \limsup_{t\to-0} \frac{\rho(t)-\rho(0)}{t}.$$

We assume that there exists a monotone decreasing sequence $\{t_n\}$ of positive numbers convergent to zero such that

$$\lim_{n\to\infty} \frac{\rho(0)-\rho(t_n)}{t_n} =: k$$

is positive and each $p_n := \exp(\rho(t_n)v(t_n))$ is a normal cut point. Thus $\lambda(t_n) > \rho(t_n)$ for each n. Let δ be a positive number less than the convexity radius at p. Without loss of generality, we may assume that the Gaussian curvature G of M satisfies $|G| \le 1$ on $B(p,\delta)$. Choose a positive number $\delta_0 < \delta$ with $\cosh 2\delta_0 < 2$. For each $q \in B(p,\delta_0)\backslash\{p\}$, let $\theta(q)$ denote the angle made by $-\dot\gamma_{v(0)}(\rho(0))$ and $\exp^{-1}(q)$, where $\exp^{-1}$ denotes the local inverse mapping of $\exp_p$ on $B(p,\delta_0)$. Let γ_n denote the unit speed minimal geodesic joining $p = \gamma_n(0)$ to p_n.

Lemma 6. *There exists a positive constant C_7 such that $\theta(p_n) \le C_7 t_n$ for any n.*

Proof. Since

$$\lim_{n\to\infty} \frac{\rho(0)-\rho(t_n)}{t_n}$$

is positive, for any sufficiently large n there exists a unique point r_n on the geodesic segment $\gamma_{v(0)}|_{(0,\rho(0))}$ which is the nearest point on the segment from p_n. Fix any sufficiently large n, so that r_n is defined and $\lambda(\tau) < \infty$ on the interval $[0,t_n]$. Then, by definition,

$$(3) \qquad d(p_n,r_n) \le \int_0^{t_n} \|Y_N(\rho(t_n);v(\tau))\|\,d\tau,$$

where $Y_N(t;v(\tau))$ is the N-Jacobi field along the geodesic $\gamma_{v(\tau)}$ defined by

(4) $$Y_N(t; v(\tau)) := \frac{\partial}{\partial \tau} \exp(tv(\tau)),$$

and

$$\| Y_N(\rho(t_n); v(\tau)) \| := \sqrt{g(Y_N(\rho(t_n); v(\tau)), Y_N(\rho(t_n); v(\tau)))}.$$

Since $Y_N(\lambda(\tau); v(\tau)) = 0$, there exists a positive constant C_3, which is independent of n, such that

(5) $$\| Y_N(\rho(t_n); v(\tau)) \| \leq C_3 |\rho(t_n) - \lambda(\tau)|.$$

Since ρ is locally Lipschitz and $\lambda'(0) = 0$, there exists a positive constant C_4, which is independent of n, such that

$$|\rho(t_n) - \rho(0)| \leq C_4 t_n, \quad |\lambda(0) - \lambda(\tau)| \leq C_4 \tau^2.$$

Thus by the triangle inequality, we get

(6) $$|\rho(t_n) - \lambda(\tau)| \leq C_4(t_n + \tau^2).$$

Combining (3), (5) and (6), we obtain

(7) $$d(p_n, r_n) \leq C_3 C_4 t_n^2 \left(1 + \frac{t_n}{3}\right) < C_3 C_4 t_n^2 (1 + t_n).$$

Without loss of generality, we may assume that the two points r_n and p_n are in the ball $B(p, \delta_0)$ and

(8) $$\frac{k}{2} \leq \frac{\rho(0) - \rho(t_n)}{t_n}.$$

Hence we get the geodesic triangle $\triangle(p\ r_n\ p_n)$ all whose edges are in $B(p, \delta_0)$. From the Rauch comparison theorem and the Toponogov comparison theorem (e.g., cf. Theorems 2.5 and 4.2 in [**10**]), there exists a geodesic triangle $\triangle(\bar{p}\ \bar{r}_n\ \bar{p}_n)$ in the 2-dimensional sphere $S^2(1)$ of constant Gaussian curvature 1 with same side lengths such that $\theta_n := \theta(p_n)$ is not greater than the inner angle $\bar{\theta}_n$ of the triangle $\triangle(\bar{p}\ \bar{r}_n\ \bar{p}_n)$ at the vertex $\bar{p}$. From the law of sines (or equivalently Clairaut's relation), we have

(9) $$\sin \bar{\theta}_n \sin d(p, p_n) \leq \sin d(p_n, r_n).$$

By the equations (7) and (8), we may assume that $\bar{\theta}_n$ is less than $\pi/2$. Since

$$\sin x \leq x \leq \frac{\pi}{2} \sin x$$

on the interval $[0, \pi/2]$, we get by (9)

(10) $$\theta_n \le \bar{\theta}_n \le \frac{\pi^2}{4}\frac{d(p_n, r_n)}{d(p, p_n)}.$$

On the other hand, from the triangle inequality,

(11) $$|\rho(0) - \rho(t_n)| \le d(p, p_n).$$

By the equations (7), (8), (10) and (11), we have

$$\theta_n \le \frac{\pi^2 C_3 C_4}{2k}(1 + t_n)t_n.$$

Hence the proof is complete. □

Lemma 7. *There exists a positive constant* C_8 *such that*

$$x_n - y_n \le C_8 \theta(p_n)$$

for any n. *Here* x_n, y_n *denote the maximum and the minimum of* $\{t > 0; \exp(\rho(t)v(t)) = p_n\}$ *respectively.*

Proof. At first, suppose that there exists a sector Σ at p whose boundary contains a subarc of $\gamma_{v(0)}$. Choose a cut point p_{n_1} from p_n's in such a way that the minimal arc $m : [0, 1] \to C_N$ joining p to p_{n_1} lies in Σ. From Lemma 5, the curve m bisects the sector at p containing itself. On the other hand, $\lim_{n\to\infty} \theta(p_n) = 0$ by Lemma 6. Thus, p_n does not lie on the curve m for any sufficiently large n. Choose any sufficiently large n satisfying $p_n \in \Sigma \cap B(p, \delta_0)\backslash m[0, 1]$ and fix it. Let $\alpha_n : [0, l_n] \to C_N$ be a unit speed minimal arc in C_N joining $p_n = \alpha_n(0)$ to $m[0, 1]$. For each $t \in (0, l_n]$, let $\Sigma_-(\alpha_n(t))$ denote the sector at $\alpha_n(t)$ such that

$$\Sigma_-(\alpha_n(t)) \supset \alpha_n(t - \delta, t)$$

for a small $\delta > 0$. Note that $\Sigma_n := \Sigma_-(\alpha_n(l_n))$ forms a sequence of sectors satisfying the four properties in Lemma 5. Since p_n is a normal cut point, we may define the sector $\Sigma_-(\alpha_n(0))$ at $\alpha_n(0)$ if we extend α_n to $(-\delta, 0]$ for some $\delta > 0$. Furthermore we may assume the sector Σ_n satisfies the property 4 in Lemma 5. Let $0 \le t_1 \le t_2 \le l_n$, and let $u_1 < \tilde{u}_1$ (respectively $u_2 < \tilde{u}_2$) denote the parameter values of $v(t)$ such that $\gamma_{v(u_1)}, \gamma_{v(\tilde{u}_1)}$ (respectively $\gamma_{v(u_2)}, \gamma_{v(\tilde{u}_2)}$) are the N-segments determining the sector $\Sigma_-(\alpha_n(t_1))$ (respectively $\Sigma_-(\alpha_n(t_2))$). Since the disc domain $D(\Sigma_n) = D(\Sigma_-(\alpha_n(l_n)))$ contains both sectors $\Sigma_-(\alpha_n(t_1))$ and $\Sigma_-(\alpha_n(t_2))$, $D(\Sigma_-(\alpha_n(t_1)))$ is a subset of $D(\Sigma_-(\alpha_n(t_2)))$. Here $D(\Sigma)$ denotes the disc domain bounded by the two N-segments determining the sector Σ together with the subarc of N cut off by these two N-segments. In particular, $\pi \circ v[u_1, \tilde{u}_1]$ is a subarc of $\pi \circ v[u_2, \tilde{u}_2]$. Thus, from Lemma 3,

(12) $$\tilde{u}_1 - u_1 \le C_1^{-1}\xi(\Sigma_-(\alpha_n(t_2)))$$

for any $0 \le t_1 \le t_2 \le l_n$. Here $\xi(\Sigma_-(\alpha_n(t)))$ denotes the inner angle of $\Sigma_-(\alpha_n(t))$ at $\alpha_n(t)$. Let b_n be the maximal number of $\{l_n \ge t \ge 0; \theta(\alpha_n(t)) = \theta_n\}$, where $\theta_n := \theta(p_n)$. Since the set of all normal cut points is open and dense in C_N, we may assume $\alpha_n(b_n)$ is a normal cut point of N. Hence $(\theta \circ \alpha_n)'(b_n)$ is nonnegative, since $\theta(\alpha_n(l_n)) > \theta_n$ if n is sufficiently large. Since α_n bisects the sector $\Sigma_-(\alpha_n(t))$ at $\alpha_n(t)$ for each t, we get

$$\frac{1}{2}\xi(\Sigma_-(\alpha_n(b_n))) \le \angle(\dot{\gamma}_n(d(p,\alpha_n(b_n))), -\dot{\gamma}_{v(u_n)}(\rho(u_n))), \tag{13}$$

where $u_n := \min\{t > 0; \exp(\rho(t)v(t)) = \alpha_n(b_n)\}$. Since $(\theta \circ \alpha_n)'(b_n) \ge 0$ and $\xi(\Sigma_-(\alpha_n(b_n)))$ is small, we may assume $(\theta \circ \gamma_{v(u_n)})'(\rho(u_n)) > 0$. Thus from Lemma 4, we get a geodesic triangle $\triangle(p\ \alpha_n(b_n)\ \gamma_{v(0)}(\rho(0)+\varepsilon_n))$, where $\varepsilon_n > 0$, in the convex ball $B(p,\delta_0)$. Therefore, from Lemma 1, we get

$$\angle(\dot{\gamma}_n(d(p,\alpha_n(b_n))), -\dot{\gamma}_{v(u_n)}(\rho(u_n))) \le C_6\theta_n, \tag{14}$$

where $C_6 := (2 - \cosh 2\delta_0)^{-1}$. Therefore by (12), (13) and (14), we obtain

$$y_n - x_n \le C_1^{-1}\xi(\Sigma_-(\alpha_n(b_n))) \le 2C_1^{-1}C_6\theta_n, \tag{15}$$

if there exists a sector Σ at p whose boundary contains a subarc of $\gamma_{v(0)}$. Suppose that there is no sector at p whose boundary contains a subarc of $\gamma_{v(0)}$. This case actually occurs (e.g. see the example constructed by Gluck and Singer in [**3**]). For each n let Σ_n be the sector at p containing p_n. By (7), the sequence $\{\Sigma_n\}$ shrinks to a subarc of $\gamma_{v(0)}$ as n goes to infinity. Thus for any sufficiently large n, the two N-segments $\gamma_{v(u_n)}, \gamma_{v(\tilde{u}_n)}$ determining Σ_n, bound a disk domain together with $\pi \circ v[u_n, \tilde{u}_n]$. Choose any such n and fix it. Let $\beta_n : [0, l_n] \to C_N$ denote the unit speed minimal arc joining $p_n = \beta_n(0)$ to p. Let $\Sigma_-(p_n)$ denote the sector at p_n disjoint from $\beta_n(0, l_n]$. Since $D(\Sigma_n)$ contains $D(\Sigma_-(p_n))$, we get $y_n - x_n \le C_1^{-1}\xi(\Sigma_n)$ by Lemma 3. Here $\xi(\Sigma_n)$ denotes the inner angle of Σ_n at p. Thus we may assume that $\theta_n < (1/2)\xi(\Sigma_n)$, otherwise we get $y_n - x_n \le 2C_1^{-1}\theta_n$. By Lemma 5, $\theta(\beta_n(t)) > (1/2)\xi(\Sigma_n)$ for any $t < l_n$ sufficiently close to l_n. Therefore there exists a maximum b_n in $\{l_n \ge t \ge 0; \theta(\beta_n(t)) = \theta_n\}$. By the similar argument to the first case, we have the equation (15). This completes the proof. □

Theorem 8. *Let N be a closed smooth submanifold of a complete 2-dimensional smooth Riemannian manifold M. For any unit speed smooth curve $w(t)$ on* Uv,

$$\lim_{t\to 0}\frac{\rho \circ w(t) - \rho \circ w(0)}{t} = 0,$$

if $\rho(w(0)) = \lambda_1(w(0)) < \infty$.

Proof. Suppose that

$$\liminf_{t\to+0}\frac{\rho\circ v(t)-\rho\circ v(0)}{t}\neq 0$$

for some unit speed smooth curve $v(t)$ on $\mathrm{U}\nu$ with $\rho(v(0))=\lambda_1(v(0))<\infty$. Thus, by the equation (1), there exists a monotone decreasing sequence $\{t_n\}$ of positive numbers convergent to zero such that

$$\lim_{n\to\infty}\frac{\rho(v(0))-\rho(v(t_n))}{t_n}$$

is positive. For simplicity, we put $\rho(t):=\rho(v(t))$, $\lambda(t):=\lambda_1(v(t))$. Since ρ is locally Lipschitz, we may assume $p_n:=\exp(\rho(t_n)v(t_n))$ is a normal cut point. If x_n and y_n denote the maximum and minimum of the set $\{t>0;\ \exp(\rho(v(t))v(t))=p_n\}$ respectively, $\gamma_{v(x_n)}$ and $\gamma_{v(y_n)}$ bound a disk domain D_n together with the subarc $\pi\circ v|_{[y_n,x_n]}$ of N for any sufficiently large n. Since $C_N\cap D_n$ is a tree for any sufficiently large n, $C_N\cap D_n$ has an endpoint $q_n:=\exp(\rho(s_n)v(s_n))$, $s_n\in(y_n,x_n)$, which is a focal point of N along any N-segment through q_n. Furthermore, for any sufficiently large n, $\rho(s_n)<\rho(t_n)$. In fact, let $c_n:[0,1]\to C_N$ denote the minimal arc joining $q_n=c_n(0)$ to p_n and $\Sigma_-(c_n(t))$ the sector at $c_n(t)$ such that

$$\Sigma_-(c_n(t))\supset c_n(t,t-\delta)$$

for a small $\delta>0$. Choose any sufficiently large n, so that the inner angle at $c_n(t)$ of the sector $\Sigma_-(c_n(t))$ is less than $\pi/2$. Thus, from the first variational formula, $d(N,c_n(t))$ is monotone increasing. This implies $\rho(s_n)=d(N,q_n)<\rho(t_n)=d(N,p_n)$. Therefore, from Lemmas 6 and 7, it follows that

$$\frac{\rho(0)-\rho(t_n)}{t_n}\leq(C_7C_8+1)\frac{\lambda(0)-\lambda(s_n)}{s_n}. \tag{16}$$

By Lemma 2 and the equation (16), we get

$$\lim_{n\to\infty}\frac{\rho(0)-\rho(t_n)}{t_n}\leq 0,$$

which is a contradiction. Hence

$$\liminf_{t\to+0}\frac{\rho(v(t))-\rho(v(0))}{t}=0$$

for any unit speed smooth curve $v(t)$ on $\mathrm{U}\nu$ with $\rho(v(0))=\lambda_1(v(0))<\infty$. If $w(t)$ denotes a smooth unit speed curve in $\mathrm{U}\nu$ with $\lambda_1(w(0))=\rho(w(0))<\infty$, then we have

$$\liminf_{t\to+0}\frac{\rho\circ w(t)-\rho\circ w(0)}{t}=\liminf_{t\to+0}\frac{\rho\circ \overline{w}(t)-\rho\circ \overline{w}(0)}{t}=0,$$

where $\overline{w}(t)=w(-t)$. Since

$$0=\liminf_{t\to+0}\frac{\rho\circ \overline{w}(t)-\rho\circ \overline{w}(0)}{t}=-\limsup_{t\to-0}\frac{\rho\circ w(t)-\rho\circ w(0)}{t},$$

we get

$$\liminf_{t\to+0}\frac{\rho\circ w(t)-\rho\circ w(0)}{t}=\limsup_{t\to-0}\frac{\rho\circ w(t)-\rho\circ w(0)}{t}=0.$$

Thus, by (1) and (2),

$$\lim_{t\to0}\frac{\rho\circ w(t)-\rho\circ w(0)}{t}=0.$$ □

Proof of Main Theorem. Let $w(t)$ be a smooth unit speed curve in $\mathrm{U}v$. From Theorem 8, ρ is differentiable at $w(0)$, if $\lambda_1(w(0))=\rho(w(0))<\infty$. Suppose that $\lambda_1(w(0))>\rho(w(0))$. Then there exist two sectors Σ_+ and Σ_- at $\exp(\rho(w(0))w(0))$ such that for sufficiently small $\delta>0$,

$$\Sigma_+\supset\{\exp(\rho(w(t))w(t));0<t<\delta\}$$

and

$$\Sigma_-\supset\{\exp(\rho(w(t))w(t));0>t>-\delta\}.$$

Let $2\theta_+$ and $2\theta_-$ be the inner angles of Σ_+ and Σ_- at $\exp(\rho(w(0))w(0))$ respectively. From Lemma 2.1 and Proposition 2.2 in [**8**], it follows that

$$\lim_{t\to+0}\frac{\rho\circ w(t)-\rho\circ w(0)}{t}=-\|Y(\rho(w(0)))\|\cot\theta_+ \tag{17}$$

and

$$\lim_{t\to-0}\frac{\rho\circ w(t)-\rho\circ w(0)}{t}=\|Y(\rho(w(0)))\|\cot\theta_-, \tag{18}$$

where $Y(t):=Y_N(t;w(0))$ denotes the N-Jacobi field along $\gamma_{w(0)}(t)$ defined in the equation (4) by the unit speed curve $w(\tau)$ in $\mathrm{U}v$. If there exist exactly two N-segments through $\exp(\rho(w(0))w(0))$, then $\theta_+=\pi-\theta_-$. Otherwise $\theta_+<\pi-\theta_-$. Therefore the proof is complete. □

The following two corollaries are ones to the Main Theorem.

Corollary 9. *Let* $\tilde{c}:(a,b)\to \mathrm{U}v$ *be a smooth unit speed curve such that each cut point* $\exp(\rho(\tilde{c}(t))\tilde{c}(t))$ *admits at most two sectors. If* $\rho\circ\tilde{c}$ *is differentiable on* (a,b), *then* $(\rho\circ\tilde{c})' := (d/dt)(\rho\circ\tilde{c})$ *is continuous on* (a,b). *Hence, if there exist at most two N-segments through* $\exp(\rho(\tilde{c}(t))\tilde{c}(t))$ *for each* $t\in(a,b)$, *then the curve* $\exp(\rho(\tilde{c}(t))\tilde{c}(t))$, $t\in(a,b)$, *is* C^1.

Proof. If $\lambda_1(\tilde{c}(t)) > \rho(\tilde{c}(t))$, then from (17) and (18), we get

$$(\rho\circ\tilde{c})'(t) = -\|Y_1(\rho(\tilde{c}(t)))\|\cot\theta(t). \tag{19}$$

Here $Y_1(t) := Y_N(t;\tilde{c}(0))$ and $2\theta(t)$ denotes the inner angle of a sector at $c(t) := \exp(\rho(\tilde{c}(t))\tilde{c}(t))$. Note that $c(t)$ is a normal cut point of N for each differentiable point t of $\rho\circ\tilde{c}$ if $\lambda_1(\tilde{c}(t)) > \rho(\tilde{c}(0))$. Thus it is clear from (19) that $(\rho\circ\tilde{c})'$ is continuous at t if $\lambda_1(\tilde{c}(t_0)) > \rho(\tilde{c}(t_0))$. Suppose that $\lambda_1(\tilde{c}(t_0)) = \rho(\tilde{c}(t_0))$. From Theorem 8, it follows that

$$(\rho\circ\tilde{c})'(t_0) = 0. \tag{20}$$

Let $\{a_n\}$ be a monotone sequence of points in (a,b) convergent to t_0 such that $\lambda_1(\tilde{c}(a_n)) > \rho(\tilde{c}(a_n))$. By Lemma 3 there exists a positive constant C_1 such that

$$|a_n - t_0| \le C_1\theta(a_n). \tag{21}$$

Here $2\theta(a_n)$ denotes the minimum of all the inner angles of the two sectors at $c(a_n)$. Since $Y_1(\rho(\tilde{c}(t_0))) = 0$, there exists a positive constant C_3 such that

$$\|Y_1(\rho(\tilde{c}(a_n)))\| \le C_3|\rho(\tilde{c}(a_n)) - \rho(\tilde{c}(t_0))|. \tag{22}$$

From the equations (19), (20), (21) and (22), we get $\lim_{n\to\infty}(\rho\circ\tilde{c})'(a_n) = 0$. Hence

$$\lim_{t\to t_0}(\rho\circ\tilde{c})'(t) = 0 = (\rho\circ\tilde{c})'(t_0).$$

Therefore $(\rho\circ\tilde{c})'$ is continuous on (a,b). □

Corollary 10. *The function* ρ *is differentiable on* $\{v\in \mathrm{U}v;\rho(v)<\infty\}$ *except a countable subset.*

Proof. From the Main Theorem, if $v(t_0)$ is a non-differentiable point of ρ, where $v(t)$, $t\in(a,b)$, denotes a unit speed smooth curve on $\mathrm{U}v$ such that $\rho(v(t))<\infty$ on (a,b), then $\lambda_1(v(t_0)) > \rho(v(t_0))$, and $\exp(\rho(v(t_0))v(t_0))$ admits at least three sectors or there exists a non-constant curve $w(s)$, $s\in(\alpha,\beta)$, in $\mathrm{U}v$ such that $\exp(\rho(w(s))w(s)) = \exp(\rho(v(t_0))v(t_0))$, for any $s\in(\alpha,\beta)$. The set S of all such cut points is a countable set (cf. [**12**]). Furthermore, for each $q\in S$, $A(q) := \{v\in \mathrm{U}v; \exp(\rho(v)v) = q, \rho(v) < \lambda_1(v)\}$ is countable. Thus $\bigcup_{q\in S} A(q)$ is also countable. □

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