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Criteria for monotonicity of operator means

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Abstract. Let $\{\psi_r\}_{r>0}$ and $\{\phi_r\}_{r>0}$ be the families of operator monotone functions on $[0,\infty)$ satisfying $\psi_r(x^r g(x)) = x^r$, $\phi_r(x^r g(x)) = x^r h(x)$, where g and h are continuous and g is increasing. Suppose σ_{ψ_a} and σ_{ϕ_r} are the corresponding operator connections. We will show that if $A^a \sigma_{\psi_a} B \geq 1$ $(a > 0)$, then $A^r \sigma_{\psi_r} B$ and $A^r \sigma_{\phi_r} B$ are both increasing for $r \geq a$, and then we will apply this to the geometric operator means to get a simple assertion from which many operator inequalities follow.

1. Introduction.

In this paper we denote bounded positive semidefinite operators on a Hilbert space by A, B, C and so on. A real valued continuous function $\varphi(x)$ on $[0,\infty)$ is called an *operator monotone function* if $0 \leq A \leq B$ implies $\varphi(A) \leq \varphi(B)$. The fact that x^a $(0 < a \leq 1)$ is operator monotone is called the *Löwner-Heinz* inequality.

In [**8**] (see p. 76 of [**9**] for the relevant topics) a quadratic operator equation $B = XAX$ was studied and it was shown that if A is nonsingular, then there is a solution T with $0 \leq T \leq 1$ if and only if $(A^{1/2}BA^{1/2})^{1/2} \leq A$ and that T is then given by the formula $T = A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2}$ if A is invertible. The solution of $B = XA^{-1}X$ is therefore given by $A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$. On the other hand, in [**7**] it was shown that if A is invertible, the maximum of all X such that

$$\begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0$$

equals $A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$, which is called the *geometric mean* of A and B and denoted by $A \sharp B$. Therefore, by using this symbol, the solution T of $B = XAX$ is given by $T = A^{-1} \sharp B$ if A is invertible. For $0 < \lambda < 1$ and for invertible A the weighted geometric mean is defined as:

$$A \underset{\lambda}{\sharp} B := A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2}.$$

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Furuta [**3**], [**4**] showed that $A \le B$ implies for $1 \le s, p$ and $0 < r$

$$A^{1+r} \le (A^{r/2}B^pA^{r/2})^{(1+r)/(p+r)}, \tag{1}$$

$$A^{1-t+r} \le \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(1-t+r)/(ps-ts+r)} \quad (0 \le t \le 1, t \le r). \tag{2}$$

Further, in [**1**], [**2**], [**10**] it was shown that $A \le B$ implies for $0 < p, r$

$$e^{rA} \le (e^{rA/2}e^{pB}e^{rA/2})^{r/(r+p)}. \tag{3}$$

These inequalities can be rewritten with the symbol $\sharp$; for instance, (1) is equivalent to $A \le A^{-r} \underset{(1+r)/(p+r)}{\sharp} B^p$.

Now let us state a simple fact on numerical weighted geometric means: For positive numbers a, b, c, x and y, if $(x^a)^{bc/(a+bc)}(y^b)^{a/(a+bc)} \le 1$, then for any d with $-a \le d \le bc$, $(x^r)^{(sc-d)/(r+sc)}(y^s)^{(r+d)/(r+sc)}$ is decreasing for $r \ge a$ and for $s \ge b$. We will show that this result is true even if x and y are replaced by A and B and that (1), (2) and (3) follow simply from it.

We study in a more general situation. Namely, we treat operator connections (or means) which include every weighted geometric mean. Kubo and Ando [**6**] defined a *connection*, which is denoted by σ, and showed that there is a one to one correspondence between σ and an operator monotone function $\varphi \ge 0$ on $[0, \infty)$ by the formula

$$A\sigma B = A^{1/2}\varphi(A^{-1/2}BA^{-1/2})A^{1/2} \tag{4}$$

if A is invertible; σ is called an *operator mean* if $A\sigma A = A$, which is equivalent to $\varphi(1) = 1$. The operator mean corresponding to $\varphi(x) = x^{1/2}$ is clearly geometric mean.

The properties of a connection σ which we will need in this paper are the following:

(i) $A \le C$ *and* $B \le D$ *imply* $A\sigma B \le C\sigma D$;

(ii) $C(A\sigma B)C = (CAC)\sigma(CBC)$ if C is invertible;

(iii) $(A_n\sigma B_n) \downarrow (A\sigma B)$ if $A_n \downarrow A$ and $B_n \downarrow B$ in the strong topology;

(iv) $A\sigma B = B\sigma' A$, where σ' is the connection corresponding to $x\varphi(1/x)$.

In this paper we write σ_φ for σ corresponding to φ. In [**11**], to extend (1) and (3) we constructed a family $\{\phi_r(x)\}_{r>0}$ of non-negative operator monotone functions which satisfies

$$\phi_r(g(x)f(x)^r) = f(x)^{c+r} \quad (0 \le c \le 1),$$

where g and f are appropriate increasing functions; here by replacing $f(x)$ by x and $g(f^{-1}(x))$ by another function $g(x)$, ϕ_r satisfies $\phi_r(g(x)x^r) = x^rx^c$. In [**12**] we also studied the operator monotone function $\phi_{r,t}(x)$ defined by

$$\phi_{r,t}(x) = x^{r/(r+t)}f(x^{t/(r+t)}), \quad \textit{i.e.}, \quad \phi_{r,t}(x^rx^t) = x^rf(x^t),$$

where $f \geq 0$ is a given operator monotone function and $r > 0$ and $t > 0$. These investigations have led us to set up a pair of operator monotone functions $\{\psi_r\}$ and $\{\phi_r\}$ with the following situation:

$$\psi_r(x^r g(x)) = x^r, \quad i.e., \quad x^{-r}\sigma_{\psi_r} g(x) = 1, \tag{5}$$

$$\phi_r(x^r g(x)) = x^r h(x), \quad i.e., \quad x^{-r}\sigma_{\phi_r} g(x) = h(x). \tag{6}$$

In this situation, ψ_r may be considered to be the subsidiary function of ϕ_r.

From now on, we assume that $\{\psi_r\}_{r>0}$ and $\{\phi_r\}_{r>0}$ are families of non-negative functions on $[0, \infty)$ satisfying (5) and (6) respectively, where g and h are continuous and g is increasing and that ψ_r and ϕ_r are both operator monotone for every r which is not less than a non-negative real number. Note that ψ_r is strictly increasing on $[0, \infty)$ with $\psi_r(0) = 0$ and $\psi_r(\infty) = \infty$, so the inverse function ψ_r^{-1} on $[0, \infty)$ exists. We remark that $h(x)$ is not necessarily increasing and that the region of r for which ψ_r is operator monotone is not necessarily coincident with that of r for which ϕ_r is: for instance, in (5) and (6) set $g(x) = x^t$ for a fixed $t > 0$ and $h(x) = x^{-1}$, then $\psi_r(x) = x^{r/(t+r)}$ is operator monotone for $r > 0$; on the other hand $\phi_r(x) = x^{(-1+r)/(t+r)}$ is operator monotone for $r \geq 1$.

In the next section we show a fundamental principle, from which we can derive easily (1), (2) and (3): to be precise, we show that if $A^a\sigma_{\psi_a}B \geq 1$ $(a > 0)$, then $A^r\sigma_{\psi_r}B$ and $A^r\sigma_{\phi_r}B$ are both increasing for $r \geq a$; we deal with geometric means and give a very useful theorem in the third section; in the last section, we explain how easily we can get (1), (2) and (3).

2. Criteria for monotonicity.

THEOREM 2.1. *Let* $\{\psi_r\}_{r\geq a}$ $(a > 0)$ *be a family of non-negative and operator monotone functions satisfying* (5). *Then the following hold*:

(a) *if* $A^a\sigma_{\psi_a}B \geq 1$, *then* $A^r\sigma_{\psi_r}B$ *is increasing for* $r \geq a$;

(b) *if* A *and* B *are invertible and if* $A^a\sigma_{\psi_a}B \leq 1$, *then* $A^r\sigma_{\psi_r}B$ *is decreasing for* $r \geq a$.

PROOF. To prove the first statement (a), it suffices to show

$$A^s\sigma_{\psi_s}B \geq 1 \quad \text{for some } s \geq a \Rightarrow A^r\sigma_{\psi_r}B \geq A^s\sigma_{\psi_s}B \quad \text{for every } r \in [s, 2s].$$

Indeed, from $A^a\sigma_{\psi_a}B \geq 1$ it follows that $A^r\sigma_{\psi_r}B$ is increasing in $[a, 2a]$ and hence not less than 1; by the mathematical induction, we can see the statement. Since $A^r = (A^s)^{r/s}$, we may show that

$$A\sigma_{\psi_s}B \geq 1 \quad \text{for some } s \geq a \Rightarrow A^{r/s}\sigma_{\psi_r}B \geq A\sigma_{\psi_s}B \quad \text{for every } r \in [s, 2s]. \tag{7}$$

Notice $(A + \varepsilon)\sigma_{\psi_s}(B + \varepsilon) \geq A\sigma_{\psi_s}B \geq 1$ for $\varepsilon > 0$. If we could show $(A + \varepsilon)^{r/s}\sigma_{\psi_r}(B + \varepsilon) \geq (A + \varepsilon)\sigma_{\psi_s}(B + \varepsilon)$, then by (iii) in the preceding section we would get (7) as $\varepsilon \to +0$. We therefore assume that A and B are invertible. Put $y = x^s$ in $\psi_s(x^s g(x)) = x^s$ and $\psi_r(x^r g(x)) = x^r$. Then, by setting $b = (r-s)/s$, we obtain

$$\psi_r(y^b \psi_s^{-1}(y)) = y^b y, \quad i.e., \quad y^{-b}\sigma_{\psi_r}\psi_s^{-1}(y) = y. \tag{8}$$

The assumption $A\sigma_{\psi_s}B \geq 1$ implies $\psi_s(A^{-1/2}BA^{-1/2}) \geq A^{-1}$. Here, denote the left-hand side by H and the right-hand side by K. Since $H \geq K$ and $0 \leq b \leq 1$, by the Löwner-Heinz inequality, $K^{-b} \geq H^{-b}$. Hence by (i) in Section 1 we have

$$K^{-b}\sigma_{\psi_r}\psi_s^{-1}(H) \geq H^{-b}\sigma_{\psi_r}\psi_s^{-1}(H) = H,$$

here the last equality follows from (8). Multiplying the above from the left and the right with $A^{1/2}$ yields, by (ii) in Section 1,

$$A^{b+1}\sigma_{\psi_r}B \geq A\sigma_{\psi_s}B.$$

Consequently, we have (7). We next show the second statement (b). Let A and B be invertible. To see

$$A^s\sigma_{\psi_s}B \leq 1 \Rightarrow A^r\sigma_{\psi_r}B \leq A^s\sigma_{\psi_s}B \quad (s \leq r \leq 2s),$$

it is sufficient to show

$$A\sigma_{\psi_s}B \leq 1 \Rightarrow A^{r/s}\sigma_{\psi_r}B \leq A\sigma_{\psi_s}B \quad (s \leq r \leq 2s).$$

It is not difficult to get this in a fashion similar to the above. □

In the second statement (b) of the above theorem, we assumed A and B are invertible, because the norm of $(A+\varepsilon)^a\sigma_{\psi_s}(B+\varepsilon)$ may not necessarily converge to that of $A^a\sigma_{\psi_s}B$ as $\varepsilon \to +0$. We do not know if the invertibility of A and B can be removed.

Theorem 2.2. *Let $\{\psi_r\}_{r \geq a}$ and $\{\phi_r\}_{r \geq a}$ $(a > 0)$ be families of non-negative operator monotone functions satisfying* (5) *and* (6). *Then the following hold*:

(a) *if $A^a\sigma_{\psi_a}B \geq 1$, then $A^r\sigma_{\phi_r}B$ is increasing for $r \geq a$*;

(b) *if A and B are invertible and if $A^a\sigma_{\psi_a}B \leq 1$, then $A^r\sigma_{\phi_r}B$ is decreasing for $r \geq a$.*

Proof. We show the second statement (b). By Theorem 2.1, from $A^a\sigma_{\psi_a}B \leq 1$ it follows that $A^s\sigma_{\psi_s}B \leq 1$ for $s \geq a$. Thus, it sufficies to show

$$A^s\sigma_{\psi_s}B \leq 1 \quad \text{for some } s \geq a \Rightarrow A^r\sigma_{\phi_r}B \leq A^s\sigma_{\phi_s}B \quad \text{for every } r \in [s, 2s].$$

Further, it is clearly sufficient to show that

$$A\sigma_{\psi_s}B \leq 1 \quad \text{for some } s \geq a \Rightarrow A^{r/s}\sigma_{\phi_r}B \leq A\sigma_{\psi_s}B \quad \text{for every } r \in [s, 2s].$$

$A\sigma_{\psi_s}B \le 1$ implies

$$\psi_s(A^{-1/2}BA^{-1/2}) \le A^{-1}.$$

Here put the left-hand side by H and the right-hand side by K. In (5) and (6), putting $y = x^s$ and $b = (r-s)/s$ we obtain

$$\phi_r(y^b\psi_s^{-1}(y)) = y^b\phi_s(\psi_s^{-1}(y)) \quad (y > 0). \tag{9}$$

Since $H \le K$ and $0 < b < 1$, we have $K^{-b} \le H^{-b}$, which gives

$$K^{-b}\sigma_{\phi_r}\psi_s^{-1}(H) \le H^{-b}\sigma_{\phi_r}\psi_s^{-1}(H) = H^{-b}\phi_r(H^b\psi_s^{-1}(H)).$$

By (9), $H^{-b}\phi_r(H^b\psi_s^{-1}(H)) = \phi_s(\psi_s^{-1}(H))$. Thus the above inequality gives

$$A^b\sigma_{\phi_r}(A^{-1/2}BA^{-1/2}) \le \phi_s(A^{-1/2}BA^{-1/2}).$$

Multiplying the above from the left and the right with $A^{1/2}$ yields

$$A^{b+1}\sigma_{\phi_r}B \le A\sigma_{\phi_s}B.$$

Consequently, the proof of the second statement is complete. To see the first statement we may assume that A and B are invertible as in the proof of Theorem 2.1. Then one can show it in the same way as above. □

THEOREM 2.3. *Let $\{\psi_r\}_{r>0}$ and $\{\phi_r\}_{r>0}$ be families of non-negative operator monotone functions satisfying* (5) *and* (6). *If $A \le B$ or if $\log A \le \log B$ for invertible A and B, then for $r > 0$*

$$A^r \le \psi_r(A^{r/2}g(B)A^{r/2}), \quad \psi_r(B^{r/2}g(A)B^{r/2}) \le B^r, \tag{10}$$

$$A^{r/2}h(B)A^{r/2} \le \phi_r(A^{r/2}g(B)A^{r/2}), \quad \phi_r(B^{r/2}g(A)B^{r/2}) \le B^{r/2}h(A)B^{r/2}. \tag{11}$$

PROOF. Suppose first $A \le B$. Since the functions given in the theorem are all continuous, by replacing A and B by $A+\varepsilon$ and $B+\varepsilon$ respectively if necessary, we may assume that A and B are both invertible. Since $B^{-a} \le A^{-a}$ for $0 < a \le 1$, $A^{-a}\sigma_{\psi_a}g(B) \ge B^{-a}\sigma_{\psi_a}g(B) = 1$ and $A^{-a}\sigma_{\phi_a}g(B) \ge B^{-a}\sigma_{\phi_a}g(B) = h(B)$. By the above theorems we get $A^{-r}\sigma_{\psi_r}g(B) \ge 1$ and $A^{-r}\sigma_{\phi_r}g(B) \ge h(B)$ for $r \ge a$; by letting a take over all $0 < a < 1$, these hold for all $r > 0$. Thus we get $A^r \le \psi_r(A^{r/2}g(B)A^{r/2})$ and $A^{r/2}h(B)A^{r/2} \le \phi_r(A^{r/2}g(B)A^{r/2})$ for $r > 0$, that is, the first inequalities of both (10) and (11). From $B^{-a} \le A^{-a}$ it also follows that $B^{-a}\sigma_{\psi_a}g(A) \le A^{-a}\sigma_{\psi_a}g(A) \le 1$ and $B^{-a}\sigma_{\phi_a}g(A) \le h(A)$. Thus we get the second inequalities of both (10) and (11).

Suppose next $\log A \le \log B$ and A and B are invertible. Upon replacing B by $B+\varepsilon$ if necessary, we may assume $\log A + \varepsilon_1 \le \log B$ for some $\varepsilon_1 > 0$.

By taking the uniform derivatives of A^t and B^t at $t=0$ there is a $\delta>0$ so that $A^a \le B^a$ for every $a \in (0,\delta)$. Thus the same argument leads us to (10) and (11). □

We have already shown (10) in [**11**]. The functions $\psi_r(x) = x^{r/(r+q)}$ and $\phi_r(x) = x^{(r+p)/(r+q)}$ satisfies the condition of the preceding theorem with $g(x) = x^q$ and $h(x) = x^p$ for $(0 < p \le q)$. So, (10) and (11) imply

$$A^r \le (A^{r/2}B^qA^{r/2})^{r/(r+q)}, \quad (B^{r/2}A^qB^{r/2})^{r/(r+q)} \le B^r,$$
$$A^{r/2}B^pA^{r/2} \le (A^{r/2}B^qA^{r/2})^{(r+p)/(r+q)}, \quad (B^{r/2}A^qB^{r/2})^{(r+p)/(r+q)} \le B^{r/2}A^pB^{r/2}.$$

One can see that these are extensions of (1) and (3); indeed, by putting $p=1$ in the third inequality one get (1).

REMARK 2.1. In the above theorems, we assumed that the families $\{\psi_r\}_{r>0}$ and $\{\phi_r\}_{r>0}$ satisfy (5) and (6) respectively. However their proofs are still valid if (8) and (9) hold. Therefore, theorems are true even if we assume that ψ_r and ϕ_r are non-negative operator monotone functions on $[0,\infty)$ with $\psi_r(0)=0$ and $\psi_r(\infty)=\infty$ and that for all r and s with $r>s>0$

$$\psi_r(\psi_s(x)^{(r-s)/s}x) = \psi_s(x)^{r/s} \quad \text{and} \quad \phi_r(\psi_s(x)^{(r-s)/s}x) = \psi_s(x)^{(r-s)/s}\phi_s(x)$$

instead of (5) and (6); because they satisfy

$$\psi_r(y^{(r-s)/s}\psi_s^{-1}(y)) = y^{r/s} \quad \text{and} \quad \phi_r(y^{(r-s)/s}\psi_s^{-1}(y)) = y^{(r-s)/s}\phi_s(\psi_s^{-1}(y)),$$

from which (8) and (9) follow.

REMARK 2.2. Let $\{A_r\}_{r>0}$ be a weakly continuous semi-group of positive semidefinite operators, that is, $A_{r+s} = A_rA_s$. Then we get $(A_r)^a = A_{ra}$ for $a>0$. Thus from Theorem 2.2 we obtain

(a) *if* $A_a\sigma_{\psi_b}B \ge 1$, *then* $A_{ar}\sigma_{\psi_{br}}B$ *is increasing for* $r \ge 1$;

(b) *if* $A_a\sigma_{\psi_b}B \le 1$ *for invertible* A_a *and* B, *then* $A_{ar}\sigma_{\psi_{br}}B$ *is decreasing for* $r \ge 1$.

3. Weighted geometric means.

Our objective in this section is to apply the results we got in the preceding section to the weighted geometric means. As we mentioned in the first section the symbols $\underset{\lambda}{\sharp}$ and σ_{x^λ} express the same weighted geometric mean for $0<\lambda\le 1$. By (iv) we have $A \underset{\lambda}{\sharp} B = B \underset{1-\lambda}{\sharp} A$.

LEMMA 3.1. *Let* $a>0$, $c>0$ *and* $c>d$. *Then the following hold*:

(a) *if A and B are invertible and if* $A^a \underset{a/(a+c)}{\sharp} B \le 1$, *then* $A^r \underset{(r+d)/(r+c)}{\sharp} B$ *is decreasing for* $r \ge \max(a, -d)$;

(b) *if* $A^a \underset{a/(a+c)}{\sharp} B \ge 1$, *then* $A^r \underset{(r+d)/(r+c)}{\sharp} B$ *is increasing for* $r \ge \max(a, -d)$.

Proof. The functions $\psi_r(x) = x^{r/(r+c)}$ and $\phi_r(x) = x^{(r+d)/(r+c)}$ satisfy (5) and (6) with $g(x) = x^c$ and $h(x) = x^d$. ψ_r is operator monotone for $r > 0$ and so is ϕ_r for $r \ge \max(0, -d)$. We show only the first statement. By Theorem 2.1 $A^{a_1} \underset{a_1/(a_1+c)}{\sharp} B \le 1$ for $a_1 := \max(a, -d)$. Thus by Theorem 2.2 $A^r \underset{(r+d)/(r+c)}{\sharp} B$ is decreasing for $r \ge a_1$. This implies the desired result. □

Theorem 3.2. *For a given* $c > 0$ *define a function* $F(r, s)$ *by*

$$F(r,s) = A^r \underset{r/(r+sc)}{\sharp} B^s \quad \text{for } r > 0, s > 0. \tag{12}$$

Then, for $r \ge a > 0$, $s \ge b > 0$ *the following hold*:

(a) *if A and B are both invertible and* $F(a,b) \le 1$, *then* $F(r,s) \le F(a,b)$;

(b) *if* $F(a,b) \ge 1$, *then* $F(r,s) \ge F(a,b)$.

Proof. We show only the first statement. From Lemma 3.1 it follows that

$$1 \ge F(a,b) \ge F(r,b) = A^r \underset{r/(r+bc)}{\sharp} B^b = B^b \underset{bc/(r+bc)}{\sharp} A^r = B^b \underset{b/(b+r/c)}{\sharp} A^r$$

$$\ge B^s \underset{s/(s+r/c)}{\sharp} A^r = A^r \underset{(r/c)/(s+r/c)}{\sharp} B^s = A^r \underset{r/(r+sc)}{\sharp} B^s = F(r,s). \qquad \square$$

By using the above theorem twice, from $F(a,b) \le 1$ it follows that $F(r_2, s_2) \le F(r_1, s_1) \le F(a,b)$ for $r_2 \ge r_1 \ge a$ and for $s_2 \ge s_1 \ge b$.

The case $\lambda = 1/2$ of the following corollary resembles the result shown in [**1**].

Corollary 3.3. *For a given λ as* $0 < \lambda < 1$ *the following hold*:

(a) *if* $A \underset{\lambda}{\sharp} B \le 1$ *for invertible A and B, then* $A^r \underset{\lambda}{\sharp} B^r$ *is decreasing for* $r \ge 1$;

(b) *if* $A \underset{\lambda}{\sharp} B \ge 1$, *then* $A^r \underset{\lambda}{\sharp} B^r$ *is increasing for* $r \ge 1$.

Proof. Define c by $\lambda = 1/(1+c)$ and use Theorem 3.2 to get this. □

Now we treat a quadratic equation $B = XAX$ given in the first section. Assume that A and B are invertible. Then the solution is given by $A^{-1} \sharp B$. By Corollary 3.3 we get:

(a) *if* $A^{-1} \sharp B \ge 1$ *then the solution* $A^{-r} \sharp B^r$ *of* $B^r = XA^rX$ *is increasing for* $r \ge 1$;

(b) *if* $A^{-1} \sharp B \le 1$ *then* $A^{-r} \sharp B^r$ *is decreasing for* $r \ge 1$.

The following is the main theorem of this section.

Theorem 3.4. *For real numbers $c > 0$ and d, define $F(r,s)$ by* (12) *and $G(r,s)$ by*

$$G(r,s) = A^r \mathop{\sharp}_{(r+d)/(r+sc)} B^s \quad \text{for } r > 0, s > 0 \quad \text{with } 0 \le \frac{r+d}{r+sc} \le 1. \tag{13}$$

Let $a > 0$, $b > 0$ and $-a \le d \le bc$. Then for $r_2 \ge r_1 \ge a$ and for $s_2 \ge s_1 \ge b$ the following hold:

(a) *if A and B are both invertible and $F(a,b) \le 1$, then $G(r_2,s_2) \le G(r_1,s_1)$*;

(b) *if $F(a,b) \ge 1$, then $G(r_2,s_2) \ge G(r_1,s_1)$.*

Proof. We show only the first statement. Theorm 3.2 implies $F(r_1,s_1) \le F(a,b) \le 1$. Therefore, it is sufficient to prove $G(r,s) \le G(a,b)$ for $r \ge a$ and for $s \ge b$. Suppose $r \ge a$ and $s \ge b$. Since $r \ge a \ge -d$, by Lemma 3.1, we have $G(r,b) \le G(a,b)$ and $F(r,b) \le F(a,b)$. The latter fact yields

$$1 \ge A^r \mathop{\sharp}_{r/(r+bc)} B^b = B^b \mathop{\sharp}_{bc/(r+bc)} A^r = B^b \mathop{\sharp}_{b/(b+r/c)} A^r,$$

from which, by Lemma 3.1, it follows that

$$B^s \mathop{\sharp}_{(s-d/c)/(s+r/c)} A^r \le B^b \mathop{\sharp}_{(b-d/c)/(b+r/c)} A^r \quad \text{because of } s \ge b \ge \frac{d}{c}.$$

Since the right-hand side in the above inequality equals

$$B^b \mathop{\sharp}_{(bc-d)/(bc+r)} A^r = A^r \mathop{\sharp}_{(r+d)/(r+bc)} B^b = G(r,b),$$

and since the left-hand side equals

$$B^s \mathop{\sharp}_{(sc-d)/(sc+r)} A^r = A^r \mathop{\sharp}_{(r+d)/(r+sc)} B^s = G(r,s),$$

the above inequality means $G(r,s) \le G(r,b)$. Consequently, we get $G(r,s) \le G(a,b)$. □

The above theorem says that if $F(a,b) \le 1$, $G(a,b) \le K$ then $G(r,s) \le K$ for $r \ge a$, $s \ge b$; moreover, if $F(a,b) = 1$ then $G(r,s)$ is constant, though this directly follows from the definitions of $F(r,s)$ and $G(r,s)$. Notice that $G(r,s) = F(r,s)$ if $d = 0$.

So far, we have seen that $F(a,b) \le 1$ (or $F(a,b) \ge 1$) has a great influence on $G(r,s)$. Now we give a sufficient condition on $G(r,s)$ in order that $F(a,b) \le 1$ (or $F(a,b) \ge 1$).

Proposition 3.5. *Let A and B be invertible. Let $a > 0$ and $c > d > 0$. Then the following hold*:

$$A^a \underset{(a+d)/(a+c)}{\sharp} B \le A^{-d} \Rightarrow A^a \underset{a/(a+c)}{\sharp} B \le 1;$$

$$A^a \underset{(a+d)/(a+c)}{\sharp} B \ge A^{-d} \Rightarrow A^a \underset{a/(a+c)}{\sharp} B \ge 1.$$

Proof. The first assumption implies

$$(A^{-a/2}BA^{-a/2})^{(a+d)/(a+c)} \le A^{-(a+d)},$$

from which, by the Löwner-Heinz inequality, it follows that

$$(A^{-a/2}BA^{-a/2})^{a/(a+c)} \le A^{-a}, \quad \text{and hence } A^a \underset{a/(a+c)}{\sharp} B \le 1. \qquad \square$$

4. Applications.

We mentioned after Theorem 2.3 that (10) and (11) are extensions of (1) and (3). However we give simple proofs of (1) and (3) to explain how Theorem 3.4 is useful, and we give an extension of (2).

(1): We may assume A and B are invertible. From $A \le B$ it follows that $A^{-a} \ge B^{-a}$ for every a with $0 < a < 1$. Substitute A^{-1} for A in (12) and (13), and put $c = 1$ and $d = 1$. Then

$$F(a,1) = A^{-a} \underset{a/(a+1)}{\sharp} B \ge B^{-a} \underset{a/(a+1)}{\sharp} B = 1, \quad G(a,1) = A^{-a} \underset{(a+1)/(a+1)}{\sharp} B = B.$$

Thus by Theorm 3.4

$$G(r,s) = A^{-r} \underset{(r+1)/(r+s)}{\sharp} B^s$$

is increasing for $r \ge a$ and for $s \ge 1$; especially, $G(r,s) \ge G(a,1) = B \ge A$. Since a is arbitrary, we have $G(r,s) \ge A$ for $r > 0, s \ge 1$. Replace p for s to get (1). $\square$

(3): We show, by using Theorem 3.4, the stronger result:

$$A \le B \Rightarrow e^{rA/2}e^{pB}e^{rA/2} \le (e^{rA/2}e^{qB}e^{rA/2})^{(r+p)/(r+q)} \quad (r > 0, q \ge p \ge 0).$$

Upon replacing B by $B + \varepsilon$ if necessary, we may assume that $A + \varepsilon \le B$ for some $\varepsilon > 0$. By taking the uniform derivatives of e^{tA} and e^{tB} at $t = 0$, there is a $\delta > 0$ so that $e^{aA} \le e^{aB}$ for all a with $0 < a < \delta$. Put

$$F(r,s) = e^{-rA} \underset{r/(r+s)}{\sharp} e^{sB}, \quad G(r,s) = e^{-rA} \underset{(r+p)/(r+s)}{\sharp} e^{sB}.$$

Then we get

$$F(a,p) = e^{-aA} \underset{a/(a+p)}{\sharp} e^{pB} \geq e^{-aB} \underset{a/(a+p)}{\sharp} e^{pB} = 1,$$

$$G(a,p) = e^{-aA} \underset{(a+p)/(a+p)}{\sharp} e^{pB} = e^{pB}.$$

Thus $G(r,s)$ is increasing for $r > 0$ and for $s \geq p$, since a is arbitrary. This implies especially $G(r,q) \geq G(r,p)$, from which we can get the desired result. □

Now we give an extension of (2). By putting $B = A$ in the following we get (2).

Proposition 4.1. *If $A \leq B \leq C$ and if B is invertible, then for $0 \leq t \leq 1$, $t \leq r$, $1 \leq p$ and $1 \leq s$*

$$A^{1-t+r} \leq \{A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}\}^{(1-t+r)/(ps-ts+r)},$$

$$\{C^{r/2}(B^{-t/2}A^pB^{-t/2})^sC^{r/2}\}^{(1-t+r)/(ps-ts+r)} \leq C^{1-t+r}. \tag{14}$$

Proof. If $t = 0$, (14) reduces to (1). So we assume $0 < t \leq 1$. We may, without loss of generality, assume A is invertible. Put

$$K = B^{-t/2}C^pB^{-t/2}.$$

Then (14) is equivalent to

$$A^{1-t} \leq A^{-r} \underset{(r+1-t)/(r+ps-ts)}{\sharp} K^s \quad (t \leq r, 1 \leq p, 1 \leq s).$$

Put

$$F(r,s) = A^{-r} \underset{r/(r+ps-ts)}{\sharp} K^s \quad \text{and} \quad G(r,s) = A^{-r} \underset{(r+1-t)/(r+ps-ts)}{\sharp} K^s.$$

$B^t \geq A^t$ yields $A^{t/2}B^{-t}A^{t/2} \leq 1$; since $x^{t/p}$ is operator concave (see [**5**]) we obtain

$$A^{t/2}B^{-t/2}(C^p)^{t/p}B^{-t/2}A^{t/2} \leq (A^{t/2}B^{-t/2}C^pB^{-t/2}A^{t/2})^{t/p},$$

from which it follows that

$$F(t,1) = A^{-t} \underset{t/p}{\sharp} K^1 \geq B^{-t/2}C^tB^{-t/2} \geq 1,$$

$$G(t,1) = A^{-t} \underset{1/p}{\sharp} K^1 \geq B^{-t/2}CB^{-t/2} \geq B^{1-t} \geq A^{1-t}.$$

By virtue of Theorem 3.4, $G(r,s)$ is therefore increasing for $r \geq t$ and for $s \geq 1$; in particular, $G(r,s) \geq A^{1-t}$. Thus we get (14). The second inequality follows from (14) by taking the inverse of it. □

COMMENTS. After this paper was accepted, the author heard that the essentially same result as Theorem 3.4 (a) has been already gotten by T. Furuta, T. Yamazaki, M. Yanagida in "*Operator functions implying generalized Furuta inequality*" Math. Inequal. Appl. (1998). Their proof depends on (1); however, ours does not depend on (1).

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