J. Math. Soc. Japan
Vol. 55, No. 1, 2003

On commuting canonical endomorphisms of subfactors

By Takashi SANO

(Received Jul. 19, 2001)

Abstract. In the Jones Index theory, Longo's sector theory has been a powerful approach to the analysis for inclusions of factors and canonical endomorphisms have played an important role. In this paper, two topics on commuting canonical endomorphisms are studied: For a composition of two irreducible inclusions of depth 2 factors, the commutativity of corresponding canonical endomorphisms is shown to be the condition for the ambient irreducible inclusion to be of depth 2, that is, to give a finite dimensional Kac algebra. And an equivalent relation between the commuting co-commuting square condition and the existence of two simultaneous commuting canonical endomorphisms is discussed.

1. Introduction.

In the Jones index theory ([**10**]), we have recognized the notion of commuting squares ([**18**], [**2**], etc.) is very important from the beginning. In particular, we have studied commuting co-commuting (or non-degenerate commuting) squares ([**21**], [**20**]). On the other hand, Longo's sector theory ([**14**], [**6**], etc.) has got a precious position in index theory for inclusions of properly infinite factors, and canonical endomorphisms have played an important role. In this article, we would like to study two kinds of inclusions of factors via "commuting canonical endomorphisms".

In §2, we study the commutativity of canonical endomorphisms appearing in a composition of inclusions of factors: let L be a properly infinite factor with $\rho_1, \rho_2 \in \mathrm{End}(L)$, and we assume that each of the inclusions

$$L \supset \rho_1(L) =: M \quad \text{and} \quad M \supset \rho_1\rho_2(L) =: N$$

is an irreducible one with finite index and of depth 2 ([**17**]). Then we show that the following are equivalent:

(1) The inclusion $L \supset N$ is irreducible and of depth 2.

(2) $\dim \mathrm{Hom}(\overline{\rho_1}\rho_1, \rho_2\overline{\rho_2}) = 1$ and the canonical endomorphisms $\overline{\rho_1}\rho_1$ and $\rho_2\overline{\rho_2}$ commute in $\mathrm{Sect}(L)$.

2000 *Mathematical Subject Classification.* 46L37.

Key Words and Phrases. canonical endomorphism, commuting square, Kac Algebra, depth 2 inclusion.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 11740101), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Izumi and Kosaki make an intensive study on the classification of finite dimensional Kac algebras and they give several concrete examples via this kind of compositions of two inclusions ([**7**], [**8**]).

In §3, we recall basic facts on commuting co-commuting squares, in particular, the existence of a simultaneous canonical endomorphism for a commuting co-commuting square shown by Guido and Longo in [**3**]. We show that these two simultaneous canonical endomorphisms for a commuting co-commuting square have the commutativity to give several properties of extensions. After that, we see an equivalent relation between the commuting square condition and the existence of two simultaneous canonical endomorphisms.

Our approach in this article depends heavily on Longo's sector theory and we shall freely use standard notations and well-known facts on the sector theory ([**14**], [**6**], [**13**], etc.).

The author would like to express his sincere gratitude to Professor Roberto Longo for the hospitality in Tor Vergata university where the second half of this work was almost completed. He is also grateful to the referee for fruitful comments.

2. Commuting canonical endomorphisms appearing in a composition of depth 2 inclusions of factors.

Let L be a properly infinite factor with $\rho_1, \rho_2 \in \mathrm{End}(L)$, and we set $\gamma_i := \rho_i \overline{\rho_i}$. In this section, we assume that each of the inclusions

$$L \supset \rho_1(L), \quad L \supset \rho_2(L)$$

is an irreducible one with finite index and of depth 2, and let us consider the endomorphism $\rho := \overline{\rho_1}\rho_2 \in \mathrm{End}(L)$ and the inclusion

$$L \supset \rho(L) \ \ (= \overline{\rho_1}\rho_2(L)).$$

In general, if γ_1 and γ_2 commute, then the odd powers of the canonical endomorphism for $\rho = \overline{\rho_1}\rho_2$ is described as the products of those for $\overline{\rho_1}$ and ρ_2. In the present case, we have more:

Theorem 2.1. *Let L be a properly infinite factor with $\rho_1, \rho_2 \in \mathrm{End}(L)$, and we assume that each of the inclusions*

$$L \supset \rho_1(L), \quad L \supset \rho_2(L)$$

is an irreducible one with finite index and of depth 2. Then the following are equivalent:

(1) *The inclusion $L \supset \overline{\rho_1}\rho_2(L)$ is irreducible and of depth 2.*

(2) *For the canonical endomorphisms $\gamma_1 = \rho_1\overline{\rho_1}$ and $\gamma_2 = \rho_2\overline{\rho_2}$, we have $\dim \mathrm{Hom}(\gamma_1, \gamma_2) = 1$ and $\gamma_1\gamma_2 = \gamma_2\gamma_1$ in $\mathrm{Sect}(L)$.*

Corollary 2.2. *Let L be a properly infinite factor with $\rho_1, \rho_2 \in \mathrm{End}(L)$, and we assume that each of the inclusions*

$$L \supset \rho_1(L) =: M \quad \text{and} \quad M \supset \rho_1\rho_2(L) =: N$$

is irreducible and of depth 2. Then the following are equivalent:

(1) *The inclusion $L \supset N$ is irreducible and of depth 2.*

(2) $\dim \mathrm{Hom}(\overline{\rho_1}\rho_1, \rho_2\overline{\rho_2}) = 1$ *and* $\overline{\rho_1}\rho_1\rho_2\overline{\rho_2} = \rho_2\overline{\rho_2}\,\overline{\rho_1}\rho_1$ *in* $\mathrm{Sect}(L)$.

Recall that for inclusions $L \supset M \supset N$ of properly infinite isomorphic factors, we have endomorphisms $\rho_1, \rho_2 \in \mathrm{End}(L)$ such that $M = \rho_1(L)$ and $N = \rho_1\rho_2(L)$ ([**15**]). We remark that $\rho_1(L) \supset \rho_1\rho_2(L)$ is isomorphic to $L \supset \rho_2(L)$ and both of the irreducibility and the depth 2 condition for the endomorphism ρ_1 correspond to those for the conjugate $\overline{\rho_1}$. Therefore, we apply Theorem 2.1 to get Corollary 2.2.

Remark 1. As in Theorem 2.1, let us consider the inclusion $L \supset \overline{\rho_1}\rho_2(L)$ and assume that each of the inclusions

$$L \supset \rho_1(L), \quad L \supset \rho_2(L)$$

is of depth 2 but not necessarily irreducible. If $\gamma_1\gamma_2 = \gamma_2\gamma_1$, then it follows from direct calculations of $\rho\bar{\rho}\rho$ that the inclusion $L \supset \rho(L)$ is of depth 2.

In order to show Theorem 2.1, we prepare lemmas. At first, recall the following: an inclusion $L \supset \varphi(L)$ is irreducible if and only if $\dim \mathrm{Hom}(\varphi, \varphi) = 1$ and an irreducible inclusion $L \supset \varphi(L)$ is of depth 2 if and only if $\dim \mathrm{Hom}(\varphi\bar{\varphi}, \varphi\bar{\varphi}) = (d\varphi)^2$, where $d\varphi$ means the statistical dimension of φ ([4, 15]). Thanks to the Frobenius reciprocity, for the endomorphism $\rho = \overline{\rho_1}\rho_2$, we have

$$\dim \mathrm{Hom}(\rho, \rho) = \dim \mathrm{Hom}(\overline{\rho_1}\rho_2, \overline{\rho_1}\rho_2) = \dim \mathrm{Hom}(\gamma_1, \gamma_2)$$

and

$$\begin{aligned} \dim \mathrm{Hom}(\rho\bar{\rho}, \rho\bar{\rho}) &= \dim \mathrm{Hom}(\overline{\rho_1}\rho_2\overline{\rho_2}\rho_1, \overline{\rho_1}\rho_2\overline{\rho_2}\rho_1) \\ &= \dim \mathrm{Hom}(\gamma_1\gamma_2, \gamma_2\gamma_1). \end{aligned}$$

By the discussion so far, we have

Lemma 2.3. *The inclusion $L \supset \rho(L) = \overline{\rho_1}\rho_2(L)$ is irreducible and of depth 2 if and only if* $\dim \mathrm{Hom}(\gamma_1, \gamma_2) = 1$ *and* $\dim \mathrm{Hom}(\gamma_1\gamma_2, \gamma_2\gamma_1) = (d\rho)^2 = (d\rho_1)^2(d\rho_2)^2$.

Next we recall the following Cauchy-Schwarz type inequality:

Lemma 2.4 ([**5**, Lemma 16]). *For $\rho_1, \rho_2 \in \mathrm{Sect}(L)$, we have*

$$(\dim \mathrm{Hom}(\rho_1, \rho_2))^2 \leq \dim \mathrm{Hom}(\rho_1, \rho_1) \times \dim \mathrm{Hom}(\rho_2, \rho_2).$$

When $d\rho_1 = d\rho_2$, *the equality occurs here if and only if* $\rho_1 = \rho_2 \in \mathrm{Sect}(L)$.

This lemma gives the following characterization on the commutativity of canonical endomorphisms:

LEMMA 2.5. *We assume that* $\rho_1, \rho_2 \in \mathrm{Sect}(L)$. *Then we have* $\gamma_1\gamma_2 = \gamma_2\gamma_1 \in \mathrm{Sect}(L)$ *if and only if* $\dim \mathrm{Hom}(\gamma_1\gamma_2, \gamma_2\gamma_1) = \dim \mathrm{Hom}(\gamma_1^2, \gamma_2^2)$.

PROOF. Assume the latter condition holds. By Lemma 2.4, we have

$$\begin{aligned}(\dim \mathrm{Hom}(\gamma_1\gamma_2, \gamma_2\gamma_1))^2 &\leq \dim \mathrm{Hom}(\gamma_1\gamma_2, \gamma_1\gamma_2) \times \dim \mathrm{Hom}(\gamma_2\gamma_1, \gamma_2\gamma_1) \\ &= (\dim \mathrm{Hom}(\gamma_1^2, \gamma_2^2))^2.\end{aligned}$$

The assumption means that this inequality becomes the equality. And remark that $d(\gamma_1\gamma_2) = d(\gamma_2\gamma_1)$ $(=(d\rho_1)^2(d\rho_2)^2)$. Hence, applying Lemma 2.4, we get the commutativity of the canonical endomorphisms γ_1 and γ_2. The other implication follows from direct calculations. □

Now we give a proof of Theorem 2.1.

PROOF. We only show the implication from (1) to (2). The depth 2 assumption of ρ_1 and ρ_2 means that

$$\rho_1\overline{\rho_1}\rho_1 = (d\rho_1)^2\rho_1, \quad \rho_2\overline{\rho_2}\rho_2 = (d\rho_2)^2\rho_2,$$

so that we have

$$\gamma_1^2 = (d\rho_1)^2\gamma_1, \quad \gamma_2^2 = (d\rho_2)^2\gamma_2.$$

Hence, we have

$$\dim \mathrm{Hom}(\gamma_1^2, \gamma_2^2) = (d\rho_1)^2(d\rho_2)^2 \dim \mathrm{Hom}(\gamma_1, \gamma_2) = (d\rho_1)^2(d\rho_2)^2$$

by the irreducibility, i.e., $\dim \mathrm{Hom}(\gamma_1, \gamma_2) = 1$. The depth 2 assumption for the irreducible inclusion $L \supset \rho(L)$ corresponds to

$$\dim \mathrm{Hom}(\gamma_1\gamma_2, \gamma_2\gamma_1) = (d\rho)^2 = (d\rho_1)^2(d\rho_2)^2$$

by Lemma 2.3. Hence, combining these equations, we have

$$\dim \mathrm{Hom}(\gamma_1\gamma_2, \gamma_2\gamma_1) = (d\rho_1)^2(d\rho_2)^2 = \dim \mathrm{Hom}(\gamma_1^2, \gamma_2^2),$$

therefore, applying Lemma 2.5, we get the commutativity of γ_1 and γ_2. □

EXAMPLE 2.6. Let P be a properly infinite factor and let H and K be finite groups with outer actions $\alpha = \{\alpha_h\}_{h \in H}$ and $\beta = \{\beta_k\}_{k \in K}$ on P. For the inclusions

$$L := P \rtimes_\alpha H \supset P \supset P^{(K,\beta)} := \{x \in P; \beta_k(x) = x \ (k \in K)\}$$

([**1**], [**7**], [**8**]), we choose endomorphisms ρ_1, ρ_2 on L satisfying $P = \rho_1(L)$ and $P^{(K,\beta)} = \rho_1\rho_2(L)$. In this case, $\rho_2\overline{\rho_2}$ $(\in \mathrm{Sect}(L))$ corresponds to $\bigoplus_{k \in K}[\beta_k]$ $(\in \mathrm{Sect}(P))$ via the identification $L \supset \rho_2(L)$ with P $(=\rho_1(L)) \supset P^{(K,\beta)}$ $(=\rho_1\rho_2(L))$, and $\overline{\rho_1}\rho_1$ may similarly be identified with $\bigoplus_{h \in H}[\alpha_h]$ via the identification $L \supset \overline{\rho_1}(L)$ with P $(=\rho_1(L)) \supset \rho_1\overline{\rho_1}(L) \cong P^{(H,\alpha)}$. Hence, we may think that ρ_1 and ρ_2 are endomorphisms on P with $\overline{\rho_1}\rho_1 = \bigoplus[\alpha_h]$ and $\rho_2\overline{\rho_2} = \bigoplus[\beta_k]$. Then the irreducibility $P \cap \rho_1\rho_2(P)' = \boldsymbol{C}$ is equivalent to

$$\{[\alpha_h]\}_{h \in H} \cap \{[\beta_k]\}_{k \in K} = \{[\mathrm{id}]\}$$

in $\mathrm{Sect}(P) = \mathrm{Out}(P) = \mathrm{Aut}(P)/\mathrm{Int}(P)$, and the commutativity of $\overline{\rho_1}\rho_1$ and $\rho_2\overline{\rho_2}$ just corresponds to

$$\{[\alpha_h][\beta_k]\}_{h \in H, k \in K} = \{[\beta_k][\alpha_h]\}_{h \in H, k \in K}.$$

Therefore, applying Corollary 2.2, we have

Proposition 2.7. *Let P be a properly infinite factor and let H and K be finite groups with outer actions $\alpha = \{\alpha_h\}_{h \in H}$ and $\beta = \{\beta_k\}_{k \in K}$ on P, and we assume that these two groups $\{\alpha_h\}_{h \in H} \cong H$ and $\{\beta_k\}_{k \in K} \cong K$ have the trivial intersection $(H \cap K = \{e\})$ in $\mathrm{Out}(P) = \mathrm{Aut}(P)/\mathrm{Int}(P)$. Then the irreducible inclusion*

$$P \rtimes_\alpha H \supset P^{(K,\beta)} := \{x \in P; \beta_k(x) = x \ (k \in K)\}$$

is of depth 2 if and only if the product HK forms a group, that is, $HK = KH$ holds in $\mathrm{Out}(P)$.

We remark that Izumi and Kosaki give an intensive study on finite dimensional Kac algebras by considering this kind of inclusions and more general ones in which commuting canonical endomorphisms appear. (See [**7**], [**8**].)

3. Simultaneous canonical endomorphisms and commuting co-commuting squares.

Let

$$\begin{array}{ccc} L & \supset & M \\ \cup & & \cup \\ N & \supset & K \end{array}$$

be inclusions of properly infinite factors with finite index, and we simply denote these by (L, M, N, K). In this section, we treat only minimal conditional expectations and denote the minimal expectation from L onto K (resp. M, N) by E_K^L (resp. E_M^L, E_N^L). We call (L, M, N, K) a commuting square ([**18**], [**2**]) if $E_M^L E_N^L = E_N^L E_M^L = E_K^L$ is satisfied. If a commuting square (L, M, N, K) satisfies $[L : M]_0 = [N : K]_0$ (and/or $[L : N]_0 = [M : K]_0$), it is said to be co-commuting or non-degenerate ([**19**], [**21**]).

For a commuting co-commuting square, Guido and Longo show the existence of a simultaneous canonical endomorphism:

PROPOSITION 3.1 ([**3**, Proposition 2.3]). *Let (L, M, N, K) be a commuting co-commuting square of properly infinite factors with finite index and $L \cap K' = \boldsymbol{C}$. Then there exists a canonical endomorphism γ_M^L for $L \supset M$ whose restriction to N is a canonical endomorphism for $N \supset K$.*

A simultaneous canonical endomorphism is obtained as follows ([**3**]): let φ be a bicyclic state for $M \supset K$ and let us consider the faithful state $\varphi \circ E_M^L = \omega_\xi \in L_*$ $(\xi \in L^2(L)_+)$. We set $e := [M\xi] = [K\xi]$ and take an isometry $v \in \langle N, e \rangle$ with $vv^* = e$. Then $\gamma_M^L := \Psi^{-1} \circ \Phi$ turns out to be a common canonical endomorphism for both $L \supset M$ and $N \supset K$, where Ψ and Φ are $*$-isomorphisms given by $\Psi = Adv : L \to Me$ and $\Phi : M \cong Me$. (See [**16**, Proposition 2.9].)

In this case, we get the inclusions

$$\begin{array}{ccccccccc} L_1 := & \langle L, e \rangle & \supset & L & \supset & M & = \gamma_1(L_1) & \supset & \gamma(L) \\ & \cup & & \cup & & \cup & & & \cup \\ N_1 := & \langle N, e \rangle & \supset & N & \supset & K & = \gamma_1(N_1) & \supset & \gamma(K), \end{array}$$

where the $*$-isomorphism $\gamma_1 : L_1 \to L$ is defined as the extension of γ with $\gamma_1(e) \in K$. Since the inclusions (L_1, L, N_1, N) is commuting and co-commuting, so is the inclusions $(M, \gamma(L), K, \gamma(N))$.

For two simultaneous canonical endomorphisms for (L, M, N, K), we have

LEMMA 3.2. *Let (L, M, N, K) be inclusions of properly infinite factors with canonical endomorphisms $\gamma_1 : L \to M$ and $\gamma_2 : L \to N$ satisfying that $\gamma_1|_N$ (resp. $\gamma_2|_M$) is a canonical endomorphism for $N \supset K$ (resp. $M \supset K$). Then the products $\gamma_1\gamma_2$ and $\gamma_2\gamma_1$ are canonical endomorphisms for $L \supset K$ and their difference is in* $\mathrm{Int}(K)$.

PROOF. We have

$$\begin{aligned} AdJ_KJ_L &= AdJ_KJ_N \circ AdJ_NJ_L \\ &= (Adu_K \circ \gamma_1|_N) \circ AdJ_NJ_L \\ &= (Adu_K \circ \gamma_1) \circ (Adu_N \circ \gamma_2) \\ &= Adu_K\gamma_1(u_N) \circ \gamma_1\gamma_2, \end{aligned}$$

where $u_K \in K$ and $u_N \in N$ are suitable unitaries. We remark that $u_K\gamma_1(u_N) \in K$. Similarly, we have

$$AdJ_KJ_L = AdJ_KJ_M \circ AdJ_MJ_L = Ad(v_K\gamma_2(u_M))\gamma_2\gamma_1$$

for unitaries $v_K \in K$ and $u_M \in M$, and we also have $v_K\gamma_2(v_M) \in K$. Hence, we get the conclusion. $\square$

Theorem 3.3. *Let (L, M, N, K) be inclusions of properly infinite factors with simultaneous canonical endomorphisms $\gamma_1 : L \to M$ and $\gamma_2 : L \to N$. Suppose that $L \cap K' = \boldsymbol{C}$. Then we have*

$$M \cap \gamma_1(N)' = \boldsymbol{C}.$$

Proof. We may write $M = \rho_1(L)$ and $N = \rho_2(L)$ for $\rho_1, \rho_2 \in \mathrm{End}(L)$. Since $\gamma_1 = \rho_1\overline{\rho_1} \in \mathrm{Sect}(L, M)$, we have

$$M \cap \gamma_1(N)' \cong \rho_1(L) \cap \rho_1\overline{\rho_1}\rho_2(L)' = \rho_1(L \cap \overline{\rho_1}\rho_2(L)').$$

Therefore, it is sufficient to show the irreducibility of $\rho := \overline{\rho_1}\rho_2 \in \mathrm{Sect}(L)$. By the Frobenius reciprocity, we have

$$\dim \mathrm{Hom}(\rho, \rho) = \dim \mathrm{Hom}(\rho_1\bar{\rho}_1\rho_2\bar{\rho}_2, \mathrm{id}_L).$$

Thanks to Lemma 3.2, we have $\rho_1\bar{\rho}_1\rho_2\bar{\rho}_2 = \gamma_1\gamma_2 = \gamma_K^L$. Hence, the irreducibility of $L \supset K$ implies the last dimension is 1; therefore, we get the conclusion. $\square$

This proposition means that an irreducible commuting co-commuting square gives another irreducible one: starting from a commuting co-commuting square, we have three kinds of other extensions. Applying Theorems 2.1 and 2.2 in [**20**], we get that if one of them is irreducible and of depth 2 then so are the others. As an application of Lemma 3.2, we have an alternative proof:

Proposition 3.4 ([**20**, Theorem 2.2]). *Let (L, M, N, K) be a commuting co-commuting square of properly infinite factors with $L \cap K' = \boldsymbol{C}$. Suppose that both of the inclusions $L \supset M$ and $L \supset N$ are of depth 2. Then so is the inclusion $L \supset K$.*

Proof. Let us denote

$$M = \rho_1(L), \quad N = \rho_2(L), \quad K = \rho_1\rho_3(L) = \rho_2\rho_4(L) = \rho(L)$$

for $\rho_1, \rho_2, \rho_3, \rho_4, \rho \in \mathrm{End}(L)$. Then the assumption means

$$\rho_1\overline{\rho_1}\rho_1 = (d\rho_1)^2\rho_1, \quad \rho_2\overline{\rho_2}\rho_2 = (d\rho_2)^2\rho_2.$$

Hence, by Lemma 3.2, we have

$$\begin{aligned}\rho\bar{\rho}\rho &= (\rho_1\overline{\rho_1}\rho_2\overline{\rho_2})\rho = \rho_1\overline{\rho_1}(\rho_2\overline{\rho_2}\rho_2)\rho_4 \\ &= \rho_1\overline{\rho_1}((d\rho_2)^2\rho_2\rho_4) = (d\rho_2)^2(\rho_1\overline{\rho_1}\rho_1)\rho_3 \\ &= (d\rho_2)^2((d\rho_1)^2\rho_1\rho_3) = (d\rho_1)^2(d\rho_2)^2\rho = (d\rho)^2\rho.\end{aligned}$$

Therefore, we are done. □

We remark that Theorem 2.1 in [**20**] and similar results in [**11**], [**23**] can be proved similarly.

In the remainder of this section, we would like to show that the existence of common canonical endomorphisms implies the commuting co-commuting property: let

$$\begin{array}{ccc} L & \supset & M \\ \cup & & \cup \\ N & \supset & K \end{array}$$

be inclusions of properly infinite factors with finite index. And assume that

(i) there is a canonical endomorphism $\gamma_1 : L \to M$ such that $\gamma_1|_N$ is a canonical endomorphism of $N \supset K$,

(ii) there is a canonical endomorphism $\gamma_2 : L \to N$ such that $\gamma_2|_M$ is a canonical endomorphism of $M \supset K$,

(iii) the inclusion $L \supset K$ is irreducible.

In this case, we have

$$[L : M]_0 = [N : K]_0 \quad (\text{and/or } [L : N]_0 = [M : K]_0).$$

And moreover, we have

Theorem 3.5. *The set of inclusions (L, M, N, K) forms a commuting co-commuting square.*

The following lemma is due to H. Kosaki.

Lemma 3.6. *There exists a non-zero isometry u in K such that $\gamma_1(x)u = ux$ for each $x \in M$.*

Proof. Since $\gamma_1 : L \to M$ is a canonical endomorphism, there is an isometry $u \in M$ such that

$$\gamma_1(x)u = ux \quad \text{for each } x \in M.$$

We have to show that u actually belongs to the smaller factor K. Since $M \supset K$, we obviously have

$$\gamma_1(x)u = ux \quad \text{for each } x \in K. \tag{3.1}$$

On the other hand, by the assumption (i) $\gamma_1|_N : N \to K$ is also a canonical endomorphism, and hence we can also find an isometry $\tilde{u}$ in K such that

$$\gamma_1(x)\tilde{u} = \tilde{u}x \quad \text{for each } x \in K. \tag{3.2}$$

Let $\iota_{K \hookrightarrow M}$ be the inclusion map considered as an element in $\mathrm{Sect}(K, M)$. Let $\gamma = \gamma_1|_K$ be the restriction considered as an element in $\mathrm{Sect}(K)$ (so that $\iota_{K \hookrightarrow M} \circ \gamma \in \mathrm{Sect}(K, M)$). Then, (3.1) and (3.2) mean

$$\iota_{K \hookrightarrow M} \circ \gamma(x)u = xu \quad \text{and} \quad \iota_{K \hookrightarrow M} \circ \gamma(x)\tilde{u} = \tilde{u}x \quad \text{for each } x \in K.$$

This means that both of $u \in M$ and $\tilde{u} \in K \subseteq M$ are intertwiners between the two K-M sectors $\iota_{K \hookrightarrow M} \circ \gamma, \iota_{K \hookrightarrow M}$. By the assumption (ii), $\gamma\ (=\gamma_1|_N) = \rho\bar{\rho}$ with $\rho \in \mathrm{Sect}(K)$ and $\gamma_1(N) = \rho(K)$, a downward basic extension of $N \supset K$. Therefore, we have

$$\begin{aligned} \dim \mathrm{Hom}(\iota_{K \hookrightarrow M} \circ \gamma, \iota_{K \hookrightarrow M}) &= \dim \mathrm{Hom}(\iota_{K \hookrightarrow M} \circ \rho\bar{\rho}, \iota_{K \hookrightarrow M}) \\ &= \dim \mathrm{Hom}(\iota_{K \hookrightarrow M} \circ \rho, \iota_{K \hookrightarrow M} \circ \rho). \end{aligned}$$

The algebra of self-intertwiners of $\iota_{K \hookrightarrow M} \circ \rho \in \mathrm{Sect}(K, M)$ is

$$M \cap (\iota_{K \hookrightarrow M} \circ \rho(K))' = M \cap \rho(K)'.$$

Since $\rho(K) = \gamma_1(N)$, the above relative commutant is one-dimensional by Theorem 3.3. Therefore, u and $\tilde{u}$ differ only by a scalar, and we conclude $u \in K$. $\square$

By the symmetric arguments, we can also take an isometry $v \in K$ such that $\gamma_2(x)v = vx$ for each $x \in N$.

PROOF. (Proof of Theorem 3.5) Taking intertwiners $u, v \in K$ as above, we set

$$E := u^*\gamma_1(\cdot)u, \quad F := v^*\gamma_2(\cdot)v.$$

By Proposition 5.1 in [**14**] and the assumption (iii), the map E is the common conditional expectation for both $L \supset M$ and $N \supset K$, and the map F is one for both $L \supset N$ and $M \supset K$. Therefore, the products EF and FE coincide with the conditional expectation E_K^L from L onto K because of the uniqueness of a conditional expectation for the irreducible inclusion $L \supset K$. Hence, we get the conclusion. $\square$

REMARK 2. Let $(H, \alpha), (K, \beta)$ be as in Proposition 2.7, and we consider the following inclusions of fixed point algebras:

$$\begin{array}{ccc} P & \supset & P^{(H,\alpha)} \\ \cup & & \cup \\ P^{(K,\beta)} & \supset & P^{(H,\alpha)} \cap P^{(K,\beta)}. \end{array}$$

We point out that these inclusions have the commuting canonical endomorphisms $\overline{\rho_1}\rho_1$ and $\rho_2\overline{\rho_2}$, and actually we have the following:

PROPOSITION 3.7. *Let P be a type II or type III factor, and let H and K be finite groups with outer actions $\alpha = \{\alpha_h\}_{h \in H}$ and $\beta = \{\beta_k\}_{k \in K}$ on P. We consider the implementing unitaries $\{u_h\}_{h \in H}$ of $\{\alpha_h\}_{h \in H}$ and $\{v_k\}_{k \in K}$ of $\{\beta_k\}_{k \in K}$ on the standard space $L^2(P)$, and we set $M := \{P \vee \{u_h\}_{h \in H}\}''$, $N := \{P \vee \{v_k\}_{k \in K}\}''$, and $L := M \vee N$. Assume that $\varepsilon(\alpha_H)$ and $\varepsilon(\beta_K)$ have the trivial intersection in $\mathrm{Out}(P)$ and that the inclusion*

$$P \supset P^{(H,\alpha)} \cap P^{(K,\beta)}$$

or $L \supset P$ has finite index. Then the extensions

$$\begin{array}{ccc} L & \supset & M \\ \cup & & \cup \\ N & \supset & P \end{array}$$

form a commuting square and $[L : P] = |\varepsilon(\alpha_H) \vee \varepsilon(\beta_K)|$. Moreover, these extensions or

$$\begin{array}{ccc} P & \supset & P^{(H,\alpha)} \\ \cup & & \cup \\ P^{(K,\beta)} & \supset & P^{(H,\alpha)} \cap P^{(K,\beta)} \end{array}$$

form a commuting co-commuting square if and only if $[L : P] = |HK|$ or $H \vee K = HK$ $(=KH)$ in $\mathrm{Out}(P)$.

We just give a sketch of a proof for a type *III* inclusion, because the argument is similar to that of Theorem 6.2 in [**21**]: let G be the group generated by $\varepsilon(\alpha_H)$ and $\varepsilon(\beta_K)$. For each $g \in G$, we choose a unitary $w_g \in L$ and an automorphism $\theta_g \in \mathrm{Aut}(P)$ such that $\theta_g = adw_g$, $\varepsilon(\theta_g) = g$, $w_e = e$, $\theta_e = e$, $w_{\varepsilon(\alpha_h)} = u_h$, $w_{\varepsilon(\beta_k)} = v_k$, and w_g is a word consisting of u_h and v_k. For a conditional expectation $E = E_P^L$, we have $E(w_g^* w_h) = 0$ $(g \neq h)$ to obtain the projection $q := \sum_g w_g e_P^L w_g^* \in \langle L, e_K^L \rangle$. The finite index assumption implies that

$$\infty > E^{-1}(1) \geq E^{-1}(JqJ) = E^{-1}\left(\sum Jw_g Je_P^L Jw_g^* J\right) = |G|,$$

hence G is finite. Each element of the *-algebra

$$L_0 := \left\{ x = \sum_{g \in G} x_g w_g : x_g \in P \right\}$$

has the unique expression; $x_g = E(xw_g^*)$, so the *-algebra L_0 is closed with respect to the strong operator topology, hence we have $L = L_0$. To see the relative commutant $L \cap P'$, we have $y(\sum_g x_g w_g) = (\sum_g x_g w_g)y$, or $yx_g = x_g \theta_g(y)$. Because of the freeness of θ_g $(g \neq e)$, $x_g = 0$ $(g \neq e)$ and $x_e \in P \cap P' = \boldsymbol{C}$, that is, $L \cap P' = \boldsymbol{C}$. This implies $w_g w_h \in P w_{gh}$, and we have a unitary representation y_g with $y_g \in P w_g$ ([**9**], [**22**]) to get the outer action γ of G on P such that $\gamma_g = ad y_g$. In this case, we can show that

$$\begin{array}{ccc} L & \supset & M \\ \cup & & \cup \\ N & \supset & P \end{array}$$

is isomorphic to

$$\begin{array}{ccc} P \rtimes_\gamma G & \supset & P \rtimes_\gamma H \\ \cup & & \cup \\ P \rtimes_\gamma K & \supset & P, \end{array}$$

and this forms a commuting square. The latter part of the proposition follows from Theorem 7.1 in [**21**].

References

[1] D. Bisch and U. Haagerup, Composition of subfactors: New examples of infinite depth subfactors, Ann. Sci. École Norm. Sup., **29** (1996), 329–383.

[2] F. Goodman, P. de la Harpe and V. F. R. Jones, Coxeter Graphes and Towers of Algebras, MSRI Publications, **14** (1989), Springer.

[3] D. Guido and R. Longo, Relativistic invariance and charge conjugation in quantum field theory, Comm. Math. Phys., **148** (1992), 521–551.

[4] F. Hiai, Minimizing indices of conditional expectations onto a subfactor, Publ. Res. Inst. Math. Sci., Kyoto Univ., **24** (1988), 673–678.

[5] J. H. Hong and H. Kosaki, The group of one-dimensional bimodules arising from composition of subfactors, J. Math. Soc. Japan, **52** (2000), 293–333.

[6] M. Izumi, Application of fusion rules to classification of subfactors, Publ. Res. Inst. Math. Sci., Kyoto Univ., **27** (1991), 953–994.

[7] M. Izumi and H. Kosaki, Finite dimensional Kac algebras arising from certain group actions on a factor, Internat. Math. Research Notices, **8** (1996), 357–370.

[8] M. Izumi and H. Kosaki, Kac algebras arising from composition of subfactors: General theory and classification, to appear in Mem. Amer. Math. Soc.

[9] V. F. R. Jones, Actions of finite groups on the hyperfinite type II_1 factor, Mem. Amer. Math. Soc., **237** (1980).

[10] V. F. R. Jones, Index for subfactors, Invent. Math., **72** (1983), 1–25.

[11] Y. Kawahigashi, Classification of paragroup actions on subfactors, Publ. Res. Inst. Math. Sci., Kyoto Univ., **31** (1995), 481–518.

[12] H. Kosaki, Sector theory and automorphisms for factor-subfactor pairs, J. Math. Soc. Japan, **48** (1996), 427–454.
[13] H. Kosaki, Type *III* Factors and Index Theory, Seoul National University GARC lecture notes, **43** (1998).
[14] R. Longo, Index of subfactors and statistics of quantum fields *I*, *II*, Comm. Math. Phys., **126** (1989), 217–247, **130** (1990), 285–309.
[15] R. Longo, Minimal index and braided subfactors, J. Funct. Anal., **109** (1992), 98–112.
[16] R. Longo and K.-H. Rehren, Nets of subfactors, Rev. Math. Phys., **7** (1995), 567–597.
[17] A. Ocneanu, Quantized group, string algebras and Galois theory for algebras, London Math. Soc. Lecture Note Ser., **136** (1988), 119–172.
[18] S. Popa, Orthogonal pairs of *-subalgebras in finite von Neumann algebras, J. Operator Theory, **16** (1986), 261–272.
[19] S. Popa, Classification of amenable subfactors of type *II*, Acta Math., **172** (1994), 163–256.
[20] T. Sano, Commuting co-commuting squares and finite dimensional Kac algebras, Pacific J. Math., **172** (1996), 243–253.
[21] T. Sano and Y. Watatani, Angles between two subfactors, J. Operator Theory, **32** (1994), 209–241.
[22] C. Sutherland, Cohomology and extensions of von Neumann algebras *I*, *II*, Publ. Res. Inst. Math. Sci., Kyoto Univ., **16** (1980), 105–133, 135–174.
[23] J. Wierzbicki, An estimation of the depth from an intermediate subfactor, Publ. Res. Inst. Math. Sci., Kyoto Univ., **30** (1994), 1139–1144.

Takashi SANO
Department of Mathematical Sciences
Faculty of Science
Yamagata University
Yamagata 990-8560
Japan