J. Math. Soc. Japan
Vol. 55, No. 1, 2003

Periodic leaves for diffeomorphisms preserving codimension one foliations

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(Received Jun. 21, 1999)
(Revised May 23, 2001)

Abstract. We consider the group of diffeomorphisms of a compact manifold M which preserve a codimension one foliation $\mathscr{F}$ on M. For the C^2 case if $\mathscr{F}$ has compact leaves with nontrivial holonomy then at least one of these leaves is periodic. Our main result is proved in the context of diffeomorphisms which preserve commutative actions of finitely generated groups on $[0,1]$. Applying this result to foliations almost without holonomy we prove the periodicity of all compact leaves with nontrivial holonomy. We also study the codimension one foliation preserving diffeomorphisms that are C^2 close to the identity.

Introduction.

Let $\mathscr{F}$ be a C^r $(1 \leq r \leq \infty)$ transversely orientable codimension one foliation of a smooth compact manifold M possibly with boundary. By $\mathrm{Diff}_+^r(M;\mathscr{F})$ we denote the subgroup of $\mathrm{Diff}^r(M)$ consisting of those diffeomorphisms which preserve both the foliation $\mathscr{F}$ (they carry leaves of $\mathscr{F}$ to leaves of $\mathscr{F}$) and the transverse orientation of $\mathscr{F}$, endowed with the C^r natural topology.

Let $f \in \mathrm{Diff}_+^r(M;\mathscr{F})$. A leaf L of $\mathscr{F}$ is *f-periodic* with *period* $k \geq 1$ if $f^k(L) = L$ and $f^i(L) \neq L$, $1 \leq \forall i \leq k-1$. If $f(L) = L$ the leaf L is also called an *f-invariant* leaf.

In this paper we present some results concerning the existence of f-periodic (or f-invariant) leaves. Our main interest is on the occurrence of such leaves in the set of compact leaves of $\mathscr{F}$ which we denote by $\mathscr{C}(\mathscr{F})$. Clearly the leaves lying in the interior of $\mathscr{C}(\mathscr{F})$ need not be periodic. We denote the frontier of $\mathscr{C}(\mathscr{F})$ (i.e. the union of the compact leaves with nontrivial holonomy) by $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$.

Our plan is as follows.

First we show that in the C^2 case (Theorem 2.1) provided that $\mathrm{Fr}(\mathscr{C}(\mathscr{F})) \neq \emptyset$ for any $f \in \mathrm{Diff}_+^2(M;\mathscr{F})$ the foliation $\mathscr{F}$ always has an f-periodic leaf in $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$.

2000 *Mathematics Subject Classification*. Primary 57R30.
Key Words and Phrases. periodic leaf, commutativity, codimension one foliation.
Partially supported by CNPq and FAPERJ of Brazil.

Having established the existence of periodic leaves in $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$, we wish to know whether all the leaves in $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$ are f-periodic. Using Theorem 2.1 and modifying $f \in \mathrm{Diff}^r_+(M;\mathscr{F})$ along the leaves of $\mathscr{F}$ we reduce this problem to the following question which is actually the main question of this paper.

Question 3.3. *Let $\mathfrak{G}$ be a finitely generated group and let $H : \mathfrak{G} \to \mathrm{Diff}^2_+([0,1])$ be a representation. If $\psi \in \mathrm{Aut}(\mathfrak{G})$ and $f \in \mathrm{Diff}^2_+([0,1])$ satisfy the relation $H_{\psi(\alpha)} = f \circ H_\alpha \circ f^{-1}$ for all $\alpha \in \mathfrak{G}$ then $\mathrm{Fix}(f) \supset \mathrm{Fr}(\mathrm{Fix}(H))$?*

As already remarked by E. Ghys and V. Sergiescu [**6**] and T. Tsuboi [**24**] an application of Kopell's Lemma gives an affirmative answer to this question for the case $\psi \equiv \mathrm{Id}_{\mathfrak{G}}$.

Our main result gives an affirmative response to Question 3.3 when the image of H is a commutative group.

Theorem 5.3. *Let $\mathfrak{G}$ be a finitely generated group and let $H : \mathfrak{G} \to \mathrm{Diff}^2_+([0,1])$ be a representation. Suppose there are $\psi \in \mathrm{Aut}(\mathfrak{G})$ and $f \in \mathrm{Diff}^2_+([0,1])$ such that $H_{\psi(\alpha)} = f \circ H_\alpha \circ f^{-1}$, $\forall \alpha \in \mathfrak{G}$. If $\mathrm{Im}(H)$ is a commutative subgroup of $\mathrm{Diff}^2_+([0,1])$ then $\mathrm{Fix}(f) \supset \mathrm{Fr}(\mathrm{Fix}(H))$.*

The proof of this theorem requires a small generalization of Kopell's Lemma which we give in §4 (Lemma 4.1).

We recall from the theory of transversely orientable codimension one foliations on compact manifolds that in foliations almost without holonomy the holonomies of compact leaves are commutative. Then Theorem 5.3 implies the following corollary.

Corollary. *Let $\mathscr{F} \in \mathrm{Fol}^2_+(M)$ be a foliation almost without holonomy and let $f \in \mathrm{Diff}^2_+(M;\mathscr{F})$. Then every leaf in $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$ is f-periodic.*

Here, $\mathrm{Fol}^r_+(M)$ denotes the space of all C^r codimension one transversely orientable and tangent to the boundary foliations on M endowed with the Epstein C^r topology [**3**]. The space $\mathrm{Diff}^r(M)$ is endowed with the natural C^r topology.

In §6 we give some results concerning the invariance of noncompact leaves in foliations almost without holonomy which will be necessary in §7. These results show that there is a strong rigidity on the transversal dynamics of diffeomorphisms preserving this type of foliations.

In §7 we study the invariance of leaves under diffeomorphisms $f \in \mathrm{Diff}^r_+(M;\mathscr{F})$ which are C^r close to Id_M. To deal with such diffeomorphisms we introduce the notion of a *strongly f-invariant* leaf, that is, an f-invariant leaf L of $\mathscr{F}$ such that for any x in L the points x and $f(x)$ are close along L, in the induced metric on L. The main results of this section are Theorems 7.2 and 7.8 stated below.

THEOREM 7.2. *Let $\mathscr{F} \in \mathrm{Fol}^2_+(M)$. There exists a neighborhood $\mathscr{V}$ of the identity in $\mathrm{Diff}^2(M)$ such that for all $f \in \mathscr{V} \cap \mathrm{Diff}^2_+(M;\mathscr{F})$ we have*:

(i) *the leaves with nontrivial holonomy are strongly f-invariant*,

(ii) *if a connected component of $M - \mathscr{C}(\mathscr{F})$ has an f-invariant leaf* (*resp. a strongly f-invariant leaf*) *then all leaves lying in this component are f-invariant* (*resp. strongly f-invariant*).

At the end of the paper we apply Theorem 2.1 together with a result of Duminy [**5**] to the compact leaf persistence problem (Definition 7.7).

THEOREM 7.8. *Let $\mathscr{F} \in \mathrm{Fol}^r_+(M)$, $r \geq 2$ and let Q be a closed manifold. Let $\mathscr{F}_Q$ denote the foliation on $Q \times M$ obtained by multiplying by Q each leaf of $\mathscr{F}$. Suppose that $\mathscr{F}$ cannot be C^r approximated by foliations defined by fibrations over S^1. Then $\mathscr{F}$ has the compact leaf persistence property if and only if the same is true for $\mathscr{F}_Q$.*

We would like to thank E. Ghys for clarifying conversations and hospitality at E.N.S. de Lyon where part of this work was developed. We also thank T. Barbot and D. Gaboriau for their suggestions.

CONVENTIONS. In this paper all manifolds are smooth connected, possibly with boundary. We shall assume that all foliations and plane fields have codimension one and are tangent to the boundary (if the boundary is non empty), unless the context clearly indicates otherwise.

Throughout this paper M denotes a compact manifold.

1. Prelimaries.

This section reviews briefly the notion of equivalence class of compact leaves and support of a class of a codimension one foliation introduced by Bonatti-Firmo [**1**], which will be used in this paper.

They define the following equivalence relation on the set of compact leaves of $\mathscr{F}$: two compact leaves L and $\tilde{L}$ of $\mathscr{F}$ are said to be *equivalent* if there exists an immersion $\imath : L \times [a,b] \to M$ with $a \leq b$ satisfying the following conditions:

1.1 for every $t \in [a,b]$ the restriction of the map $\imath$ to $L \times \{t\}$ is an embedding of L in M,

1.2 $\imath(L \times \{a\}) = L$ and $\imath(L \times \{b\}) = \tilde{L}$,

1.3 for each $x \in L$ the path $\imath_x : (a,b) \to M$ defined by $\imath_x(t) = \imath(x,t)$ is transverse to $\mathscr{F}$.

In this case we say that the immersion $\imath$ *realizes the equivalence* of the compact leaves L and $\tilde{L}$. The equivalence class of L will be denoted by $[L]$.

Observe that if the immersion $\imath$ is not an embedding then M fibers over

S^1 with fiber L, and each compact leaf of $\mathscr{F}$ is a fiber of the fibration. Furthermore, there exists a vector field transverse to both the foliation $\mathscr{F}$ and to the fibers.

The equivalence classes of compact leaves satisfy:

1.4 $\mathscr{F}$ has finitely many equivalence classes of compact leaves,

1.5 if Λ consists of all immersions which realise some equivalence of compact leaves equivalent to L then the subset of the manifold M given by $\bigcup_{\iota \in \Lambda} \iota(L \times [a,b])$ is a compact subset of M, saturated by $\mathscr{F}$ and contains all the compact leaves equivalent to L. This compact set will be referred as the *support* of L and will be denoted by $\mathrm{supp}[L]$,

1.6 there exists $\iota_0 \in \Lambda$ such that $\mathrm{supp}[L] = \iota_0(L \times [a,b])$,

1.7 there exists a neighborhood U of $\mathrm{supp}[L]$ in M such that every leaf of $\mathscr{F}$ meeting U has a compact leaf equivalent to L in its adherence,

1.8 if L and $\tilde{L}$ are non equivalent compact leaves of the foliation $\mathscr{F}$ then $\mathrm{supp}[L] \cap \mathrm{supp}[\tilde{L}] = \emptyset$.

The proofs of 1.4 to 1.8 can be found in [**1**] or in [**4**].

2. Periodic compact leaves.

If a foliation $\mathscr{F}$ is defined by a fibration $L \to M \xrightarrow{\pi} S^1$ (the leaves of $\mathscr{F}$ are the fibers of π) then the transversal dynamic properties of $f \in \mathrm{Diff}^r_+(M;\mathscr{F})$ are completely determined by the induced diffeomorphism $g \in \mathrm{Diff}^r_+(S^1)$. In this case the f-periodic leaves correspond to the g-periodic points. If $r \geq 2$ it is well known that either all the orbits of g are dense or g has a periodic point and all periodic points have the same period.

In this section we analyse the existence of compact periodic leaves for a foliation $\mathscr{F}$ which is not defined by a fibration. The leaves lying in the interior of $\mathscr{C}(\mathscr{F})$ need not be periodic, however we do not know if $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$ can have non periodic leaves. This question will be considered in §3 and §5.

For a compact leaf L we have $f(\mathrm{supp}[L]) = \mathrm{supp}[f(L)]$ for $f \in \mathrm{Diff}^r_+(M;\mathscr{F})$. Thus both the f-period of $\mathrm{supp}[L]$ and the f-period of $[L]$ are well defined. Hence in view of 1.4, if $\mathrm{supp}[L]$ is not a fibration over S^1 then its boundary leaves are f-periodic. The following example shows that the f-period can change from one class to another.

Example. Let α and β be the generators of $\pi_1(T^2)$ and consider the foliation on $[0,1] \times T^2$ defined by the suspension of a representation $H : \pi_1(T^2) \to \mathrm{Diff}^\infty_+([0,1])$ satisfying:

$H_\beta = \mathrm{Id}$, $\mathrm{Fix}(H_\alpha) = \{0,1\}$ and H_α is C^∞-flat on $\mathrm{Fix}(H_\alpha)$.

On each boundary component of $[0,1] \times T^2$ we paste a solid torus foliated by a C^∞-flat Reeb component in such way that α is null homotopic in both solid torus. This process gives a foliation $\mathscr{F} \in \mathrm{Fol}_+^\infty(S^2 \times S^1)$ with nontrivial holonomies H_α and H_β. With small perturbations of α and β we construct closed curves Γ_1 and Γ_2 transverse to $\mathscr{F}$ such that:

– Γ_1 generates $\pi_1(S^2 \times S^1)$,
– Γ_2 is null homotopic in $S^2 \times S^1$.

For such an $\mathscr{F}$ the turbulization along the closed transversals Γ_1 and Γ_2 produces a foliation $\mathscr{G}$ on $S^2 \times S^1$ with two Reeb components $\Re(\Gamma_i)$; $i = 1, 2$ lying in small neighborhoods of Γ_i; $i = 1, 2$.

Let $\tilde{\mathscr{G}}$ and $\tilde{\Re}(\Gamma_1)$ be the lifts to the double cover of $S^2 \times S^1$ of $\mathscr{G}$ and $\Re(\Gamma_1)$ respectively. If $\tilde{\Re}_1(\Gamma_2)$ and $\tilde{\Re}_2(\Gamma_2)$ denote the two distinct lifts of $\Re(\Gamma_2)$ then the nontrivial covering map f preserves $\tilde{\mathscr{G}}$ and the f-period of the class $[\partial\tilde{\Re}(\Gamma_1)]$ is 1 while the f-period of $\partial\tilde{\Re}_i(\Gamma_2)$ is 2.

To assure the existence of periodic compact leaves we can restrict ourselves to the group of diffeomorphisms which preserve both $\mathscr{C}(\mathscr{F})$ (instead of $\mathscr{F}$) and the transverse orientations of $\mathscr{F}$ in $\mathscr{C}(\mathscr{F})$, i.e. the group of diffeomorphisms of M which carry leaves in $\mathscr{C}(\mathscr{F})$ into leaves in $\mathscr{C}(\mathscr{F})$ preserving the transverse orientations of $\mathscr{F}$ in $\mathscr{C}(\mathscr{F})$. For this we need the following extension of the definition of equivalence classes of compact leaves.

Two compact leaves L and L' are in the same *extended equivalence class* if they satisfy 1.1, 1.2 and

1.3* for each $x \in L$ the path $\iota_x : (a, b) \to M$ is transverse to the compact leaves of $\mathscr{F}$.

The extended equivalence class of L which we will denote by $[L]_e$ satisfies 1.4 and 1.5. Thus the support of the extended class, $\mathrm{supp}[L]_e$, is well defined and satisfies 1.6 and 1.8. Moreover, $\mathrm{supp}[L]_e$ is a finite union of:

– some supports of equivalence classes (not extended) of compact leaves diffeomorphic to L, and
– foliated products $L \times [0, 1]$ where $L \times \{0\}$ and $L \times \{1\}$ are the unique compact leaves of $\mathscr{F}$ in $L \times [0, 1]$ and $\mathscr{F}|_{L \times [0,1]}$ is not defined by a suspension.

In each support (not extended) $L \times [a, b]$ the foliation is given by a suspension. Thus, its compact leaves are fibers of the trivial fibration $L \times [a, b] \to [a, b]$. Moreover, we can extend these fibrations in order to obtain a locally trivial fibration on $\mathrm{supp}[L]_e$. This extension is a consequence of the following remark: a C^r foliation on $L \times [-\varepsilon, 0] \cup L \times [1, 1 + \varepsilon]$ with trivial holonomy and having $L \times \{0\}$ and $L \times \{1\}$ as compact leaves can be extended (by extending the trivial holonomy to the interval $[-\varepsilon, 1 + \varepsilon]$) to a C^r foliation by compact leaves without holonomy on $L \times [0, 1]$.

Of course, if $f \in \mathrm{Diff}^r(M)$ preserves $\mathscr{C}(\mathscr{F})$ then for each compact leaf L of $\mathscr{F}$ we have $f(\mathrm{supp}[L]_e) = \mathrm{supp}[f(L)]_e$.

Theorem 2.1. *Let* $\mathscr{F} \in \mathrm{Fol}_+^2(M)$ *and let* $f \in \mathrm{Diff}^2(M)$. *Suppose that* f *preserves both* $\mathscr{C}(\mathscr{F})$ *and the transverse orientations of* $\mathscr{F}$ *in* $\mathscr{C}(\mathscr{F})$. *If* $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$ *is non empty then* $\mathscr{F}$ *has an* f*-periodic compact leaf in* $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$. *Furthermore, if* L *and* L' *are* f*-periodic compact leaves then*

- L *and* L' *have the same period when* $[L]_e = [L']_e$,
- L *and* $\mathrm{supp}[L]_e$ *have the same period when* $\mathrm{supp}[L]_e \neq M$.

Proof. Suppose $\mathrm{Fr}(\mathscr{C}(\mathscr{F})) \neq \emptyset$. We have to consider two cases.

Case 1: $\mathrm{supp}[L]_e$ *fibers over* $[a,b]$ *with* $a \leq b$, *for every compact leaf* L.

Let $\mathrm{supp}[L_1]_e, \dots, \mathrm{supp}[L_k]_e$ be the supports of the extended equivalence classes of the compact leaves of $\mathscr{F}$ and denote by L_i^+ and L_i^- the leaves in the frontier of $\mathrm{supp}[L_i]_e$; $i = 1, \dots, k$. Since f preserves the extended supports, $f^n(L_j^+)$ and $f^n(L_j^-)$ lie in $\bigcup_{i=1}^k \{L_i^+ \cup L_i^-\}$ and thus L_i^+ and L_i^-; $i = 1, \dots, k$ are f-periodic. If one of these leaves lies in $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$ the proof is finished. If not, $M = \mathrm{supp}[L]_e = L \times [a,b]$ for some compact leaf L. In this case the connected component of $M - \mathrm{Fr}(\mathscr{C}(\mathscr{F}))$ which contains $L \times \{a\}$ is f-invariant. In particular the boundary leaves of this component are f-invariant and one of them necessarily lies in $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$.

Case 2: $\mathrm{supp}[L]_e$ *fibers over* S^1, *for some compact leaf* L.

Let $L \to M \xrightarrow{\pi} S^1$ be the fibration defined by $\mathrm{supp}[L]_e$. Fix a section Γ of π which we identify with the base S^1.

Since the compact leaves of $\mathscr{F}$ are fibers of π and since f preserves the compact leaves of $\mathscr{F}$, there exist a finite number of disjoint compact arcs $A_1, \dots, A_s$ in Γ satisfying

- $f|_\Gamma$ is transverse to the fibers of π in $\Gamma - \bigcup_{i=1}^s \mathrm{Int}(A_i)$,
- $\mathscr{C}(\mathscr{F}) \cap (\bigcup_{i=1}^s A_i) = \emptyset$.

Now, increasing the size of the segments A_i if necessary, we can modify $f|_\Gamma$ on each A_i to produce a C^2 embedding $g : \Gamma \to M$ such that

- $f|_\Gamma \equiv g$ on $\Gamma - \bigcup_{i=1}^s \mathrm{Int}(A_i)$,
- $\{\bigcup_{i=1}^s g(A_i)\} \cap \mathscr{C}(\mathscr{F}) = \emptyset$,
- g is transverse to the fibers of π.

It follows that $\tilde{f} = \pi \circ g$ is a C^2 orientation preserving diffeomorphism of S^1. The conclusion follows from Denjoy's theorem and Lemma 2.2 below taking into account that the compact subset $K = \mathscr{C}(\mathscr{F}) \cap \Gamma$ of Γ is invariant under $\tilde{f}$ and $\mathscr{F}$ is not a fibration.

The last two assertions follow from properties of orientation preserving diffeomorphisms of the circle and of the interval. □

Given a representation $H : \mathfrak{G} \to \mathrm{Diff}_+^r(S^1)$ of a group $\mathfrak{G}$ we denote by $\mathcal{O}_H(x)$ the orbit of $x \in S^1$ and by $\mathscr{C}(H)$ the set of points $x \in S^1$ for which $\mathcal{O}_H(x)$ is finite. If $\mathcal{O}_H(x)$ is finite and $y \notin \mathcal{O}_H(x)$ then in each connected component of

$S^1 - \mathcal{O}_H(x)$ we have at least one point of $\mathcal{O}_H(y)$. Thus, $\sharp(\mathcal{O}_H(x)) \le \sharp(\mathcal{O}_H(y))$. Therefore $\sharp(\mathcal{O}_H(x)) = \sharp(\mathcal{O}_H(y))$ when $\mathcal{O}_H(x)$ and $\mathcal{O}_H(y)$ are finite. In particular $\mathcal{C}(H)$ is compact.

Lemma 2.2. *Let $H : \mathfrak{G} \to \mathrm{Diff}^1_+(S^1)$ be a representation of a group $\mathfrak{G}$. Let K be a compact subset of S^1 such that $\mathrm{Fr}(K) \neq \emptyset$. Suppose that K is invariant under H, i.e. $H_\alpha(K) = K$, $\forall \alpha \in \mathfrak{G}$. If $\mathcal{C}(H) \neq \emptyset$ then $\mathcal{C}(H) \cap \mathrm{Fr}(K) \neq \emptyset$.*

Proof. Suppose that $\mathcal{C}(H) \neq \emptyset$ and let $x \in \mathcal{C}(H)$. We have to consider two cases: (1) $x \in \mathrm{Int}(K)$ and (2) $x \notin K$. In the first case the connected component of K which contains x is a compact interval whose extremities are in $\mathcal{C}(H)$. In the second case the connected component of $S^1 - K$ which contains x is an open interval whose extremities are in $\mathcal{C}(H)$. $\square$

3. Periodic leaves and invariant fixed points.

In this section we describe a modification of the diffeomorphism f along the leaves of $\mathcal{F}$ which allows us to reduce the study of periodic compact leaves to the study of invariant fixed points of a group action on $[0,1]$.

First observe that we can assume that $\mathcal{F}$ is given by the suspension of a representation of a group on $\mathrm{Diff}^r_+([0,1])$. In fact, by Theorem 2.1 the foliation $\mathcal{F}$ has a compact f-periodic leaf L. Replacing f with f^k, where k is the period of L, the compact f-periodic leaves lying in $\mathrm{supp}[L]$ become f^k-invariant leaves. If $\mathrm{supp}[L]$ fibers over S^1 then by cutting M along L, we can assume that $\mathrm{supp}[L]$ fibers over $[0,1]$.

Thus consider $\mathcal{F} \in \mathrm{Fol}^r_+([0,1] \times F)$ transverse to $[0,1] \times \{x\}$ for all $x \in F$, where F is a closed $(n-1)$-manifold. Under these conditions we have the following lemma.

Lemma 3.1. *For each $f \in \mathrm{Diff}^r_+([0,1] \times F; \mathcal{F})$ and $p \in F$ there is a diffeomorphism $g \in \mathrm{Diff}^r_+([0,1] \times F; \mathcal{F})$ satisfying*:

(i) *the maps $f|_{\{0\}\times F}$ and $g|_{\{0\}\times F}$ are C^r-isotopic in $\{0\} \times F$,*

(ii) $(0,p) \in \mathrm{Fix}(g)$,

(iii) $g([0,1] \times \{x\}) = [0,1] \times \{g(x)\}$, *for all $x \in F$,*

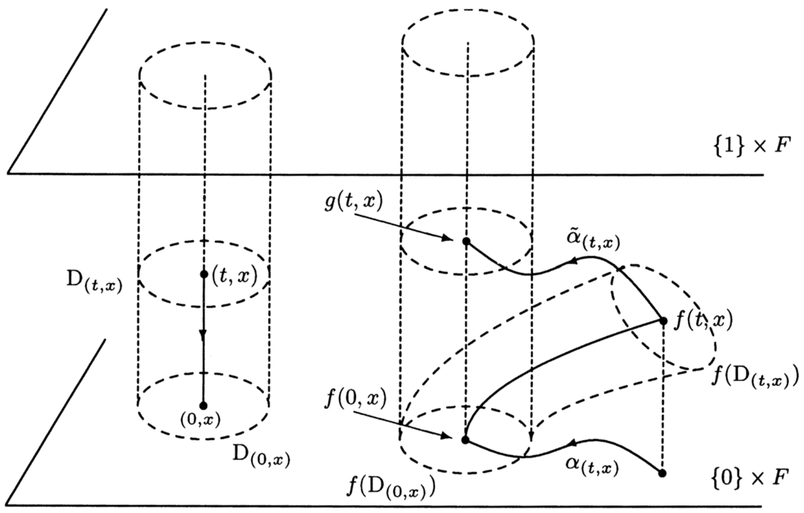
(iv) *$f(z)$ and $g(z)$ lie in the same leaf of $\mathcal{F}$ for each $z \in [0,1] \times F$.*

Proof. Let $p \in F \equiv \{0\} \times F$ be such that $f(p) \neq p$ and let α be an embedded path on F from p to $f(p)$. The foliation $\mathcal{F}$ restricted to a convenient small neighborhood of α (in $[0,1] \times F$) is a product foliation. Thus, isotoping f along the leaves of $\mathcal{F}$ by an isotopy supported on this neighborhood we can obtain a diffeomorphism fixing the point p.

Thus, without loss of generality we may assume that $f(p) = p$.

Without changing $f|_{\{0\}\times F}$ we shall construct the required diffeomorphism, as follows:

Let $(t,x) \in [0,1] \times F$ and denote by π_2 the natural projection of $[0,1] \times F$ on F. Let $\alpha_{(t,x)}$ be the path on F defined as $\alpha_{(t,x)}(s) = \pi_2 \circ f((1-s)t, x)$ for all $s \in [0,1]$. We define $g(t,x) = \tilde{\alpha}_{(t,x)}(1)$ where $\tilde{\alpha}_{(t,x)}$ is the lift of $\alpha_{(t,x)}$ starting at $f(t,x)$ on the leaf passing through $f(t,x)$.



It is easy to verify that g is a C^r map satisfying (i), (ii), (iii) and (iv). Now it remains to show that g is a diffeomorphism.

First we prove that g is a local diffeomorphism in the leaf direction. For this fix $(t,x) \in [0,1] \times F$ and let $\mathrm{D}_{(0,x)}$ be a compact $(n-1)$-disk on F centered at $(0,x)$. The leaf passing through (t,x) of the restricted foliation $\mathscr{F}|_{[0,1]\times \mathrm{D}_{(0,x)}}$ is a $(n-1)$-disk diffeomorphic to $\mathrm{D}_{(0,x)}$ under the projection π_2. It will be denoted by $\mathrm{D}_{(t,x)}$.

Now let $\Theta : f(\mathrm{D}_{(t,x)}) \to f(\mathrm{D}_{(0,x)})$ be the diffeomorphism defined by the composite

$$f(\mathrm{D}_{(t,x)}) \xrightarrow{f^{-1}} \mathrm{D}_{(t,x)} \xrightarrow{\pi_2} \mathrm{D}_{(0,x)} \xrightarrow{f} f(\mathrm{D}_{(0,x)}).$$

Let L be the leaf of $\mathscr{F}$ passing through $f(t,x)$ and $g(t,x)$. The leaf of $\mathscr{F}|_{[0,1]\times f(\mathrm{D}_{(0,x)})}$ passing through $g(t,x)$ is also a $(n-1)$-disk diffeomorphic to $f(\mathrm{D}_{(0,x)})$ under the projection π_2 whose inverse in this case will be denoted by π^{-1}.

Consequently, the map $g : \mathrm{D}_{(t,x)} \to L$ is given by the composite

$$\mathrm{D}_{(t,x)} \xrightarrow{f} f(\mathrm{D}_{(t,x)}) \xrightarrow{\Theta} f(\mathrm{D}_{(0,x)}) \xrightarrow{\pi^{-1}} L$$

and we conclude that g is a local diffeomorphism in the leaf direction.

In a neighborhood of (t,x) in $[0,1] \times \{x\}$ the map g is given by the composite of f and the holonomy diffeomorphism of the path $\tilde{\alpha}_{(t,x)}$ relative to the transversals $f([0,1] \times \{x\})$ and $[0,1] \times \{f(0,x)\}$. Therefore, g is a local diffeomorphism.

Since $g|_{\{0\} \times F} = f|_{\{0\} \times F}$ is a diffeomorphism of F, we conclude that g is a diffeomorphism and the proof is finished. $\square$

It follows from (iv) that f and g coincide on the quotient space $([0,1] \times F)/\mathscr{F}$. Consequently, they have the same invariant leaves.

From now on we only consider diffeomorphisms $f \in \mathrm{Diff}^r_+([0,1] \times F; \mathscr{F})$ satisfying:

3.2. *$f([0,1] \times \{p\}) = [0,1] \times \{p\}$ for some point $p \in F$.*

In this situation we have the following property:

Given $z \in [0,1] \times \{p\}$ and a loop α in F based at p, let $\tilde{\alpha}_z$ be the lifting of α to a leaf of $\mathscr{F}$ such that $\tilde{\alpha}_z(0) = z$. Since both $f(\tilde{\alpha}_z)$ and $[\widetilde{f(\alpha)}]_{f(z)}$ depend continuously on z, its projections on F are homotopic.

Consequently the holonomy maps H_α and $H_{f(\alpha)}$ of $\mathscr{F}$ associated to the loops α and $f(\alpha)$ satisfy:

$$H_{f(\alpha)} = f \circ H_\alpha \circ f^{-1}.$$

Therefore the problem about the periodicity of compact leaves which are contained in $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$ can be reduced to the following question:

Question 3.3. *Let $\mathfrak{G}$ be a finitely generated group and let $H : \mathfrak{G} \to \mathrm{Diff}^2_+([0,1])$ be a representation. If $\psi \in \mathrm{Aut}(\mathfrak{G})$ and $f \in \mathrm{Diff}^2_+([0,1])$ satisfy the relation $H_{\psi(\alpha)} = f \circ H_\alpha \circ f^{-1}$ for all $\alpha \in \mathfrak{G}$ then $\mathrm{Fix}(f) \supset \mathrm{Fr}(\mathrm{Fix}(H))$?*

The following remark will be used in §6.

Remark 3.4. Let $\mathscr{F} \in \mathrm{Fol}^r_+(M)$, let $f \in \mathrm{Diff}^r_+(M; \mathscr{F})$ and let L be an f-invariant compact leaf of $\mathscr{F}$. It is easy to construct an isotopy $\{f_t\}_{t \in [0,1]}$ of f along the leaves of $\mathscr{F}$ and such that the map f_1 preserves the fibers of a small tubular neighborhood of L.

4. On Kopell's Lemma.

In this section we prove a more general version of the famous Kopell's Lemma which we will use in the next section. The proof we give here is essentially an extension of the proof of Lemma 1.a given by Kopell in [**14**, pp. 168–169].

Lemma 4.1. *Let f be a C^2 contraction to 0 on $[0,a]$, $a > 0$. Suppose there is a sequence $g_n \in \mathrm{Diff}^1_+([f(a),a])$; $n \in \boldsymbol{N} \cup \{0\}$ such that the map $g : (0,a] \to (0,a]$*

defined by $g(x) = f^n \circ g_n \circ f^{-n}(x)$, $\forall x \in [f^{n+1}(a), f^n(a)]$ *is a* C^1*-diffeomorphism. Then* g *extends to a* C^1*-diffeomorphism on* $[0,a]$ *if and only if* $g_n \to \mathrm{Id}$ *in the* C^1*-topology.*

Before proving Lemma 4.1 we recall the following two crucial results stated by Kopell in [**14**, pp. 168–169].

If f is a C^2-contraction to 0 on $[0,a]$ then:

(I) $A(x,y;n) = (f^n)'(y)/((f^n)'(x))$ *converges uniformly on compact subsets of* $(0,a] \times (0,a]$ *to a continuous strictly positive function* $A(x,y)$,

(II) *the differential equation*

$$y' = \frac{1}{A(x,y)}; \quad x, y \in (0,a] \tag{4.1}$$

has the following property: *if* g *is a solution of* 4.1 *such that* $g(x_o) = x_o$ *for some* $x_o \in (0,a]$ *then* $g \equiv \mathrm{Id}$.

PROOF OF LEMMA 4.1. From the definition of g we have that

$$g(f^n(x)) = f^n(g_n(x)), \quad \forall x \in [f(a),a], n \in \boldsymbol{N}.$$

Differentiation yields

$$g'(f^n(x)).(f^n)'(x) = (f^n)'(g_n(x)).g_n'(x) \tag{4.2}$$

for all $x \in [f(a),a]$, $n \in \boldsymbol{N}$.

Then, we get

$$g_n'(x) = g'(f^n(x))\frac{(f^n)'(x)}{(f^n)'(g_n(x))}, \quad \forall x \in [f(a),a], n \in \boldsymbol{N}. \tag{4.3}$$

Hence g_n is a solution of the differential equation

$$y' = \frac{g'(f^n(x))}{A(x,y;n)}; \quad x, y \in [f(a),a] \tag{4.4}$$

satisfying the initial condition $g_n(a) = a$.

We observe that $g'(f^n(x))/A(x,y;n)$ is a continuous map of $(x,y) \in [f(a),a]^2$ and has continuous partial derivative with respect to y, which assures the existence and uniqueness of the solutions of 4.4.

Suppose g extends to a C^1-diffeomorphism of $[0,a]$. Since $f^n(a) \in \mathrm{Fix}(g)$ we have $g'(0) = 1$ which implies that $g'(f^n(x)) \to 1$ uniformly on $[f(a),a]$. Thus, $g'(f^n(x))/A(x,y;n)$ converges to $1/A(x,y)$ on $[f(a),a] \times [f(a),a]$. We will show that the differential equation

$$y' = \frac{1}{A(x,y)}; \quad x, y \in (0,a] \tag{4.5}$$

also satisfies the property of existence and uniqueness of solutions.

Indeed, if g_1 and g_2 are solutions of 4.5 which coincide at a point x_o then g_2' never vanishes and we have $g_2^{-1} \circ g_1(x_o) = x_o$. On the other side, we have

$$\begin{aligned}(g_2^{-1} \circ g_1)'(x) &= (g_2^{-1})'(g_1(x)).g_1'(x) \\ &= \frac{1}{g_2'(g_2^{-1} \circ g_1(x))} g_1'(x) = \frac{A(g_2^{-1} \circ g_1(x), g_1(x))}{A(x, g_1(x))} \\ &= \frac{1}{A(x, g_1(x)).A(g_1(x), g_2^{-1} \circ g_1(x))} = \frac{1}{A(x, g_2^{-1} \circ g_1(x))}.\end{aligned}$$

Thus $g_2^{-1} \circ g_1$ is a solution of 4.5 having x_o as a fixed point. By (II) it follows that $g_1 \equiv g_2$ on a neighborhood of x_o and hence $g_1 \equiv g_2$ on their common definition interval.

The equations 4.4 and 4.5 satisfy the existence and uniqueness property of maximal solutions. Thus $g_n \to \mathrm{Id}$ uniformly on $[f(a), a]$. Returning to 4.3 we see that $g_n' \to 1$ uniformly on $[f(a), a]$ and consequently $g_n \to \mathrm{Id}$ in the C^1-topology.

Conversely, suppose that $g_n \to \mathrm{Id}$ in the C^1-topology. From 4.2 we obtain

$$g'(f^n(x)) = A(x, g_n(x); n).g_n'(x), \quad \forall x \in [f(a), a], n \in \boldsymbol{N}.$$

Taking the limit when $n \to \infty$ on both sides we have that $g'(x) \to 1$ as $x \to 0$ and the proof is finished. $\square$

5. Commutative holonomy and invariant fixed points.

In this section we prove Theorem 5.3. The generalized version of Kopell's Lemma proved in §4 will play an important role in this proof.

If $\psi \equiv \mathrm{Id}_{\mathfrak{G}}$ then Question 3.3 has an affirmative answer without any assumption on $\mathfrak{G}$.

Lemma 5.1. *Let $\mathfrak{G}$ be a group and let $H : \mathfrak{G} \to \mathrm{Diff}_+^2([0,1])$ be a representation. If $f \in \mathrm{Diff}_+^2([0,1])$ is such that $H_\alpha \circ f = f \circ H_\alpha$ for all α in $\mathfrak{G}$ then* $\mathrm{Fix}(f) \supset \mathrm{Fr}(\mathrm{Fix}(H_\alpha))$, $\forall \alpha \in \mathfrak{G}$.

Furthermore, in each connected component C of $[0,1] - \mathrm{Fix}(H)$ we have that if $f(z) = H_\gamma(z) \neq z$ for some $z \in C$ and $\gamma \in \mathfrak{G}$ then $H_\gamma \equiv f$ on C. In particular $f|_C \equiv \mathrm{Id}$ or $\mathrm{Fix}(f|_C) = \varnothing$.

The proof of this lemma is an application of Kopell's Lemma. Moreover, it has a similar version for C^2 diffeomorphism of S^1 by replacing $\mathrm{Fix}(H_\alpha)$ with periodic points of H_α, $\mathrm{Fix}(H)$ with points with finite orbit by $\mathfrak{G}$, and $\mathrm{Fix}(f)$ with periodic points of f.

Here is the codimension one foliation version of Lemma 5.1.

THEOREM 5.2. *Let* $\mathscr{F} \in \mathrm{Fol}^2_+(M)$, *let* $f \in \mathrm{Diff}^2_+(M;\mathscr{F})$ *and let* L *be a compact* f*-invariant leaf of* $\mathscr{F}$. *If* $f_* : \pi_1(L) \to \pi_1(L)$ *is the identity homomorphism then every leaf in* $\mathrm{supp}[L]$ *with nontrivial holonomy is* f*-invariant. Furthermore, if a connected component* C *of* $M - \mathscr{C}(\mathscr{F})$ *contained in* $\mathrm{supp}[L]$ *has an* f*-invariant leaf then all the leaves in* C *are also* f*-invariant.*

We now prove the main result of this section.

THEOREM 5.3. *Let* $\mathfrak{G}$ *be a finitely generated group and let* $H : \mathfrak{G} \to \mathrm{Diff}^2_+([0,1])$ *be a representation. Suppose there are* $\psi \in \mathrm{Aut}(\mathfrak{G})$ *and* $f \in \mathrm{Diff}^2_+([0,1])$ *such that* $H_{\psi(\alpha)} = f \circ H_\alpha \circ f^{-1}$, $\forall \alpha \in \mathfrak{G}$. *If* $\mathrm{Im}(H)$ *is a commutative subgroup of* $\mathrm{Diff}^2_+([0,1])$ *then* $\mathrm{Fix}(f) \supset \mathrm{Fr}(\mathrm{Fix}(H))$.

PROOF. First note that $\mathfrak{G}/\ker(H)$ is a commutative finitely generated torsion free group. Then it is isomorphic to $\boldsymbol{Z}^k$ for some $k \in \boldsymbol{N} \cup \{0\}$. On the other hand, our hypothesis implies that $\psi(\ker(H)) = \ker(H)$. So, it suffices to show the lemma for $\mathfrak{G} = \boldsymbol{Z}^k$.

Let $a, b \in \mathrm{Fix}(H)$ such that $a < b$ and $(a,b) \cap \mathrm{Fix}(H) = \emptyset$. We need to show that $a, b \in \mathrm{Fix}(f)$.

Suppose $f(b) \neq b$. Since $f(\mathrm{Fix}(H)) = \mathrm{Fix}(H)$, we can assume, without loss of generality, that $f(t) < t$, $0 < \forall t < 1$. In particular, $f(b) \le a < b$. From the equality $H_{\psi(\alpha)} = f \circ H_\alpha \circ f^{-1}$, $\forall \alpha \in \boldsymbol{Z}^k$ we have

$$H_\alpha = f^n \circ H_{\psi^{-n}(\alpha)} \circ f^{-n}, \quad \forall \alpha \in \boldsymbol{Z}^k, n \in \boldsymbol{Z}.$$

Using Lemma 4.1 one sees that $H_{\psi^{-n}(\alpha)} \to \mathrm{Id}$ on $[f(b), b]$ in the C^1-topology, as $|n| \to \infty$. In particular, the same is true on $[a,b] \subset [f(b), b]$.

Now fix $h \in \mathrm{Im}(H)$ with $h|_{[a,b]} \neq \mathrm{Id}$. From Szekeres [**23**] there exists a C^1 vector field X on $[a,b)$ with flow (X_t) such that $X_1 = h$. For each $g \in \mathrm{Diff}^1_+([a,b])$ commuting with h let $\tau(g) \in \boldsymbol{R}$ be the unique real number given by Kopell's result such that $g = X_{\tau(g)}$. The relation $g = X_{\tau(g)}$ defines a continuous homomorphism

$$\tau : \{H_\alpha|_{[a,b]} ; \alpha \in \boldsymbol{Z}^k\} \to \boldsymbol{R}.$$

Hence, $\omega = \tau \circ H : \boldsymbol{Z}^k \to \boldsymbol{R}$ is a non trivial homomorphism such that

$$\omega(\psi^n(\alpha)) \to 0 \quad \text{as } |n| \to \infty$$

for each $\alpha \in \boldsymbol{Z}^k$. According Lemma 5.4 below this is a contradiction and the proof of the theorem is complete. □

Lemma 5.4. *Let $\omega : \boldsymbol{R}^k \to \boldsymbol{R}$ be a linear form. Suppose that there exists an isomorphism $\psi : \boldsymbol{R}^k \to \boldsymbol{R}^k$ such that $\omega(\psi^n(\alpha)) \to 0$, $\forall \alpha \in \boldsymbol{Z}^k$ as $|n| \to \infty$. Then ω is trivial.*

Proof. Complexifying ω and ψ it is easy to verify that $\omega(\psi^n(\alpha)) \to 0$ as $|n| \to \infty$ for every α in $\boldsymbol{C}^k$.

We use the Jordan form of ψ to prove that ω is trivial. It suffices to show the case when ψ has Jordan form $[\psi] = \lambda I_s + E_{1,s}$ in a basis $\{v_1, \dots, v_s\}$, where I_s is the identity matrix and $(E_{1,s})_{i,j} = \delta_{i,j+1}$, $\forall 1 \le i, j \le s$ and $0 \ne \lambda \in \boldsymbol{C}$. Replacing ψ by ψ^{-1} if necessary we can assume that $|\lambda| \ge 1$.

For $n \ge s-1$ we have:

$$\begin{aligned}(\lambda I_s + E_{1,s})^n &= \sum_{i=0}^{s-1} \binom{n}{i} \lambda^{n-i} (E_{1,s})^i \\ &= \lambda^n I_s + \binom{n}{1} \lambda^{n-1} E_{1,s} + \cdots + \binom{n}{s-1} \lambda^{n-s+1} (E_{1,s})^{s-1}.\end{aligned}$$

Thus

$$\psi^n(v_i) = \lambda^n v_i + \binom{n}{1} \lambda^{n-1} v_{i-1} + \cdots + \binom{n}{i-1} \lambda^{n-i+1} v_1$$

for all $n \ge s-1$ and $1 \le i \le s$. We conclude that

$$\begin{aligned}\omega(\psi^n(v_i)) = \lambda^n \omega(v_i) + \binom{n}{1} \lambda^{n-1} \omega(v_{i-1}) + \cdots + \\ + \binom{n}{i-1} \lambda^{n-i+1} \omega(v_1)\end{aligned} \tag{5.1}$$

for all $n \ge s-1$ and $1 \le i \le s$.

As $n \to \infty$ we obtain:

$$\omega(\psi^n(v_1)) \to 0 \Rightarrow |\lambda^n| \, \|\omega(v_1)\| \to 0 \Rightarrow \omega(v_1) = 0.$$

Returning to 5.1 we conclude by finite induction on i that $\omega(v_i) = 0$, $1 \le \forall i \le s$. This completes the proof of the lemma. $\square$

Here is the S^1 version of Theorem 5.3. Recall that $\mathscr{C}(H)$ denotes the set of points $x \in S^1$ for which $\mathcal{O}_H(x)$ is finite.

Theorem 5.5. *Let $\mathfrak{G}$ be a finitely generated group and let $H : \mathfrak{G} \to \mathrm{Diff}^2_+(S^1)$ be a representation. Suppose there are $\psi \in \mathrm{Aut}(\mathfrak{G})$ and $f \in \mathrm{Diff}^2_+(S^1)$ such that $H_{\psi(\alpha)} = f \circ H_\alpha \circ f^{-1}$, $\forall \alpha \in \mathfrak{G}$. If $\mathrm{Im}(H)$ is a commutative subgroup of $\mathrm{Diff}^2_+(S^1)$ then every point in $\mathrm{Fr}(\mathscr{C}(H))$ is f-periodic.*

PROOF. If $\mathrm{Fr}(\mathscr{C}(H)) \neq \emptyset$ then by Theorem 2.1 there is an f-periodic point x_o in $\mathrm{Fr}(\mathscr{C}(H))$ with period $k \geq 1$.

Let $I_{x_o} = \{\alpha \in \mathfrak{G}; H_\alpha(x_o) = x_o\}$ be the isotropy group of x_o. The relation

$$H_{\psi^k(\alpha)} = f^k \circ H_\alpha \circ f^{-k}, \quad \forall \alpha \in \mathfrak{G}$$

assures that $\psi^k(I_{x_o}) = I_{x_o}$. On the other hand, I_{x_o} acts trivially on $\mathscr{C}(\mathscr{H})$. Restricting the action to the subgroup I_{x_o} and replacing f with f^k, ψ with ψ^k and $\mathrm{Diff}^2_+(S^1)$ with $\mathrm{Diff}^2_+([0,1])$, Theorem 5.3 gives that $\mathrm{Fr}(\mathrm{Fix}(H|_{I_{x_o}})) \subset \mathrm{Fix}(f^k)$, which implies that the set of periodic points of f contains $\mathrm{Fr}(\mathscr{C}(H))$. □

To complete this section we give the foliation version of Theorem 5.3.

THEOREM 5.6. *Let $\mathscr{F} \in \mathrm{Fol}^2_+(M)$, let $f \in \mathrm{Diff}^2_+(M; \mathscr{F})$ and let L be an f-periodic compact leaf. Suppose the holonomy of $[L]$ is commutative. Then every leaf equivalent to L lying in $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$ is also f-periodic.*

COROLLARY. *Let $\mathscr{F} \in \mathrm{Fol}^2_+(M)$ be a foliation almost without holonomy and let $f \in \mathrm{Diff}^2_+(M; \mathscr{F})$. Then every leaf in $\mathrm{Fr}(\mathscr{C}(\mathscr{F}))$ is f-periodic.*

PROOF. The hypothesis implies that each equivalence class of compact leaves has commutative holonomy. □

6. Invariant non compact leaves.

In this section we analyse the invariance of leaves lying in a connected component C of $M - \mathscr{C}(\mathscr{F})$ without holonomy. This means that we will be working with foliations almost without holonomy.

We begin by briefly recalling important well known results due to Sacksteder, Hector, Moussu and Imanishi for a foliation $\mathscr{F} \in \mathrm{Fol}^2_+(M)$ without holonomy in $N = \mathrm{Int}(M)$. For details see [**8**], [**12**], [**13**].

Let $\mathscr{F} \in \mathrm{Fol}^2_+(M)$ be without holonomy in $N = \mathrm{Int}(M)$. By a Sacksteder result there is a topological flow $\{\varphi_t\}_{t \in \boldsymbol{R}}$ on N transverse to $\mathscr{F}$ and preserving $\mathscr{F}$. Furthermore, $\{\varphi_t\}$ acts transitively on the quotient space $N/\mathscr{F}$ of N by the leaves of $\mathscr{F}$.

Fix a base point p in N. Let $(\tilde{N}, \tilde{p}) \xrightarrow{\pi} (N, p)$ be the universal covering of N and denote by $\tilde{\mathscr{F}}$ and by $\{\tilde{\varphi}_t\}_{t \in \boldsymbol{R}}$ the lifts to $\tilde{N}$ of $\mathscr{F}$ and $\{\varphi_t\}_{t \in \boldsymbol{R}}$ respectively. We also denote by L_q (resp. $\tilde{L}_{\tilde{q}}$) the leaf of $\mathscr{F}$ (resp. $\tilde{\mathscr{F}}$) passing through q (resp. $\tilde{q}$).

REMARK 6.1. The map $\psi : \tilde{L}_{\tilde{p}} \times \boldsymbol{R} \to \tilde{N}$ defined by $\psi(x, t) = \tilde{\varphi}_t(x)$ is a homeomorphism which carries leaves $\tilde{L}_{\tilde{p}} \times \{t\}$ of the product foliation to leaves of $\tilde{\mathscr{F}}$.

Remark 6.2. By Remark 6.1 each loop $\gamma \in \Omega(N; p)$ is homotopic either to a loop on L_p or to a loop transverse to $\mathscr{F}$. Loops which are transverse to $\mathscr{F}$ are homotopic to the product of two paths γ_s and γ_{L_p} where $\gamma_s = \{\varphi_{ts}(p); t \in [0,1]\}$ with $\varphi_s(p) \in L_p$ $(s \neq 0)$ and γ_{L_p} is a path on L_p from $\varphi_s(p)$ to p.

Moreover, since $\mathscr{F}|_N$ has no holonomy, the inclusion of L_p in N induces an injective homomorphism of $\pi_1(L_p; p)$ in $\pi_1(N; p)$ whose image we shall identify with $\pi_1(L_p; p)$.

We identify the leaf space $\tilde{N}/\tilde{\mathscr{F}}$ with $\boldsymbol{R}$ via the map $T = \pi_2 \circ \psi^{-1}$ where π_2 is the standard projection of $\tilde{L}_{\tilde{p}} \times \boldsymbol{R}$ on the second factor.

Remark 6.3. The action of $\pi_1(N; p)$ on $\tilde{N}$ preserves $\tilde{\mathscr{F}}$, so it acts on $\tilde{N}/\tilde{\mathscr{F}} \approx \boldsymbol{R}$. By Remark 6.2, $\pi_1(N; p)$ acts on $\tilde{N}/\tilde{\mathscr{F}}$ by translations. This action is described by the homomorphism $\phi : \pi_1(N; p) \to \boldsymbol{R}$ given by $\phi(\alpha) = T(\alpha \cdot \tilde{p})$ where $\alpha \cdot \tilde{x}$ denotes the action of $\alpha \in \pi_1(N; p)$ on $\tilde{x} \in \tilde{N}$. We have that $\ker(\phi) = \pi_1(L_p; p)$.

The quotient group $\pi_1(N; p)/\pi_1(L_p; p)$ is isomorphic with the subgroup

$$K = \{\tau \in \boldsymbol{R}; \varphi_\tau(p) \in L_p\}$$

of $\boldsymbol{R}$. The group K is called the *group of periods* of $\mathscr{F}$ and $\{\varphi_t\}_{t \in \boldsymbol{R}}$.

Remark 6.4. $\bar{K} = \boldsymbol{R}$ if and only if all the leaves of $\mathscr{F}$ are dense in N. If $\bar{K} \neq \boldsymbol{R}$ then $\mathscr{F}$ is defined by a fibration over S^1.

We continue to assume that if $f \in \mathrm{Homeo}_+(N; \mathscr{F})$ has f-invariant leaves then f has at least a fixed point, say p, lying in one of those leaves. This is true up to an isotopy along the leaves of $\mathscr{F}$ as already noted in §3. Hence $f_* : \pi_1(N; p) \to \pi_1(N; p)$ defines an isomorphism

$$\bar{f}_* : \pi_1(N; p)/\pi_1(L_p; p) \to \pi_1(N; p)/\pi_1(L_p; p).$$

In what follows we keep the notation above.

Proposition 6.5. *Let $\mathscr{F} \in \mathrm{Fol}^2_+(M)$ and let $f \in \mathrm{Homeo}_+(M; \mathscr{F})$. Suppose that $\mathscr{F}$ restricted to $N = \mathrm{Int}(M)$ has trivial holonomy and has dense leaves. Then the following conditions are equivalent*:

(i) *all leaves in N are f-invariant,*

(ii) *there is an f-invariant leaf in N and*

$$\bar{f}_* : \pi_1(N)/\pi_1(L) \to \pi_1(N)/\pi_1(L)$$

is the identity homomorphism.

Proof. Let L be an f-invariant leaf of $\mathscr{F}$ and let $p \in L$ be a fixed point of f. Fix $\tilde{p} \in \pi^{-1}(p)$ and let $\tilde{f} \in \mathrm{Homeo}_+(\tilde{N}; \tilde{\mathscr{F}})$ be the lift of f to $\tilde{N}$ with

$\tilde{f}(\tilde{p}) = \tilde{p}$. We denote by $\tilde{f}_T : \boldsymbol{R} \to \boldsymbol{R}$ the projection of $\tilde{f}$ on $\boldsymbol{R}$ via T. By the lifting properties of $\tilde{f}$ we have that

$$\tilde{f}(\alpha \cdot x) = f_*(\alpha) \cdot \tilde{f}(x), \quad \forall \alpha \in \pi_1(N; p), x \in \tilde{N}$$

hence, on the quotient we obtain

$$\tilde{f}_T(\tau + t) = \bar{f}_*(\tau) + \tilde{f}_T(t), \quad \forall t \in \boldsymbol{R}, \tau \in K \subset \boldsymbol{R}.$$

Since $\tilde{f}_T(0) = 0$ we conclude that $\tilde{f}_T(\tau) = \bar{f}_*(\tau)$, $\forall \tau \in K$.

Suppose that $\bar{f}_* \equiv \mathrm{Id}$. Then, $\tilde{f}_T(\tau) = \tau$, $\forall \tau \in K$ and as K is dense in $\boldsymbol{R}$ we have that $\tilde{f}_T \equiv \mathrm{Id}_{\boldsymbol{R}}$. Thus every leaf of $\tilde{\mathscr{F}}|_{\tilde{N}}$ is $\tilde{f}$-invariant, hence every leaf of $\mathscr{F}|_N$ is f-invariant.

Conversely, suppose that every leaf in $\mathscr{F}|_N$ is f-invariant. In this case, $\tilde{f}_T(T(\tilde{F})) - T(\tilde{F}) \in K$ for each leaf $\tilde{F}$ of $\tilde{\mathscr{F}}$. Since $\tilde{f}_T(0) = 0$, $\tilde{f}_T - \mathrm{Id}$ is continuous and $K \neq \boldsymbol{R}$, we have $\tilde{f}_T(T(\tilde{F})) - T(\tilde{F}) = 0$ and so $\tilde{f}_T \equiv \mathrm{Id}_{\boldsymbol{R}}$. Thus $\bar{f}_* \equiv \mathrm{Id}$, which completes the proof. □

We remark that the union of invariant leaves need not be closed in M. The following example shows that the invariance of all leaves in $\mathscr{F}|_{\mathrm{Int}(M)}$ does not imply the invariance of the boundary leaves.

EXAMPLE. Consider the foliation $\mathscr{F}$ on $T^2 \times S^1$ obtained by multiplying by S^1 each leaf of a fixed foliation on the torus T^2. Let (Id, φ) be the double covering of $T^2 \times S^1$ where $\varphi : S^1 \to S^1$ is the double covering of S^1, and let $\tilde{\mathscr{F}}$ be the (Id, φ)-lift of $\mathscr{F}$. It is easy to see that all leaves of $\tilde{\mathscr{F}}$ are invariant under the nontrivial covering map $\tilde{f} \in \mathrm{Aut}((\mathrm{Id}, \varphi))$.

Now choose a closed curve Γ in $T^2 \times \{1\} \subset T^2 \times S^1$ transverse to $\mathscr{F}$ and apply the turbulization process to introduce a Reeb component in a small tubular neighborhood of Γ. Let $\mathscr{F}_1$ be this new foliation and let $\tilde{\mathscr{F}}_1$ be its (Id, φ)-lift. All leaves of $\tilde{\mathscr{F}}_1$ are $\tilde{f}$-invariant except the leaves on the solid torus associated to the Reeb components. The desired example is obtained by taking out the interior of the Reeb components of $\tilde{\mathscr{F}}_1$.

We close this section by treating the case where $\mathscr{F}|_{\mathrm{Int}(M)}$ has no dense leaves. In this situation we must have $\partial M \neq \emptyset$.

PROPOSITION 6.6. *Let $\mathscr{F} \in \mathrm{Fol}_+^2(M)$ and let $f \in \mathrm{Diff}_+^2(M; \mathscr{F})$. Suppose that $\mathscr{F}$ restricted to $N = \mathrm{Int}(M)$ satisfies the following properties*:

(i) *without holonomy*,
(ii) *without dense leaves*,
(iii) *without compact leaves.*

If $\mathscr{F}$ has an f-invariant leaf in N then all leaves in N are f-invariant.

Proof. In the absence of holonomy and dense leaves the foliation $\mathscr{F}|_N$ is defined by a fibration $\xi : N \to S^1$. Fix a section $\Gamma \approx S^1$ of ξ. The projection of f on Γ along the leaves defines a C^2 preserving orientation diffeomorphism f_Γ of Γ having a fixed point $p \in \Gamma$ corresponding to the invariant leaf. We shall see that the dynamics of $\mathscr{F}$ close to ∂M implies $f_\Gamma \equiv \mathrm{Id}_\Gamma$.

Suppose there is a (compact) boundary leaf L such that $f(L) = L$. Fix a collar neighborhood $U = [0,1) \times L$ of L in M. As already noted in §3 (Remark 3.4), we can assume up to an isotopy of f along the leaves of $\mathscr{F}$ that:

- f has a fixed point $q \in L$,
- in a small neighborhood of L in U the map f preserves the foliation defined by the fibers of U.

Consequently the relation

$$H_{f(\alpha)} = f \circ H_\alpha \circ f^{-1}, \quad \forall \alpha \in \Omega(L;q) \tag{6.1}$$

holds in a small neighborhood of the origin in $[0,1)$.

Since $\mathscr{F}|_N$ has no dense leaves, the holonomy of L is cyclic and generated by a C^2-contraction H_β defined in a neighborhood of the origin in $[0,1)$.

Let $\mathscr{H}$ be the normal subgroup of $\pi_1(L;q)$ consisting of the homotopy classes whose holonomy is trivial. We have $\pi_1(L;q)/\mathscr{H} \approx \boldsymbol{Z}$.

In view of 6.1 we can take the quotient of $f_* : \pi_1(L;q) \to \pi_1(L;q)$ by $\mathscr{H}$ obtaining an isomorphism $\bar{f}_* : \pi_1(L;q)/\mathscr{H} \to \pi_1(L;q)/\mathscr{H}$ such that $\bar{f}_*(\bar{\beta}) = \bar{\beta}$, where $\bar{\beta}$ is the class of β in $\pi_1(L;q)/\mathscr{H}$. Hence $H_{f(\beta)} = H_\beta$ in a neighborhood $[0,\varepsilon)$ of the origin in $[0,1)$ and then $f \circ H_\beta = H_\beta \circ f$ in $[0,\varepsilon)$. Therefore, f induces a diffeomorphism $\tilde{f}$ on the quotient $(0,\varepsilon)/H_\beta \approx S^1$ of $(0,\varepsilon)$ by the action of H_β.

According to Szekeres and Kopell there exists a C^1 vector field X on $[0,\varepsilon)$ and $t_o \in \boldsymbol{R}$ such that H_β and f are the time one and the time t_o maps of X respectively. Therefore, $\tilde{f}$ is represented by a rotation.

The restriction of ξ to a fundamental domain of H_β is a k-sheeted covering $\tilde{\xi} : (0,\varepsilon)/H_\beta \to \Gamma \approx S^1$ of Γ for some $k \in \boldsymbol{N}$.

Fix $\tilde{p} \in (0,\varepsilon)/H_\beta$ such that $\tilde{\xi}(\tilde{p}) = p$. Let $\tilde{f}_\Gamma$ be the lift of f_Γ having $\tilde{p}$ as a fixed point. Then $\tilde{f} = \mathrm{R} \circ \tilde{f}_\Gamma$ where $\mathrm{R} \in \mathrm{Aut}(\tilde{\xi})$. Since $\tilde{f}_\Gamma$ is a lift of the orientation preserving diffeomorphism f_Γ, it follows that $\tilde{f}_\Gamma$ commutes with R and we have:

$$(\tilde{f})^k = \mathrm{R}^k \circ (\tilde{f}_\Gamma)^k = (\tilde{f}_\Gamma)^k. \tag{6.2}$$

Thus $(\tilde{f}_\Gamma)^k$ is conjugated to a rotation. Since $\tilde{f}_\Gamma$ has a fixed point, it must be the identity map.

In the general case there is an $s \in \boldsymbol{N}$ such that $f^s(L) = L$ and 6.2 gives $(\tilde{f^s})^k = (\tilde{f^s_\Gamma})^k$. This finishes the proof. □

7. Invariant leaves under diffeomorphisms close to the identity.

We begin this section by giving a construction which will facilitate the study of invariant leaves under diffeomorphisms close to identity. This construction enables us to reduce this problem to that of existence of fixed points.

Let $\mathscr{F} \in \mathrm{Fol}^r_+(M)$. Fix a C^∞ vector field X on M transverse to $\mathscr{F}$ and a C^∞ riemannian metric on M. Let $\mathfrak{A} = \{A_i\}$ and $\mathfrak{B} = \{B_k\}$ be two finite cubic covers of M bi-regular for $\mathscr{F}$ and X such that if $A_i \cap A_j \neq \emptyset$ then $A_i \cup A_j \subset B_k$ for some $B_k \in \mathfrak{B}$.

A *cubic cover of M bi-regular for $\mathscr{F}$ and X* is a cover of M by open sets diffeomorphic to $(0,1)^{n-1} \times (0,1)$ such that the foliations $\mathscr{F}$ and X are defined by the first and the second factors of $(0,1)^{n-1} \times (0,1)$ respectively.

Now let $f \in \mathrm{Diff}^r_+(M;\mathscr{F})$ be close to the identity in $\mathrm{Diff}^r(M)$ so that for each $x \in M$ the points x and $f(x)$ lie in the same bi-regular neighborhood A_i of $\mathfrak{A}$. Taking f sufficiently C^r close to Id_M the *projection of $f(x)$ along the leaves of $\mathscr{F}|_{A_i}$* on the orbit of $X|_{A_i}$ passing through x defines a C^r diffeomorphism f_X of M which is C^r close to Id_M and preserves the leaves of $\mathscr{F}$. Furthermore, all the orbits of X are invariant under f_X.

The construction above holds also for foliations close to the foliation $\mathscr{F}$. In fact, the following result is well known.

7.1. *Let $\varepsilon > 0$ and let $\mathscr{V}_1$ be a neighborhood of Id_M in $\mathrm{Diff}^r(M)$. There exist neighborhoods $\mathscr{V}$ of Id_M in $\mathrm{Diff}^r(M)$ and $\mathscr{U}$ of $\mathscr{F}$ in $\mathrm{Fol}^r_+(M)$ satisfying the following properties at any $\tilde{\mathscr{F}} \in \mathscr{U}$ and $\tilde{f} \in \mathscr{V} \cap \mathrm{Diff}^r_+(M;\tilde{\mathscr{F}})$:*

(7.1.1) *the projection $\tilde{f}_X$ of $\tilde{f}$ along the leaves of $\tilde{\mathscr{F}}$ is well defined,*

(7.1.2) $\tilde{f}_X \in \mathscr{V}_1 \cap \mathrm{Diff}^r_+(M;\tilde{\mathscr{F}})$,

(7.1.3) $d_X(x;\tilde{f}_X(x))$, $d_{\tilde{\mathscr{F}}}(\tilde{f}(x);\tilde{f}_X(x)) < \varepsilon$, $\forall x \in M$, *where d_X and $d_{\tilde{\mathscr{F}}}$ denote the induced distance on the X-orbits and on the leaves of $\tilde{\mathscr{F}}$ respectively.*

We observe that for convenient choices of $\varepsilon > 0$ and of $\mathscr{V}_1$ the map $\tilde{f}_X$ does not depend on the covers $\mathfrak{A}$ and $\mathfrak{B}$, moreover $\tilde{f}_X$ is the end point of a C^r isotopy of $\tilde{f}$ along the leaves of $\tilde{\mathscr{F}}$. To see this, consider the codimension one plane field tangent to $\mathscr{F}$ and approximate it by a C^∞ plane field σ. The restriction of the exponential map to the sub-bundle σ of TM gives small embedded $(n-1)$-disks $\mathrm{D}_x(\mathscr{F})$ in M centered at x. These disks are not contained in the leaf $L_x(\mathscr{F})$ of $\mathscr{F}$ passing through x, but they can be projected on $L_x(\tilde{\mathscr{F}})$ along the X-orbits. Thus we can assume that $\tilde{f}(x)$ is in the disk centered at $\tilde{f}_X(x)$. Now the isotopy is natural. The compactness of M is fundamental to control the minimum ray of the embedded disks $\mathrm{D}_x(\mathscr{F})$.

More generally:

Choosing the neighborhoods $\mathscr{V}$ and $\mathscr{U}$ in 7.1 *sufficiently small, we can guarantee that*

(7.1.4) *there exists a C^r isotopy from $\tilde{f}$ to $\tilde{f}_X$ along the leaves of $\tilde{\mathscr{F}}$ and lying in $\mathscr{V}_1$.*

From now on given $\mathscr{F} \in \mathrm{Fol}^r_+(M)$ and a vector field X on M transverse to $\mathscr{F}$ we consider only perturbations $\tilde{\mathscr{F}}$ of $\mathscr{F}$ and $\tilde{f}$ of Id_M for which X is transverse to $\tilde{\mathscr{F}}$ and the projection $\tilde{f}_X$ of $\tilde{f}$ along the leaves of $\tilde{\mathscr{F}}$ is well defined. We also assume that the holonomy of $\mathscr{F}$ and of its perturbated are defined in segments contained in the X-orbits.

Let $\tilde{f} \in \mathrm{Diff}^r_+(M; \tilde{\mathscr{F}})$ be such that $\tilde{f}_X$ has a fixed point $p \in M$. In this case the leaf $\tilde{L}_p$ of $\tilde{\mathscr{F}}$ passing through p is contained in $\mathrm{Fix}(\tilde{f}_X)$ and the points p and $\tilde{f}(p)$ are close along $\tilde{L}_p$. The leaves of $\tilde{\mathscr{F}}$ passing through fixed points of $\tilde{f}_X$ will be called *strongly $\tilde{f}$-invariant* leaves.

Theorem 7.2. *Let $\mathscr{F} \in \mathrm{Fol}^2_+(M)$. There exists a neighborhood $\mathscr{V}$ of the identity in $\mathrm{Diff}^2(M)$ such that for all $f \in \mathscr{V} \cap \mathrm{Diff}^2_+(M; \mathscr{F})$ we have:*

(i) *the leaves of $\mathscr{F}$ with nontrivial holonomy are strongly f-invariant,*

(ii) *if a connected component of $M - \mathscr{C}(\mathscr{F})$ has an f-invariant leaf (resp. strongly f-invariant leaf) then all leaves lying in this component are f-invariant (resp. strongly f-invariant).*

This theorem together with (7.1.4) implies that for $r \geq 2$ if $\mathscr{F} \in \mathrm{Fol}^r_+(M)$ has nontrivial holonomy in all connected components of $M - \mathscr{C}(\mathscr{F})$ then every $f \in \mathrm{Diff}^r_+(M; \mathscr{F})$ sufficiently close to the identity is C^r isotopic to the identity map under an isotopy along the leaves of $\mathscr{F}$. In fact, on the connected components of $M - \mathscr{C}(\mathscr{F})$ this isotopy is given by Theorem 7.2 and by (7.1.4). It is easy to extend it to $\mathrm{Int}(\mathscr{C}(\mathscr{F}))$. This extension is essentially the isotopy to the identity of a diffeomorphism of $[0, 1]$ close to the identity.

In [**15**] J. Leslie proves that for $\mathscr{F} \in \mathrm{Fol}^\infty_+(M)$ and $\partial M \neq \varnothing$ the connected component $\mathfrak{D}(M; \mathscr{F})$ of $\mathrm{Diff}^\infty_+(M; \mathscr{F}) \subset \mathrm{Diff}^r(M)$ containing the identity map admits a structure of an infinite dimensional manifold. Clearly the leaves with nontrivial holonomy are f-invariant for all $f \in \mathfrak{D}(M; \mathscr{F})$. In particular this holds for f close to Id_M in the topology of the manifold $\mathfrak{D}(M; \mathscr{F})$. As remarked in the previous paragraph, if $\mathscr{F}$ has nontrivial holonomy in all connected components of $M - \mathscr{C}(\mathscr{F})$ then every $f \in \mathrm{Diff}^\infty_+(M; \mathscr{F})$ sufficiently C^∞ close to Id_M is contained in $\mathfrak{D}(M; \mathscr{F})$. In this case the group $\mathfrak{D}(M; \mathscr{F})$ is a closed subgroup of $\mathrm{Diff}^\infty(M)$.

Before giving its proof we will use Theorem 7.2 to prove the following result.

Theorem 7.3. *Let $\mathscr{F} \in \mathrm{Fol}^r_+(M)$, $r \geq 2$, let X be a vector field transverse* *to $\mathscr{F}$ and let $\mathscr{V}_1$ be a neighborhood of* Id_M *in* $\mathrm{Diff}^r(M)$. *There exists a neighborhood $\mathscr{V}$ of* Id_M *in* $\mathrm{Diff}^r(M)$ *such that for all $f \in \mathscr{V} \cap \mathrm{Diff}^r(M;\mathscr{F})$ we have*:

(i) *f_X is well defined and $f_X \in \mathscr{V}_1 \cap \mathrm{Diff}^r_+(M;\mathscr{F})$,*

(ii) *if $g \in \mathscr{V} \cap \mathrm{Diff}^r_+(M;\mathscr{F})$ then $f_X \circ g_X \equiv g_X \circ f_X$ on each connected component of $M - \mathscr{C}(\mathscr{F})$,*

(iii) *there exists a C^r isotopy $\{F_t\}_{t\in[0,1]}$ of $f_X \equiv F_0$ supported on $\overline{\mathrm{Int}(\mathscr{C}(\mathscr{F}))}$ such that $F_1 \equiv \mathrm{Id}$ on $\overline{\mathrm{Int}(\mathscr{C}(\mathscr{F}))}$ and $F_t \in \mathscr{V}_1 \cap \mathrm{Diff}^r(M;\mathscr{F})$, $\forall t \in [0,1]$.*

PROOF. The item (i) was already shown. In order to prove (ii) fix a connected component C of $M - \mathscr{C}(\mathscr{F})$. By Theorem 7.2 we have only to consider the case where $\mathscr{F}$ has no holonomy in C. There are two possibilities.

Case 1: *$\bar{C}$ is a boundaryless manifold.*

In this case all leaves are dense. Let $\mathscr{V}$ be a neighborhood of Id_M where (i) holds. Fix an X-orbit Γ ($\boldsymbol{R}$ or S^1) and let $\mathfrak{H}$ be the subgroup of $\mathrm{Diff}^r_+(\Gamma)$ generated by the set $\{f_X|_\Gamma; f \in \mathscr{V}\}$. We claim that $\mathfrak{H}$ is a commutative group. Indeed, let $g \in \mathfrak{H}$ and suppose that g has a fixed point $x \in \Gamma$. All the X-orbits are invariant by the diffeomorphism g considered as an element of $\mathrm{Diff}^r_+(M;\mathscr{F})$. Thus, $g \equiv \mathrm{Id}$ on a neighborhood of x in L_x, so we must have $g \equiv \mathrm{Id}$ on L_x. Since L_x is dense, we have $g \equiv \mathrm{Id}$ on M. In particular, $g \equiv \mathrm{Id}_\Gamma$. Therefore the action of $\mathfrak{H}$ is free of fixed points on Γ.

Case 2: *$\bar{C}$ is a manifold with non empty boundary.*

Let L be a boundary leaf of $\bar{C}$. Since $\mathscr{F}|_C$ is without holonomy the compact leaf L has nontrivial commutative holonomy. Fix a holonomy map H_α of L given by a contraction defined in a segment $\Sigma \subset \bar{C}$ starting at $p \in L$ and lying on an X-orbit. Let $f, g \in \mathscr{V} \cap \mathrm{Diff}^r_+(M;\mathscr{F})$, where $\mathscr{V}$ is the neighborhood given in Theorem 7.2. From Theorem 7.2 we have $f_X \equiv \mathrm{Id}_L \equiv g_X$ on L. Thus, f_X and g_X commute with H_α in a neighborhood of p in Σ. By Kopell's lemma, the commutator $[f_X, g_X]$ has a fixed point in C. Again by Theorem 7.2 we have $[f_X, g_X] \equiv \mathrm{Id}_C$.

In order to prove (iii) fix a connected component C of $\mathrm{Int}(\mathscr{C}(\mathscr{F}))$. We know that C is the interior of $[0,1] \times L$ where L is a compact leaf of $\mathscr{F}$, and $\mathscr{F}|_C$ is the product foliation defined by the second factor. Since the boundary leaves $L_0 = \{0\} \times L$ and $L_1 = \{1\} \times L$ of $\overline{\mathrm{Int}(\mathscr{C}(\mathscr{F}))}$ have nontrivial holonomy, by Theorem 7.2 we have $f_X|_{L_i} \equiv \mathrm{Id}_{L_i}$, $i = 0, 1$.

We first prove that f_X is C^r-flat at L_i. The only nontrivial situation is when L_i lies in the boundary of a connected component of $M - \mathscr{C}(\mathscr{F})$ without holonomy. Since f_X commutes with the holonomies of L_i which are C^r-flat, Kopell's result implies that f_X is also C^r-flat at L_i.

Each segment $\{x\} \times [0,1]$ is invariant under f_X. This allows us to isotope $f_X|_C$ to Id_C. As remarked before this isotopy is essentially the isotopy to the identity of a diffeomorphism of $[0,1]$ close to the identity. This finishes the proof of the theorem. $\square$

The proof of Theorem 7.2 requires the following two lemmas.

Lemma 7.4. *Let $\mathscr{F} \in \mathrm{Fol}^2_+(M)$. There exist neighborhoods $\mathscr{V}$ of the identity in $\mathrm{Diff}^2(M)$ and $\mathscr{U}$ of $\mathscr{F}$ in $\mathrm{Fol}^2_+(M)$ such that the following holds. For all $\tilde{\mathscr{F}} \in \mathscr{U}$ and $\tilde{f} \in \mathscr{V} \cap \mathrm{Diff}^2_+(M;\tilde{\mathscr{F}})$ we have that if a connected component of $M - \mathscr{C}(\tilde{\mathscr{F}})$ has a strongly $\tilde{f}$-invariant leaf then all leaves lying in this component are also strongly $\tilde{f}$-invariant.*

Proof. Consider neighborhoods $\mathscr{V}$ and $\mathscr{U}$ of Id_M and $\mathscr{F}$ respectively given by 7.1. Let $\tilde{\mathscr{F}} \in \mathscr{U}$, let $\tilde{f} \in \mathscr{V} \cap \mathrm{Diff}^2_+(M;\tilde{\mathscr{F}})$ and let $\tilde{C}$ be a connected component of $M - \mathscr{C}(\tilde{\mathscr{F}})$. It suffices to show that $\tilde{C} \cap \mathrm{Fix}(\tilde{f}_X)$ is an open subset of $\tilde{C}$.

Let $z \in \tilde{C} \cap \mathrm{Fix}(\tilde{f}_X)$. The leaf $\tilde{L}_z$ of $\tilde{\mathscr{F}}$ passing through z is contained in $\mathrm{Fix}(\tilde{f}_X)$ and consequently $\mathrm{Cl}(\tilde{L}_z) \subset \mathrm{Fix}(\tilde{f}_X)$. If $\tilde{L}_z$ is not dense in $\tilde{C}$ then $\mathrm{Cl}(\tilde{L}_z)$ contains a leaf $\tilde{L}$ with nontrivial holonomy (a compact boundary leaf or a leaf in an exceptional minimal set).

Fix $p \in \tilde{L}$ and let $\Sigma \subset \mathrm{Cl}(\tilde{C})$ be a small segment on an X-orbit, starting at p. There exist a loop $\alpha \in \Omega(\tilde{L};p)$ and a point $q \in \tilde{L}_z \cap \Sigma$ sufficiently close to p such that $\tilde{H}_\alpha(q)$ lies between p and q on Σ. Hence $\tilde{H}^n_\alpha(q)$ is defined for all $n \in \mathbf{N}$ and $\tilde{H}^n_\alpha(q) \to q'$ as $n \to \infty$. Thus $q' \in \mathrm{Fix}(\tilde{f}_X) \cap \mathrm{Fix}(\tilde{H}_\alpha)$. Since $\tilde{f}_X \equiv \mathrm{Id}$ on $\tilde{L}$, it commutes with the holonomies of $\tilde{L}$ in small neighborhoods of p in Σ. By Kopell's result, $\tilde{f}_X \equiv \mathrm{Id}$ on a neighborhood of q in Σ. Consequently, $\tilde{f}_X \equiv \mathrm{Id}$ on a neighborhood of z in $\tilde{C}$. $\square$

Lemma 7.5. *Let $\mathscr{F} \in \mathrm{Fol}^2_+(M)$. There exists a neighborhood $\mathscr{V}$ of the identity in $\mathrm{Diff}^2(M)$ such that every connected component of $M - \mathscr{C}(\mathscr{F})$ with nontrivial holonomy contains a strongly f-invariant leaf for all $f \in \mathscr{V} \cap \mathrm{Diff}^2_+(M;\mathscr{F})$.*

Proof. Suppose that $\mathscr{C}(\mathscr{F})$ is non empty and let $[L_1], \ldots, [L_s]$, $s \geq 1$ be the equivalence classes of compact leaves of $\mathscr{F}$. Fix a vector field X on M transverse to $\mathscr{F}$ and transverse to the fibers of the fibrations defined by the supports of the classes $[L_i]$; $1 \leq i \leq s$. Choose a neighborhood $\mathscr{V}$ of the identity in $\mathrm{Diff}^2(M)$ such that for $f \in \mathscr{V} \cap \mathrm{Diff}^2_+(M;\mathscr{F})$ we have:

- $\mathrm{supp}[L_i]$; $1 \leq i \leq s$ are f-invariant,
- f_X is defined,
- $f_* : \pi_1(L_i) \to \pi_1(L_i)$ is the identity homomorphism for $i = 1, \ldots, s$.

By Theorem 5.2 the leaves in $\bigcup_{i=1}^s \mathrm{supp}[L_i]$ with nontrivial holonomy are f_X-invariant, which means that they are strongly f-invariant leaves. So, the lemma is proved for connected components of $M - \mathscr{C}(\mathscr{F})$ contained in $\bigcup_{i=1}^s \mathrm{supp}[L_i]$.

Let now $C \subset M - \bigcup_{i=1}^{s} \text{supp}[L_i]$ be a connected component of $M - \mathscr{C}(\mathscr{F})$ having a leaf L with nontrivial holonomy. Thus there is a loop α on L whose associated holonomy H_α has a one side isolated fixed point p. Choosing $\mathscr{V}$ even smaller if necessary, we have that $f_X(p) = p$ for all $f \in \mathscr{V} \cap \text{Diff}_+^2(M; \mathscr{F})$ and the leaf passing through p is strongly f-invariant.

Since there are only finitely many connected components in $M - \bigcup_{i=1}^{s} \text{supp}[L_i]$, the proof is finished for $\mathscr{C}(\mathscr{F}) \neq \emptyset$.

The proof for the case $\mathscr{C}(\mathscr{F}) = \emptyset$ is the same as above for a component lying in $M - \bigcup_{i=1}^{s} \text{supp}[L_i]$. □

Proof of Theorem 7.2. Recall that $M - \bigcup_{i=1}^{s} \text{supp}[L_i]$ is a finite disjoint union of connected components $\{C_j\}_{j=1}^{k}$ of $M - \mathscr{C}(\mathscr{F})$.

In order to prove (i) choose $f \in \text{Diff}_+^2(M; \mathscr{F})$ sufficiently C^2 close to the identity so that Theorem 5.2 applies to each $\text{supp}[L_i]$; $i = 1, \ldots, s$ and Lemmas 7.4 and 7.5 apply to each C_j; $j = 1, \ldots, k$.

The item (ii) for the case of strongly invariant leaves follows from Lemma 7.4. It remains to prove (ii) for f-invariant leaves which are not strongly f-invariant. By (i) these leaves lie in connected components of $M - \mathscr{C}(\mathscr{F})$ without holonomy. In this case the theorem follows from Propositions 6.5 and 6.6. □

Theorem 7.6 below is an application of the following result of G. Duminy [**5**].

Theorem 7.6. *Let $\mathscr{F} \in \text{Fol}_+^2(M)$. There exist neighborhoods $\mathscr{V}$ of* Id *in* $\text{Diff}^2(M)$ *and $\mathscr{U}$ of $\mathscr{F}$ in* $\text{Fol}_+^2(M)$ *such that if $\tilde{\mathscr{F}} \in \mathscr{U}$ has a compact leaf then either $\tilde{\mathscr{F}}$ is a fibration or $\tilde{\mathscr{F}}$ has a compact leaf with nontrivial holonomy whose orbit by the action of the group generated by $\mathscr{V} \cap \text{Diff}_+^2(M; \tilde{\mathscr{F}})$ is a finite union of compact leaves.*

Theorem (G. Duminy). *There exists a neighborhood $\mathscr{V}$ of the identity in* $\text{Diff}_+^2(S^1)$ *such that the group generated by any family of elements of $\mathscr{V}$ either has a finite orbit or has all orbits dense.*

Proof of Theorem 7.6. Fix a vector field X transverse to $\mathscr{F}$ having a closed orbit Γ. Choose neighborhoods $\mathscr{V}$ of the identity in $\text{Diff}^2(M)$ and $\mathscr{U}$ of $\mathscr{F}$ in $\text{Fol}_+^2(M)$ such that:

- $\tilde{f}_X$ is defined for all $\tilde{\mathscr{F}} \in \mathscr{U}$ and $\tilde{f} \in \mathscr{V} \cap \text{Diff}_+^2(M; \tilde{\mathscr{F}})$,
- the restriction of $\tilde{f}_X$ to Γ lies in the neighborhood of the identity $\mathscr{V}' \subset \text{Diff}_+^2(\Gamma)$ given by Duminy's theorem.

Let $\tilde{\mathscr{F}} \in \mathscr{U}$ and suppose $\mathscr{C}(\tilde{\mathscr{F}}) \neq \emptyset$. We have two possibilities:

(i) $\mathscr{C}(\tilde{\mathscr{F}}) \cap \Gamma = \emptyset$.

In this case the connected component $\tilde{C}$ of $M - \mathscr{C}(\tilde{\mathscr{F}})$ which contains Γ is invariant under $\tilde{f} \in \mathscr{V} \cap \mathrm{Diff}_{+}^{2}(M;\tilde{\mathscr{F}})$. Consequently its boundary leaves satisfy the theorem.

(ii) $\mathscr{C}(\tilde{\mathscr{F}}) \cap \Gamma \neq \emptyset$.

If the action of the group $\tilde{\mathscr{H}}_{\Gamma}$ generated by

$$\{\tilde{f}_X|_{\Gamma}; \tilde{f} \in \mathscr{V} \cap \mathrm{Diff}_{+}^{2}(M;\tilde{\mathscr{F}})\}$$

has all orbits dense then $\tilde{\mathscr{F}}$ is a fibration. If not, there is $x_o \in \Gamma$ whose orbit by the action of $\tilde{\mathscr{H}}_{\Gamma}$ is finite. In this case the proof follows from Lemma 2.2. □

It would seem that the proof of Theorem 7.6 should not require a result as strong as the result of Duminy. More precisely, we would like to answer the following question.

Question. Let $\mathscr{F} \in \mathrm{Fol}_{+}^{2}(M)$ and $\mathfrak{G}$ be a finitely generated subgroup of $\mathrm{Diff}_{+}^{2}(M;\mathscr{F})$. Suppose $\mathscr{F}$ has a compact leaf L with nontrivial holonomy such that supp$[L]$ is a fibration over S^1. Then does $\mathscr{F}$ have a compact leaf with nontrivial holonomy whose orbit by the action of $\mathfrak{G}$ is a finite union of compact leaves? The same question with $\mathfrak{G} \subset \mathrm{Diff}_{+}^{2}(M;\mathscr{F})$ not finitely generated.

In particular, consider $f \in \mathrm{Diff}_{+}^{2}(S^1)$ such that $\emptyset \neq \mathrm{Fix}(f) \neq S^1$. We know that if $g \in \mathrm{Diff}_{+}^{2}(S^1)$ commutes with f then all points in $\mathrm{Fr}(\mathrm{Fix}(f))$ are g-periodic with the same period. Is it true that there exists $x_o \in \mathrm{Fr}(\mathrm{Fix}(f))$ whose orbit by the centralizer of f is finite?

To conclude this section we give an application of Theorem 7.6 to the persistence of compact leaves.

Definition 7.7. A foliation $\mathscr{F} \in \mathrm{Fol}^{r}(M)$ has the *compact leaf persistence property* if every foliation C^r close to $\mathscr{F}$ has at least a compact leaf.

We shall denote by $\mathscr{F}_Q$ the foliation in $\mathrm{Fol}^{r}(Q \times M)$ obtained by multiplying each leaf of $\mathscr{F} \in \mathrm{Fol}^{r}(M)$ by Q.

If $\mathscr{F}$ has the compact leaf persistence property then the same holds for the foliation $\mathscr{F}_{S^1}$ of $S^1 \times M$, unless $\mathscr{F}$ can be approximated by foliations given by fibrations over S^1. This follows from Theorem 2.1 and from the following fact: the foliations close to $\mathscr{F}_{S^1}$ are given by suspending foliations $\tilde{\mathscr{F}}$ close to $\mathscr{F}$ by diffeomorphisms $\tilde{f} \in \mathrm{Diff}_{+}^{r}(M;\tilde{\mathscr{F}})$ close to the identity.

Theorem 7.8. *Let $\mathscr{F} \in \mathrm{Fol}_{+}^{r}(M)$, $r \geq 2$ and let Q be a closed manifold. Suppose that $\mathscr{F}$ cannot be C^r approximated by foliations defined by fibrations over S^1. Then $\mathscr{F}$ has the compact leaf persistence property if and only if the same property holds for $\mathscr{F}_Q$.*

PROOF. If $\mathscr{F}_Q$ has the compact leaf persistence property it is easy to see that $\mathscr{F}$ has the same property.

For the converse, assume that $\mathscr{F}$ has the compact leaf persistence property. Let X be a vector field transverse to the foliation $\mathscr{F}$.

If $\tilde{\mathscr{G}} \in \mathrm{Fol}^r_+(Q \times M)$ is sufficiently C^r close to $\mathscr{F}_Q$ we can define the perturbed holonomies $\tilde{f}_\alpha \in \mathrm{Diff}^r_+(M; \tilde{\mathscr{G}}|_{\{q\}\times M})$ associated to the homotopy class of each loop $\alpha \in \Omega(Q; q)$ as follows:

Let $x \in M$ and let $\tilde{\alpha}$ denote the path in a leaf of $\tilde{\mathscr{G}}$ starting at (q, x) obtained by lifting the loop $(\alpha(t), x)$; $t \in [0, 1]$ along the orbits of the vector field $(0, X)$ in $Q \times M$. We define $\tilde{f}_\alpha(x) = \pi_2(\tilde{\alpha}(1))$ where π_2 is the standard projection of $Q \times M$ on M.

Fix loops $\alpha_1, \dots, \alpha_k \in \Omega(Q; q)$ whose homotopy classes generate $\pi_1(Q; q)$ and let $\mathscr{V}$ and $\mathscr{U}$ be as in Theorem 7.6. Choose $\tilde{\mathscr{G}}$ sufficiently C^r close to $\mathscr{F}_Q$ so that:

- $\tilde{\mathscr{G}}|_{\{q\}\times M}$ lies in $\mathscr{U}$,
- $\tilde{f}_{\alpha_i}$; $1 \le i \le k$ lie in $\mathscr{V}$,
- $\tilde{\mathscr{G}}|_{\{q\}\times M}$ has a compact leaf with nontrivial holonomy.

It follows that $\tilde{\mathscr{G}}|_{\{q\}\times M}$ has a compact leaf $\tilde{L}$ whose orbit by the action of the subgroup generated by $\{\tilde{f}_{\alpha_i};\ 1 \le i \le k\}$ is a finite union of compact leaves $L_1, \dots, L_s$.

Let $\tilde{F}$ be the leaf of $\tilde{\mathscr{G}}$ containing $\tilde{L}$. We claim that $\tilde{F}$ is compact. For this, let $\tilde{l}$ be a leaf of $\tilde{\mathscr{G}}|_{\{q\}\times M}$ contained in $\tilde{F}$. Consider a path $\tilde{\beta}$ in $\tilde{F}$ from $\tilde{L}$ to $\tilde{l}$ transverse to the second factor of $Q \times M$. The path $\tilde{\beta}$ projects to a closed loop β on Q such that $\tilde{f}_\beta(\tilde{L}) = \tilde{l}$. Since $\pi_1(Q; q)$ is generated by the homotopy classes of $\alpha_1, \dots, \alpha_k$ we have that $\tilde{l} \subset \bigcup_{j=1}^s \tilde{L}_j$. Hence $\tilde{F} \cap \{q\} \times M = \bigcup_{j=1}^s \tilde{L}_j$. Since $\tilde{F}$ is a fibration over Q with fiber $\tilde{L}$, and $\tilde{F}$ intersects $\{q\} \times M$ finitely many times, we conclude that $\tilde{F}$ is compact and the proof is finished. □

Let $\pi : E \to M$ be a fibration with fibre Q. Suppose that $\mathscr{F}$ has the compact leaf persistence property and it cannot be C^r approximated by foliations defined by fibrations (over S^1). In general, we do not know if $\pi^*(\mathscr{F})$ has the compact leaf persistence property. If $H_1(Q; \boldsymbol{R}) = \{0\}$ then $\pi^*(\mathscr{F})$ has this property. This is a consequence of the following D. Henč's result [**10**]: if $H_1(Q; \boldsymbol{R}) = \{0\}$ then, up to isomorphisms C^r close to the identity map, all the small C^r perturbations of $\pi^*(\mathscr{F})$ are obtained as a pull-back of small C^r perturbations of $\mathscr{F}$.

References

[1] C. Bonatti et S. Firmo, Feuilles compactes d'un feuilletage générique en codimension un, Ann. Sci. École Norm. Sup. (4), **t. 24** (1994), 406–462.

[2] C. Camacho and A. Lins Neto, Geometric Theory of Foliations, Birkhäuser, Boston, 1985.
[3] D. B. A. Epstein, A topology for the space of foliations, Geom. Topol., Rio de Janeiro, Springer Lecture Notes in Math., **597** (1977), 132–150.
[4] S. Firmo and A. Sarmiento, Codimension one foliations without unstable compact leaves, Topology Appl., **81** (1997), 247–267.
[5] E. Ghys, Sur les groupes engendrés par des difféomorphismes proche de l'identité, Bol. Soc. Brasil. Mat., vol. 24 (2) (1993), 137–178.
[6] E. Ghys et V. Sergiescu, Sur un groupe remarquable de difféomorphismes du cercle, Comment. Math. Helv., vol. 62 (1987), 185–239.
[7] C. Godbillon, Dynamical Systems on Surfaces, Springer-Verlag, 1983.
[8] C. Godbillon, Feuilletages—Études Géométriques, Birkhäuser, 1991.
[9] A. Haefliger, Variétés Feuilletées, Ann. Scuola Norm. Sup. Pisa Cl. Sci., vol. 16 (1962), 367–397.
[10] D. Henč, Transverse intersection and stability of foliations, Bol. Soc. Brasil. Mat., vol. 13, no. 1 (1982), 1–18.
[11] M. W. Hirsch, Stability of compact leaves of foliations, Dynamical Systems, Bahia 1971, Academic Press, 1973, 135–153.
[12] G. Hector and U. Hirsch, Introduction to the geometry of foliations, Parts A and B, Friedr. Vieweg und Sohn, Braunschweig, 1981 and 1983.
[13] H. Imanishi, On the theorem of Denjoy-Sacksteder for codimension one foliations without holonomy, J. Math. Kyoto Univ., vol. 14 (1974), 607–634.
[14] N. Kopell, Commuting Diffeomorphisms, Proc. Sympos. Pure Math., vol. 14 (1970), 165–184.
[15] J. Leslie, Two classes of classical subgroups of Diff(M), J. Differential Geom., vol. 5 (1971), 427–435.
[16] J. Leslie, A remark on the group of automorphisms of a foliation having a dense leaf, J. Differential Geom., vol. 7 (1972), 597–601.
[17] S. P. Novikov, Topology of foliations, Trans. Moscow Math. Soc., (1965), 268–304.
[18] G. Reeb, Sur certaines propriétés topologiques des variétés feuilletées, Act. Sc. et Ind., Hermann, Paris, 1952.
[19] R. Sacksteder, On the existence of exceptional leaves in foliations of codimension one, Ann. Inst. Fourier (Grenoble), vol. 14 (2) (1964), 221–226.
[20] R. Sacksteder, Foliations and pseudogroups, Amer. J. Math., **87** (1965), 79–102.
[21] P. Schweitzer, Codimension one foliations without compact leaves, Comment. Math. Helv., vol. 70 (1995), 171–209.
[22] F. Sergeraert, Feuilletages et difféomorphismes infinimente tangents à l'identité, Invent. Math., vol. 39 (1977), 253–275.
[23] G. Szekeres, Regular iteration of real and complex functions, Acta Math., vol. 100 (1958), 203–258.
[24] T. Tsuboi, On 2-cycles of BDiff(S^1) which are represented by foliated S^1-bundles over T^2, Ann. Inst. Fourier (Grenoble), vol. 31 (2) (1981), 1–59.

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