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Phantom maps and injectivity of forgetful maps

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Abstract. In this note, we attack a question about the injectivity of the forgetful map posed ten years ago by Tsukiyama. We show that we can insert the forgetful map in an exact sequence and that the problem can be reduced to the computation of the sequence which turns out unexpectedly to be related to the phantom map problem and the famous Halperin conjecture in rational homotopy theory.

1. Introduction.

Among many research topics in homotopy theory, there are two interesting ones: phantom map theory and forgetful maps which forget the G-action. An interesting relation between these two concepts have been observed in this paper which is used to attack the forgetful map problem in [**12**] posed by Tsukiyama ten years ago.

A pair of maps f, g from a CW complex to a topological space is called a *phantom pair* [**18**] if the restrictions of f, g to the n-skeleton of the complex are homotopic for all $n \geq 0$. In this case, we call the map f a *phantom map with respect to* g which is denoted by a g-phantom map. The set of homotopy classes of g-phantom maps from X to Y is denoted by $Ph^g(X, Y)$. It is clear that the concept of g-phantom map is homotopy invariant. Especially, if g is a constant map, then the g-phantom map becomes just a *phantom map* and $Ph^g(X, Y)$ is just $Ph(X, Y)$. If g is the identity map, then f is a g-phantom map if and only if f is a weak identity as defined by Roitberg [**20**].

Historically Adams and Walker [**1**] found the first nontrivial phantom map and Gray made the first detailed study of phantom maps in his Ph.D. thesis [**8**]. Later many other authors, McGibbon, Meier, Møller, Oda-Shitanda, Roitberg, Sullivan, Zabrodsky and etc. contributed a lot of ideas to this area, see [**14**] for a comprehensive survey of this area. Among many others, they proved the following which are crucial to our later applications.

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THEOREM 1.1 ([**14**]). *Let X and Y be nilpotent CW complexes with finite type. Then the set $Ph(X, Y)$ of all phantom maps from X to Y is either one point set or uncountable.*

Let X_τ denote the homotopy fiber of the rationalization map $r : X \to X_{(0)}$. In general, $Ph(X, Y)$ is a proper subset of $[X, Y]$. But if we use the fact $Ph(X, Y) = \mathrm{Im}\{r^* : [X_{(0)}, Y] \to [X, Y]\}$, then they are equal under some conditions and even more is true by Theorem 5.2 in [**14**] as follows.

THEOREM 1.2. *If X and Y are 1-connected CW complexes of finite type. If $[X_\tau, Y] = [\Sigma X_\tau, Y] = *$, then there are bijections of pointed sets*

$$Ph(X, Y) = [X, Y] \approx [X_{(0)}, Y] \approx \prod_{n>0} H^n(X, \pi_{n+1}(Y) \otimes R)$$

where R is the group of real numbers which is a rational vector space.

If we combine the results in [**14**], then we have the following theorem.

THEOREM 1.3 ([**14**]). *Let $X = \Sigma^m Z$ and $Y = \Omega^n(K)$ $m, n \geq 0$ where K is a 1-connected finite CW complex. If*

(i) *Z is BG and G is a connected compact Lie group, or*

(ii) *Z is a 1-connected finite Postnikov space, i.e. $\pi_j Z = 0$ for j sufficiently large, then there are bijections of pointed sets*

$$Ph(X, Y) = [X, Y] \approx [X_{(0)}, Y] \approx \prod_{i>0} H^i(X, \pi_{i+1}(Y) \otimes R).$$

In the above theorem the target space is the (iterated) loop space of a finite CW complex. To deal with some essential infinite space, Zabrodsky [**28**] extended the above theorem as follows

THEOREM 1.4. *The equation in Theorem 1.2 remains true if $X = \Sigma^m K \cdot (H, l+3)$ and $Y = \Omega^n B\,\mathrm{aut}(P)$ where $m, l, n \geq 0$, H is a finitely generated abelian group and P is a 1-connected finite CW complex.*

Theorem 1.3 and 1.4 say, in some cases, all maps are phantom maps and the homotopy classes of them can be calculated. The general phantom pair is studied only briefly by Oda-Shitanda [**18**] and seems to be forgotten later. Roitberg ([**20**] and [**21**]) has studied the weak identities and posed several interesting open questions about them and later Shitanda has also some related works on it.
According to our point of view, the main problem one faces with the general phantom pair is the following

QUESTION 1.5. Let $g_1, g_2 : X \to Y$ be two maps. What is the relation between $Ph^{g_1}(X, Y) = \{g_1\}$ and $Ph^{g_2}(X, Y) = \{g_2\}$?

A well known result in this direction is

Theorem 1.6 ([**18**]). *If Y is an H-space with inverse, then for any two maps $g_1, g_2 : X \to Y$ two equations in Question* 1.5 *are equivalent.*

For our applications, we will extend Theorem 1.6 to the case when Y is not an H-space. Actually what we need is about somewhat more general notion, see Theorem 2.6, 2.7 for details.

Now let us turn to a forgetful map. Given a principal G-bundle $q : P \to B$, let $\mathrm{Aut}^G(P)$ be the group of G-equivariant homotopy self-equivalences of the fibration. Let $\mathrm{Aut}(P)$ be the group of unbased homotopy self-equivalences of the space P. The following question was posed by Tsukiyama in [**12**].

Question 1.7. When is the natural map

$$F : \mathrm{Aut}^G(P) \to \mathrm{Aut}(P)$$

which forgets the G-action, a monomorphism?

In [**26**], Tsukiyama constructed an example which answers the Question 1.7 negatively and in [**27**], he gave a sufficient condition which answers the question positively. His example is the following:

Example 1.8. Given a connected compact Lie group G which is not a torus and a maximal torus T, then there is a principal G-bundle $G \to G/T \to BT$ over BT which is classified by the natural map $Bi : BT \to BG$ where $i : T \to G$ is the inclusion of T into G. Then $\mathrm{Aut}(G/T)$ is finite and there is an exact sequence

$$0 \to \pi_1(map(BT, BG), Bi) \to \mathrm{Aut}^G(G/T) \to \mathrm{Aut}(BT).$$

Since $\pi_1(map(BT, BG), Bi)$ is uncountable, $\mathrm{Aut}^G(G/T)$ is uncountable and hence the forgetful map $F : \mathrm{Aut}^G(G/T) \to \mathrm{Aut}(G/T)$ is not injective.

One of the main results in this paper is the following

Theorem 1.9. *Let $q : P \to B$ be a principal G-bundle with P a homotopy type of* 1*-connected finite CW complex. Then there is an exact sequence*

$$\pi_1(\mathrm{aut}(P)) \to \pi_1(map_*(BG, B\,\mathrm{aut}(P)), c) \to \mathrm{Aut}^G(P) \xrightarrow{F} \mathrm{Aut}(P)$$

where $c : BG \to B\,\mathrm{aut}(P)$ is determined by the given principal bundle.

Remark 1.10. In the above theorem, the calculation of the kernel of F is in some sense equivalent to the calculation of $\pi_1(map_*(BG, B\,\mathrm{aut}(P)), c)$. If this group is trivial, then the forgetful map is injective. On the other hand, if the group is uncountable, the kernel of F is also uncountable since $\pi_1(\mathrm{aut}(P)) = \pi_1(map(P, P), \mathrm{id})$ is countable.

The following theorem shows that, in some case, the kernel of F is either zero or uncountable.

Theorem 1.11. *Let $q : P \to B$ be as above, $k : B \to BG$ the classifying map, $\bar{k} : \bar{B} \to BG$ the associated fibration with fiber $\bar{P}$ and $c : BG \to B\,\mathrm{aut}(P)$ the classifying map of $\bar{k}$. Then the following statements are true*:

(i) *If $Ph_1^c(BG, B\,\mathrm{aut}(P)) \neq 0$, the kernel of F is uncountable.*

(ii) *If G is a connected compact Lie group and c is a phantom map, then the kernel of F is either zero or uncountable.*

(iii) *If G is 1-connected $K(H, m)$ for a finitely generated abelian group H, then the kernel of F is either zero or uncountable.*

According to Tsukiyama ([**26**] and [**27**]), it is possible that the kernel of forgetful map is zero or uncountable. The above theorem says in some case that these are the only possibilities. Furthermore we will show that the results in phantom map theory and rational calculation which is usually not so difficult can be used to decide when the kernel of forgetful map is zero or uncountable. The above theorem leads to another natural question.

Question 1.12. Is it possible that the kernel of forgetful map is non-trivial finite or countable?

Now we will give concrete conditions for the injectivity or non-injectivity of forgetful map.

Theorem 1.13. *Let P be a 1-connected finite CW complex. Then the following statements are equivalent.*

(i) *There is a connected compact Lie group G and a principal G-bundle such that the total space has the homotopy type of P, the classifying map c is a phantom map and the associated forgetful map has uncountable kernel.*

(ii) $\bigoplus_{i>0} \pi_{2i+1}(map(P_{(0)}, P_{(0)}), \mathrm{id})$ *is nontrivial.*

Before giving concrete examples of principal bundles with non-injective forgetful maps, we recall some backgrounds. A 1-connected CW complex X such that $\dim H^*(X) < \infty$ is called *rationally elliptic* [**23**] if $\dim \pi_*(X) \otimes Q < \infty$. Let $(X, *)$ be any pointed space. The *Gottlieb group* (or *evaluation subgroup*) [**9**] is defined by

$$G_n(X) = \mathrm{Im}\{ev_* : \pi_n(map(X, X), \mathrm{id}) \to \pi_n(X)\}$$

where $ev : map(X, X)_{\mathrm{id}} \to X$ is defined by $ev(f) = f(*)$.

The Gottlieb groups are extremely difficult to compute in general. However for rational spaces there have been some remarkable results on the Gottlieb groups.

The following theorem was proved by Félix and Halperin in [**7**, Theorem III].

Theorem 1.14. *If X is a* 1*-connected finite CW complex*, *then* $G_{even}(X_{(0)}) = 0$.

Moreover, if X is rationally nontrivial and elliptic, then we have the following result from [**23**, Theorem 4.1].

Theorem 1.15. *If X is a* 1*-connected finite CW complex which is rationally nontrivial and elliptic*, *then* $G_*(X_{(0)}) \neq 0$.

Now the following theorem follows immediately from Theorem 1.13, 1.14 and Theorem 1.15.

Theorem 1.16. *Let P be a* 1*-connected finite CW complex which is rationally nontrivial and elliptic. Then there are a compact Lie group G and a principal G-bundle with total space homotopy equivalent to P such that the forgetful map has uncountable kernel.*

Remark 1.17. We do not know the existence of 1-connected CW complex P such that $\bigoplus_{i>0} \pi_{2i+1}(map(P_{(0)}, P_{(0)}), \mathrm{id}) = 0$.

If we assume $G = K(H, 2m)$ where H is a finitely generated abelian group and $m \geq 1$, we have the following

Theorem 1.18. *Let P be a* 1*-connected finite CW complex. Then the following statements are equivalent.*

(i) *For any principal $K(H, 2m)$-bundle where $m \geq 1$, H a finitely generated abelian group and total space homotopy equivalent to P, the associated forgetful map is injective.*
(ii) $\bigoplus_{i>1} \pi_{2i}(map(P_{(0)}, P_{(0)}), \mathrm{id}) = 0$.

Now we want to give some examples with injective forgetful maps. Again we first recall some backgrounds. A 1-connected CW complex X is said to be of *type F_0* if X is rationally elliptic and $H^{odd}(X; Q) = 0$. A fibration $P \xrightarrow{j} E \to B$ with a path connected base is *orientable* if it is totally non-cohomologous to zero, i.e. if the map $j^* : H^*(E; Q) \to H^*(P; Q)$ is surjective. One of the most beautiful conjectures in rational homotopy theory is the following

Conjecture 1.19 ([**6**]). *Let $P \to E \to B$ be an orientable fibration such that the fiber P is homotopy equivalent to a CW complex of type F_0. Then the Serre spectral sequence* (*with rational coefficients in Q*) *of the fibration collapses at the E^2 term.*

In [**17**, Theorem A], Meier found the relation between Halperin conjecture and the group $\pi_{even}(map(P_{(0)}, P_{(0)}), \mathrm{id})$ as follows.

THEOREM 1.20. *Let P be of type F_0. Then the following statements are equivalent.*

(i) *The Serre spectral sequence of every orientable fibration with fiber P collapses at the E^2 term.*

(ii) $\pi_{even}(map(P_{(0)}, P_{(0)}), \mathrm{id}) = 0$.

The Halperin conjecture have been verified for a couple of special cases. The results obtained so far can be stated as follows.

THEOREM 1.21 ([**2**], [**13**], [**22**], [**25**]). *If P satisfies one of the following conditions*

(i) *P is a Kähler manifold,*

(ii) *$H^*(P; Q)$ as an algebra has at most 3 generators,*

(iii) *$P = G/U$ for G a compact Lie group and U a closed subgroup of maximal rank,*

then the Halperin conjecture is true.

Comparing Theorem 1.18 and 1.20 we obtain immediately the following

THEOREM 1.22. *Let P be a 1-connected finite CW complex of type F_0. Then the following statements are equivalent.*

(i) *The Halperin conjecture is true for P.*

(ii) *For any principal $K(H, 2m)$-bundle where $m \geq 1$, H a finitely generated abelian group and total space homotopy equivalent to P, the associated forgetful map is injective and $\pi_2(map(P_{(0)}, P_{(0)}), \mathrm{id}) = 0$.*

COROLLARY 1.23. *Let P be a 1-connected finite CW complex satisfying one of the conditions of Theorem 1.21. Then for any principal $K(H, 2m)$-bundle where $m \geq 1$, H a finitely generated abelian group and total space homotopy equivalent to P, the associated forgetful map is injective.*

In section 2, we will introduce the phantom element which is a generalization of phantom pair and use this concept to prove Theorem 2.9 and Theorem 2.10. In section 3, we will study the forgetful map and try to insert it into an exact sequence and prove Theorem 1.9. In section 4, the results of previous sections will be applied here to get the precise information about the forgetful map and prove Theorem 1.11, 1.13, 1.18. In this paper all our basic spaces will be assumed to be 1-connected CW complexes of finite type. We will also use the following notations:

(i) X^n is the n-skeleton of X.

(ii) $map(X, Y)$ is the space of continuous mappings from X to Y.

(iii) $map_*(X, Y)$ is the subspace of pointed mappings from (X, x_0) to (Y, y_0).

(iv) $r : X \to X_{(0)}$ is the rationalization.

(v) Let X_τ be the homotopy fiber of r. Then $X_\tau \to X \to X_{(0)}$ is a cofibration up to homotopy.

(vi) $\hat{e} : Y \to \hat{Y}$ is the profinite completion [**24**] and $\rho : Y_\rho \to Y$ is the homotopy fiber of $\hat{e}$.

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2. Phantom elements.

Let X be a CW complex and Y be a space. A map $f : X \to Y$ is called a *phantom map* if $f|_{X^n}$ (the restriction of f to the n-th skeleton of X) is homotopic to the constant map for all $n \geq 0$. A pair of maps $f, g : X \to Y$ is called a *phantom pair* [**18**] if $f|_{X^n}$ is homotopic to $g|_{X^n}$ for all $n \geq 0$. For a fixed map $g : X \to Y$ we denote by $Ph^g(X, Y)$ the set of homotopy classes of maps f such that f and g are a phantom pair. Each element of $Ph^g(X, Y)$ is also called a *g-phantom map*.

Here we generalize the concept of phantom pair as follows.

Definition 2.1. Let X be a CW complex, Y a space and $g : X \to Y$ be any based map. Then $\alpha \in \pi_j(map_*(X, Y), g)$ is called a *g-phantom element* if $(i_n^*)_*(\alpha) = 0$ for each $n \geq 0$ where $(i_n^*)_* : \pi_j(map_*(X, Y), g) \to \pi_j(map_*(X^n, Y), g|_{X^n})$ is the homomorphism induced by the inclusion $i_n : X^n \to X$. Denote the set of all g-phantom elements by

$$Ph_j^g(X, Y) = \{\alpha \in \pi_j(map_*(X, Y), g) \mid \alpha \text{ is a } g\text{-phantom element}\}.$$

If $j = 0$, then α is a g-phantom element if and only if it represents the homotopy class of a map which is a g-phantom map. If g is the constant map, then a g-phantom map is the same as the phantom map. Since Adams and Walker [**1**] found the first essential phantom map, this area has attracted interests of many mathematicians.

Let us recall some basic results about homotopy of a sequence of fibration at first. Let

$$\cdots \longrightarrow X_{n+1} \xrightarrow{v_{n+1}} X_n \xrightarrow{v_n} X_{n-1} \longrightarrow \cdots$$

be a sequence of fibration of spaces and X be the inverse limit of the above inverse system. If we choose base points $x_n \in X_n$ such that $v_n(x_n) = x_{n-1}$, it was shown by Bousfield and Kan in [**5**] that there exists the following short exact sequence for $j \geq 0$

$$* \to \underset{\longleftarrow n}{\lim}{}^1 \pi_{j+1}(X_n, x_n) \to \pi_j(\underset{\longleftarrow}{\lim}\ X_n, x) \to \underset{\longleftarrow}{\lim}\ \pi_j(X_n, x_n) \to *.$$

THEOREM 2.2. *Let X, Y be nilpotent CW complexes of finite type and $g : X \to Y$ be any map. Then for all $j \geq 0$, we have*

$$Ph_j^g(X, Y) = \underset{\longleftarrow n}{\lim}{}^1 \pi_{j+1}(map_*(X^n, Y), g|_{X^n}).$$

PROOF. Let X^n be the n-skeleton of X. Then

$$\cdots \to map_*(X^{n+1}, Y) \to map_*(X^n, Y) \to map_*(X^{n-1}, Y) \to \cdots$$

is a sequence of fibration with $map_*(X, Y)$ as the inverse limit. By the Bousfield and Kan's result and the given fibration, there exists a short exact sequence

$$* \to \underset{\longleftarrow n}{\lim}{}^1 \pi_{j+1}(map_*(X^n, Y), g|_{X^n}) \to \pi_j(map_*(X, Y), g)$$

$$\to \underset{\longleftarrow n}{\lim}\ \pi_j(map_*(X^n, Y), g|_{X^n}) \to *$$

for all $j \geq 0$ and any map $g : X \to Y$. By the definition of phantom elements, the proof is complete. □

A natural problem about $Ph_j^g(X, Y)$ is its cardinality. For this we have the following

THEOREM 2.3 ([**14**]). *The first derived inverse limit of an inverse system of countable groups is either one point set or uncountable.*

COROLLARY 2.4. *Let X, Y be nilpotent CW complexes of finite type and $g : X \to Y$ be any map. Then $Ph_j^g(X, Y)$ is either one point set or uncountable for all $j \geq 0$.*

Another natural question is the extended version of Question 1.5.

QUESTION 2.5. For two maps $f, g : X \to Y$, what is the relation between $Ph_j^g(X, Y)$ and $Ph_j^f(X, Y)$?

The first result in this direction is an extension of the result of Oda-Shitanda [**18**].

THEOREM 2.6. *Let X be a nilpotent CW complex and Y be an H-space with inverse. If $f, g : X \to Y$ are any two maps, then $Ph_j^g(X, Y) = Ph_j^f(X, Y)$.*

PROOF. Let Y be an H-space with a multiplication $\mu : Y \times Y \to Y$ and an inverse $\nu : Y \to Y$ such that the composites of

$$Y \xrightarrow{(1,\nu)} Y \times Y \xrightarrow{\mu} Y,$$

$$Y \xrightarrow{(\nu,1)} Y \times Y \xrightarrow{\mu} Y$$

are homotopic to the constant map. For the n-skeleton X^n, let us define h_n by the composite

$$\begin{aligned} map_*(X^n, Y)_{f^n} &\xrightarrow{(1,\nu_*)} map_*(X^n, Y)_{f^n} \times map_*(X^n, Y)_{\nu \circ f^n} \\ &\longrightarrow map_*(X^n, Y \times Y)_{(f^n, \nu \circ f^n)} \xrightarrow{\mu_*} map_*(X^n, Y)_*. \end{aligned}$$

Then for any map $f : X \to Y$, it is easy to show that the following diagram commutes up to homotopy

$$\begin{tikzcd} map_*(X^{n+1}, Y)_{f^{n+1}} \arrow[r, "i_n^*"] \arrow[d, "h_{n+1}"] & map_*(X^n, Y)_{f^n} \arrow[d, "h_n"] \\ map_*(X^{n+1}, Y)_* \arrow[r, "i_n^*"'] & map_*(X^n, Y)_* \end{tikzcd}$$

where f^n is the restriction of f to X^n and $i_n : X^n \to X^{n+1}$ is the inclusion map. The proof is complete by Theorem 2.2. □

Theorem 2.7 ([**19**]). *Let X, Y be nilpotent CW complexes of finite type and $f : X \to Y$ be any map. Then the following statements are equivalent.*

(i) *$\alpha \in \pi_j(map_*(X, Y), f)$ is a phantom element.*

(ii) *$(\hat{e}_*)_*(\alpha) = 0$ or $(\tau^*)_*(\alpha) = 0$*

where $(\hat{e}_*)_* : \pi_j(map_*(X, Y), f) \to \pi_j(map_*(X, \hat{Y}), \hat{f})$ is the induced map by the profinite completion $\hat{e}$ and $(\tau^*)_* : \pi_j(map_*(X, Y), f) \to \pi_j(map_*(X_\tau, Y), f_\tau)$ is the induced map by the inclusion $X_\tau \to X$.

Proof. We will only prove (ii) $\Rightarrow$ (i) which is necessary for our applications in this paper. For the proof of the other part, see [**19**]. Let $\alpha \in \pi_j(map_*(X, Y), f)$ and $(\hat{e}_*)_*(\alpha) = 0$. If we consider the following commutative diagram

$$\begin{tikzcd} \pi_j(map_*(X, Y), f) \arrow[r, "(i_n^*)_*"'] \arrow[d, "(\hat{e}_*)_*"] & \pi_j(map_*(X^n, Y), f|_{X^n}) \arrow[d, "(\hat{e}_*)_*"] \\ \pi_j(map_*(X, \hat{Y}), \hat{f}) \arrow[r, "(i_n^*)_*"'] & \pi_j(map_*(X^n, \hat{Y}), \hat{f}|_{X^n}) \end{tikzcd}$$

then we have $(\hat{e}_*)_* \circ (i_n^*)_*(\alpha) = (i_n^*)_* \circ (\hat{e}_*)_*(\alpha) = 0$. In [**24**], Sullivan showed that if Y is a nilpotent space, $\hat{e} : Y \to \hat{Y}$ and h, g are any two maps from a finite CW complex Z to Y such that $\hat{e} \circ g \simeq \hat{e} \circ h$, then $g \simeq h$. By the result of Sullivan, it follows immediately that the map $\hat{e}_* : map_*(X^n, Y)_{f|_{X^n}} \to map_*(X^n, \hat{Y})_{\hat{f}|_{X^n}}$ has the above property. Thus the induced homomorphism

$$(\hat{e}_*)_* : \pi_j(map_*(X^n, Y), f|_{X^n}) \to \pi_j(map_*(X^n, \hat{Y}), \hat{f}|_{X^n})$$

is a monomorphism. Thus $(i_n^*)_*(\alpha) = 0$ and hence α is a phantom element.

Let $(\tau^*)_*(\alpha) = 0$. Then we have $(\hat{e}_*)_* \circ (\tau^*)_*(\alpha) = 0 \in \pi_j(map_*(X_\tau, \hat{Y}), \hat{f}_\tau)$. By Proposition 2.1 of [**18**], $map_*(X_{(0)}, \hat{Y})$ is weakly contractible and hence the induced map $(\tau^*)_* : \pi_j(map_*(X, \hat{Y}), \hat{f}) \to \pi_j(map_*(X_\tau, \hat{Y}), \hat{f}_\tau)$ is an isomorphism for $j > 0$. Since $(\hat{e}_*)_* \circ (\tau^*)_* = (\tau^*)_* \circ (\hat{e}_*)_*$, we have $(\hat{e}_*)_*(\alpha) = 0 \in \pi_j(map_*(X, \hat{Y}), \hat{f})$. By the first part, α is a phantom element. □

Proposition 2.8. *Let X, Y be 1-connected CW complexes of finite type such that $map_*(X_\tau, Y) \overset{w}{\simeq} *$. If $f : X \to Y$ is a phantom map, then*

$$Ph_j^f(X, Y) = \pi_j(map_*(X, Y), f) = [\Sigma^j X_{(0)}, Y].$$

Proof. To show the first equality, it suffices to prove $\pi_j(map_*(X_\tau, Y), f_\tau) = 0$ by Theorem 2.7. From Theorem 5.1 of [**14**], f_τ is homotopic to the constant map and hence

$$\pi_j(map_*(X_\tau, Y), f_\tau) \cong \pi_j(map_*(X_\tau, Y), *) \cong [\Sigma^j X_\tau, Y] = 0.$$

Thus the proof of first equality is complete.

The second equality follows from the proof of Theorem D(i) in Zabrodsky [**28**] since the proof depends only on the condition $map_*(X_\tau, Y) \overset{w}{\simeq} *$. □

Theorem 2.9. *Let $X = K(H, m)$, $Y = B\,\mathrm{aut}(P)$ and $f : X \to Y$ be any map where $m \geq 3$, H is a finitely generated abelian group and P is a 1-connected finite CW complex. Then $Ph_j^f(X, Y) = \pi_j(map_*(X, Y), f) = [\Sigma^j X_{(0)}, Y]$.*

Proof. By 1.3.1 and Corollary C′ in [**28**], it is easy to show that $map_*(X_\tau, Y)$ is weakly contractible. It also follows by Theorem 1.4 that any map $f : X \to Y$ is a phantom map. Thus we can apply Proposition 2.8 to get

$$Ph_j^f(X, Y) = \pi_j(map_*(X, Y), f).$$

To prove the last equality, note that if X is 1-connected, then $map_*(X, Y)$ is homotopy equivalent to $map_*(X, \tilde{Y})$ where $\tilde{Y}$ is the universal covering of Y. Since $\tilde{Y}$ is a 1-connected space, thus the proof is completed by Proposition 2.8. □

THEOREM 2.10. *Let* $X = BG$ *and* $Y = B\,\mathrm{aut}(P)$ *where* G *is a connected compact Lie group and* P *is a* 1*-connected finite CW complex. If* $f : X \to Y$ *is a phantom map and* $j \geq 1$, *then* $Ph_j^f(X, Y) = \pi_j(map_*(X, Y), f) = [\Sigma^j X_{(0)}, Y]$.

PROOF. To prove the first equality, it suffices by the proof of Proposition 2.8 to show that

$$[\Sigma^j X_\tau, Y] = 0.$$

If we use Theorem C(c) in [**28**], it is easy to show

$$[\Sigma^j X_\tau, Y] = [\Sigma^{j-1} BG_\tau, \Omega Y] = [\Sigma^{j-1} BG_\tau, \mathrm{aut}(P)] = 0.$$

The proof of the second equality follows from the same argument as in the proof of Theorem 2.9. □

3. Forgetful map and its description.

Let us consider the principal G-bundle $q : P \to B$ with structure group G where G acts on P freely. For each such bundle one can consider the space $\mathrm{aut}^G(P)$ of unbased G-equivariant homotopy self-equivalences of P and the group

$$\mathrm{Aut}^G(P) = \pi_0(\mathrm{aut}^G(P))$$

which is called the group of G-equivariant homotopy self-equivalences of the bundle.

On the other hand, we can also consider the space $\mathrm{aut}(P)$ of unbased homotopy self-equivalences of the space P and the group

$$\mathrm{Aut}(P) = \pi_0(\mathrm{aut}(P))$$

which is called the group of unbased homotopy self-equivalences of P. There have been an extensive study on these two subjects (see [**12**]) and the extensive references there. In [**12**], Tsukiyama posed the following question.

When is the natural map $F : \mathrm{Aut}^G(P) \to \mathrm{Aut}(P)$, which forgets the G-action, a monomorphism?

No progress has been made except two recent papers by Tsukiyama ([**26**] and [**27**]). In this paper, we will try to attack this question. Our approach is based on the identification of the space of G-equivariant homotopy self-equivalences as the loop space on a mapping space and the recent results on the Sullivan conjecture. What is most interesting about our results is the relation between the injectivity of F and the existence of the phantom map between appropriate spaces.

In [**26**] and [**27**], Tsukiyama used an indirect approach to attack the question and got partial results on it. In this section, based on a simple but crucial observation, we will identify the homomorphism F as the homomorphism induced on π_0 by a map whose homotopy fiber can be determined explicitly and thus we can determine the kernel of F under a reasonable condition.

Now let G be a topological group, $q : P \to B$ be a principal G-bundle and $k : B \to BG$ be the classifying map. For the map k, we can take $\bar{k} : \bar{B} \to BG$ as a fibration via the standard factorization of a map into the composite of a homotopy equivalence and a Hurewicz fibration. Given fibration $\bar{k} : \bar{B} \to BG$, we can form the group $\mathrm{Aut}_{BG}(\bar{B})$ the group of homotopy classes of self homotopy equivalences of $\bar{B}$ over BG. The following proposition is a well known result, see [**3**], [**4**], [**10**], [**11**].

Proposition 3.1. *Let $c : BG \to B\,\mathrm{aut}(P)$ be the classifying map for the fibration $\bar{k} : \bar{B} \to BG$. Then there is a natural isomorphism*

$$\mathrm{Aut}^G(P) \cong \mathrm{Aut}_{BG}(\bar{B}) \cong \pi_1(map(BG, B\,\mathrm{aut}(P)), c).$$

If the above isomorphism is natural in object G, then the map F will be naturally isomorphic to the map $\pi : \pi_1(map(BG, B\,\mathrm{aut}(P)), c) \to \pi_1(B\,\mathrm{aut}(P))$ whose kernel can be computed explicitly by the evaluation fibration

$$map_*(BG, B\,\mathrm{aut}(P))_c \to map(BG, B\,\mathrm{aut}(P))_c \to B\,\mathrm{aut}(P).$$

A careful check confirms the above speculation and leads to the following

Theorem 3.2. *Let $q : P \to B$ be a principal G-bundle and $c : BG \to B\,\mathrm{aut}(P)$ be the classifying map for the fibration $\bar{k} : \bar{B} \to BG$. Then there is a commutative diagram*

$$\begin{array}{ccc} \pi_0\,\mathrm{aut}_{BG}(\bar{B}) & \xrightarrow[\cong]{} & \pi_0\,\mathrm{aut}^G(P) \\ \downarrow & & \downarrow \\ \pi_0\,\mathrm{aut}(\bar{P}) & \xrightarrow[\cong]{} & \pi_0\,\mathrm{aut}(P), \end{array}$$

where $\bar{P}$ is the fiber of $\bar{k}$ which is homotopy equivalent to P and the two horizontal maps are isomorphisms. It follows from the above diagram that there is an exact sequence

$$\pi_1(\mathrm{aut}(P)) \to \pi_1(map_*(BG, B\,\mathrm{aut}(P)), c) \to \mathrm{Aut}^G(P) \xrightarrow{F} \mathrm{Aut}(P).$$

This theorem follows directly from the following lemmas and the fact that the homotopy fiber of the fibration $\mathrm{aut}_{BG}(\bar{B}) \to \mathrm{aut}(\bar{P})$ is $\Omega map_*(BG, B\,\mathrm{aut}(P))_c$ by Corollary 5.7 and Proposition 5.8 in [**4**]. Let $q : P \to B$ be a principal G-bundle and $k : X \to B$ be a map, then there is an associated principal G-bundle over X defined by

$$\begin{tikzcd} k^*(P) \arrow[r] \arrow[d, "k^*(q)"] & P \arrow[d, "q"] \\ X \arrow[r, "k"'] & B. \end{tikzcd}$$

Lemma 3.3. *Let $\pi : EG \to BG$ be the universal principal G-bundle. Then the rule that takes k to $k^*(\pi)$ defines a natural bijection from $[B, BG]$, the set of free homotopy classes of maps from B to BG, to the set of isomorphism classes of G-bundles over B.*

Proof. This is well known. □

Lemma 3.4. *If $q : P \to B$ is a principal G-bundle and $g : X \to B$ is a homotopy equivalence, then the induced bundle map from $g^*(q) : g^*(P) \to X$ to q is a homotopy equivalence between two principal bundles.*

Proof. This is (1.9) of [**3**]. □

Lemma 3.5. *Let $q : P \to B$ and $k : B \to BG$ be as above. Then there is a commutative diagram*

$$\begin{tikzcd} \mathrm{aut}^G(k^*(EG)) \arrow[r] \arrow[d] & \mathrm{aut}^G(P) \arrow[d] \\ \mathrm{aut}(k^*(EG)) \arrow[r] & \mathrm{aut}(P), \end{tikzcd}$$

where the two vertical maps are forgetful maps and the horizontal maps are homotopy equivalences which are defined in the proof.

Proof. By Lemma 3.4, there is a principal bundle isomorphism $h : P \to k^*(EG)$ over B. Define the horizontal maps by the rule $x \mapsto h^{-1} \circ x \circ h$. It is obvious that the diagram is commutative. □

Lemma 3.6. *Let $q : P \to B$, $k : B \to BG$ and $\bar{k} : \bar{B} \to BG$ be as above. Then there is a commutative diagram up to homotopy*

$$\begin{tikzcd} \mathrm{aut}^G(\bar{k}^*(EG)) \arrow[r] \arrow[d] & \mathrm{aut}^G(k^*(EG)) \arrow[d] \\ \mathrm{aut}(\bar{k}^*(EG)) \arrow[r] & \mathrm{aut}(k^*(EG)), \end{tikzcd}$$

where the two vertical maps are forgetful maps and the horizontal maps are homotopy equivalences which are defined in the proof.

Proof. By Lemma 3.4, there is a homotopy equivalence h between two principal bundles $k^*(\pi) : k^*(EG) \to B$ and $\bar{k}^*(\pi) : \bar{k}^*(EG) \to \bar{B}$. If we define the horizontal maps as in the proof of the above lemma, then the diagram is easily seen to be commutative up to homotopy. □

Lemma 3.7. *Let $q : P \to B$, $k : B \to BG$ and $\bar{k} : \bar{B} \to BG$ be as above. Then there is a commutative diagram up to homotopy*

$$\begin{array}{ccc} \pi_0 \operatorname{aut}_{BG}(\bar{B}) & \longrightarrow & \pi_0 \operatorname{aut}^G(\bar{k}^*(EG)) \\ \downarrow & & \downarrow \\ \pi_0 \operatorname{aut}(\bar{P}) & \longrightarrow & \pi_0 \operatorname{aut}(\bar{k}^*(EG)) \end{array}$$

where $\bar{P}$ is the fiber of the fibration $\bar{k}$ which is homotopy equivalent to P, the right vertical map is a forgetful map, the left vertical map is the function sending a map to the map which induced on the fiber of fibration $\bar{k}$ at a based point of BG and the horizontal maps are defined in the proof.

Proof. By definition, $\bar{k}^*(EG) = \{(b, e) \in \bar{B} \times EG \mid \bar{k}(b) = \pi(e)\}$ and $\bar{P} = \{(b, *) \in \bar{B} \times EG \mid \bar{k}(b) = *\}$. Now there is the obvious map $f : \bar{P} \to \bar{k}^*(EG)$ defined by $f((b, *)) = (b, *)$ which is a homotopy equivalence by the general property of pullback.

Given a self homotopy equivalence $h \in \operatorname{aut}_{BG}(\bar{B})$ there exists a map $\bar{h} : \bar{k}^*(EG) \to \bar{k}^*(EG)$ defined by $\bar{h}(b, e) = (h(b), e)$ which makes the pair $(\bar{h}, h)$ a principal bundle map. The given map h induces also an obvious map $\tilde{h} \in \operatorname{aut}(\bar{P})$ defined by $\tilde{h}(b, *) = (h(b), *)$. It is easy to check that $\bar{h} \circ f = f \circ \tilde{h}$. If we define the horizontal maps in the above diagram by

$$\pi_0 \operatorname{aut}_{BG}(\bar{B}) \to \pi_0 \operatorname{aut}^G(\bar{k}^*(EG)), h \mapsto (\bar{h}, h)$$

$$\pi_0 \operatorname{aut}(\bar{P}) \to \pi_0 \operatorname{aut}(\bar{k}^*(EG)), g \mapsto f \circ g \circ f^{-1},$$

then it is easy to check that the diagram is commutative which is what we want to prove. □

4. Applications to forgetful maps.

In Section 3, we have embedded the forgetful map into the exact sequence

$$\pi_1(\operatorname{aut}(P)) \to \pi_1(map_*(BG, B\operatorname{aut}(P)), c) \to \operatorname{Aut}^G(P) \xrightarrow{F} \operatorname{Aut}(P).$$

In this section, we will apply the phantom map theory to extract information about the forgetful map. If

$$\pi_1(map_*(BG, B\operatorname{aut}(P)), c) = 0,$$

then we know that $\mathrm{Ker}\, F = 0$ from the above exact sequence. If

$$\pi_1(map_*(BG, B\,\mathrm{aut}(P)), c) \neq 0,$$

then we can not say anything about the $\mathrm{Ker}\, F$. If P is a finite CW complex, then $\pi_1(\mathrm{aut}(P))$ is a countable group. Therefore, if $\pi_1(map_*(BG, B\,\mathrm{aut}(P)), c)$ is uncountable, then $\mathrm{Ker}\, F$ is uncountable. This is the relation what we find between phantom map theory and the injectivity of forgetful map. From the above discussion, we have the following which is Theorem 1.11 in the introduction.

Theorem 4.1. *Let $q : P \to B$ be a principal G-bundle with P a 1-connected finite CW complex, $k : B \to BG$ the classifying map, $\bar{k} : \bar{B} \to BG$ the associated fibration with fiber $\bar{P}$ and $c : BG \to B\,\mathrm{aut}(P)$ the classifying map of $\bar{k}$. Then the following statements are true*:

(i) *If $Ph_1^c(BG, B\,\mathrm{aut}(P)) \neq 0$, then $\mathrm{Ker}\, F$ is uncountable.*

(ii) *If c is a phantom map and G is a connected compact Lie group, then $\mathrm{Ker}\, F$ is either zero or uncountable.*

(iii) *If G is 1-connected $K(H, m)$ for a finitely generated abelian group H, then $\mathrm{Ker}\, F$ is either zero or uncountable.*

Proof. Since $Ph_1^c(BG, B\,\mathrm{aut}(P))$ is a subgroup of $\pi_1(map_*(BG, B\,\mathrm{aut}(P)), c)$ and is either one point or uncountable by Corollary 2.4, (i) is clear.

If we use $Ph_1^c(BK(H, m), B\,\mathrm{aut}(P)) = Ph_1^c(K(H, m+1), B\,\mathrm{aut}(P))$ and Theorem 2.9 for G to be a 1-connected $K(H, m)$ and if we use Theorem 2.10 for c a phantom map and G a connected compact Lie group, we can show

$$Ph_1^c(BG, B\,\mathrm{aut}(P)) = \pi_1(map_*(BG, B\,\mathrm{aut}(P)), c).$$

This implies $\pi_1(map_*(BG, B\,\mathrm{aut}(P)), c)$ (and hence $\mathrm{Ker}\, F$) is either zero or uncountable. □

Lemma 4.2. *If the map c in Theorem 4.1 is a phantom map, then F is injective (has uncountable kernel, respectively) if and only if $[BG_{(0)}, \mathrm{aut}(P)]$ is trivial (nontrivial, respectively).*

Proof. By Theorem 2.10, we have

$$Ph_1^c(BG, B\,\mathrm{aut}(P)) = \pi_1(map_*(BG, B\,\mathrm{aut}(P)), c) = [\Sigma^1 BG_{(0)}, B\,\mathrm{aut}(P)].$$

Thus the kernel of forgetful map is either zero or uncountable. It is zero if and only if $\pi_1(map_*(BG, B\,\mathrm{aut}(P)), c) = 0$. Since

$$\pi_1(map_*(BG, B\,\mathrm{aut}(P)), c) = [\Sigma^1 BG_{(0)}, B\,\mathrm{aut}(P)] = [BG_{(0)}, \mathrm{aut}(P)],$$

it completes the proof. □

Theorem 4.3. *Let P be a* 1*-connected finite CW complex. Then the following statements are equivalent.*

(i) *There is a connected compact Lie group G and a principal G-bundle where the total space has the homotopy type of P and the classifying map c is a phantom map such that the associated forgetful map has uncountable kernel.*

(ii) $\bigoplus_{i>0} \pi_{2i+1}(map(P_{(0)}, P_{(0)}), \mathrm{id})$ *is nontrivial.*

Proof. To prove this theorem, we first note that given map $c_0 : BG \to B\,\mathrm{aut}(P)$ then there is a principal G-bundle $q : P' \to B$ such that P' is the same homotopy type to P and the natural associated map $c : BG \to B\,\mathrm{aut}(P)$ of the associated fibration $\bar{k} : \bar{B} \to BG$ is homotopic to c_0. Therefore it is sufficient to take $c_0 = *$ and choose a Lie group G such that $[BG_{(0)}, \mathrm{aut}(P)] = [BG_{(0)}, map(P, P)_{\mathrm{id}}] \neq 0$.

According to Theorem 1.2, $[BG_{(0)}, map(P, P)_{\mathrm{id}}] \neq 0$ if $H^t(BG, Q) \neq 0$ and $\pi_{t+1}(map(P_{(0)}, P_{(0)}), \mathrm{id}) \neq 0$. Therefore it remains to show $H^t(BG, Q) \neq 0$ for some even integer t satisfying $\pi_{t+1}(map(P_{(0)}, P_{(0)}), \mathrm{id}) \neq 0$. Let t_0 be the smallest positive even integer such that $\pi_{t+1}(map(P_{(0)}, P_{(0)}), \mathrm{id}) \neq 0$. There exists of course a compact Lie group G such that $H^{t_0}(BG, Q) \neq 0$. It follows from the discussion above that there exists a principal G-bundle such that the total space has the homotopy type of P and the associated forgetful map has uncountable kernel by the above lemma.

Conversely, if we assume $\bigoplus_{i>0} \pi_{2i+1}(map(P_{(0)}, P_{(0)}), \mathrm{id}) = 0$, then it is easy to see that $[BG_{(0)}, \mathrm{aut}(P)] = 0$ for any connected compact Lie group. This completes the proof by the above lemma. □

Corollary 4.4. *Let P be a* 1*-connected finite CW complex. Then for all principal G-bundle $q : P \to B$ such that the structure group is a connected compact Lie group and the associated map c is a phantom map, the associated forgetful map is injective if and only if* $\bigoplus_{i>0} \pi_{2i+1}(map(P_{(0)}, P_{(0)}), \mathrm{id}) = 0$.

By the same way to Theorem 4.3, we have the following theorem for $G = K(H, m)$.

Theorem 4.5. *Let P be a* 1*-connected finite CW complex. Then the following statements are equivalent.*

(i) *There is a principal $K(H, 2m+1)$ ($K(H, 2m)$, respectively)-bundle where $m \geq 1$, H a finitely generated abelian group and the total space has the homotopy type of P such that the associated forgetful map has uncountable kernel.*

(ii) $\bigoplus_{i>1} \pi_{2i+1}(map(P_{(0)}, P_{(0)}), \mathrm{id})$ ($\bigoplus_{i>1} \pi_{2i}(map(P_{(0)}, P_{(0)}), \mathrm{id})$, *respectively*) *is nontrivial.*

Corollary 4.6. *Let P be a* 1*-connected finite CW complex. Then the following statements are equivalent.*

(i) *For every principal $K(H, 2m+1)$-bundle where $m \geq 1$, H a finitely generated abelian group and total space homotopy equivalent to P, the associated forgetful map is injective.*

(ii) $\bigoplus_{i>1} \pi_{2i+1}(map(P_{(0)}, P_{(0)}), \mathrm{id}) = 0.$

The following corollary is Theorem 1.18 in the introduction.

Theorem 4.7. *Let P be a* 1*-connected finite CW complex. Then the following statements are equivalent.*

(i) *For any principal $K(H, 2m)$-bundle where $m \geq 1$, H a finitely generated abelian group and total space homotopy equivalent to P, the associated forgetful map is injective.*

(ii) $\bigoplus_{i>1} \pi_{2i}(map(P_{(0)}, P_{(0)}), \mathrm{id}) = 0.$

Let us conclude this paper with another question motivated by the results obtained in this paper.

Question 4.8. Is it possible that for every 1-connected CW complex P there exist a compact Lie group G and a principal G-bundle such that the total space has the homotopy type of P and the associated forgetful map F has uncountable kernel?

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