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# Finite order automorphisms and dimension groups of Cantor minimal systems

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**Abstract.** We compute the dimension group of the skew product extension of a Cantor minimal system associated with a finite group valued cocycle. Using it, we study finite subgroups in the commutant group of a Cantor minimal system and prove that a finite subgroup of the kernel of the mod map must be cyclic. Moreover, we give a certain obstruction for finite subgroups of commutant groups to have non-zero intersection to the kernel of mod maps. We also give a necessary and sufficient condition for dimension groups so that the kernel of the mod map can include a finite order element.

## 1. Introduction.

When $X$ is the Cantor set and $\phi$ is a homeomorphism on $X$ which has no non-trivial invariant closed subset, the topological dynamical system $(X, \phi)$ is called a Cantor minimal system. Cantor minimal systems are analogues of ergodic systems in the topological setting and several authors have studied them in the various ways. Among many subjects about Cantor minimal systems, the study of automorphism groups is one of the most mysterious parts. We denote by $C(\phi)$ the set of all homeomorphisms on $X$ commuting with $\phi$ for a Cantor minimal system $(X, \phi)$ and call it the automorphism group or commutant group of $(X, \phi)$. The symbol $C(\phi)$ usually means the set of all continuous maps $\gamma : X \to X$ which commute with $\phi$, but we restrict our attention only on the set of commuting homeomorphisms in this paper. As $\phi$ is minimal, $C(\phi)$ acts on $X$ freely. We denote by $\boldsymbol{Z}\phi$ the subgroup generated by $\phi$ in $C(\phi)$. In the present paper, we give a new direction in the study of commutant groups, by using dimension groups of Cantor minimal systems.

The notion of dimension group was introduced for Cantor minimal systems in [**HPS**] and this new invariant threw a new light on the study of Cantor minimal systems. Let $(X, \phi)$ be a Cantor minimal system and

$$B_\phi = \{f - f \circ \phi^{-1}; f \in C(X, \boldsymbol{Z})\}$$

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be the coboundary subgroup of integer valued continuous functions $C(X, \boldsymbol{Z})$. The dimension group $K^0(X, \phi)$ of a Cantor minimal system $(X, \phi)$ is the quotient of $C(X, \boldsymbol{Z})$ by $B_\phi$. Define the positive cone

$$K^0(X, \phi)^+ = \{[f] \in K^0(X, \phi); f \in C(X, \boldsymbol{Z})^+\},$$

where the bracket means the quotient map. The dimension group $K^0(X, \phi)$ is an ordered group with this positive cone and the equivalence class $[1]$ of the constant function one is called the order unit of $K^0(X, \phi)$. It was proved in [**GPS**] that $K^0(X, \phi)$, as an ordered group with a distinguished order unit, characterizes the strong orbit equivalence class of $(X, \phi)$. The dimension group $K^0(X, \phi)$ is order isomorphic to the $K_0$-group of the $C^*$-algebra $C^*(X, \phi)$. In this paper, however, we don't deal with $C^*$-algebras.

One of purposes of this paper is to compute the dimension group of the Cantor minimal system $(Y, \psi)$ arising from the skew product extension of a Cantor minimal system $(X, \phi)$ associated with a finite group valued cocycle. The dimension group of a Cantor minimal system $(X, \phi)$ is usually computed as the inductive limit system arising from the ordered Bratteli diagram of $(X, \phi)$ ([**HPS**]). Unfortunately, however, there is no explicit way to write down the ordered Bratteli diagram of the skew product system $(Y, \psi)$ by means of the ordered Bratteli diagram of the original system $(X, \phi)$ and the cocycle. In Theorem 2.5, we will describe the dimension group $K^0(Y, \psi)$ as the quotient of the restricted dimension group by the canonical infinitesimal subgroup.

Our main tool for the study of commutants is the mod map. Since $\gamma \in C(\phi)$ satisfies $g \circ \gamma^{-1} \in B_\phi$ for all $g \in B_\phi$,

$$\operatorname{mod}(\gamma)([f]) = [f \circ \gamma^{-1}]$$

gives rise to an order automorphism of $K^0(X, \phi)$ preserving the order unit ([**GPS2**]). It can be easily checked that the mod map is a group homomorphism from $C(\phi)$ to $\operatorname{Aut}(K^0(X, \phi))$. We define $T(\phi) = C(\phi) \cap \ker \operatorname{mod}$. In Section 3, we will prove that every finite subgroup of $T(\phi)$ is cyclic. Moreover, we will show that if a finite subgroup $G$ of $C(\phi)$ includes an element of prime order $p$ and it is in $T(\phi)$, then the $p$-Sylow group of $G$ is cyclic. It should be remarked that every finite group can be embedded into the commutant group of a Cantor minimal system, which was shown in [**LM**]. Next, we will consider when the kernel of the mod map can contain a finite order automorphism, and give a necessary and sufficient condition for dimension groups. The invariant $\eta$ defined in [**M**] will be computed for finite order elements. Several examples of finite subgroups of $C(\phi)$ and $T(\phi)$ are given in Section 4. In our examples, every $T(\phi)$ is abelian. As mentioned above, $T(\phi)$ cannot contain non-abelian finite groups. But, we have no idea to deal with infinite order elements of $C(\phi)$. It's an interesting open problem whether or not the subgroup $T(\phi)$ is always abelian.

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## 2. Dimension groups of skew products.

In this section, we define skew product extensions of Cantor minimal systems associated with finite group valued cocycles, and give an algorithm of computing the dimension group of skew product systems.

DEFINITION 2.1. Let $(X, \phi)$ be a Cantor minimal system and $G$ be a finite group.

A continuous map $c : X \to G$ is called a $G$-valued cocycle.

$G$-valued cocycles $c$ and $c'$ are cohomologous if there is a continuous map $b : X \to G$ such that $b(x)c(x)b(\phi(x))^{-1} = c'(x)$ holds for all $x \in X$.

Let $c : X \to G$ be a cocycle. We set $Y = X \times G$ and a homeomorphism $\psi$ on $Y$ such as $\psi(x, g) = (\phi(x), gc(\phi(x)))$. This dynamical system $(Y, \psi)$ is called the skew product extension or extension, simply, of $(X, \phi)$ associated with the $G$-valued cocycle $c$. Of course, cohomologous cocycles determine isomorphic systems.

Let $(Y, \psi)$ be the extension of $(X, \phi)$ associated with a cocycle $c$. Obviously, there is a factor map from $(Y, \psi)$ to $(X, \phi)$. For each element $h \in G$, let $\gamma_h$ be the homeomorphism on $Y$ which sends $(x, g)$ to $(x, hg)$. Then $\{\gamma_g\}_{g \in G}$ forms a subgroup of $C(\psi)$ isomorphic to $G$. We call this finite subgroup a canonical commutant of the skew product extension.

Notice that the extension $(Y, \psi)$ is not always minimal. For example, when $(X, \phi)$ is an odometer system and $G$ is a finite group and not cyclic, the skew product extension $(Y, \psi)$ is never minimal for any $G$-valued cocycle, because every cocycle $c : X \to G$ is cohomologous to a cocycle $c'$ whose range contains only the identity $e$ and a single element $g$. The reader may refer to Section VIII.4 of [**D**] for odometer systems.

LEMMA 2.2. *Let $(Y, \psi)$ be a Cantor minimal system and $G \subset C(\psi)$ be a finite subgroup. Then, there exist a Cantor minimal system $(X, \phi)$ and a cocycle $c : X \to G$, such that the extension of $(X, \phi)$ by the cocycle $c$ is isomorphic to $(Y, \psi)$ and the subgroup $G$ coincides with the canonical commutant under this isomorphism.*

PROOF. Let $\{\gamma_g\}_{g \in G}$ be the finite subgroup of $C(\psi)$. We can consider the quotient system $(X, \phi)$ of $(Y, \psi)$ by the action of $\{\gamma_g\}_g$. Denote the factor map by $\pi$. Since $Y$ is the Cantor set, there exists a clopen subset $Y_0$ such that the restriction $\tilde{\pi}$ of $\pi$ on $Y_0$ is a homeomorphism onto $X$. Set $Y_g = \gamma_g(Y_0)$ for each $g \in G$. Then, $\{Y_g\}_g$ is a clopen partition of $Y$ and a map $\rho$ sending

$(x, g) \in X \times G$ to $\gamma_g(\tilde{\pi}^{-1}(x))$ is a homeomorphism. The minimal homeomorphism $\rho^{-1} \circ \psi \circ \rho$ on $X \times G$ determines a cocycle $c : X \to G$. It is clear that $\{\gamma_g\}_g$ coincides with the canonical commutant. $\square$

We fix a Cantor minimal system $(X, \phi)$ and a finite group valued cocycle $c : X \to G$. Let $(Y, \psi)$ be the extension. We would like to compute the dimension group $K^0(Y, \psi)$. In order to do this, at first, we must represent the system $(X, \phi)$ by using an ordered Bratteli diagram ([**HPS**]) and fix the notation.

Let $B = (V, E, \leq)$ be a simple ordered Bratteli diagram associated with $(X, \phi)$, where $V = \bigcup_{n=0}^{\infty} V_n$ and $E = \bigcup_{n=1}^{\infty} E_n$ are the sets of vertices and edges. We denote the range and source map by $r$ and $s$. For every $v \in V \setminus V_0$, a linear order is defined on $r^{-1}(v)$. Let $(e_1, e_2, \ldots, e_m)$ be the ordered list of the edges in $r^{-1}(v)$. We define a map $\theta$ from $V \setminus V_0$ to the set of finite sequences consisting of vertices such as $\theta(v) = (s(e_1), s(e_2), \ldots, s(e_m))$. In this paper, we use the map $\theta$ to describe the partial order on $E$. We can identify $(X, \phi)$ with the infinite path space of $B$ and the Bratteli-Vershik map on it. For each $n \in \boldsymbol{N}$, the vertex set $V_n$ corresponds to towers of Kakutani-Rohlin partitions of $X$. We denote this partition by

$$\mathscr{P}_n = \{X(n, v, k); v \in V_n, 1 \leq k \leq h(v)\},$$

where each $X(n, v, k)$ is the clopen set of level $k$ in the tower corresponding to $v$ and $h : V \to \boldsymbol{N}$ is the height of the tower. Here, we have $\phi(X(n, v, k)) = X(n, v, k+1)$ for $v \in V_n$ and $0 \leq k \leq h(v) - 1$. We set

$$X_n = \bigcup_{v \in V_n} X(n, v, h(v))$$

and call it the top set of $n$-th step. Then, we also get

$$\phi(X_n) = \bigcup_{v \in V_n} X(n, v, 1).$$

The sequence of the top sets $\{X_n\}_n$ is decreasing and shrinks to one point set $\{x_{\max}\}$. We also have that $\mathscr{P}_{n+1}$ is a finer partition than $\mathscr{P}_n$ and $\{\mathscr{P}_n\}_n$ generates the topology of $X$.

Let us recall the way of computing the dimension group $K^0(X, \phi)$. We denote by $C(\mathscr{P}_n)$ the set of integer valued functions on $X$ which are constant on each clopen set of $\mathscr{P}_n$. The characteristic functions on the clopen sets contained in the same tower of $\mathscr{P}_n$ give the same element of $K^0(X, \phi)$. Therefore, we can view that $C(\mathscr{P}_n)$ forms a free abelian group $\boldsymbol{Z}^{V_n}$. We denote the canonical basis of $\boldsymbol{Z}^{V_n}$ by the same symbols $\{v; v \in V_n\}$ as vertices. For each $n \in \boldsymbol{N}$, the edge set $E_n$ determines the incidence matrix $A_n$ from $\boldsymbol{Z}^{V_n}$ to $\boldsymbol{Z}^{V_{n+1}}$, which is given by

$$A_n(v, w) = \sharp\{k; 1 \leq k \leq h(w), X(n+1, w, k) \subset X(n, v, h(v))\}$$

for $v \in V_n$ and $w \in V_{n+1}$. Hence, the dimension group is computed such as

$$K^0(X,\phi) = \lim A_n : \boldsymbol{Z}^{V_n} \to \boldsymbol{Z}^{V_{n+1}}.$$

If we set $u_n = \sum_{v \in V_n} h(v)v \in \boldsymbol{Z}^{V_n}$ for all $n \in \boldsymbol{N}$, we get $u_n A_n = u_{n+1}$ and $\{u_n\}_n$ is the order unit of $K^0(X,\phi)$.

We would like to consider the dimension group of the skew product $(Y,\psi)$. We may assume that the Kakutani-Rohlin partition $\mathscr{P}_1$ is finer than the clopen partition determined by the cocycle $c$. We can define a map $d$ from $V \setminus V_0$ to $G$ as follows;

$$d(v) = c(x)c(\phi(x))c(\phi^2(x)) \cdots c(\phi^{h(v)-1}(x)),$$

where $x$ is an arbitrary point in $X(n,v,1)$. We call $d(v)$ the label of the vertex $v$ determined by the cocycle $c$. Let $w$ be a vertex of $V_{n+1}$. If $\theta(w) = (v_1, v_2, \dots, v_m)$, we get

$$d(w) = d(v_1)d(v_2) \cdots d(v_m)$$

from the definition of $d$.

Lemma 2.3. *In the above setting, the skew product $(Y,\psi)$ is a Cantor minimal system if and only if $\{d(v); v \in V_n\}$ generates $G$ for every $n \in \boldsymbol{N}$.*

Proof. Assume $\{d(v); v \in V_n\}$ does not generate $G$ for some $n$. We can replace the cocycle $c$ to $c'$, in the same cohomology class, whose range generates a proper subgroup of $G$, and so the extension $(Y,\psi)$ is not minimal.

Let us prove the converse. We assume that $d(V_n)$ generates $G$ for all $n$. We can identify $X$ and $x_{\max} \in X$ with the infinite path space of $B = (V, E, \leq)$ and the unique maximal path. It suffices to show that the set

$$H = \{g \in G; (x_{\max}, g) \in \overline{Orb_\psi((x_{\max}, e))}\}$$

coincides with $G$. Since $H$ is a subgroup of $G$, it is enough to show that $H$ includes a generating set of $G$. Let $x_{\max} = (e_1, e_2, \dots)$ and $\phi(x_{\max}) = (f_1, f_2, \dots)$ be the unique maximal and minimal paths. We also assume that all maximal edges of $E_n$ have the same source vertex as $e_n$. Take $n \in \boldsymbol{N} \setminus \{1,2\}$ and set $\theta(r(f_n)) = (v_1, v_2, \dots, v_m)$. When we define

$$h_i = \sum_{j=1}^{i} h(v_j), \quad g_i = d(v_1)d(v_2) \cdots d(v_i)$$

for $i = 1,2,\dots,m$, we have $\psi^{h_i}(x_{\max}, e) = (\phi^{h_i}(x_{\max}), g_i)$ and the first $n-2$ edges of $\phi^{h_i}(x_{\max})$ agree with $(e_1, e_2, \dots, e_{n-2})$. We set $K_n = \{g_i \in G; i = 1,2,\dots,m\}$. From the assumption, $K_n$ is a generating set of $G$. Because $G$ is a finite set, there exists a generating set $K$ of $G$ such that $K_n = K$ holds for infinitely many $n$. Then we get $(x_{\max}, g) \in \overline{Orb_\psi((x_{\max}, e))}$ for all $g \in K$, and so the proof is completed. □

From now on, we assume that the skew product $(Y, \psi)$ is a Cantor minimal system. We define Kakutani-Rohlin partitions $\{\mathcal{Q}_n\}_n$ for $(Y, \psi)$ as the following;

$$Y(n, v, k, g) = X(n, v, k) \times \{g\},$$

$$\mathcal{Q}_n = \{Y(n, v, k, g); v \in V_n, 1 \le k \le h(v), g \in G\},$$

and we set the top set as

$$Y_n = X_n \times G = \bigcup_{v \in V_n, g \in G} Y(n, v, h(v), g).$$

The partition $\mathcal{Q}_n$ consists of $\sharp(V_n \times G)$ towers, and $\{\mathcal{Q}_n\}_n$ generates the topology of $Y$. However, we should note that the intersection of the top sets $\{Y_n\}_n$ is not one point. It is equal to $\{x_{\max}\} \times G$. Therefore, we cannot use the same method as the case of $(X, \phi)$. We need a proposition obtained by Putnam in order to compute the dimension group $K^0(Y, \psi)$.

Proposition 2.4 ([**P**, Theorem 4.1]). *Let $(Y, \psi)$ be a Cantor minimal system and $\{y_i\}_{i=1}^m$ be a finite subset of $Y$ lying in distinct orbits. When we define*

$$D = C(Y, \mathbf{Z})/\{f - f \circ \psi^{-1}; f \in C(Y, \mathbf{Z}), f(y_i) = 0 \textit{ for all } i = 1, 2, \ldots, m\},$$

*it is an ordered group. Moreover, the sequence*

$$0 \to \mathbf{Z} \xrightarrow{j} \mathbf{Z}^m \xrightarrow{\delta} D \xrightarrow{q} K^0(Y, \psi) \to 0$$

*is exact, where the map $j$ sends the generator of $\mathbf{Z}$ to $(1, 1, \ldots, 1)$ and $q$ is the natural quotient map. The map $\delta$ is given as follows; for $u = (u_i)_i \in \mathbf{Z}^m$, we take $f \in C(Y, \mathbf{Z})$ such as $f(y_i) = u_i$ and define $\delta(u)$ to be the equivalence class of $f - f \circ \psi^{-1}$.*

We would like to apply the proposition above for the extension $(Y, \psi)$ and the finite subset $\{x_{\max}\} \times G$. Let us define

$$K^0(Y, \psi; G) = C(Y, \mathbf{Z})/\{f - f \circ \psi^{-1}; f \in C(Y, \psi), f(x_{\max}, g) = 0 \text{ for all } g \in G\},$$

and call it the restricted dimension group of $(Y, \psi)$. The restricted dimension group $K^0(Y, \psi; G)$ can be represented as an inductive limit sequence by means of the Kakutani-Rohlin partitions $\{\mathcal{Q}_n\}_n$. At first, we give this description of $K^0(Y, \psi; G)$.

Since the partition $\mathcal{Q}_n$ consists of $\sharp(V_n \times G)$ towers, we can consider that $C(\mathcal{Q}_n)$ forms a free abelian group $\mathbf{Z}^{V_n \times G} \cong \mathbf{Z}^{V_n} \otimes \mathbf{Z}[G]$. We denote the canonical basis of $\boldsymbol{Z}^{V_n} \otimes \boldsymbol{Z}[G]$ by $\{v \otimes g; v \in V_n, g \in G\}$, and choose the characteristic function on $Y(n, v, h(v), g)$ as the representative of $v \otimes g$. Let us consider the incidence matrix $B_n$ from $n$-th step to $n+1$-th step. If $v \in V_n$ and $w \in V_{n+1}$ satisfy $X(n+1, w, k) \subset X(n, v, h(v))$ for some $k \in \{1, 2, \dots, h(w)\}$, then, for all $g \in G$, $Y(n+1, w, k, g) \subset Y(n, v, h(v), g)$ holds and there exists $h \in G$ such that $\psi^{h(w)-k}(Y(n+1, w, k, g)) = Y(n+1, w, h(w), gh)$. Of course, if $k = h(w)$, $h$ is the identity. Otherwise, the element $h$ is given by

$$h = c(x)c(\psi(x))c(\psi^2(x)) \cdots c(\psi^{h(w)-k-1}(x)),$$

where $x$ is an arbitrary point in $X(n+1, w, k+1)$. We would like to write down the incidence matrix as the matrix the size of $V_n \times V_{n+1}$ which has entries in the group ring $\boldsymbol{Z}[G]$. Let $w$ be a vertex in $V_{n+1}$ and assume $\theta(w) = (v_1, v_2, \dots, v_m)$. For a vertex $v \in V_n$, we define non-negative integers $\{\lambda_g\}_g$ as follows;

$$\lambda_g = \sharp\{i; 1 \leq i \leq m, v = v_i, g = d(v_{i+1})d(v_{i+2}) \cdots d(v_m)\},$$

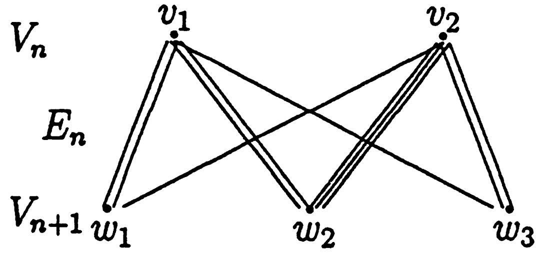
and set $B_n(v, w) = \sum_{g \in G} \lambda_g g$. From the argument above, we see that the matrix $B_n$ represents the connecting map from $n$-th step to $n+1$-th step. Then, we get

$$K^0(Y, \psi; G) = \lim B_n : \boldsymbol{Z}^{V_n} \otimes Z[G] \to \boldsymbol{Z}^{V_{n+1}} \otimes \boldsymbol{Z}[G],$$

where vectors are considered as row vectors of $\boldsymbol{Z}[G]^{V_n}$ and the product of vectors and matrices are computed in the obvious way. The positive cone is obtained by the inductive limit of the canonical positive cone of $\boldsymbol{Z}^{V_{n+1}} \otimes \boldsymbol{Z}[G]$, and the order unit is given by $\sum_{g \in G} u_n \otimes g \in \boldsymbol{Z}^{V_{n+1}} \otimes \boldsymbol{Z}[G]$ at $n$-th step.

For example, consider the case of the diagram in Fig. 1, where the partial order in the edge set $E_n$ is expressed by the map $\theta$. If the label of $v_i$ is $g_i \in G$ for each $i = 1, 2$, then we have $d(w_1) = g_1 g_1 g_2$, $d(w_2) = g_1 g_2 g_2 g_1 g_2$ and $d(w_3) = g_1 g_2 g_2$. The usual incidence matrix $A_n : \boldsymbol{Z}^{V_n} \to \boldsymbol{Z}^{V_{n+1}}$ and the incidence matrix $B_n : \boldsymbol{Z}[G]^{V_n} \to \boldsymbol{Z}[G]^{V_{n+1}}$ of the extension are given by

$$A_n = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}, \quad B_n = \begin{bmatrix} g_1 g_2 + g_2 & g_2 g_2 g_1 g_2 + g_2 & g_2 g_2 \\ e & g_2 g_1 g_2 + g_1 g_2 + e & g_2 + e \end{bmatrix}.$$




$$\theta(w_1) = (v_1, v_1, v_2)$$
$$\theta(w_2) = (v_1, v_2, v_2, v_1, v_2)$$
$$\theta(w_1) = (v_1, v_2, v_2)$$

Figure 1.

We can obtain $A_n$ from $B_n$, sending each entry of $B_n$ by the canonical ring homomorphism from $\boldsymbol{Z}[G]$ to $\boldsymbol{Z}$.

In order to compute the dimension group $K^0(Y,\psi)$, we must describe explicitly the image of the map $\delta$ in Proposition 2.4. We can realize that $\delta$ is the map from $\boldsymbol{Z}[G]$. For $g \in G$ and $n \in \boldsymbol{N}$, let $f$ be the characteristic function on the clopen set $X_n \times \{g\}$. Then, we have $f(x_{\max}, g) = 1$ and $f(x_{\max}, h) = 0$ for $h \neq g$, and so the image of $g \in \boldsymbol{Z}[G]$ by the map $\delta$ is given by the equivalence class of $f - f \circ \psi^{-1}$ in $K^0(Y,\psi;G)$. When $x_v \in X$ is a point in $X(n,v,1)$ for each $v \in V_n$, the function $f \circ \psi^{-1}$ is the characteristic function on the clopen set $\bigcup_{v \in V_n} Y(n,v,1,gc(x_v))$. Since $\psi^{h(v)-1}(Y(n,v,1,gc(x_v)) = Y(n,v,h(v),gd(v))$ for every $v \in V_n$, we can conclude that $\delta(g)$ equals

$$e(n,g) = \sum_{v \in V_n} v \otimes g - v \otimes gd(v) \in \boldsymbol{Z}^{V_n} \otimes \boldsymbol{Z}[G]$$

at $n$-th step. The equation $e(n,g)B_n = e(n+1,g)$ can be checked easily, and so $\{e(n,g)\}_n$ determines a subgroup $\boldsymbol{Z}$ in $K^0(Y,\psi;G)$. We denote by $Z_n$ the subgroup of $\boldsymbol{Z}^{V_n} \otimes \boldsymbol{Z}[G]$ generated by $\{e(n,g); g \in G\}$. Remark that $\sum_g e(n,g)$ equals zero for all $n$.

Theorem 2.5. *Keep the above notation. The dimension group $K^0(Y,\psi)$ of the skew product $(Y,\psi)$ is obtained as the quotient of*

$$K^0(Y,\psi;G) = \lim B_n : \boldsymbol{Z}^{V_n} \otimes \boldsymbol{Z}[G] \to \boldsymbol{Z}^{V_{n+1}} \otimes \boldsymbol{Z}[G]$$

*by the subgroup $\{Z_n\}_n$ which is isomorphic to $\boldsymbol{Z}^{\sharp G-1}$.*

Let us consider the action on $K^0(Y,\psi)$ induced by the canonical commutant $\{\gamma_g\}_g$. In the following argument, we use the description of $K^0(Y,\psi)$ in the above theorem. Because $\gamma_h$ transfers $Y(n,v,h(v),g)$ to $Y(n,v,h(v),hg)$, the induced action on $\boldsymbol{Z}^{V_n} \otimes \boldsymbol{Z}[G]$ sends $v \otimes g$ to $v \otimes hg$ for all $v \in V_n$ and $g \in G$. Assume $\gamma_h \in T(\psi)$, that is, $\mathrm{mod}(\gamma_h) = id$. Then, for all $n \in \boldsymbol{N}$, $v \in V_n$ and $g \in G$, there exists $m$ such that

$$(v \otimes (g - hg))B_n B_{n+1} \cdots B_{n+m}$$

is contained in $Z_{n+m+1}$. The following lemma is needed to prove the main theorem in the next section.

Lemma 2.6. *Let $(Y,\psi)$ be the skew product as above and $H$ be a normal subgroup of $G$ such that* $\mathrm{mod}(\gamma_h) = id$ *for all $h \in H$. By telescoping the ordered Bratteli diagram $B = (V,E,\leq)$, we may assume that $(v \otimes (g - hg))B_n \in Z_{n+1}$ for all $v \in V_n$, $g \in G$ and $h \in H$.*

*For $w \in V_{n+1}$, let $K$ be the cyclic subgroup of $G$ generated by $d(w)$. Then,* *we have* $A_n(v,w) \in l\boldsymbol{Z}$ *for all* $v \in V_n$, *where the natural number* $l$ *is the index of the subgroup* $H \cap K$ *in* $H$.

Proof. Let $v$ be a vertex of $V_n$, and $B_n(v,w) = \sum_{g \in G} \lambda_g g$. We will prove $\sum_{g \in G} \lambda_g \in l\boldsymbol{Z}$. From the assumption, there exists $\{\mu_g\}_{g \in G} \subset \boldsymbol{Z}$ such that

$$(v \otimes (e-h))B_n = \sum_{g \in G} \mu_g e(n+1, g),$$

and so we have

$$(e-h)\sum_{g \in G} \lambda_g g = \sum_{g \in G} \mu_g (g - gd(w)),$$

which implies $\lambda_g - \lambda_{h^{-1}g} = \mu_g - \mu_{gd(w)^{-1}}$ for all $g \in G$. When we write $\tilde{\lambda}_g = \sum_{k \in K} \lambda_{gk}$, we get $\tilde{\lambda}_g = \tilde{\lambda}_{hg}$ and this equation holds for all $h \in H$. Let $G = \bigcup_{i=1}^{m} Hg_iK$ be the decomposition into double cosets by $H$ and $K$. Since $H$ is a normal subgroup of $G$, each double coset consists of $l$ left $K$-cosets. Hence, we get $\sum_{g \in G} \lambda_g = l \sum_{i=1}^{m} \tilde{\lambda}_{g_i} \in l\boldsymbol{Z}$. □

## 3. Commutants and dimension groups.

We will prove the main theorems about finite subgroups of the commutant group in this section. When a Cantor minimal system $(Y, \psi)$ has a finite subgroup $G$ in $C(\psi)$, by Lemma 2.2, we can regard it as the skew product extension obtained by a Cantor minimal system $(X, \phi)$ and a cocycle $c$. Moreover, the dimension group $K^0(Y, \psi)$ can be described as in Theorem 2.5 by means of the ordered Bratteli diagram of $(X, \phi)$. We often use this identification in the following argument.

Lemma 3.1. *Let* $(Y, \psi)$ *be a Cantor minimal system and* $\tau \in C(\psi)$ *be an element of finite order* $n$. *If* $\gamma \in T(\psi)$ *satisfies* $\gamma \circ \tau \circ \gamma^{-1} = \tau^k$ *for some* $k$, *then* $k$ *must be one.*

Proof. We may assume there exist a Cantor minimal system $(X, \phi)$ and a continuous function $f : X \to \boldsymbol{Z}/n\boldsymbol{Z}$ such that

$$Y = X \times \boldsymbol{Z}/n\boldsymbol{Z}, \quad \psi(x,l) = (\phi(x), l + f(x)), \quad \tau(x,l) = (x, l+1)$$

for all $x \in X$ and $l \in \boldsymbol{Z}/n\boldsymbol{Z}$. Let $\pi$ be the factor map from $(Y, \psi)$ to $(X, \phi)$. From the assumption we have $\pi \circ \gamma \circ \tau = \pi \circ \gamma$, and so there exists $\tilde{\gamma} \in C(\phi)$ such that $\tilde{\gamma} \circ \pi = \pi \circ \gamma$. We denote by $\pi^*$ the injection from $K^0(X, \phi)$ to $K^0(Y, \psi)$ induced by $\pi$ ([**GW**, Proposition 3.1]). It can be easily seen that $\mathrm{mod}(\gamma) \circ \pi^* = \pi^* \circ \mathrm{mod}(\tilde{\gamma})$, hence we get $\mathrm{mod}(\tilde{\gamma}) = id$, thus $\tilde{\gamma} \in T(\phi)$. Therefore we can find a continuous function $g : X \to \boldsymbol{Z}/n\boldsymbol{Z}$ such as

$$f - f \circ \tilde{\gamma} = g - g \circ \phi.$$

We define a homeomorphism $\gamma_0$ on $Y = X \times \boldsymbol{Z}/n\boldsymbol{Z}$ by $\gamma_0(x,l) = (\tilde{\gamma}(x), l + g(x))$ for $(x,l) \in Y$. It is clear that $\gamma_0$ commutes with $\tau$. We can also check that $\pi \circ \gamma^{-1} \circ \gamma_0 = \pi$ and

$$\begin{aligned} \psi \circ \gamma_0(x,l) &= (\phi \circ \tilde{\gamma}(x), l + g(x) + f(\tilde{\gamma}(x))) \\ &= (\tilde{\gamma} \circ \phi(x), l + f(x) + g(\phi(x))) \\ &= \gamma_0 \circ \psi(x,l) \end{aligned}$$

for all $(x,l) \in Y$. Then we conclude that there exists $m$ such that $\gamma^{-1} \circ \gamma_0 = \tau^m$ holds, which implies that $\gamma$ commutes with $\tau$. $\square$

The following lemma shows that if $T(\psi)$ contains a finite abelian subgroup $H$, then $H$ is cyclic.

Lemma 3.2. *Let $(Y,\psi)$ be a Cantor minimal system and $G$ be a finite $p$-subgroup of $C(\psi)$ for a prime $p$. If a normal subgroup $H$ of $G$ is included in $T(\psi)$ and the quotient of $G$ by $H$ is cyclic, then $G$ is also cyclic.*

Proof. Let $\rho$ be the quotient map from $G$ to $G/H$. We may consider that $(Y,\psi)$ is the extension of $(X,\phi)$ by a $G$-valued cocycle $c$, and $(X,\phi)$ is represented by an ordered Bratteli diagram. We use the same notation as in Section 2. Because $\{\gamma_h\}_h \in H$ is contained in $T(\phi)$, we may assume $(v \otimes (e-h))B_1 \in Z_2$ for all $v \in V_1$ and $h \in H$. Since $(Y,\psi)$ is minimal, $d(V_2)$ generates $G$, which implies that there exists a vertex $w \in V_2$ such that $\rho(d(w))$ is a generator of the cyclic group $G/H$. Let $K$ be the cyclic subgroup of $G$ generated by $d(w)$. By Lemma 2.6, we have $A_1(v,w) \in l\boldsymbol{Z}$ for every $v \in V_1$, where $l$ is the index of $H \cap K$ in $H$. On the other hand, when $\theta(w)$ is equal to $(v_1, v_2, \dots, v_m)$, $d(w)$ is the product of $d(v_i)$'s, and so

$$\rho(d(w)) = \sum_{i=1}^{m} \rho(d(v_i)) = \sum_{v \in V_1} A_1(v,w)\rho(d(v)).$$

If $l$ is a multiple of $p$, $\rho(d(w))$ cannot be a generator of $G/H$. Hence, $l$ is one and $H$ is included in $K$, which means that $G$ is a cyclic group generated by $d(w)$. $\square$

Using the above lemmas, we can prove the following.

Theorem 3.3. *If $(Y,\psi)$ is a Cantor minimal system and $H$ is a finite subgroup of $T(\psi)$, then $H$ is a cyclic group.*

Proof. We say that a group $G$ has a property $M$ if there exists a Cantor minimal system $(Y, \psi)$ such that $G$ is contained in $T(\psi)$. The proof is by contradiction. Assume that $H$ is a finite non-cyclic group of minimum order which has the property $M$. It is clear that all proper subgroups of $H$ are cyclic. Take a Cantor minimal system $(Y, \psi)$ which satisfies $H \subset T(\psi)$. Assume $K$ is a non-trivial normal subgroup of $H$. Then we can consider the quotient system $(X, \phi)$ of $(Y, \psi)$ by the action of $K$. By the same argument as in Lemma 3.1, we can deduce that the quotient group $H/K$ is contained in $T(\phi)$, that is, $H/K$ has property $M$. From the assumption we infer that $H/K$ is cyclic. Choose a generator of $H/K$ and its lift $h \in H$. By applying Lemma 3.1 to a generator of $K$ and $h$ in $H$, we obtain that $H$ is an abelian group. However, we know that a finite abelian group having property $M$ must be cyclic by Lemma 3.2, and it contradicts to the assumption of $H$. As a consequence, $H$ is a simple group whose every proper subgroup is cyclic. In light of Exercise 7.(a) in Section 2.2 of [**S**], we get a contradiction. □

For substitution minimal subshifts of constant length, the automorphism groups were computed in [**LM**] and [**HP**], and it was also shown that there is a substitution minimal subshift $(X, \phi)$ for an arbitrary finite group $G$ such that the commutant group $C(\phi)$ is isomorphic to $G \oplus \boldsymbol{Z}\phi$. In these papers, actually, measure-theoretic automorphims were investigated and we can deduce the same results directly in the topological dynamical setting. However, the above theorem says that the same statement never hold for $T(\phi)$. We will show that there exists a Cantor minimal system $(Y, \psi)$ for every $n \in \boldsymbol{N}$, which satisfies $C(\psi) = T(\psi) = \boldsymbol{Z}/n\boldsymbol{Z} \oplus \boldsymbol{Z}\psi$, in the next sectinon.

Lemma 3.4. *Let $(Y, \psi)$ be a Cantor minimal system and $Q$ be the quaternion group. If $C(\psi)$ contains $Q$, then $Q \cap T(\psi)$ is trivial.*

Proof. Assume $Q$ is generated by $a, b, c \in Q$ satisfying $a = b^2 = c^2$, $a^2 = e$ and $bc = acb$. We may consider that $(Y, \psi)$ is the extension of a Cantor minimal system $(X, \phi)$ associated with a $Q$-valued cocycle. We use the same notation as in Section 2. The proof is by contradiction. If $Q \cap T(\psi)$ is non-trivial, $a \in Q$ must be in $T(\psi)$. We may assume that $(v \otimes (e - a))B_1$ is included in $Z_2$ for all $v \in V_1$. Since $d(V_2)$ generates $Q$, we may also assume that there exist vertices $w_1, w_2 \in V_2$ such that $d(w_1) = b$ and $d(w_2) = c$. Let $v \in V_1$ be an arbitrary vertex. When we set $B_1(v, w_i) = \sum_{g \in Q} \lambda_g^i g$ for $i = 1, 2$, there exists $\{v_g\}_g \subset \boldsymbol{Z}$ such that

$$\sum_{g \in Q} (e - a)\lambda_g^i g = \sum_{g \in Q} v_g g(e - d(w_i))$$

holds for $i = 1, 2$. From these equations, we have

$$
\begin{aligned}
(\lambda_e^1 - \lambda_a^1) &+ (\lambda_b^1 - \lambda_{ab}^1) + (\lambda_c^1 - \lambda_{ac}^1) + (\lambda_{abc}^1 - \lambda_{bc}^1) \\
&= v_e - v_{ab} + v_b - v_e + v_c - v_{bc} + v_{abc} - v_c \\
&= v_b - v_{abc} + v_{abc} - v_{ab} + v_{abc} - v_{ab} + v_{ab} - v_{bc} \\
&= (\lambda_b^2 - \lambda_{ab}^2) + (\lambda_{abc}^2 - \lambda_{bc}^2) + (\lambda_{abc}^2 - \lambda_{bc}^2) + (\lambda_{ab}^2 - \lambda_b^2) \in 2\boldsymbol{Z}.
\end{aligned}
$$

Consequently we get $A_1(v, w_1) = \sum_{g \in Q} \lambda_g^1 \in 2\boldsymbol{Z}$. Because $v$ is an arbitrary vertex of $V_1$, we obtain a contradiction as in Lemma 3.2. □

In the next section, we will give some examples of finite subgroups of $C(\psi)$ and $T(\psi)$. The following theorem, however, shows that there exists an obstruction for a finite group $G$ which has non-trivial intersection with $T(\psi)$.

Theorem 3.5. *Let $(Y, \psi)$ be a Cantor minimal system, $G$ be a finite subgroup of $C(\psi)$ and $p$ be a prime. If a $p$-Sylow group of $G$ has non-trivial intersection with $T(\psi)$, then the $p$-Sylow group is cyclic.*

Proof. We may assume $G$ is a $p$-group which has non-zero intersection with $T(\psi)$. From Lemma 3.2, we see that all the abelian subgroups of $G$ are cyclic. Hence, by using of (4.4) of Section 4.4 in [**S**], we deduce that $G$ is a cyclic group or a generalized quaternion group. If $G$ is a generalized quaternion group, it contains the quaternion group $Q$ and $Q$ has non-trivial intersection with $T(\psi)$. By means of Lemma 3.4, we get a contradiction. □

We would like to consider a finite cyclic subgroup $\boldsymbol{Z}/m\boldsymbol{Z}$ of $T(\psi)$. In the rest of this section, we denote a generator of $\boldsymbol{Z}/m\boldsymbol{Z}$ by $a$ and the canonical ring homomorphism from $\boldsymbol{Z}[\boldsymbol{Z}/m\boldsymbol{Z}]$ to $\boldsymbol{Z}$ by $\rho$ and set $P = \sum_{j=0}^{m-1} a^j$.

Let $(Y, \psi)$ be the Cantor minimal system obtained as the extension of $(X, \phi)$ associated with a cocycle $c : X \to \boldsymbol{Z}/m\boldsymbol{Z}$. We denote the generator of the canonical commutant by $\gamma \in C(\psi)$. If a function $f \in C(Y, \boldsymbol{Z})$ is fixed by $\mathrm{mod}(\gamma)$ in $K^0(Y, \psi)$, there exists $r \in \boldsymbol{Z}[\boldsymbol{Z}/m\boldsymbol{Z}]$ such that

$$[f - f \circ \gamma^{-1}] = \delta(r),$$

where the bracket means the equivalence class in the restricted dimension group $K^0(Y, \psi; G)$ and $\delta$ is as in Section 2. Then, we can define the map $\tilde{\eta}(\gamma)$ from $\ker(id - \mathrm{mod}(\gamma))$ to $\boldsymbol{Z}/m\boldsymbol{Z}$ by

$$\tilde{\eta}(\gamma)([f]) = \rho(r) + m\boldsymbol{Z}.$$

The following is another obstruction for actions of commutants. Note that we can identify the dimension group $K^0(X, \phi)$ with $\mathrm{Im} \sum_{j=0}^{m-1} \mathrm{mod}(\gamma)^j$.

Lemma 3.6. *When* $(Y, \psi)$ *is a Cantor minimal system and* $\gamma \in C(\psi)$ *is an element of order* $m$, *we have*

$$\ker(id - \operatorname{mod}(\gamma))/\operatorname{Im} \sum_{j=0}^{m-1} \operatorname{mod}(\gamma)^j \cong \boldsymbol{Z}/m\boldsymbol{Z},$$

*and*

$$\ker \sum_{j=0}^{m-1} \operatorname{mod}(\gamma)^j = \operatorname{Im}(id - \operatorname{mod}(\gamma)).$$

Proof. Let $(Y, \psi)$ be the skew product extension of a Cantor minimal system $(X, \phi)$ associated with a $\boldsymbol{Z}/m\boldsymbol{Z}$-valued cocycle and $\tilde{\eta}(\gamma)$ be as above. We use the same notation as in Section 2. Let $\lambda_v \in \{0, 1, \dots, m-1\}$ be the natural number such that $d(v) = a^{\lambda_v}$. For every $n$, we set

$$s_n = \sum_{v \in V_n} v \otimes (e + a + \cdots + a^{\lambda_v})$$

in $K^0(Y, \psi; G)$. It is not hard to check that the quotient image of $s_n$ in $K^0(Y, \psi)$ drops into $\ker(id - \operatorname{mod}(\gamma))$, and its value by $\tilde{\eta}(\gamma)$ is one in $\boldsymbol{Z}/m\boldsymbol{Z}$. Therefore, $\tilde{\eta}(\gamma)$ is surjective. We would like to show the injectivity. Let $s = \sum_{v \in V_n} v \otimes s_v$ be an arbitrary element of $\boldsymbol{Z}^{V_n} \otimes \boldsymbol{Z}[\boldsymbol{Z}/m\boldsymbol{Z}]$, and assume

$$\sum_{v \in V_n} v \otimes (e - a)s_v = \sum_{v \in V_n} v \otimes (e - d(v))r$$

for some $r \in \boldsymbol{Z}[\boldsymbol{Z}/m\boldsymbol{Z}]$. Then, $s_v$ equals $(e + a + \cdots + a^{\lambda_v})r$ modulo a scalar multiple of $P$. If $\rho(r)$ is zero modulo $m$, there exists $r' \in \boldsymbol{Z}[\boldsymbol{Z}/m\boldsymbol{Z}]$ such that $r$ is equal to $(e - a)r'$ modulo a scalar multiple of $P$. Because $v \otimes P$ is in $\operatorname{Im} \sum_{j=0}^{m-1} \operatorname{mod}(\gamma)^j$, we can see that $s$ equals

$$\sum_{v \in V_n} v \otimes (e - d(v))r'$$

modulo $\operatorname{Im} \sum_{j=0}^{m-1} \operatorname{mod}(\gamma)^j$, which is zero in $K^0(Y, \psi)$.

The other equation is proved in a similar fashion. □

Remark. The above lemma gives a quite strong restriction for the existence of finite order automorphisms. For example, we have the following.

(i) If $K^0(Y, \psi)$ is isomorphic to $\boldsymbol{Z}[1/r] \oplus \boldsymbol{Z}$ for an odd number $r$ as an abelian group, there does not exist an order two element in $C(\psi)$.

(ii) If $K^0(Y, \psi)$ is isomorphic to $\boldsymbol{Z}^{s+1}$ as an abelian group and $s$ is not divisible by a prime $p$, then there does not exist an element of order $p$ in $C(\psi)$.

We would like to consider when there exists a finite order element in the kernel of the mod map. We say a triple $(G, G^+, u)$ is a dimension group in an abstract sense, if $(G, G^+)$ is an unperforated ordered group satisfying the Riesz interpolation property and $u$ is a distinguished element of $G^+ \setminus \{0\}$ called the order unit ([**GPS**, Section 1]). For a Cantor minimal system $(X, \phi)$, of course, $(K^0(X, \phi), K^0(X, \phi)^+, [1])$ becomes a dimension group in this meaning. Two dimension groups are said to be isomorphic, when there is an isomorphism preserving the positive cones and the order units.

Theorem 3.7. *When $(G, G^+, u)$ is a simple dimension group except $\boldsymbol{Z}$, and $m$ is a natural number, the following are equivalent.*

(i) *There exists a Cantor minimal system $(Y, \psi)$ such that $(K^0(Y, \psi), K^0(Y, \psi)^+, [1])$ is isomorphic to $(G, G^+, u)$ and $T(\psi)$ has an element of order $m$.*

(ii) *$G/mG$ is isomorphic to $\boldsymbol{Z}/m\boldsymbol{Z}$ as an abelian group and the order unit $u$ of $G$ is divisible by $m$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Lemma 3.6.

Let us prove the other implication (ii) $\Rightarrow$ (i). Take an element $u_0 \in G^+$ such as $u = mu_0$. For the simple dimension group $(G, G^+, u_0)$, there exists a Bratteli diagram $(V, E)$ and $G$ is order isomorphic to the inductive limit

$$\lim A_n : \boldsymbol{Z}^{V_n} \to \boldsymbol{Z}^{V_{n+1}},$$

where $A_n$ denotes the incidence matrix determined by $E_n$. Since $G/mG$ is isomorphic to $\boldsymbol{Z}/m\boldsymbol{Z}$, by telescoping the diagram $(V, E)$, we can choose a representative $s_n = \sum_{v \in V_n} \lambda_v v \in \boldsymbol{Z}^{V_n}$ of a generator of $G/mG$ satisfying

$$1 \leq \lambda_v \leq m, \quad s_n A_n \equiv s_{n+1} \pmod{m}$$

for every $n \in \boldsymbol{N}$. We may further assume that there is $c_v \in \{0, 1, 2, \ldots, m-1\}$ such as

$$v A_n \equiv c_v s_{n+1} \pmod{m}$$

for every $v \in V_n$ and $n \in \boldsymbol{N}$, and it follows that

$$\sum_{v \in V_n} \lambda_v c_v \equiv 1 \pmod{m}.$$

Let $X$ be the infinite path space of $(V, E)$. Although a partial order on $E$ has not yet been defined, we can find a cocycle $c : X \to \boldsymbol{Z}/m\boldsymbol{Z}$ which determines the label $d : V_n \to \boldsymbol{Z}/m\boldsymbol{Z}$ such that $d(v) = a^{\lambda_v}$.

We would like to define a partial order on the edge set $E$ and construct a Cantor minimal system $(X, \phi)$ so that the canonical commutant of the skew product extension $(Y, \psi)$ of $(X, \phi)$ associated with $c$ is included in $T(\psi)$. In this case, we must note that the dimension group of $(Y, \psi)$ is automatically isomorphic to $(G, G^+, u)$. By telescoping, we may assume that each entry of $A_n$ is not less than $m^2 + m$. Fix a natural number $n$ and an arbitrary vertex $v_0 \in V_n$. We will define a linear order on each $r^{-1}(w)$ for $w \in V_{n+1}$, so that $v_0$ is the source vertex of the minimum and maximum edges of $r^{-1}(w)$ for all $w$ and the incidence matrix $B_n$ from $\boldsymbol{Z}[\boldsymbol{Z}/m\boldsymbol{Z}]^{V_n}$ to $\boldsymbol{Z}[\boldsymbol{Z}/m\boldsymbol{Z}]^{V_{n+1}}$ satisfies

$$(v \otimes (e - a))B_n \in Z_{n+1}$$

for every $v \in V_n$.

We construct a finite set $\Gamma_w$ of directed edges on the vertex set $\boldsymbol{Z}/m\boldsymbol{Z}$ for each $w \in V_{n+1}$. Because $\sum_{v \in V_n} \lambda_v c_v$ equals one modulo $m$, we can define a finite set $\Gamma_v$ of directed edges over $\boldsymbol{Z}/m\boldsymbol{Z}$ for every $v \in V_n$, as follows;

$$\sharp\Gamma_v = c_v, \quad s(x)r(x)^{-1} = a^{\lambda_v} \quad \text{for all } x \in \Gamma_v,$$

and for each $b \in \boldsymbol{Z}/m\boldsymbol{Z}$

$$\sharp\left(r^{-1}(b) \cap \bigcup_{v \in V_n} \Gamma_v\right) - \sharp\left(s^{-1}(b) \cap \bigcup_{v \in V_n} \Gamma_v\right) = \begin{cases} 1 & b = e \\ -1 & b = a \\ 0 & \text{otherwise} \end{cases}.$$

Moreover, we can associate an element $r_v \in \boldsymbol{Z}[\boldsymbol{Z}/m\boldsymbol{Z}]$ with $v \in V_n$ by

$$r_v = \sum_{x \in \Gamma_v} r(x).$$

For each $v \in V_n$ and $w \in V_{n+1}$, we set

$$\Gamma_{v,w} = \Gamma_v \times \{1, 2, \ldots, \lambda_w\}$$

and

$$r(x, k) = r(x)a^{k-1}, \quad s(x, k) = s(x)a^{k-1}$$

for all $(x, k) \in \Gamma_{v,w}$. Let $\Gamma_v'$ be the set of directed edges over the vertex set $\boldsymbol{Z}/m\boldsymbol{Z}$ such as

$$\Gamma_v' = \{x_0, x_1, \ldots, x_{m-1}\},$$
$$r(x_k) = a^k, \quad s(x_k) = a^{k+\lambda_v}.$$

Since $A_n(v, w)$ is equal to $\sharp\Gamma_{v,w}$ modulo $m$, by adding some disjoint copies of $\Gamma_v'$ to $\Gamma_{v,w}$, we get the set $\Gamma_{v,w}'$ consisting of $A_n(v, w)$ directed edges. The element $r_{v,w} \in \boldsymbol{Z}[\boldsymbol{Z}/m\boldsymbol{Z}]$ associated with $\Gamma_{v,w}$ is defined by

$$r_{v,w} = \sum_{y \in \Gamma'_{v,w}} r(y)$$

and it equals to

$$(e + a + \cdots + a^{\lambda_w - 1}) r_v$$

modulo a scalar multiple of $P$ and satisfies $\rho(r_{v,w}) = A_n(v,w)$. Let $\Gamma_w$ be the disjoint union of $\Gamma'_{v,w}$ for all $v \in V_n$. For $b \in \boldsymbol{Z}/m\boldsymbol{Z}$, we have

$$\begin{aligned}
&\sharp\{x \in \Gamma_w; r(x) = b\} - \sharp\{x \in \Gamma_w; s(x) = b\} \\
&= \sum_{k=1}^{\lambda_w} \sum_{v \in V_n} \sharp(r^{-1}(ba^{-k+1}) \cap \Gamma_v) - \sharp(s^{-1}(ba^{-k+1}) \cap \Gamma_v) \\
&= \begin{cases} 1 & b = e,\ b \neq d(w) \\ -1 & b = d(w),\ b \neq e \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

Since the directed graph $\Gamma_w$ includes at least two copies of $\Gamma'_v$ for every $v \in V_n$, thanks to the unicursal theorem, we can find a directed path $\vec{y} = (y_1, y_2, \ldots, y_l)$ which starts from $d(w)$ and ends at $e$ and in which each element of $\Gamma_w$ appears exactly once. Moreover, we can choose the path so that $y_1$ and $y_l$ is contained in $\Gamma'_{v_0,w}$. We can obtain from the directed path $\vec{y}$ the ordered list $(e_1, e_2, \ldots, e_l)$ of edges in $r^{-1}(w)$ such that the source vertex of $e_k$ in the Bratteli diagram $(V,E)$ is $v \in V_n$ when $y_k$ is in $\Gamma'_{v,w}$. In this way, the partial order on $E_n$ is well defined and it gives a simple order on the Bratteli diagram $(V,E)$. It is clear that $B_n(v,w)$ is equal to $r_{v,w}$, and so we get

$$(v \otimes (e - a)) B_n = \sum_{w \in V_{n+1}} w \otimes (e - d(w)) r_v \in Z_n,$$

there by completing the proof. □

Let us consider the invariant $\eta$ defined in [**M**]. The homomorphism $\eta$ was defined on $T(\psi)$ and takes its value in $\mathrm{Ext}(K^0(Y,\psi), \boldsymbol{Z})$. The Ext group is the cokernel of the natural map from $\mathrm{Hom}(K^0(Y,\psi), \boldsymbol{R})$ to $\mathrm{Hom}(K^0(Y,\psi), \boldsymbol{R}/\boldsymbol{Z})$. In Section 5 of [**M**], we constructed a representative $\Phi(\gamma)$ of $\eta(\gamma)$ in $\mathrm{Hom}(K^0(Y,\psi), \boldsymbol{R}/\boldsymbol{Z})$ for $\gamma \in T(\phi)$, by fixing an invariant measure $\mu$ on $Y$.

Let $(Y,\psi)$ be the Cantor minimal system obtained as the extension of $(X,\phi)$ associated with a cocycle $c: X \to \boldsymbol{Z}/m\boldsymbol{Z}$. Assume the canonical commutant $\gamma$ is in $T(\psi)$. Then, the map $\tilde{\eta}(\gamma)$ is identified with the quotient map from $K^0(Y,\psi)$ to $K^0(Y,\psi)/mK^0(Y,\psi) \cong \boldsymbol{Z}/m\boldsymbol{Z}$.

Lemma 3.8. *In the above setting, $\tilde{\eta}(\gamma)$ is a representative of $\eta(\gamma)$.*

Proof. Fix a $\psi$-invariant measure $\mu$ on $Y$. By taking the average of $\{\mu \circ \gamma^j\}_{j=0}^m$, we may assume that $\mu$ is also invariant under the action of the canonical commutant. We denote $\boldsymbol{Z}/m\boldsymbol{Z}$ by $G$. Let $\Phi(\gamma)$ be the element of $\mathrm{Hom}(K^0(Y,\psi), \boldsymbol{R}/\boldsymbol{Z})$ determined by $\mu$ (Section 5 of [**M**]). It suffices to show that $\tilde{\eta}(\gamma)([f])$ coincides with $\Phi(\gamma)([f])$ for the characteristic function $f$ on the clopen set $Y(n, v, h(v), b)$ for all $n \in \boldsymbol{N}$, $v \in V_n$ and $b \in G$.

We can assume that there exists $\{v_k\}_k \subset \boldsymbol{Z}$ such that $(v \otimes (b - ab))B_n$ is equal to $\sum_{k \in G} v_k e(n, k)$. Then, by definition, we have

$$\tilde{\eta}(\gamma)([f]) = \frac{1}{m} \sum_{k \in G} v_k + \boldsymbol{Z}.$$

Let $(v \otimes b)B_n = \sum_{w \in V_{n+1}, k \in G} \lambda_{w,k} w \otimes k$. We define the function $\tilde{f} \in C(Y, \boldsymbol{Z})$ by

$$\tilde{f} = \sum_{w \in V_{n+1}, k \in G} \lambda_{w,k} \chi_{Y(n+1, w, h(w), k)},$$

where $\chi$ means a characteristic function. Since we have $[f] = [\tilde{f}]$ in $K^0(Y, \psi)$, there exists a function $F_0 \in C(Y, \boldsymbol{Z})$ such that $f - \tilde{f} = F_0 - F_0 \circ \psi^{-1}$, which implies

$$f - f \circ \gamma^{-1} = F_0 - F_0 \circ \gamma^{-1} - (F_0 - F_0 \circ \gamma^{-1}) \circ \psi^{-1} + \tilde{f} - \tilde{f} \circ \gamma^{-1},$$

and $\mu(F_0 - F_0 \circ \gamma^{-1}) = 0$ by the definition of $\mu$. Moreover, we have

$$\begin{aligned} \tilde{f} - \tilde{f} \circ \gamma^{-1} &= \sum_{w \in V_{n+1}, k \in G} (\lambda_{w,k} - \lambda_{w, a^{-1}k}) \chi_{Y(n+1, w, h(w), k)} \\ &= \sum_{w \in V_{n+1}, k \in G} (v_k - v_{kd(w)^{-1}}) \chi_{Y(n+1, w, h(w), k)}, \end{aligned}$$

and it is equal to $F_1 - F_1 \circ \psi^{-1}$, where $F_1$ is a function such as

$$F_1 = \sum_{w \in V_{n+1}, k \in G} \sum_{j=1}^{h(w)} v_k \chi_{Y(n+1, w, h(w), kd(w))} \circ \psi^j.$$

Therefore, we get

$$\begin{aligned} \Phi(\gamma)([f]) = \mu(F_1) + \boldsymbol{Z} &= \sum_{w \in V_{n+1}, k \in G} h(w) v_k \mu(Y(n+1, w, h(w), kd(w))) + \boldsymbol{Z} \\ &= \frac{1}{m} \sum_{w \in V_{n+1}, k \in G} h(w) v_k \mu(X(n+1, w, h(w)) \times G) + \boldsymbol{Z} \\ &= \frac{1}{m} \sum_{k \in G} v_k + \boldsymbol{Z}, \end{aligned}$$

and so $\tilde{\eta}(\gamma)$ is equal to $\Phi(\gamma)$. $\square$

The following corollary says that $\eta$ is either zero or injective on finite order elements of $T(\psi)$.

Corollary 3.9. *When* $(Y,\psi)$ *is a Cantor minimal system and* $\gamma \in T(\psi)$ *is an element of finite order, we have* $\eta(\gamma)=0$ *if and only if* $\mathrm{Hom}(K^0(Y,\psi),\boldsymbol{Z}) \neq 0$, *and in this case* $\mathrm{Hom}(K^0(Y,\psi),\boldsymbol{Z})$ *is isomorphic to* $\boldsymbol{Z}$.

Proof. Let $\gamma$ be an element of order $m$. If $\eta(\gamma)=0$, the representative $\tilde{\eta}(\gamma)$ has a lifting to $\mathrm{Hom}(K^0(Y,\psi),\boldsymbol{R})$. Since $\tilde{\eta}(\gamma)$ is a surjection to $\boldsymbol{Z}/m\boldsymbol{Z}$, the lifting gives a non-trivial element of $\mathrm{Hom}(K^0(Y,\psi),\boldsymbol{Z})$.

Conversely, if $\mathrm{Hom}(K^0(Y,\psi),\boldsymbol{Z})$ is non-zero, we get a surjection from $K^0(Y,\psi)$ to $\boldsymbol{Z}/m\boldsymbol{Z}$. Because $\tilde{\eta}(\gamma)$ coincides with the quotient map to $K^0(Y,\psi)/mK^0(Y,\psi) \cong \boldsymbol{Z}/m\boldsymbol{Z}$, it should have a lifting to $\mathrm{Hom}(K^0(Y,\psi),\boldsymbol{Z})$. Therefore, $\eta(\gamma)$ is zero in the Ext group. It is easy to check $\mathrm{Hom}(K^0(Y,\psi),\boldsymbol{Z}) \cong \boldsymbol{Z}$. □

## 4. Examples.

(**1**) Let $n,m$ be natural numbers and $l=nm$. We give a Cantor minimal system $(Y,\psi)$ such that $C(\psi)=\boldsymbol{Z}/l\boldsymbol{Z} \oplus \boldsymbol{Z}\psi$ and $T(\psi)=\boldsymbol{Z}/m\boldsymbol{Z} \oplus \boldsymbol{Z}\psi$, by using a bijective substitution ([**LM**]). For a finite or infinite word $x$, we denote the $i$-th letter of $x$ by $x_i$. Let $\xi$ be a substitution of constant length $l+m$ on the alphabet set $L=\{0,1,2,\dots,l-1\}$ such as

$$\xi(0)_i = \begin{cases} i-k & (k-1)(n+1)+1 \le i \le k(n+1),\ k=1,2,\dots,m-1 \\ i-m & (m-1)(n+1)+1 \le i \le l+m-1 \\ 0 & i=l+m \end{cases}$$

and $\xi(j)_i = \xi(0)_i + j$ for $j=0,1,\dots,l-1$ and $i=1,2,\dots,l+m$, where the addition is understood modulo $l$. For example,

$$\xi(0)=01233450, \quad \xi(1)=12344501, \quad \xi(2)=23455012,$$
$$\xi(3)=34500123, \quad \xi(4)=45011234, \quad \xi(5)=50122345,$$

when $n=3$ and $m=2$. Note that Morse substitution is obtained when we put $n=1$ and $m=2$. Let $(Y,\psi)$ be the substitution subshift determined by $\xi$. We refer the reader to [**DHS**] for basic facts about substitution subshifts and Bratteli diagrams. We define a homeomorphism $\gamma$ on $L^{\boldsymbol{Z}}$ such as $\gamma(x)_i = x_i+1$ for $x \in L^{\boldsymbol{Z}}$ and $i \in \boldsymbol{Z}$. It is easy to check that $\gamma(Y)=Y$ and $\gamma \in C(\psi)$. By Theorem 5 of [**LM**], $C(\psi)$ is isomorphic to $\boldsymbol{Z}/l\boldsymbol{Z} \oplus \boldsymbol{Z}$ and $\gamma$ is a generator of $\boldsymbol{Z}/l\boldsymbol{Z}$. The substitution rule $\xi$ can be extended to a continuous map $\xi: Y \to Y$. Let $y \in Y$ be the fixed point of $\xi$ such that $y_{-1}=y_0=0$. In the case of $n=3$ and $m=2$, the infinite sequence $y$ is

$$\cdots 5012234501233450 \,|\, 01233450123445012345501234500123345 00123\cdots,$$

where the vertical bar separates $y_{(-\infty,-1]}$ from $y_{[0,\infty)}$.

We define a map $\pi : Y \to \{0,1\}^{\boldsymbol{Z}}$ such as

$$\pi(x)_i = \begin{cases} 0 & x_i - x_{i-1} = 0 \\ 1 & x_i - x_{i-1} = 1, 1-l \end{cases}$$

for $x \in Y$. It can be easily seen that $\pi$ is a well-defined factor map from $(Y,\psi)$ to the subshift on $\{0,1\}^{\boldsymbol{Z}}$. We denote by $(X,\phi)$ the image of $(Y,\psi)$ by $\pi$. We can see that $\pi(x) = \pi(x')$ if and only if $x$ and $x'$ have the same $\gamma$-orbit in $Y$, and so the system $(X,\phi)$ is the quotient system of $(Y,\psi)$ by the action of $\gamma$. By the definition of $\pi$, we get

$$\pi(y) = \cdots 1 \,|\, \underbrace{0\overbrace{11\cdots1}^{n}0\overbrace{11\cdots1}^{n}\cdots0\overbrace{11\cdots1}^{n}}_{(n+1)\times m}1\overbrace{11\cdots1}^{n}\cdots.$$

Define the substitution rule $\zeta$ of constant length $l+m$ by

$$\zeta(0) = \underbrace{0\overbrace{11\cdots1}^{n}0\overbrace{11\cdots1}^{n}\cdots0\overbrace{11\cdots1}^{n}}_{(n+1)\times(m-1)}0\overbrace{11\cdots1}^{n}$$

$$\zeta(1) = \underbrace{0\overbrace{11\cdots1}^{n}0\overbrace{11\cdots1}^{n}\cdots0\overbrace{11\cdots1}^{n}}_{(n+1)\times(m-1)}1\overbrace{11\cdots1}^{n}$$

on the alphabet set $\{0,1\}$. Let $z \in \{0,1\}^{\boldsymbol{Z}}$ be the fixed point of $\zeta$ such that $z_{-1} = 1$ and $z_0 = 0$, where $\zeta$ is extended to a continuous map on $\{0,1\}^{\boldsymbol{Z}}$. Then, we have

$$z = \lim_{k\to\infty} \phi^{1+n+n(l+m)+\cdots+n(l+m)^k}(\pi(y)),$$

which implies that $(X,\phi)$ is the substitution subshift associated with $\zeta$. When we put

$$c(x) = \begin{cases} e & x_0 = 0 \\ a & x_0 = 1 \end{cases}$$

for $x \in X$, where $a$ denotes the generator of $\boldsymbol{Z}/l\boldsymbol{Z}$, the system $(Y,\psi)$ is the extension of $(X,\phi)$ associated with the cocycle $c$.

We would like to compute the dimension groups of $(X,\phi)$ and $(Y,\psi)$. Since the substitution rule $\zeta$ is proper, we can easily write down the ordered Bratteli diagram $(V,E,\leq)$ of $(X,\phi)$ ([**DHS**, Proposition 16]). Every vertex set $V_n$

consists of two points, namely $v_{n,0}$ and $v_{n,1}$, which correspond to the alphabets 0 and 1. For example, since 0 appears $m$ times in the word $\zeta(0)$, the number of edges between $v_{n,0}$ and $v_{n+1,0}$ equals to $m$. The dimension group $K^0(X,\phi)$ is the inductive limit of $\boldsymbol{Z}^2$ with the incidence matrix

$$A = \begin{bmatrix} m & m-1 \\ l & l+1 \end{bmatrix}.$$

Therefore, we have

$$K^0(X,\phi) \cong \boldsymbol{Z}\left[\frac{1}{l+m}\right] \oplus \boldsymbol{Z},$$

$$(p,q) \in K^0(X,\phi)^+ \quad \text{if and only if } p = q = 0 \text{ or } p + \frac{(m-1)q}{l+m-1} > 0$$

and the order unit is $(1,0)$.

Let us consider the skew product extension $(Y,\psi)$. The label of each vertex is determined by the cocycle $c$ and we get $d(v_{n,0}) = e$, $d(v_{n,1}) = a$ for all $n$. In order to compute the restricted dimension group, we would like to write the connecting matrix $B$. If $n = 3$ and $m = 2$, we have

$$\zeta(0) = 01110111, \quad \zeta(1) = 01111111,$$

and we can make the following tables;

$$\begin{array}{rcccccccc} \zeta(0) = & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ & e & a^5 & a^4 & a^3 & a^3 & a^2 & a & e \end{array}$$

and

$$\begin{array}{rcccccccc} \zeta(1) = & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & a & e & a^5 & a^4 & a^3 & a^2 & a & e. \end{array}$$

Hence, the 0–0 entry of the matrix $B$ is equal to $e + a^3$, which is the sum of elements under the letter 0 appearing $\zeta(0)$. The 1–0 entry of $B$ equals $a^5 + a^4 + a^3 + a^2 + a + e$ by the same reason. The other entries can be obtained in a similar way. In general, we have

$$B = \begin{bmatrix} \sum_{j=1}^{m} a^{jn} & \sum_{j=2}^{m} a^{jn+1} \\ \sum_{j=1}^{l} a^{j} & e + \sum_{j=1}^{l} a^{j} \end{bmatrix}$$

and the restricted dimension group $K^0(Y,\psi;\boldsymbol{Z}/l\boldsymbol{Z})$ is obtained as the inductive limit of $\boldsymbol{Z}^2 \otimes \boldsymbol{Z}[\boldsymbol{Z}/l\boldsymbol{Z}]$ with the incidence matrix $B$. It is easy to check that $(e - a^n, 0)B$ and $(0, e - a^n)B$ are contained in the subspace spanned by

$\{(0, a^k - a^{k+1})\}_k$, and so we have $\gamma^n \in T(\psi)$. Let $(Z, \tau)$ be the quotient system of $(Y, \psi)$ by the action of $\gamma^n$. The dimension group $K^0(Z, \tau)$ is naturally isomorphic to the subgroup $mK^0(Y, \psi)$ of $K^0(Y, \psi)$, because $\gamma^n$ acts identically on $K^0(Y, \psi)$. Since the dimension group is torsion free, it suffices to compute $K^0(Z, \tau)$ instead of $K^0(Y, \psi)$. We let $c'$ be the composition of $c$ and the quotient map from $\boldsymbol{Z}/l\boldsymbol{Z}$ to $\boldsymbol{Z}/n\boldsymbol{Z}$. The system $(Z, \tau)$ is isomorphic to the extension of $(X, \phi)$ associated with the cocycle $c'$. We denote $\boldsymbol{Z}/n\boldsymbol{Z}$ by $G$. The restricted dimension group $K^0(Z, \tau; G)$ is obtained by the inductive limit of $\boldsymbol{Z}^2 \otimes \boldsymbol{Z}[G]$ with the incidence matrix

$$C = \begin{bmatrix} me & (m-1)b \\ m\sum_{j=1}^{n} b^j & e + m\sum_{j=1}^{n} b^j \end{bmatrix},$$

where $b$ is the generator of $G$. By using Theorem 2.5, we can express the dimension groups $K^0(Z, \tau)$ and $K^0(Y, \psi)$ within the abelian group $\boldsymbol{Z}[1/(l+m)] \oplus \boldsymbol{Z} \oplus (\boldsymbol{Z}[1/m] \otimes \boldsymbol{Z}[G])$ such as

$$\left\{ \left( \frac{p}{(l+m)^k}, q, \frac{1}{m^k} r(e-b) \right); \begin{array}{c} p, q \in \boldsymbol{Z}, k \in \boldsymbol{N}, r \in \boldsymbol{Z}[G], \\ p + lq = (l+m-1)\rho(r) \end{array} \right\}$$

with the strict ordering from the first coordinate, where $\rho$ is the canonical ring homomorphism from $\boldsymbol{Z}[G]$ to $\boldsymbol{Z}$. The order unit of $K^0(Z, \tau)$ and $K^0(Y, \psi)$ are given by $(n(l+m-1), 0, 0)$ and $(l(l+m-1), 0, 0)$ respectively. The canonical commutant $\gamma \in C(\psi)$ acts on the dimension group $K^0(Y, \psi)$ as the multiplication by $b$ in the last coordinate, and so $T(\psi)$ is isomorphic to $\boldsymbol{Z}/m\boldsymbol{Z} \oplus \boldsymbol{Z}\psi$. By Corollary 3.9, we can see that $\eta(\gamma^n)$ is zero.

When $\xi$ is the Morse substitution, $(Y, \psi)$ and $(X, \phi)$ are isomorphic to the systems described in Example (4) of [**M**].

(**2**) Let $S_3$ be the symmetric group of degree three generated by $a$ and $b$ satisfying $a^3 = b^2 = e$ and $bab = a^{-1}$. The element $a$ forms the normal subgroup $\boldsymbol{Z}/3\boldsymbol{Z}$ of $S_3$. We construct a Cantor minimal system $(Y, \psi)$ such as $C(\psi) = S_3 \oplus \boldsymbol{Z}\psi$ and $T(\psi) = \boldsymbol{Z}/3\boldsymbol{Z} \oplus \boldsymbol{Z}\psi$.

Define a proper substitution $\xi$ on the alphabet set $L = \{0, 1, 2\}$ such that

$$\xi(0) = 010021112112$$

$$\xi(1) = 000002101212$$

$$\xi(2) = 010210121112,$$

and let $(X, \phi)$ be the associated substitution subshift. Thanks to Proposition 16 of [**DHS**], we obtain the stationary ordered Bratteli diagram $(V, E, \leq)$ which represents $(X,\phi)$. The vertex set $V_n$ is canonically identified with $L$, and so we put $V_n = \{v_{n,0}, v_{n,1}, v_{n,2}\}$ for each $n \in \boldsymbol{N}$. The incidence matrix from $\boldsymbol{Z}^{V_n}$ to $\boldsymbol{Z}^{V_{n+1}}$ is represented as

$$A = \begin{bmatrix} 3 & 6 & 3 \\ 6 & 3 & 6 \\ 3 & 3 & 3 \end{bmatrix}$$

for all $n \in \boldsymbol{N}$ under the identification of $V_n$ with $L$. The dimension group $K^0(X,\phi)$ is obtained as the inductive limit of $\boldsymbol{Z}^3$ with the incidence matrix $A$. Hence, we get

$$K^0(X,\phi) \cong \left\{ \left( \frac{p}{12^n}, \frac{q}{(-3)^n} \right); p, q \in \boldsymbol{Z}, p \equiv 2q \ (\mathrm{mod}\ 5) \right\}$$

with the strict ordering from the first coordinate and the order unit is $(20,0)$.

Let $Z = \{0,1,\dots,11\}^{\boldsymbol{Z}}$ and $\tau$ be the odometer system on $Z$. Because $\sharp r^{-1}(v_{n,i}) = 12$ for all $n \in \boldsymbol{N}\backslash\{1\}$ and $i = 0,1,2$, we can construct a factor map $\rho$ from $(X,\phi)$ to $(Z,\tau)$ in the same way of Section 2 of [**GJ**]. The same construction of factor maps can be found in Section 7 of [**M**], too. For every point $z \in Z$, the preimage $\rho^{-1}(z)$ consists of at most three points. Because the substitution rule $\xi$ satisfies

$$\{\xi(0)_i, \xi(1)_i, \xi(2)_i\} = \{0,1,2\} \quad \text{if and only if } i = 5,6,8,$$

$\rho^{-1}(z)$ includes three distinct points if and only if the tail of $z \in Z$ consists of only 4, 5 and 7. When $\gamma$ is in $C(\phi)$, there is $\sigma \in C(\tau)$ satisfying $\sigma \circ \rho = \rho \circ \gamma$ and the subset

$$\{z \in Z; \sharp\rho^{-1}(z) = 3\}$$

must be preserved by $\sigma$. It follows that $\sigma$ is a power of $\tau$. On the other hand, if $\gamma \in C(\phi)$ satisfies $\rho \circ \gamma = \rho$, $\gamma$ is the identity because there exists a point $z \in Z$ such that $\sharp\rho^{-1}(z) = 1$. As a consequence, we obtain $C(\phi) = \boldsymbol{Z}\phi$.

We set a $S_3$-valued cocycle $c$ on $X$ as follows;

$$c(x) = \begin{cases} a & x_0 = 0 \\ b & x_0 = 1 \\ a^{-1} & x_0 = 2 \end{cases}$$

for all $x \in X$. By a straightforward computation, we have

$$d(v_{n,0}) = abaaa^{-1}bbba^{-1}bba^{-1} = a,$$

$$d(v_{n,1}) = aaaaaa^{-1}baba^{-1}ba^{-1} = b,$$

$$d(v_{n,2}) = abaa^{-1}baba^{-1}bbba^{-1} = a^{-1}$$

for all $n \in \boldsymbol{N}$, and so the extension $(Y, \psi)$ associated with the cocycle $c$ is a Cantor minimal system by Lemma 2.3. By the same argument in the last paragraph, we see that the automorphism group $C(\psi)$ is isomorphic to $S_3 \oplus \boldsymbol{Z}\psi$.

The restricted dimension group $K^0(Y, \psi; S_3)$ is computed as the inductive limit of $\boldsymbol{Z}^3 \otimes \boldsymbol{Z}[S_3]$ by the incidence matrix

$$B = \begin{bmatrix} e + ab + a^2b & e + b + 2ab + 2a^2b & e + a + ab \\ 2a + a^2 + b + ab + a^2b & a + a^2 + b & a + 2a^2 + b + ab + a^2b \\ e + a^2 + a^2b & e + ab + a^2b & e + ab + a^2b \end{bmatrix}.$$

To get the first column of this matrix $B$, we need to make the following table;

$$\begin{array}{rcccccccccccc} \xi(0) = & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 2 \\ & e & b & a^2b & ab & a^2b & a & a^2b & a & a^2 & ab & a^2 & e. \end{array}$$

The tables for $\xi(1)$ and $\xi(2)$ can be wrote in the same way. By summing up the elements under the letter 0 in $\xi(0)$, the 0–0 entry $e + ab + a^2b$ of $B$ is obtained. The other entries of $B$ are also obtained in this method. When we put

$$v(g) = (g - ga, g - gb, g - ga^{-1}) \in \boldsymbol{Z}^3 \otimes \boldsymbol{Z}[S_3]$$

for $g \in S_3$, we can check the equations

$$(e - a, 0, 0)B = v(e) + v(ab),$$

$$(0, e - a, 0)B = -v(e) + v(a)$$

and

$$(0, 0, e - a)B = -v(a) - v(b),$$

which imply $\gamma_a \in T(\psi)$. In the same method as the example (1), we have

$$K^0(Y, \psi) \cong \left\{ \left( \frac{p}{12^l}, \frac{q}{(-3)^l}, \frac{r}{(-2)^l} \right); p, q, r \in \boldsymbol{Z}, p \equiv 2q \pmod 5 \right\}$$

with the strict ordering from the first coordinate and the order unit is $(10, 0, 0)$. The element $b \in S_3$ induces $\mathrm{mod}(\gamma_b)$ which changes the signal of the last coordinate.

Because there are no non-trivial homomorphisms from $K^0(Y, \psi)$ to $\boldsymbol{Z}$, by Corollary 3.9, $\eta(\gamma_a)$ is not zero in $\mathrm{Ext}(K^0(Y, \psi), \boldsymbol{Z})$. Since $(Y, \psi)$ is not strong orbit equivalent to odometer systems, this example gives a positive answer to the problem (1) raised in [**M**].

For the dihedral group $D_n = \langle a, b; e = a^n = b^2, bab = a^{-1} \rangle$ of order $2n$, we can construct a Cantor minimal system $(Y, \psi)$ such that $C(\psi) = D_n \oplus \boldsymbol{Z}\psi$ and

$T(\psi) = \boldsymbol{Z}/n\boldsymbol{Z} \oplus \boldsymbol{Z}\psi$, whenever $n$ is odd. Theorem 3.5 tells us that we can never construct it for even $n$.

(**3**) Let $D_6$ be the dihedral group as above. We will show that there is a Cantor minimal system $(Y, \psi)$ satisfying $C(\psi) = D_6 \oplus \boldsymbol{Z}\psi$ and $T(\psi) = \boldsymbol{Z}/3\boldsymbol{Z} \oplus \boldsymbol{Z}\psi$.

We define a proper substitution $\xi$ of constant length such as

$$\xi(0) = 010121122102122$$

$$\xi(1) = 001220000122122$$

$$\xi(2) = 011001201001212$$

on the alphabet set $\{0, 1, 2\}$. We denote by $(X, \phi)$ the substitution minimal subshift determined by $\xi$. When the cocycle $c$ is defined by

$$c(x) = \begin{cases} a & x_0 = 0 \\ b & x_0 = 1 \\ a^{-1} & x_0 = 2 \end{cases}$$

for all $x \in X$, we can show that the extension $(Y, \psi)$ is minimal and $C(\psi)$ is isomorphic to $D_3 \oplus \boldsymbol{Z}\psi$ in the same way as the example (2). Moreover, we have $\gamma_a^2 \in T(\psi)$ and $\eta(\gamma_a^2)$ is not zero. We omit the computation.

(**4**) We give a Cantor minimal system $(Y, \psi)$ such that $\ker \eta$ is isomorphic to $\boldsymbol{Z} \oplus \boldsymbol{Z}\psi$.

Define an ordered Bratteli diagram $(V, E, \leq)$ as follows; the vertex set $V_n$ $(n \neq 0)$ consists of two vertices, namely $v_{n,0}, v_{n,1}$, and the incidence matrix from $n$-th step to $n+1$-th step is given by $A^{n+1}$, where $A$ is

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The partial order on the edge set $E$ is defined by

$$\theta(v_{n+1,0}) = (\overbrace{v_{n,0}, v_{n,0}, \dots, v_{n,0}}^{3^n}, \overbrace{v_{n,0}, \dots, v_{n,0}}^{m_n+1}, \overbrace{v_{n,1}, \dots, v_{n,1}}^{m_n}, \overbrace{v_{n,1}, v_{n,1}, \dots, v_{n,1}}^{3^n})$$

and

$$\theta(v_{n+1,1}) = (\overbrace{v_{n,0}, v_{n,0}, \dots, v_{n,0}}^{3^n}, \overbrace{v_{n,1}, \dots, v_{n,1}}^{m_n+1}, \overbrace{v_{n,0}, \dots, v_{n,0}}^{m_n}, \overbrace{v_{n,1}, v_{n,1}, \dots, v_{n,1}}^{3^n}),$$

for $n \in \boldsymbol{N}$, where $m_n$ is $(3^n - 1)/2$. When we denote the infinite path space by $X$ and the Bratteli-Vershik map by $\phi$, we can see that $(X, \phi)$ is a Cantor minimal system, since $(V, E, \leq)$ is simple. The dimension group $K^0(X, \phi)$ is isomorphic to $\boldsymbol{Z}[1/3] \oplus \boldsymbol{Z}$, where the unique state is given by $\boldsymbol{Z}[1/3] \oplus \boldsymbol{Z} \ni (p, q) \mapsto p + q/2$

and the order unit is $(1,0)$. We can construct a factor map from $(X,\phi)$ to an odometer system of type $3^\infty$. By the same reason as the above examples, we have $C(\phi) \cong \boldsymbol{Z}\phi$.

Let $G$ be the projective limit of $\boldsymbol{Z}/3^n\boldsymbol{Z}$ and $a$ be $(1,1,\dots) \in G$. The addition by $a$ is the odometer system of type $3^\infty$ on $G$. We denote by $\rho_n$ the canonical projection from $G$ to $\boldsymbol{Z}/3^n\boldsymbol{Z}$. Define a $G$-valued cocycle $c$ on $X$ such as $c(x) = a$ if the infinite path $x$ goes through $v_{1,0}$ and $c(x) = a^{-1}$ otherwise. The extension $(Y,\psi)$ of $(X,\phi)$ associated with the $G$-valued cocycle $c$ can be defined in the same way as the case of finite groups and the canonical commutant $\{\gamma_g\}_{g\in G} \in C(\psi)$ is obtained. Let $(Y_n,\psi_n)$ be the extension of $(X,\phi)$ associated with the $\boldsymbol{Z}/3^n\boldsymbol{Z}$-valued cocycle $\rho_n \circ c$. By sending $(x,g) \in Y$ to $\{(x,\rho_n(g))\}_n$, we get the isomorphism from $(Y,\psi)$ to the projective limit of $(Y_n,\psi_n)$. By Proposition 3.7 of [**M**], $(Y,\psi)$ is a Cantor minimal system and the dimension group $K^0(Y,\psi)$ is the inductive limit of $K^0(Y_n,\psi_n)$. Moreover, we have $C(\psi) = G \oplus \boldsymbol{Z}\psi$.

We would like to show that the canonical commutant $\gamma_g$ is in $T(\psi)$ for every $g \in G$. Take an element $g \in G$. When we write $\gamma_n$ as the generator of the canonical commutant of $(Y_n,\psi_n)$, the restriction of $\mathrm{mod}(\gamma_g)$ on $K^0(Y_n,\psi_n)$ is a power of $\mathrm{mod}(\gamma_n)$. Therefore, it suffices to show that $\gamma_n$ acts identically on $K^0(Y_n,\psi_n)$ for all $n \in \boldsymbol{N}$. The label of the vertices $v_{m,0}$ and $v_{m,1}$ determined by the cocycle $\rho_n \circ c$ are $\rho_n(a)$ and $\rho_n(a^{-1})$ respectively. We write $B_l \in M_2(\boldsymbol{Z}[\boldsymbol{Z}/3^n\boldsymbol{Z}])$ as the incidence matrix of the extension $(Y_n,\psi_n)$ from $l$-th step to $l+1$-th step. When $l$ is larger than $n$, the matrix $B_l$ is given by

$$\begin{bmatrix} \sum_{j=0}^{m_n} \rho_n(a)^{-j} & \sum_{j=0}^{m_n-1} \rho_n(a)^{j} \\ \sum_{j=0}^{m_n-1} \rho_n(a)^{-j} & \sum_{j=0}^{m_n} \rho_n(a)^{j} \end{bmatrix},$$

modulo scalar multiples of $\sum_{j=1}^{3^n} \rho_n(a)^j$. Then,

$$(e - \rho_n(a), 0)B_l \in Z_{l+1}$$

and

$$(0, e - \rho_n(a))B_l \in Z_{l+1}$$

are easily checked for all $l > n$, where $Z_l$ is a subspace of $\boldsymbol{Z}^2 \otimes \boldsymbol{Z}[\boldsymbol{Z}/3^n\boldsymbol{Z}]$ spanned by

$$(\rho_n(a^j) - \rho_n(a^{j+1}), \rho_n(a^j) - \rho_n(a^{j-1})) \quad j = 1,\dots,3^n.$$

As a consequence, $\gamma_n \in T(\psi_n)$ is derived, and so $T(\psi) = C(\psi) = G \oplus \boldsymbol{Z}\psi$.

The dimension group $K^0(Y,\psi)$ is isomorphic to $\boldsymbol{Z}[1/3] \oplus \boldsymbol{Z}[1/3]$ with the unique state given by $(p,q) \mapsto p + q/2$ and the order unit is $(1,0)$. The invariant $\eta$ argued in Lemma 3.8 takes its value in

$$\mathrm{Ext}\left(\boldsymbol{Z}\left[\frac{1}{3}\right]\oplus\boldsymbol{Z}\left[\frac{1}{3}\right],\boldsymbol{Z}\right)\cong\mathrm{Ext}\left(\boldsymbol{Z}\left[\frac{1}{3}\right],\boldsymbol{Z}\right)\oplus\mathrm{Ext}\left(\boldsymbol{Z}\left[\frac{1}{3}\right],\boldsymbol{Z}\right).$$

Let $\kappa$ be the natural quotient map from $G$ to $\mathrm{Ext}(\boldsymbol{Z}[1/3],\boldsymbol{Z})$ whose kernel is generated by $a\in G$. For all $g\in G$, we have

$$\eta(\gamma_g)=(0,\kappa(g)),$$

which implies $\ker\eta$ is isomorphic to $\boldsymbol{Z}^2$ generated by $\gamma_a$ and $\psi$. This example shows that the value of $\eta$ can be zero for non-trivial infinite order elements, even if $\mathrm{Hom}(K^0(Y,\psi),\boldsymbol{Z})$ is zero.

We remark that all Cantor minimal systems in the above examples are orbit equivalent to odometer systems by Theorem 2.2 of [**GPS**].

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