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Partition properties on $\mathscr{P}_\kappa\lambda$

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Abstract. Menas [**13**] showed there exist $2^{2^{\lambda^{<\kappa}}}$ normal ultrafilters on $\mathscr{P}_\kappa\lambda$ with the partition property if κ is $2^{\lambda^{<\kappa}}$-supercompact. We first show that λ-supercompactness of κ implies the existence of a normal ultrafilter on $\mathscr{P}_\kappa\lambda$ with the partition property. We also prove by a similar technic that $\mathrm{part}^*(\kappa,\lambda)$ holds if κ is λ-ineffable with $\mathrm{cf}(\lambda) \geq \kappa$. Note that Magidor [**11**] showed κ is λ-ineffable if $\mathrm{part}^*(\kappa,\lambda)$ holds, and we proved the converse under some additional assumption in [**7**].

1. Introduction.

There are several combinatorial properties related to supercompactness such as partition property and ineffability. In fact, Menas [**13**, Theorem 3] proved there exist $2^{2^{\lambda^{<\kappa}}}$ normal ultrafilters on $\mathscr{P}_\kappa\lambda$ with the partition property if κ is $2^{\lambda^{<\kappa}}$-supercompact, and Magidor [**11**] proved that κ is λ-ineffable if $\mathrm{part}^*(\kappa,\lambda)$, and that κ is supercompact if κ is θ-ineffable for all $\theta \geq \kappa$.

It is well known that every normal ultrafilter on κ has the partition property as well as $\mathrm{part}^*(\kappa,\kappa)$ holds whenever κ is ineffable. On the other hand Solovay proved the existence of normal ultrafilters on $\mathscr{P}_\kappa\lambda$ without the partition property for some κ and λ.

Thus it is natural to ask: (1) Does $\mathscr{P}_\kappa\lambda$ carry a normal ultrafilter with the partition property if κ is λ-supercompact?

(2) Does $\mathrm{part}^*(\kappa,\lambda)$ hold whenever κ is λ-ineffable?

In this paper we give affirmative answers to both questions. First we reduce the assumption in Menas' theorem to show:

THEOREM 3.1. *If κ is λ-supercompact, then there exists a normal ultrafilter on $\mathscr{P}_\kappa\lambda$ with the partition property.*

In [**7**], we gave a partial answer for the question (2) under an additional assumption. By the same idea as used for proving the above theorem we eliminate the assumption in [**7**] to prove:

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Theorem 5.1. *If* κ *is* λ*-ineffable and* $\mathrm{cof}(\lambda) \geq \kappa$, *then* $\mathrm{part}^*(\kappa, \lambda)$ *holds.*

The paper consists of five sections. In the next section, we give some notations and definitions. The above theorems are proved in sections 3 and 5. Section 4 is devoted to give some lemmas.

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2. Notations and definitions.

We use standard $\mathscr{P}_\kappa\lambda$-combinatorial terminologies (e.g., see [**8**]). Throughout this paper, κ denotes a regular uncountable cardinal, $\lambda \geq \kappa$ a cardinal, and all ideals and filters are assumed to be κ-complete. Let $\mathscr{I}$ be an ideal on a set S. $\mathscr{I}^*$ denotes the dual filter and $\mathscr{I}^+$ denotes the set $\mathscr{P}(S) \backslash \mathscr{I}$. For any subset $S' \subset S$, $\mathscr{I} \restriction S'$ denotes $\{X \subset S' \mid X \in \mathscr{I}\}$. For any function $f : S \to T$, $f_*(\mathscr{I})$ denotes the ideal $\{X \subset T \mid f^{-1}X \in \mathscr{I}\}$ on T.

Let A be a set such that $\kappa \subset A$. $\mathscr{P}_\kappa A$ denotes the set $\{x \subset A \big| |x| < \kappa\}$. For each $x \in \mathscr{P}_\kappa A$, Q_x denotes the set $\{s \subset x \big| |s| < |x \cap \kappa|\}$. For any $x, y \in \mathscr{P}_\kappa A$, $x \prec y$ means that $x \in Q_y$. Let Y be a subset of $\mathscr{P}_\kappa A$. Y is said to be *unbounded* if for any $x \in \mathscr{P}_\kappa A$ there exists a $y \in Y$ such that $x \subset y$. Y is called a *club* if Y is unbounded and closed under $\subset$-increasing chains with length $< \kappa$. Y is said to be *stationary* if $X \cap C \neq \phi$ for any club $C \subset \mathscr{P}_\kappa A$. Let $\mathrm{NS}_{\kappa, A}$ denote the set of all non-stationary subsets of $\mathscr{P}_\kappa A$.

A function $f : Y \to A$ is said to be *regressive* if $f(x) \in x$ holds for all $x \in Y$. Let $\mathscr{I}$ be an ideal on $\mathscr{P}_\kappa A$. $\mathscr{I}$ is said to be *normal* if it contains all bounded subsets, and for any $X \in \mathscr{I}^+$ and regressive function $f : X \to A$ there exists an $a \in A$ such that $f^{-1}\{a\} \in \mathscr{I}^+$. $\mathscr{I}$ is said to be *strongly normal* if for any $X \in \mathscr{I}^+$ and function $f : X \to \mathscr{P}_\kappa A$ such that $f(x) \prec x$ for $x \in X$ there is a $Y \in \mathscr{I}^+ \restriction X$ such that $f \restriction Y$ is constant. It is known that $\mathrm{NS}_{\kappa, A}$ is the smallest normal ideal on $\mathscr{P}_\kappa A$. A filter on $\mathscr{P}_\kappa A$ is *normal* if the dual ideal of it is a normal ideal. We say κ is A*-supercompact* if there exists a normal ultrafilter on $\mathscr{P}_\kappa A$. For any ultrafilter U on $\mathscr{P}_\kappa A$, M_U denotes the ultrapower of the universe by U.

For each function $\tau : \mathscr{P}_\kappa A \to \mathscr{P}_\kappa A$, $\mathrm{cl}(\tau)$ denotes the set $\{x \in \mathscr{P}_\kappa A \mid \forall t \in Q_x(\tau(t) \in Q_x)\}$. For each $\tau : A \times A \to \mathscr{P}_\kappa A$, $\mathrm{cl}(\tau)$ denotes the set $\{x \in \mathscr{P}_\kappa A \mid \forall \alpha, \beta \in x(\tau(\alpha, \beta) \subset x)\}$. It is known [**12**] for any $X \subset \mathscr{P}_\kappa A$, X contains a club if and only if there exists a $\tau : A \times A \to \mathscr{P}_\kappa A$ such that $\mathrm{cl}(\tau) \subset X$. For any $B \supset A$, the function $p : \mathscr{P}_\kappa B \to \mathscr{P}_\kappa A$ which is defined by $p(x) = x \cap A$ is called the *projection* from $\mathscr{P}_\kappa B$ to $\mathscr{P}_\kappa A$.

Let $Y \subset \mathscr{P}_\kappa A$. $[Y]^2$ denotes the set $\{(x, y) \in Y \times Y \mid x \subsetneq y\}$. For any function $f : [Y]^2 \to 2$, a subset H of Y is said to be *homogeneous* for f if $|f``[H]^2| =$

1. An ultrafilter U on $\mathscr{P}_\kappa A$ has the *partition property* if for any $X \in U$ and any $f : [X]^2 \to 2$ there exists $H \in U$ such that $H \subset X$ and H is homogeneous for f. We say that Y has the *partition property* if for any $f : [Y]^2 \to 2$ there exists a stationary subset H of Y such that H is homogeneous for f. Y is said to be *ineffable* (*almost ineffable*) if for any $\{s_x \subset x \mid x \in Y\}$ there exists an $S \subset A$ such that $\{x \in Y \mid s_x = S \cap x\}$ is stationary (unbounded). Set

$$\mathrm{NP}_{\kappa,A} = \{X \subset \mathscr{P}_\kappa A \mid X \text{ does not have the partition property}\},$$

$$\mathrm{NIn}_{\kappa,A} = \{X \subset \mathscr{P}_\kappa A \mid X \text{ is not ineffable}\}, \text{ and}$$

$$\mathrm{NAIn}_{\kappa,A} = \{X \subset \mathscr{P}_\kappa A \mid X \text{ is not almost ineffable}\}.$$

Carr [**3**], [**4**] showed that $\mathrm{NP}_{\kappa,A}$, $\mathrm{NIn}_{\kappa,A}$, and $\mathrm{NAIn}_{\kappa,A}$ are normal, and that these ideals are strongly normal if $|A|^{<\kappa} = |A|$.

We say that part$^*(\kappa, A)$ holds if $\mathrm{NP}_{\kappa,A}$ is a proper ideal, that κ is *A-ineffable* if $\mathrm{NIn}_{\kappa,A}$ is a proper ideal, and that κ is *almost A-ineffable* if $\mathrm{NAIn}_{\kappa,A}$ is a proper ideal. It is known that for any $B \supset A$, $\mathrm{NIn}_{\kappa,A} \subset p_*(\mathrm{NIn}_{\kappa,B})$, where p denotes the projection from $\mathscr{P}_\kappa B$ to $\mathscr{P}_\kappa A$.

3. Normal ultrafilters with the partition property.

Concerning normal ultrafilters on $\mathscr{P}_\kappa\lambda$ without the partition property, Solovay proved:

Theorem 1 (Menas [**13**]). *If κ is λ-supercompact and ν is λ-supercompact for some $\kappa < \nu \leq \lambda$, then there exists a normal ultrafilter on $\mathscr{P}_\kappa\lambda$ without the partition property.*

Kunen proved:

Theorem 2 (Kunen-Pelletier [**9**]). *Assume that there exists a normal ultrafilter on $\mathscr{P}_\kappa\lambda$ without the partition property. Then the least such $\lambda > \kappa$ is weakly Π^2_1-indescribable and inaccessible.*

On the other hand, Menas proved that:

Theorem 3 (Menas [**13**]). *If κ is $2^{\lambda^{<\kappa}}$ supercompact, then there exist $2^{2^{\lambda^{<\kappa}}}$ normal ultrafilters on $\mathscr{P}_\kappa\lambda$ with the partition property.*

In this section, we prove:

Theorem 3.1. *If κ is λ-supercompact, then there exists a normal ultrafilter on $\mathscr{P}_\kappa\lambda$ with the partition property.*

The proof will be done by a slightly different argument from that in Menas

[**13**]. We first reduce this theorem to a certain lemma (Lemma 3.4, below). The following two lemmas are due to Menas [**12**], [**13**].

Lemma 3.2 (Menas [**12**]). *If κ is λ-supercompact, then there exists a normal ultrafilter U on $\mathscr{P}_\kappa\lambda$ such that*

$$M_U \models \text{“}\kappa \text{ is not } \lambda\text{-supercompact”}.$$

Lemma 3.3 (Menas [**13**]). *Let U be a normal ultrafilter on $\mathscr{P}_\kappa\lambda$. Then, the following* (a) *and* (b) *are equivalent.*

(a) *U has the partition property.*

(b) *There exists an $X \in U$ such that $x \prec y$ for all $(x, y) \in [X]^2$.*

By these results, Theorem 3.1 directly follows the next lemma.

Lemma 3.4. *Suppose that*

(1) $M_U \models$ “*κ is not λ-supercompact*”.

Then, there exists an $X \in U$ such that

(2) *$x \prec y$ for all $(x, y) \in [X]^2$.*

In order to prove this lemma, we need the notion of ω-Jonsson functions and some known results. Let S be an infinite set. We denote by ${}^\omega S$ the set of functions from ω to S. A function F from ${}^\omega S$ to S is called an *ω-Jonsson function* for S if $F``{}^\omega T = S$ for any $T \subset S$ with $|T| = |S|$. Concerning ω-Jonsson functions, Erdös-Hajnal (e.g., see [**8**, Theorem 23.13]) proved:

Lemma 3.5 (Erdös-Hajnal). *For any infinite set S, there exists an ω-Jonsson function for S.*

Solovay proved:

Lemma 3.6 (Solovay [**12**]). *Let U be a normal ultrafilter on $\mathscr{P}_\kappa\lambda$ and $F : {}^\omega\lambda \to \lambda$ an ω-Jonsson function. Then*

$$\{x \in \mathscr{P}_\kappa\lambda \mid F \restriction {}^\omega x \text{ is } \omega\text{-Jonsson}\} \in U.$$

The next lemma is due to Magidor.

Lemma 3.7 (Magidor [**10**]). *If κ is $<\lambda$-supercompact and λ is θ-supercompact, then κ is θ-supercompact.*

Proof of Lemma 3.4. Suppose that U is a normal ultrafilter on $\mathscr{P}_\kappa\lambda$ which satisfies (1). Let δ be the largest strong limit cardinal $\leq \lambda$. Define δ_i (for $i < \omega$) by $\delta_0 = \delta$, and $\delta_{i+1} = 2^{\delta_i}$. Let $n < \omega$ be such that $\delta_n \leq \lambda < \delta_{n+1}$. Note that

$$M_U \models \text{“}\kappa \text{ is } \alpha\text{-supercompact, for any } \alpha \in [\kappa, \delta).$$

Take ω-Jonsson functions F and F_i for λ and δ_i (for $i \le n$). Define $X_0 \subset \mathscr{P}_\kappa\lambda$ by:

$$x \in X_0 \quad \text{if and only if } x \in \mathscr{P}_\kappa\lambda \text{ and the following hold.}$$

(3) $x \cap \kappa$ is inaccessible and $x \cap \kappa$ is not x-supercompact,
(4) $x \cap \kappa$ is $x \cap \alpha$-supercompact for all $\alpha \in x \cap [\kappa, \delta)$,
(5) $\mathrm{ot}(x \cap \delta)$ is a strong limit cardinal and $\mathrm{ot}(x \cap \delta_i)$ is a cardinal for $i \le n$,
(6) $2^{|x \cap \delta_i|} = |x \cap \delta_{i+1}|$ for $i < n$ and $|x| \le 2^{|x \cap \delta_n|}$,
(7) For $i \le n$, $F_i \restriction {}^{\omega}(x \cap \delta_i)$ is an ω-Jonsson function for $x \cap \delta_i$,
(8) $F \restriction {}^{\omega}x$ is an ω-Jonsson function for x.

By Lemma 3.6 and the fact that $[\langle \mathrm{ot}(x \cap \alpha) \mid x \in \mathscr{P}_\kappa\lambda \rangle]_U$ represents α in M_U for any $\alpha \le \lambda$, it holds that $X_0 \in U$.

Claim 1. *If $(x, y) \in [X_0]^2$ and $x \cap \delta_n \ne y \cap \delta_n$ then $x \prec y$.*

Proof of Claim 1. To get a contradiction, assume that

$$(x, y) \in [X_0]^2 \text{ and } x \cap \delta_n \ne y \cap \delta_n \text{ and } x \prec y \text{ does not hold.}$$

Since $y \cap \kappa$ is a strong limit cardinal, it holds that $y \cap \kappa \le |x \cap \delta_0|$. Since $x \cap \kappa$ is $x \cap \alpha$-supercompact for all $\alpha \in x \cap [\kappa, \delta)$, we have that

$$x \cap \kappa \text{ is } y \cap \alpha\text{-supercompact for all } \alpha \in [x \cap \kappa, y \cap \kappa).$$

By this, Lemma 3.7, and the fact that $y \cap \kappa$ is $y \cap \alpha$-supercompact for all $\alpha \in y \cap [\kappa, \delta)$, we have that

$$x \cap \kappa \text{ is } y \cap \alpha\text{-supercompact for all } \alpha \in [x \cap \kappa, \delta) \cap y.$$

By this, since $x \cap \kappa$ is not x-supercompact, it holds that $|y \cap \delta| \le |x|$. Since $|y \cap \delta|$ is a strong limit cardinal, we have that $|y \cap \delta| \le |x \cap \delta|$. By this and (7), $x \cap \delta = y \cap \delta$. This implies that $x \cap \delta_n = y \cap \delta_n$. This contradicts the assumption. (Claim 1)□

In case that $\lambda = \delta_n$, $X = X_0$ satisfies (2) by Claim 1. Let $\delta_n < \lambda$. Define $g : \kappa \to \kappa$ and $f_i : \kappa \to \kappa$ (for $i \le n+1$) by

$$g(\alpha) = \begin{cases} \text{the least } \beta \ge \alpha \text{ such that } \alpha \text{ is not } \beta\text{-supercompact}, & \text{if such } \beta < \kappa \text{ exists,} \\ 0, & \text{otherwise,} \end{cases}$$

$$f_0(\alpha) = \text{the largest strong limit cardinal} \le g(\alpha),$$

$$f_{i+1}(\alpha) = 2^{f_i(\alpha)}, \quad \text{for } i \le n.$$

Note that $f_0(x \cap \kappa) = \mathrm{ot}(x \cap \delta)$ for all $x \in X_0$. So it holds that

$$f_n(x \cap \kappa) = \mathrm{ot}(x \cap \delta_n) \quad \text{and} \quad \mathrm{ot}(x) < f_{n+1}(x \cap \kappa) \quad \text{for all } x \in X_0.$$

For each $\alpha < \kappa$, take an injection $\Gamma_\alpha : f_{n+1}(\alpha) \to \mathscr{P}(f_n(\alpha))$. For each $x \in X_0$, define π_x and a_x by

$$\pi_x : \mathrm{ot}(x \cap \delta_n) \to x \cap \delta_n \text{ is the order isomorphism,}$$

$$a_x = \pi_x``\Gamma_{\mathrm{ot}(x \cap \kappa)}(\mathrm{ot}(x)).$$

Since $a_x \subset x \cap \delta_n$ for all $x \in X_0$, there exists an $A \subset \delta_n$ such that

$$X = \{x \in X_0 \mid a_x = A \cap x\} \in U.$$

We claim that X satisfies (2). To get a contradiction, assume that there exists $(x, y) \in [X]^2$ such that $x \prec y$ does not hold. By Claim 1, it holds that $x \cap \delta_n = y \cap \delta_n$. So we have that $\pi_x = \pi_y$. Set $\alpha = x \cap \kappa$ $(= y \cap \kappa)$, $\xi = \mathrm{ot}(x)$, and $\eta = \mathrm{ot}(y)$. Since $\xi \neq \eta$, we have that $\Gamma_\alpha(\xi) \neq \Gamma_\alpha(\eta)$. So $a_x = \pi_x``\Gamma_\alpha(\xi) = \pi_y``\Gamma_\alpha(\xi) \neq \pi_y``\Gamma_\alpha(\eta) = a_y$. This contradicts the fact $a_x = A \cap x = A \cap y = a_y$.

(Lemma 3.4 and Theorem 3.1) □

Define the Mitchell ordering $\lhd$ on the set of normal ultrafilters on $\mathscr{P}_\kappa\lambda$ by:

$$F \lhd U \quad \text{if and only if } F \in M_U.$$

Similar to normal ultrafilters on measurable cardinals (see Mitchell [**14**]), $\lhd$ is well-founded ordering and it can be defined

$$\mathrm{o}(U) = \sup\{\mathrm{o}(F) + 1 \mid F \lhd U\}, \quad \text{for all normal ultrafilters } U \text{ on } \mathscr{P}_\kappa\lambda.$$

Using this, Theorem 3.1 can be restated as:

$$\text{If } \mathrm{o}(U) = 0, \quad \text{then } U \text{ has the partition property.}$$

So the following question is natural.

QUESTION. Can we find γ such that $\sup\{o(U) \mid U$ has the partition property$\} \leq \gamma < \min\{o(U) \mid U$ does not have the partition property$\}$?

4. Several lemmas.

In this section we will state some lemmas which will be used in the next section.

4.1. The λ-Shelah property.

The λ-Shelah property was introduced by Carr [**2**]. A subset $X \subset \mathscr{P}_\kappa\lambda$ has the *Shelah property* if for any $\{f_x : x \to x \mid x \in X\}$ there exists a function $f : \lambda \to \lambda$ such that

$$\forall x \in \mathscr{P}_\kappa\lambda \ \exists y \in X \quad (x \subset y \text{ and } f_y \restriction x = f \restriction x).$$

Set $\mathrm{NSh}_{\kappa,\lambda} = \{X \subset \mathscr{P}_\kappa\lambda \mid X$ does not have the Shelah property$\}$. It is known that $\mathrm{NSh}_{\kappa,\lambda}$ is a normal ideal on $\mathscr{P}_\kappa\lambda$ and $\mathrm{NSh}_{\kappa,\lambda} \subset \mathrm{NAIn}_{\kappa,\lambda}$. We say that κ is λ-*Shelah* if $\mathrm{NSh}_{\kappa,\lambda}$ is a proper ideal.

The following two lemmas are due to Carr [**3**], [**4**].

Lemma 4.1 (Carr [**4**]). $\{x \in \mathscr{P}_\kappa\lambda \mid x \cap \kappa$ *is an inaccessible cardinal*$\} \in \mathrm{NSh}^*_{\kappa,\lambda}$.

Lemma 4.2 (Carr [**3**]). *If κ is $2^{\lambda^{<\kappa}}$-Shelah, then κ is λ-supercompact.*

Furthermore we need

Lemma 4.3 (Johnson [**5**]). *Let $\delta \leq \lambda$ and F be an ω-Jonsson function for δ. Then,*

$$\{x \in \mathscr{P}_\kappa\lambda \mid F \restriction {}^{\omega}(x \cap \delta) \text{ is an } \omega\text{-Jonsson function for } x \cap \delta\} \in \mathrm{NSh}^*_{\kappa,\lambda}.$$

The following lemma is due to Abe [**1**].

Lemma 4.4 (Abe [**1**, Corollary 3.4]). *Let γ, δ be cardinals such that $2^\gamma = \delta \leq \lambda$. Then,*

$$\{x \in \mathscr{P}_\kappa\lambda \mid 2^{|x \cap \gamma|} = |x \cap \delta|\} \in \mathrm{NSh}^*_{\kappa,\lambda}.$$

A similar argments give proofs of the following lemmas.

Lemma 4.5. *If δ is a cardinal $\leq \lambda$, then*

$$\{x \in \mathscr{P}_\kappa\lambda \mid \mathrm{ot}(x \cap \delta) \text{ is a cardinal}\} \in \mathrm{NIn}^*_{\kappa,\lambda}.$$

Lemma 4.6. *If γ is a strong limit cardinal $\leq \lambda$, then*

$$\{x \in \mathscr{P}_\kappa\lambda \,\|\, x \cap \gamma| \text{ is a strong limit cardinal}\} \in \mathrm{NIn}^*_{\kappa,\lambda}.$$

Lemma 4.7. *If $\gamma \leq \lambda \leq 2^\gamma$, then* $\{x \in \mathscr{P}_\kappa\lambda \,\|\, x| \leq 2^{|x \cap \gamma|}\} \in \mathrm{NIn}^*_{\kappa,\lambda}$.

4.2. The correspondence between $\mathscr{P}_\kappa\lambda$ and $\mathscr{P}_\kappa\lambda^{<\kappa}$.

Let $\theta = \lambda^{<\kappa}$ and $p : \mathscr{P}_\kappa\theta \to \mathscr{P}_\kappa\lambda$ be the projection. Take a bijection $h : \mathscr{P}_\kappa\lambda \to \theta$ and define $q : \mathscr{P}_\kappa\theta \to \mathscr{P}_\kappa\lambda$ by

$$q(y) = \bigcup h^{-1}y \quad \text{for } y \in \mathscr{P}_\kappa\theta,$$

where $h^{-1}y$ denotes the set $\{x \in \mathscr{P}_\kappa\lambda \mid h(x) \in y\}$. Set

$$Y_0 = \{y \in \mathscr{P}_\kappa\theta \mid p(y) = q(y) \text{ and } h``Q_{p(y)} = y\}.$$

The next lemma is due to Abe [**1**, Proposition 1.2].

Lemma 4.8 (Abe [**1**, Proposition 1.2]). $Y_0 \in \mathrm{WNS}^*_{\kappa,\theta}$, *where* $\mathrm{WNS}_{\kappa,\theta}$ *denotes the smallest strongly normal ideal on* $\mathscr{P}_\kappa\theta$.

The next lemma appeared in [**6**].

Lemma 4.9. $\{x \in \mathscr{P}_\kappa\lambda \mid x \cap \kappa$ *is almost x-ineffable*$\} \in p_*(\mathrm{NIn}_{\kappa,\theta})^*$.

The next lemma is essentially due to Carr [**4**, Proposition 4.1 (1)].

Lemma 4.10. *Let* $X \in p_*(\mathrm{NIn}_{\kappa,\theta})^+$. *Then, for any* $\{a_x \subset Q_x \mid x \in X\}$ *there exists an* $A \subset \mathscr{P}_\kappa\lambda$ *such that*

$$\forall \tau : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda \ \exists x \in X \cap \mathrm{cl}(\tau) \quad (a_x = A \cap Q_x).$$

Proof. Set $Y = p^{-1}X \cap Y_0$. By Lemma 4.8, $Y \in \mathrm{NIn}^+_{\kappa,\theta}$. For each $y \in Y$, set $b_y = h``a_{p(y)}$. Since $Y \subset Y_0$, $b_y \subset y$ for all $y \in Y$. So there exists a $B \subset \theta$ such that

$$Y' = \{y \in Y \mid b_y = B \cap y\} \in \mathrm{NS}^+_{\kappa,\theta}.$$

Set $A = h^{-1}B$. We claim that A is as required. To show this, let $\tau : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$. Let $\tau' = h \circ \tau \circ h^{-1} : \theta \to \theta$. Since $Y' \in \mathrm{NS}^+_{\kappa,\theta}$, there exists a $y \in Y' \cap \mathrm{cl}(\tau')$. It is easy to check that $p(y) \in X \cap \mathrm{cl}(\tau)$ and $a_{p(y)} = A \cap Q_{p(y)}$. □

Lemma 4.11. *Suppose that* $\mathrm{part}^*(\kappa, \lambda)$ *fails. Then,*

$$\{x \in \mathscr{P}_\kappa\lambda \mid \mathrm{part}^*(x \cap \kappa, x) \ \textit{fails}\} \in p_*(\mathrm{NIn}_{\kappa,\theta})^*.$$

Proof. To get a contradiction, assume that

$$X = \{x \in \mathscr{P}_\kappa\lambda \mid \mathrm{part}^*(x \cap \kappa, x) \ \text{holds}\} \in p_*(\mathrm{NIn}_{\kappa,\theta})^+.$$

Let $X' = \{x \in X \mid x \cap \kappa$ is inaccessible$\}$. By Lemma 4.1, $X' \in p_*(\mathrm{NIn}_{\kappa,\theta})^+$. Since $\mathrm{part}^*(\kappa, \lambda)$ fails, there exists a function $f : [\mathscr{P}_\kappa\lambda]^2 \to 2$ such that

$$\forall H \in \mathrm{NS}^+_{\kappa,\lambda} \quad (H \text{ is not homogeneous for } f).$$

For each $x \in X'$, take $H_x \in \mathrm{NS}^+_{\kappa\cap x, x}$ and $e_x < 2$ such that $f``[H_x]^2 = \{e_x\}$. By Lemma 4.10, there exists an $H \subset \mathscr{P}_\kappa\lambda$ and $e < 2$ such that

$$(\star) \qquad \forall \tau : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda \ \exists x \in X' \cap \mathrm{cl}(\tau) \quad (H_x = H \cap Q_x \text{ and } e_x = e).$$

It is easy to check that H is homogeneous for f. We have to show that H is stationary. Let C be a club of $\mathscr{P}_\kappa\lambda$. Take a function $\tau : \mathscr{P}_\kappa\lambda \to C$ such that $x \subset \tau(x)$ for each $x \in \mathscr{P}_\kappa\lambda$. By $(\star)$, there exists an $x \in X' \cap \mathrm{cl}(\tau)$ such that $H_x = H \cap Q_x$. Since $x \in \mathrm{cl}(\tau)$, it holds that $C \cap Q_x$ is a club in Q_x. So it holds that $\emptyset \neq H_x \cap C \cap Q_x \subset H \cap C$. □

5. Proof of Theorem 5.1.

In this section, we prove:

Theorem 5.1. *If* κ *is* λ*-ineffable and* $\mathrm{cof}(\lambda) \geq \kappa$ *then* $\mathrm{part}^*(\kappa, \lambda)$ *holds.*

Proof. To get a contradiction, assume that κ is λ-ineffable, $\mathrm{cof}(\lambda) \geq \kappa$, and $\mathrm{part}^*(\kappa,\lambda)$ fails. Since κ is λ-Shelah, by a result of Johnson [**5**], it holds that $\lambda^{<\kappa} = \lambda$. Let δ be the largest strong limit cardinal $\leq \lambda$. Define δ_i (for $i < \omega$) by $\delta_0 = \delta$ and $\delta_{i+1} = 2^{\delta_i}$. Let $n < \omega$ be such that $\delta_n \leq \lambda < \delta_{n+1}$. Take ω-Jonsson functions F for λ and F_i for δ_i for each $i \leq n$. Let X be the set of all $x \in \mathscr{P}_\kappa\lambda$ which satisfy:

(1) $x \cap \kappa$ is an inaccessible cardinal,
(2) $\mathrm{ot}(x \cap \delta)$ is a strong limit cardinal,
(3) $\mathrm{ot}(x \cap \delta_{i+1}) = 2^{\mathrm{ot}(x\cap\delta_i)}$ for all $i < n$ and $\mathrm{ot}(x) \leq 2^{\mathrm{ot}(x\cap\delta_n)}$,
(4) $F \restriction {}^\omega x$ and $F_i \restriction {}^\omega(x \cap \delta_i)$ are ω-Jonsson functions for x and $x \cap \delta_i$, for $i \leq n$, respectively,
(5) $x \cap \kappa$ is almost x-ineffable,
(6) $\mathrm{part}^*(x \cap \kappa, x)$ fails.

By Lemmas 4.1, 4.6, 4.4, 4.7, 4.3, 4.9 and 4.11, it holds that $X \in p_*(\mathrm{NIn}_{\kappa,\lambda\leq\kappa})^* = \mathrm{NIn}^*_{\kappa,\lambda}$. Since κ is λ-ineffable, we have that $X \in \mathrm{NIn}^+_{\kappa,\lambda}$. By Lemma 4.2 and (5) above, every $x \in X$ satisfies:

(7) $x \cap \kappa$ is $x \cap \alpha$-supercompact for all $\alpha \in x \cap [\kappa, \delta)$.

The next claim is crucial.

Claim 2. *If $(x, y) \in [X]^2$ and $x \cap \delta_n \neq y \cap \delta_n$, then $x \prec y$.*

Proof of Claim 2. To get a contradiction, assume that there exists $(x, y) \in [X]^2$ such that

$$x \cap \delta_n \neq y \cap \delta_n \text{ and } x \prec y \text{ does not hold.}$$

By (3) it holds that $x \cap \delta \neq y \cap \delta$. Since $F \restriction {}^\omega(y \cap \delta)$ is ω-Jonsson, it holds that $|x \cap \delta| < |y \cap \delta|$. Since $|y \cap \kappa|$ and $|y \cap \delta|$ are strong limit cardinals, we have that

(8) $2^{|x|} < |y \cap \delta|$ and $y \cap \kappa \leq |x \cap \delta|$.

By (7), (8), and Lemma 3.7, $x \cap \kappa$ is x-supercompact. This contradicts that $\mathrm{part}^*(x \cap \kappa, x)$ fails. (Claim 2)□

We complete the proof by showing that $X \in \mathrm{NP}^+_{\kappa,\lambda}$. The proof is divided into two cases.

Case 1. $\lambda = \delta_n$.

By Claim 2 it holds that $\forall (x, y) \in [X]^2\ (x \prec y)$. So $X \in \mathrm{NP}^+_{\kappa,\lambda}$ follows from Carr's theorem [**4**, Theorem 4.2 (1)]. But for the reader's convenience we give a proof. Let $f : [X]^2 \to 2$. For $x \in X$ define $a_x \subset Q_x$ by

$$a_x = \{t \in Q_x \mid t \in X \text{ and } f(t, x) = 0\}.$$

Then, by Lemma 4.10 there exists an $A \subset \mathscr{P}_\kappa\lambda$ such that

$$\forall \tau : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda\ \exists x \in X \cap \mathrm{cl}(\tau) \quad (a_x = A \cap Q_x).$$

Set $X' = \{x \in X \mid a_x = A \cap Q_x\}$. Note that $X' \in \mathrm{NS}^+_{\kappa,\lambda}$. It is easy to check that

$$\forall(x,y) \in [X' \cap A]^2 \quad (f(x,y) = 0) \quad \text{and} \quad \forall(x,y) \in [X' \backslash A]^2 \quad (f(x,y) = 1).$$

So $X' \cap A$ or $X' \backslash A$ is as required.

Case 2. $\delta_n < \lambda$.

Define g, $f_i : \kappa \to \kappa$ (for $i \le n+1$) by

$$g(\alpha) = \begin{cases} \text{the smallest } \beta \ge \alpha \text{ such that } \alpha \text{ is not } \beta\text{-supercompact}, & \text{if such } \beta \text{ exists,} \\ 0, & \text{otherwise,} \end{cases}$$

$$f_0(\alpha) = \text{the largest strong limit cardinal} \le g(\alpha),$$

$$f_{i+1}(\alpha) = 2^{f_i(\alpha)}, \quad \text{for all } \alpha < \kappa \text{ and } i \le n.$$

For any $x \in X$, since $\mathrm{ot}(x \cap \delta) \le g(x \cap \kappa) \le \mathrm{ot}(x)$, it holds that

$$f_0(x \cap \kappa) = \mathrm{ot}(x \cap \delta) \quad \text{and} \quad f_n(x \cap \kappa) = \mathrm{ot}(x \cap \delta_n) \le \mathrm{ot}(x) \le f_{n+1}(x \cap \kappa).$$

For each $\alpha < \kappa$, take an injection $\Gamma_\alpha : f_{n+1}(\alpha) + 1 \to \mathscr{P}(f_n(\alpha))$. For each $x \in X$, define π_x and s_x by

$$\pi_x : \mathrm{ot}(x \cap \delta_n) \to x \cap \delta_n \text{ is the order isomorphism, and}$$

$$s_x = \pi_x``\Gamma_{x \cap \kappa}(\mathrm{ot}(x)) \ (\subset x \cap \delta_n).$$

To show that $X \in \mathrm{NP}^+_{\kappa,\lambda}$, let $f : [X]^2 \to 2$. For $x \in X$ set

$$a_x = \{t \in Q_x \mid t \in X \text{ and } f(t,x) = 0\}.$$

Since $X \in p_*(\mathrm{NIn}_{\kappa,\theta})^+$, there exist $S \subset \delta_n$ and $A \subset \mathscr{P}_\kappa\lambda$ such that

$$X' = \{x \in X \mid s_x = S \cap x \text{ and } a_x = A \cap Q_x\} \in \mathrm{NS}^+_{\kappa,\lambda}.$$

Claim 3. $\forall(x,y) \in [X']^2 \ (x \cap \delta_n \ne y \cap \delta_n)$.

Proof of Claim 3. To get a contradiction, assume that

$$(x,y) \in [X']^2 \quad \text{and} \quad x \cap \delta_n = y \cap \delta_n.$$

Note that $s_x = s_y$. Set $\alpha = x \cap \kappa$ $(= y \cap \kappa)$, $\xi = \mathrm{ot}(x)$, and $\eta = \mathrm{ot}(y)$. Since $|x| < |y|$, it holds that $\xi < \eta$. Since $x \cap \delta_n = y \cap \delta_n$, it holds that $\pi_x = \pi_y$. Since $\Gamma_\alpha(\xi) \ne \Gamma_\alpha(\eta)$, we have that

$$s_x = \pi_x``\Gamma_\alpha(\xi) \ne \pi_y``\Gamma_\alpha(\eta) = s_y.$$

This is a contradiction. (Claim 3)□

By Claims 2 and 3, it holds that

$$\forall(x,y)\in[X']^2 \quad (x \prec y).$$

So $X'\cap A$ or $X'\setminus A$ is a desired stationary homogeneous set for f. (Theorem 5.1)□

Corollary 5.2. *Let $\kappa\le\lambda<\mu$. If* part$^*(\kappa,\mu)$ *holds, then* part$^*(\kappa,\lambda)$ *holds.*

Proof. Let $\theta=\lambda^{<\kappa}$. Note that $\mathrm{cof}(\theta)\ge\kappa$. Using the corespondence between $\mathscr{P}_\kappa\lambda$ and $\mathscr{P}_\kappa\theta$, it is not difficult to check that part$^*(\kappa,\theta)$ implies part$^*(\kappa,\lambda)$. So, it suffices to show that part$^*(\kappa,\theta)$ holds. By Magidor [**11**] it holds that κ is μ-ineffable. Then, by Johnson [**5**] it holds that $\theta\le\lambda^+\le\mu$. So κ is θ-ineffable. Hence part$^*(\kappa,\theta)$ holds. □

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