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Factorization in analytic crossed products

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Abstract. Let M be a von Neumann algebra, let α be a $*$-automorphism of M, and let $M \rtimes_\alpha \boldsymbol{Z}$ be the crossed product determined by M and α. In this paper, considering the Cholesky decomposition for a positive operator in $M \rtimes_\alpha \boldsymbol{Z}$, we give a factorization theorem for positive operators in $M \rtimes_\alpha \boldsymbol{Z}$ with respect to analytic crossed product $M \rtimes_\alpha \boldsymbol{Z}_+$ determined by M and α. And we give a necessary and sufficient condition that every positive operator in $M \rtimes_\alpha \boldsymbol{Z}$ can be factored by the form A^*A, where A belongs to $M \rtimes_\alpha \boldsymbol{Z}_+ \cap (M \rtimes_\alpha \boldsymbol{Z}_+)^{-1}$.

1. Introduction.

Let $B(\mathscr{H})$ be a set of all bounded linear operators on a Hilbert space $\mathscr{H}$. The problem of factorization of operators with respect to a subalgebra $\mathfrak{A}$ of $B(\mathscr{H})$ consists in writing a positive operator C in the form A^*A with A in $\mathfrak{A}$. If $\mathfrak{A} = B(\mathscr{H})$, then this problem is trivial, however if $\mathfrak{A} \subsetneq B(\mathscr{H})$, then it becomes complicated. Arveson ([**2**]) has introduced the notion of the outer operator in analogy with the outer functions in Hardy spaces. He showed that each positive invertible operator in $B(\mathscr{H})$ can be factored by the form A^*A, where A belongs to $\mathfrak{A}$ and the inverse A^{-1} is also in $\mathfrak{A}$. The factorization of a positive invertible finite matrix C as A^*A with A and its inverse in upper triangular form is known as the Cholesky decomposition. Power ([**9**], [**10**], [**11**]) has found a constructive Hilbert space version of the Cholesky decomposition to be of fundamental significance in the analysis of analytic operator algebras and the factorization property. He proved that every positive operator C has a factorization $C = A^*A$ with A outer in a nest algebra if and only if the nest is well-ordered. Factorization problems for other types of nest algebras are also studied by many authors (cf. [**1**], [**5**]–[**8**], etc). McAsey, Muhly and the second author in [**12**]–[**14**] studied such a factorization problem with respect to an analytic crossed product. Let M be a von Neumann algebra, let α be a $*$-automorphism of M, and let $M \rtimes_\alpha \boldsymbol{Z}$ be the crossed product determined by M and α. They showed that every positive

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invertible operator in $M \rtimes_{\alpha} \boldsymbol{Z}$ can be factored by the form A^*A, where A belongs to the analytic crossed product $M \rtimes_{\alpha} \boldsymbol{Z}_+$ determined by M and α and the inverse A^{-1} also is in $M \rtimes_{\alpha} \boldsymbol{Z}_+$.

In this paper, we consider the Cholesky decomposition for a positive operator in $M \rtimes_{\alpha} \boldsymbol{Z}$, and we investigate the factorization problem with respect to an analytic crossed product $M \rtimes_{\alpha} \boldsymbol{Z}_+$.

2. Preliminaries and definitions.

Let M be a von Neumann algebra and let α be a $*$-automorphism of M. We regard M as acting on the non-commutative L^2-space $L^2(M)$ in the sense of Haagerup (cf. [**4**]). For $x \in M$, let ℓ_x (resp. r_x) be the operator on $L^2(M)$ defined by the formula $\ell_x y = xy$ (resp. $r_x y = yx$), $y \in L^2(M)$. Then ℓ (resp. r) is a faithful normal representation (resp. anti-representation) of M on the Hilbert space $L^2(M)$. Put

$$\ell(M) = \{\ell_x \mid x \in M\} \quad \text{and} \quad r(M) = \{r_x \mid x \in M\},$$

respectively. If J is defined on $L^2(M)$ by the formula $Jy = y^*$, $y \in L^2(M)$, then J is a conjugate linear isometric involution on $L^2(M)$. Let $L^2(M)_+$ be the cone of all positive operators in $L^2(M)$. Since the quadruple $\{\ell(M), L^2(M), J, L^2(M)_+\}$ is a standard form of M in the sense of Haagerup ([**3**]), the von Neumann algebra $\ell(M)$ and $r(M)$ are commutants of one another, and $J\ell(M)J = r(M)$. Moreover, by [**3**, Theorem 3.2], there exists a unitary operator u on $L^2(M)$ such that $\ell_{\alpha(x)} = u\ell_x u^*$ and $r_{\alpha(x)} = ur_x u^*$, $x \in M$. To construct a crossed product, we consider the Hilbert space $\boldsymbol{L}^2$ defined by

$$\boldsymbol{L}^2 = \left\{ f : \boldsymbol{Z} \to L^2(M) \,\middle|\, \sum_{n \in \boldsymbol{Z}} \|f(n)\|_2^2 < \infty \right\},$$

where $\|\cdot\|_2$ is the norm of $L^2(M)$. For each $x \in M$, we define operators L_x, R_x, L_δ and R_δ on $\boldsymbol{L}^2$ by the formulae

$$(L_x f)(n) = \ell_x f(n), \quad (R_x f)(n) = r_{\alpha^n(x)} f(n)$$
$$(L_\delta f)(n) = uf(n-1) \quad \text{and} \quad (R_\delta f)(n) = f(n-1),$$

where $f \in \boldsymbol{L}^2$ and $n \in \boldsymbol{Z}$. Put $L(M) = \{L_x \mid x \in M\}$ and $R(M) = \{R_x \mid x \in M\}$. We set $\mathfrak{L} = \{L(M), L_\delta\}''$ and $\mathfrak{R} = \{R(M), R_\delta\}''$, and define the left (resp. right) analytic crossed product $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) to be the σ-weakly closed subalgebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) generated by $L(M)$ (resp. $R(M)$) and L_δ (resp. R_δ). Furthermore, we define

$$\boldsymbol{H}^2 = \{f \in \boldsymbol{L}^2 \mid f(n) = 0, n < 0\},$$

and let P be the projection from $\boldsymbol{L}^2$ onto $\boldsymbol{H}^2$. We refer the reader to [**12**]–[**16**] for discussions of these algebras including some of their elementary properties.

3. Factorizations.

We start with some general constructions for positive operator matrices. Although the following lemma is well-known, it contains an important idea of our approach. So we shall give full details of proof.

LEMMA 3.1. *Let $\mathscr{H}$ be a Hilbert space with orthogonal decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and C be a positive operator on $\mathscr{H}$ with the matrix form*

$$C = \begin{pmatrix} c & b^* \\ b & a \end{pmatrix} \begin{matrix} \mathscr{H}_1 \\ \oplus \\ \mathscr{H}_2 \end{matrix}$$

with respect to $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Then there exists the operator $c_1 = \lim_{n\to\infty} b^* \cdot (a + n^{-1}I_1)^{-1} b$ in the strong operator topology and

$$C_1 = \begin{pmatrix} c_1 & b^* \\ b & a \end{pmatrix} \leq C.$$

In particular, C_1 is minimal amongst those positive operators that agree with C on the subspace $\mathscr{H}_2$.

PROOF. If a is invertible, then the operator matrix

$$A = \begin{pmatrix} I_1 & 0 \\ -a^{-1}b & I_2 \end{pmatrix}$$

is also invertible and

$$A^*CA = \begin{pmatrix} c - b^*a^{-1}b & 0 \\ 0 & a \end{pmatrix}.$$

Hence C is a positive operator if and only if $c - b^*a^{-1}b \geq 0$.

In general, for each $n \in \boldsymbol{N}$, applying the preceding operation to the positive operator

$$C + n^{-1}I = \begin{pmatrix} c + n^{-1}I_1 & b \\ b^* & a + n^{-1}I_2 \end{pmatrix},$$

we have $b^*(a + n^{-1}I_2)^{-1}b \leq c + n^{-1}I_1$. Since $\{b^*(a + n^{-1}I_2)^{-1}b\}$ is a bounded increasing sequence of positive operators, it converges in the strong operator topology to an operator $c_1 \leq c$. Putting

$$C_1 = \begin{pmatrix} c_1 & b^* \\ b & a \end{pmatrix} \begin{matrix} \mathscr{H}_1 \\ \oplus \\ \mathscr{H}_2 \end{matrix}$$

the positive operator C_1 satisfies the required minimality condition. □

The minimality of the positive operator C_1 in Lemma 3.1 is important for our discussion. So we give the following:

Definition 3.2. Let C be a positive operator in $B(\mathscr{H})$ and let $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Then a positive operator C_1 is said to be the $\mathscr{H}_2$-minimal part of C if $C_1|_{\mathscr{H}_2} = C|_{\mathscr{H}_2}$ and

$$P_{\mathscr{H}_1} C_1 P_{\mathscr{H}_1} = \operatorname*{s\text{-}lim}_{t \to 0} P_{\mathscr{H}_1} C (tP_{\mathscr{H}_2} + P_{\mathscr{H}_2} C P_{\mathscr{H}_2})^{-1} C P_{\mathscr{H}_1},$$

where $P_{\mathscr{H}_i}$ is the projection from $\mathscr{H}$ onto $\mathscr{H}_i$ $(i = 1, 2)$. Moreover if $C = C_1$, then we say that C is $\mathscr{H}_2$-minimal.

Let C be a positive operator such that

$$C = \begin{pmatrix} c & b^* \\ b & a \end{pmatrix} \begin{matrix} \mathscr{H}_1 \\ \oplus \\ \mathscr{H}_2 \end{matrix}$$

with respect to $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Let e_t denote the spectral projection for the operator a corresponding to the interval (t, ∞). Then, for each $t > 0$, we have

$$\begin{aligned} \|b^* a^{-1/2} e_t\|^2 &= \lim_{n \to \infty} \|b^*(a + n^{-1})^{-1/2} e_t (a + n^{-1})^{-1/2} b\| \\ &\leq \lim_{n \to \infty} \|b^*(a + n^{-1})^{-1}\| \\ &\leq \|c_1\|. \end{aligned} \tag{3.1}$$

Therefore $b^* a^{-1/2} e_t$ converges to an operator d in the strong topology as $t \to 0$. For each $x \in \mathscr{H}_1$, we have

$$\begin{aligned} d a^{-1/2} x &= \lim_{t \to 0} b^* a^{-1/2} e_t a^{-1/2} x \\ &= b^* e_{0+} x. \end{aligned}$$

Furthermore, the inequality

$$0 \leq b^*(a + n^{-1} I_1)^{-1} b \leq c + n^{-1} I_2$$

implies that

$$0 \leq b^*(na + I_1)^{-1} b \leq n^{-1} c + n^{-2} I_2.$$

Taking the strong limit as $n \to \infty$, we see that

$$b^*(I_1 - e_{0+})b = 0,$$

and so we have

$$\begin{aligned} da^{1/2} &= b^* e_{0+} = b^* e_{0+} + b^*(I_1 - e_{0+}) \\ &= b^*. \end{aligned}$$

Since $a^{1/2}d^* = b$, the map $a^{-1/2} : b\mathscr{H}_2 \to d^*\mathscr{H}_2$ is well-defined such that $d^* = a^{-1/2}b$. From the inequality (3.1), we have that $c_1 \geq (b^* a^{-1/2} e_t) a^{-1/2} b$. Taking the strong limit as $t \to 0$, we see that $dd^* \leq c_1$.

On the other hand, since

$$\begin{pmatrix} dd^* & b^* \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ d^* & a^{1/2} \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ d^* & a^{1/2} \end{pmatrix} \geq 0,$$

by the minimality of C_1, we have $dd^* \geq c_1$. Thus we obtain that $c_1 = dd^*$, and so C_1 has the following matrix representation:

$$C_1 = \begin{pmatrix} dd^* & b^* \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ d^* & a^{1/2} \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ d^* & a^{1/2} \end{pmatrix}.$$

From the present argument, we see that for each positive operator C on the Hilbert space $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ is always factored by the form

$$C = \begin{pmatrix} 0 & 0 \\ d^* & a^{1/2} \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ d^* & a^{1/2} \end{pmatrix} + \begin{pmatrix} c - dd^* & 0 \\ 0 & 0 \end{pmatrix},$$

and we note that the matrix $\begin{pmatrix} 0 & 0 \\ d^* & a^{1/2} \end{pmatrix}$ has the lower triangular form and the matrix $\begin{pmatrix} c - dd^* & 0 \\ 0 & 0 \end{pmatrix}$ is positive.

Suppose now that a Hilbert space $\mathscr{H}$ have the decomposition

$$\mathscr{H} = \sum_{n=-\infty}^{\infty} \oplus \mathscr{H}_n$$

and let

$$\mathscr{M}_n = \sum_{k=-\infty}^{n} \oplus \mathscr{H}_k \quad \text{and} \quad \mathscr{N}_n = \sum_{k=n}^{\infty} \oplus \mathscr{H}_k.$$

For each positive operator C on $\mathscr{H}$, let $C^{(n+1)}$ be the $\mathscr{N}_{n+1}$-minimal part of C. Since $C - C^{(n+1)}$ is a positive operator, we can also construct the $\mathscr{N}_n$-minimal part of $C - C^{(n+1)}$ denoted by C_n. Repeating this way, we have the operator

C_k $(k < n)$ as the $\mathcal{N}_k$-minimal part of $C - (C_{k+1} + \cdots + C_n + C^{(n+1)})$. Putting $R^{(k-1)} = C - (C_k + C_{k+1} + \cdots + C_n + C^{(n+1)})$, we obtain the decomposition

$$C = R^{(k-1)} + C_k + C_{k+1} + \cdots + C_n + C^{(n+1)}$$

which we call the *Cholesky decomposition* of C with respect to $\mathcal{H} = \mathcal{M}_{k-1} \oplus \mathcal{H}_k \oplus \cdots \oplus \mathcal{H}_n \oplus \mathcal{N}_{n+1}$. The following lemma appears in [**11**].

Lemma 3.3. *Keep the notation as above. Then, for each* $k, n \in \boldsymbol{Z}$ $(k < n)$, *the operator* $C_k + C_{k+1} + \cdots + C_n + C^{(n+1)}$ *is the* $\mathcal{N}_k$*-minimal part of* C.

Now we return to the context and notation of analytic crossed products. Applying the Cholesky decomposition for positive operators in $\mathfrak{L}$, we have the following:

Theorem 3.4. *For each positive operator* C *in* $\mathfrak{L}$, *there exists a positive operator* C_∞ *in* $\mathfrak{L}$ *and an operator* A *in* $\mathfrak{A}$ *such that* $C = A^*A + C_\infty$.

Proof. Putting $\mathcal{H}_n = L^2(M)$ $(\forall n \in \boldsymbol{Z})$, we may write

$$\boldsymbol{L}^2 = \sum_{n=-\infty}^{\infty} \oplus \mathcal{H}_n.$$

Thus, considering the decomposition

$$\boldsymbol{L}^2 = \mathcal{M}_{-(n+1)} \oplus \mathcal{H}_{-n} \oplus \cdots \oplus \mathcal{H}_n \oplus \mathcal{N}_{n+1},$$

we have the Cholesky decomposition of C as follows:

$$C = R^{-(n+1)} + C_{-n} + \cdots + C_n + C^{(n+1)}.$$

It is clear that $R^{-(n+1)}$ converges to zero in the strong topology as $n \to \infty$. Since $C^{(n)} \geq C^{(n+1)}$ and $C^{(n)}$ is bounded, there exists the limit C_∞ of $\{C^{(n)}\}$ in the strong topology as $n \to \infty$ such that $C = \sum_{k=-\infty}^{\infty} C_k + C_\infty$. By Lemma 3.3, the operator $\sum_{k=n}^{\infty} C_k + C_\infty$ is the $\mathcal{N}_n$-minimal part of C. For each $n \in \boldsymbol{Z}$, there exists an operator

$$A_n = \begin{pmatrix} 0 & 0 & 0 \\ d_n^* & a_n^{1/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{M}_{n-1} \\ \oplus \\ \mathcal{H}_n \\ \oplus \\ \mathcal{N}_{n+1} \end{matrix}$$

with respect to $\boldsymbol{L}^2 = \mathcal{M}_{n-1} \oplus \mathcal{H}_n \oplus \mathcal{N}_{n+1}$ such that $C_n = A_n^* A_n$. Since

$$\left(\sum_{k=-n}^{n} A_k\right)^* \left(\sum_{k=-n}^{n} A_k\right) = \sum_{k=-n}^{n} A_k^* A_k$$

$$= \sum_{k=-n}^{n} C_k,$$

we have $\|\sum_{k=-n}^{n} A_k\|^2 \le \|C\| < \infty$, it follows that $\{\sum_{k=-n}^{n} A_k\}$ converges to an operator A in the weak operator topology as $n \to \infty$ such that $A^*A = \sum_{n=-\infty}^{\infty} C_n$. We note that the operator A has the lower triangular form with respect to the decomposition

$$\boldsymbol{L}^2 = \cdots \oplus \mathscr{H}_{-n} \oplus \cdots \oplus \mathscr{H}_n \oplus \cdots.$$

We next show that C_∞ belongs to $\mathfrak{L}$. For each $f \in \mathscr{N}_n$, $R_\delta f \in \mathscr{N}_{n+1}$, it follows that

$$\begin{aligned} R_\delta^* C^{(n+1)} R_\delta f &= R_\delta^* C R_\delta f \\ &= R_\delta^* R_\delta^* C f \\ &= Cf. \end{aligned}$$

By the minimality of $C^{(n)}$, we have $R_\delta^* C^{(n+1)} R_\delta \ge C^{(n)}$. Similarly, for each $f \in \mathscr{N}_{n+1}$, we have $R_\delta C^{(n)} R_\delta^* \ge C^{(n+1)}$, this implies $R_\delta C^{(n+1)} R_\delta^* = C^{(n)}$. Thus we see that $R_\delta C_\infty R_\delta^* = C_\infty$. Moreover, for each $n \in \boldsymbol{Z}$ and each unitary operator w in M, we have

$$\begin{aligned} R_w^* C^{(n)} R_w f &= R_w^* C R_w f \\ &= R_w^* R_w C f \\ &= Cf \quad (\forall f \in \mathscr{N}_n). \end{aligned}$$

Therefore we have $R_w^* C^{(n)} R_w \ge C^{(n)}$. Replacing w with w^*, we also see that $R_w C^{(n)} R_w^* \ge C^{(n)}$, so that $R_w^* C_\infty R_w = C_\infty$. Hence C_∞ commutes with all generators of $\mathfrak{R}$, and so C_∞ belongs to $\mathfrak{L}$ which is the commutant of $\mathfrak{R}$.

Next we claim that A belongs to $\mathfrak{L}_+$. Indeed, since $C_n = C^{(n)} - C^{(n+1)}$ and $R_\delta C^{(n+1)} R_\delta^* = C^{(n)}$, we have

$$\begin{aligned} R_\delta C_n R_\delta^* &= R_\delta (C^{(n)} - C^{(n+1)}) R_\delta^* \\ &= R_\delta C^{(n)} R_\delta^* - R_\delta C^{(n+1)} R_\delta^* \\ &= C^{(n-1)} - C^{(n)} \\ &= C_{n-1}. \end{aligned}$$

Now we consider the matrix forms of C_{n-1} and C_n as follows:

$$C_{n-1} = \left(\begin{array}{ccc|c} d_{n-1}d_{n-1}^* & b_{n-1}^* & 0 & 0 \\ b_{n-1} & a_{n-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} \mathscr{M}_{n-2} \\ \oplus \\ \mathscr{H}_{n-1} \\ \oplus \\ \mathscr{H}_n \\ \oplus \\ \mathscr{N}_{n+1} \end{array}, \quad C_n = \left(\begin{array}{cc|c} d_n d_n^* & b_n^* & 0 \\ b_n & a_n & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \begin{array}{c} \mathscr{M}_{n-1} \\ \oplus \\ \mathscr{H}_n \\ \oplus \\ \mathscr{N}_{n+1} \end{array}.$$

From the relation $R_\delta C_n R_\delta^* = C_{n-1}$ it follows that $a_n = a_{n-1}$, $b_n = b_{n-1}$ so that

$$\begin{aligned} d_{n-1} &= a_{n-1}^{-1/2} b_{n-1} \\ &= a_n^{-1/2} b_n \\ &= d_n. \end{aligned}$$

Thus we have $R_\delta A_{n-1} R_\delta^* = A_n$. Taking a strong limit as $n \to \infty$, we have that $R_\delta A R_\delta^* = A$. Furthermore, for every unitary operator $v \in M$, the operator R_v has the following matrix form:

$$R_v = \begin{pmatrix} \ddots & & & & 0 \\ & r_{\alpha^{-1}(v)} & & & \\ & & r_v & & \\ & & & r_{\alpha(v)} & \\ 0 & & & & \ddots \end{pmatrix} \begin{matrix} \vdots \\ \mathscr{H}_{-1} \\ \oplus \\ \mathscr{H}_0 \\ \oplus \\ \mathscr{N}_1 \\ \vdots \end{matrix}.$$

Since $R_v C_n R_v^* = C_n$ and considering the matrix form of C_n with respect to $\mathscr{M}_{n-1} \oplus \mathscr{H}_n \oplus \mathscr{N}_{n+1}$ as above, we see that

$$r_{\alpha^n}(v) a_n = a_n r_{\alpha^n}(v) \quad \text{and} \quad r_{\alpha^n}(v) b_n \begin{pmatrix} \ddots & & 0 \\ & r_{\alpha^{n+2}}(v^*) & \\ 0 & & r_{\alpha^{n+1}}(v^*) \end{pmatrix} = b_n.$$

Hence we see that

$$\begin{aligned} & r_{\alpha^n}(v) d_n \begin{pmatrix} \ddots & & 0 \\ & r_{\alpha^{n+2}}(v^*) & \\ 0 & & r_{\alpha^{n+1}}(v^*) \end{pmatrix} \\ &= r_{\alpha^n}(v) a_n^{-1/2} b_n \begin{pmatrix} \ddots & & 0 \\ & r_{\alpha^{n+2}}(v^*) & \\ 0 & & r_{\alpha^{n+1}}(v^*) \end{pmatrix} \\ &= a_n^{-1/2} r_{\alpha^n}(v) b_n \begin{pmatrix} \ddots & & 0 \\ & r_{\alpha^{n+2}}(v^*) & \\ 0 & & r_{\alpha^{n+1}}(v^*) \end{pmatrix} \\ &= a_n^{-1/2} b_n = d_n. \end{aligned}$$

Thus we have $R_v A_n R_v^* = A_n$ so that $R_v A R_v^* = A$. Hence A commutes with all generators of $\mathfrak{R}$, and so A belongs to $\mathfrak{L}$. Moreover, since A has the lower triangular form with respect to

$$\mathscr{H} = \cdots \oplus \mathscr{H}_{-n} \oplus \cdots \oplus \mathscr{H}_n \oplus \cdots,$$

we see that A belongs to $\mathfrak{L}_+$. □

As in [**11**, Lemma 5], we have the following:

PROPOSITION 3.5. *Let C be a positive operator in $\mathfrak{L}$ with the decomposition $C = A^*A + C_\infty$ as in* Theorem 3.5. *If there are B in $\mathfrak{L}_+$ and a positive operator D in $\mathfrak{L}$ such that $C = B^*B + D$, then $A^*A \geq B^*B$.*

PROOF. Since, by Lemma 3.3, $A^*P_{\boldsymbol{H}^2}A + C_\infty = \sum_{n=0}^\infty C_n + C_\infty$ is the $\boldsymbol{H}^2$-minimal part of C, we see that

$$A^*P_{\boldsymbol{H}^2}A + C_\infty \leq B^*P_{\boldsymbol{H}^2}B + D.$$

Thus, for all n, we have

$$\begin{aligned} R_\delta^{-n}A^*P_{\boldsymbol{H}^2}AR_\delta^n + C_\infty &= R_\delta^{-n}(A^*P_{\boldsymbol{H}^2}A + C_\infty)R_\delta^n \\ &\leq R_\delta^{-n}(B^*P_{\boldsymbol{H}^2}B + D)R_\delta^n \\ &\leq R_\delta^{-n}B^*P_{\boldsymbol{H}^2}BR_\delta^n + D. \end{aligned}$$

Taking the limit in the strong topology as $n \to \infty$, we see that $R_\delta^{-n}A^*P_{\boldsymbol{H}^2}AR_\delta^n$ and $R_\delta^{-n}B^*P_{\boldsymbol{H}^2}BR_\delta^n$ converge to 0 respectively. Hence we have $C_\infty \leq D$, and it follows that $A^*A \geq B^*B$. □

In Theorem 3.4, we have an interest in the condition for the factorization $C = A^*A$. As an analogue of Arveson's terminology of outer operator, we introduce the concept of the outer operator in analytic crossed products, and we consider the problem.

DEFINITION 3.6. An operator A in $\mathfrak{L}_+$ is called outer if the range projection E_A of A lies in $L(M)$, and $A\boldsymbol{H}^2$ is dense in $[A\boldsymbol{L}^2] \cap \boldsymbol{H}^2$ where $[A\boldsymbol{L}^2]$ denotes the closed subspace spanned by $A\boldsymbol{L}^2$.

We note that if A is an outer operator, then E_A belongs to $\mathfrak{L}_+$. Thus we see that $E_A\boldsymbol{H}^2 \subset \boldsymbol{H}^2$, it follows that E_A commutes with $P_{\boldsymbol{H}^2}$.

The following lemma which appeared in [**11**] essentially characterizes the outer operators.

LEMMA 3.7. *Let A be an operator in $\mathfrak{L}_+$ such that E_A belongs to $L(M)$. Then the following conditions are equivalent*:

(i) $[A\boldsymbol{H}^2] = [A\boldsymbol{L}^2] \cap \boldsymbol{H}^2$.
(ii) $E_{P_{H^2}A(I-P_{H^2})} \leq E_{AP_{H^2}}$.
(iii) $A^*P_{\boldsymbol{H}^2}A$ *is* $\boldsymbol{H}^2$*-minimal.*

Now we give a necessary and sufficient condition on a positive operator C for the existence of a factorization $C = A^*A$ with an outer operator A in $\mathfrak{L}_+$.

Theorem 3.8. *For each positive operator* C *in* $\mathfrak{L}$, *we put*

$$C' = \text{s-}\lim_{t\to 0}(P_{\boldsymbol{H}^2}^{\perp}CP_{\boldsymbol{H}^2})^*(tP_{\boldsymbol{H}^2} + P_{\boldsymbol{H}^2}CP_{\boldsymbol{H}^2})^{-1}(P_{\boldsymbol{H}^2}^{\perp}CP_{\boldsymbol{H}^2}).$$

Then C *admits a factorization* $C = A^*A$, *with an outer operator* A, *if and only if the operator* $R_\delta^{-n}C'R_\delta^n$ *converges in the strong operator topology to* 0 *as* $n \to \infty$.

Moreover if C *is invertible in* $\mathfrak{L}$, *then* C *satisfies this conditions.*

Proof. Suppose that $T = A^*A$ for some outer operator A in $\mathfrak{L}_+$. The equation $T = (I - P_{\boldsymbol{H}^2})A^*(I - P_{\boldsymbol{H}^2})A(I - P_{\boldsymbol{H}^2}) + A^*P_{\boldsymbol{H}^2}A$ implies that

$$C = \begin{pmatrix} (I - P_{\boldsymbol{H}^2})C(I - P_{\boldsymbol{H}^2}) & (I - P_{\boldsymbol{H}^2})(A^*P_{\boldsymbol{H}^2}A)P_{\boldsymbol{H}^2} \\ P_{\boldsymbol{H}^2}(A^*P_{\boldsymbol{H}^2}A)(I - P_{\boldsymbol{H}^2}) & P_{\boldsymbol{H}^2}(A^*P_{\boldsymbol{H}^2}A)P_{\boldsymbol{H}^2} \end{pmatrix}.$$

Since A is outer, by (iii) of Lemma 3.7, we see that

$$C' = (I - P_{\boldsymbol{H}^2})(A^*P_{\boldsymbol{H}^2}A)(I - P_{\boldsymbol{H}^2}).$$

Thus, for each $f \in \boldsymbol{L}^2$, we have

$$\begin{aligned}
\|R_\delta^{-n}C'R_\delta^n f\| &= \|R_\delta^{-n}P_{\boldsymbol{H}^2}^{\perp}A^*P_{\boldsymbol{H}^2}AP_{\boldsymbol{H}^2}^{\perp}R_\delta^n f\| \\
&= \|R_\delta^{-n}P_{\boldsymbol{H}^2}^{\perp}A^*P_{\boldsymbol{H}^2}R_\delta^n A R_\delta^{-n}P_{\boldsymbol{H}^2}^{\perp}R_\delta^n f\| \\
&\leq \|A\|\,\|P_{\boldsymbol{H}^2}R_\delta^n A R_\delta^{-n}P_{\boldsymbol{H}^2}^{\perp}R_\delta^n f\| \\
&\leq \|A\|(\|P_{\boldsymbol{H}^2}R_\delta^n A R_\delta^{-n}P_{\boldsymbol{H}^2}^{\perp}R_\delta^n f - P_{\boldsymbol{H}^2}R_\delta^n Af\| + \|P_{\boldsymbol{H}^2}R_\delta^n Af\|) \\
&\leq \|A\|^2\|P_{\boldsymbol{H}^2}^{\perp}R_\delta^n f - f\| + \|A\|\,\|R_\delta^{-n}P_{\boldsymbol{H}^2}R_\delta^n Af\| \\
&\to 0 \quad (n \to \infty).
\end{aligned}$$

Conversely, we assume that

$$\text{s-}\lim_{n\to\infty} R_\delta^{-n}C'R_\delta^n = 0. \tag{3.2}$$

The positive operator C can be factored in the form $A^*A + C_\infty$ as in Theorem 3.4. Let P_{-n} be the projection form $\boldsymbol{L}^2$ onto $R_\delta^{-n}\boldsymbol{H}^2$, and let $P_{-n}^{\perp} = I - P_{-n}$. Then we see that

$$R_\delta^{-n}P_{\boldsymbol{H}^2}R_\delta^n = P_{-n} \quad (\forall n \in \boldsymbol{N}).$$

Since $C = P_{-n}^{\perp}CP_{-n}^{\perp} + P_{-n}^{\perp}CP_{-n} + P_{-n}CP_{-n}^{\perp} + P_{-n}CP_{-n}$, we have

$$\text{s-}\lim_{t\to 0} P_{-n}CP_{-n}^{\perp}\{tP_{-n}^{\perp} + P_{-n}^{\perp}CP_{-n}^{\perp}\}^{-1}P_{-n}^{\perp}CP_{-n} = R_{\delta}^{-n}C'R_{\delta}^{n}.$$

Thus $C^{(n)}$ has the following matrix form:

$$C^{(n)} = \begin{pmatrix} P_{-n}^{\perp}CP_{-n}^{\perp} & P_{-n}^{\perp}CP_{-n} \\ P_{-n}CP_{-n}^{\perp} & R_{\delta}^{-n}C'R_{\delta}^{n} \end{pmatrix}.$$

Hence,

$$\begin{aligned} \|C^{(n)}f\|^2 &= \|P_{-n}^{\perp}CP_{-n}^{\perp}f + P_{-n}^{\perp}CP_{-n}f\|^2 + \|P_{-n}CP_{-n}^{\perp}f + R_{\delta}^{-n}C'R_{\delta}^{n}f\|^2 \\ &\leq \|P_{-n}^{\perp}Cf\|^2 + (\|C\|\,\|P_{-n}^{\perp}f\| + \|R_{\delta}^{-n}C'R_{\delta}^{n}f\|)^2. \end{aligned}$$

This follows that $\|C^{(n)}f\|^2 \to 0$ $(n \to \infty)$ by hypothesis (3.2). This implies that $C_{\infty} = 0$, and so we have the factorization $C = A^*A$. In this case, the operator

$$A^*P_{\boldsymbol{H}^2}^{\perp}A = \sum_{n=0}^{\infty} C_n$$

is $\boldsymbol{H}^2$-minimal part of C. Thus, by (iii) of Lemma 3.6, we see that A is outer.

We next assume that C is invertible. Since T_C is invertible in $B(\boldsymbol{H}^2)$, by Lemma 3.1, we see that

$$C' = H_C T_C^{-1}(H_{C^*})^*.$$

Thus, for each $f \in \boldsymbol{L}^2$, we have

$$\begin{aligned} \|R_{\delta}^{-n}C'R_{\delta}^{n}f\| &= \|R_{\delta}^{-n}H_C T_C^{-1}(H_{C^*})^*R_{\delta}^{n}f\| \\ &\leq \|H_C T_C^{-1}\|\,\|(H_{C^*})^*R_{\delta}^{n}f\| \\ &= \|H_C T_C^{-1}\|\,\|P_{\boldsymbol{H}^2}C(I - P_{\boldsymbol{H}^2})R_{\delta}^{n}f\| \\ &\leq \|H_C T_C^{-1}\|(\|P_{\boldsymbol{H}^2}CR_{\delta}^{n}f\| + \|P_{\boldsymbol{H}^2}CP_{\boldsymbol{H}^2}R_{\delta}^{n}f\|) \\ &= \|H_C T_C^{-1}\|(\|P_{\boldsymbol{H}^2}R_{\delta}^{n}Cf\| + \|R_{\delta}^{-n}P_{\boldsymbol{H}^2}CP_{\boldsymbol{H}^2}R_{\delta}^{n}f\|) \\ &= \|H_C T_C^{-1}\|(\|R_{\delta}^{-n}P_{\boldsymbol{H}^2}R_{\delta}^{n}Cf\| + \|R_{\delta}^{-n}P_{\boldsymbol{H}^2}CP_{\boldsymbol{H}^2}R_{\delta}^{n}f\|) \\ &\to 0 \quad (n \to \infty). \end{aligned}$$

This implies that $R_{\delta}^{-n}C'R_{\delta}^{n}$ converges to zero in the strong operator topology. □

Factorization problems with respect to an analytic crossed product have been studied in [**12**]–[**14**]. They showed that every invertible element T in $\mathfrak{L}$ can be decomposed as $T = UA$ where U is a unitary operator and A is an invertible operator in $\mathfrak{L}_+$. Moreover they proved that every invertible positive operator in $\mathfrak{L}$ can be factored in the form A^*A, where A belongs to $(\mathfrak{L}_+) \cap (\mathfrak{L}_+)^{-1}$. As a corollary of Theorem 3.5, we can obtain the same result.

Corollary 3.9 ([**14**, Corollary 5.3]). *Every positive invertible operator in* $\mathfrak{L}$ *can be factored in the form* A^*A, *where* A *is outer in* $\mathfrak{L}_+ \cap (\mathfrak{L}_+)^{-1}$.

Finally, we show that the factorization of positive operators in Corollary 3.9 is unique as following:

Proposition 3.10. *Let* $C = A^*A$ *be the factorization in* Corollary 3.9. *If there exists an operator* B *in* $\mathfrak{L}_+ \cap (\mathfrak{L}_+)^{-1}$ *such that* $C = B^*B$, *then there is a unitary operator* U *in* $\mathfrak{L}_+ \cap (\mathfrak{L}_+)^{-1}$ *such that* $B = UA$.

Proof. Since $A^*A = B^*B$, we see that $\|Af\| = \|Bf\|$ for each f in $\boldsymbol{L}^2$. Since A and B are invertible, there exists a unitary operator U such that $B = UA$. Moreover, we see that $U = BA^{-1} \in \mathfrak{L}_+ \cap (\mathfrak{L}_+)^{-1}$ because A and B belong to $\mathfrak{L}_+ \cap (\mathfrak{L}_+)^{-1}$. □

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