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On the Cauchy problem for non linear PDEs in the Gevrey class with shrinkings

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Abstract. In the article, the Cauchy problem of the form

$$(*) \qquad \partial_2 u(x,t) = f(u(x,t), \partial_1^p u(x, \alpha(t)t), x, t), \quad u(x,0) = 0$$

or of the form

$$(\dagger) \qquad \partial_2 u(x,t) = f(u(x,t), \partial_1^p u(\alpha(x,t)x, t), x, t), \quad u(x,0) = 0$$

is studied. In $(*)$ and $(\dagger)$ $u(x,t)$ denotes a real valued unknown function of the real variables x and t. p denotes a fixed positive integer. It is assumed that $f(u,v,x,t)$ is continuous in (u,v,x,t) and Gevrey in (u,v,x). $\alpha(t)$ in $(*)$ and $\alpha(x,t)$ in $(\dagger)$ are called shrinkings, since they satisfy the conditions $\sup|\alpha(t)| < 1$ and $\sup|\alpha(x,t)| < 1$, respectively.

1. Introduction.

In the paper [**3**] the local Cauchy problems of the form

$$(1.1) \qquad \partial_2 u(x,t) = f(u(x,t), \partial_1^p u(x, \alpha(t)t), x, t), \quad u(x,0) = 0$$

and

$$(1.2) \qquad \partial_2 u(x,t) = f(u(x,t), \partial_1^p u(\alpha(x,t)x, t), x, t), \quad u(x,0) = 0$$

were considered. In (1.1) and (1.2) $u(x,t)$ denotes a real valued unknown function of the real variables x and t. The variable x is called the *space variable* and t the *time variable*. ∂_i, $i = 1$ or 2, denotes partial differentiation with respect to the ith variable. p is an arbitrarily fixed positive integer. f and α are given continuous functions. In particular, α is called a *shrinking*, since it satisfies the inequality of the form $0 < |\alpha(t)| \le m$ or $0 < |\alpha(x,t)| \le m$, where m is a positive constant less than 1. As was mentioned in [**3**], a shrinking seems to compensates the loss of smoothness caused by differentiation with respect to the space variable. In virtue of this property of a shrinking we can consider a local Cauchy problem such as (1.1) or (1.2) in which the maximum order of differentiation with respect to the

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space variable is greater than the corresponding order with respect to the time variable.

Kawagishi [**3**] solved the problems (1.1) and (1.2) under the condition that $f(u, v, x, t)$ is analytic in (u, v, x). Roughly speaking, [**3**] can be said to be a generalization to the non-linear case of the results of Augustynowicz *et al.* [**1**], [**2**] for linear differential equations. The purpose of the present note is to generalize the results of [**3**] to the case where $f(u, v, x, t)$ is a Gevrey type function of (u, v, x).

In §2 we shall begin with the definition of a Gevrey function and list up necessary properties of Gevrey functions. In this section we shall quote some results that were given in Yamanaka [**6**], [**7**]. Although some of them may not be considered quite suitable for our present situation without modification, we shall make no modification in order not to make the note longer.

The existence of a solution of the Cauchy problem (1.1) will be given in §3 as Theorem 3.1. The uniqueness of the solution of the same Cauchy problem will be shown in another theorem, Theorem 3.2, in the same section. Our result in §3 can be said to be a partial generalization of a result of Matsumoto [**4**][1)].

In §4 we shall study the properties of the composition of a Gevrey function and a real analytic function. This is a preparation for §5.

In §5, the final section, we shall treat the Cauchy problem of the form (1.2). The existence of a solution of the Cauchy problem (1.2) will be given as Theorem 5.1. The uniqueness of the solution of the same Cauchy problem will be shown in another theorem, Theorem 5.2.

2. Gevrey functions, 1.

2.1. Gevrey functions of one variable.

We denote by $\boldsymbol{Z}_+$ the set of all non-negative integers. Let I be an interval in $\boldsymbol{R}$ and λ a constant greater than 1. We fix such a λ throughout this paper. If $w : I \to \boldsymbol{R}$ is a C^∞ function and there are two positive constants C, M such that the inequality

$$|w^{(k)}(x)| \le CM^k (k!)^\lambda$$

holds for all $x \in I$ and all $k \in \boldsymbol{Z}_+$, then w is called a Gevrey function on I of order λ. As is easily seen, a C^∞ function $w : I \to \boldsymbol{R}$ is a Gevrey function of order λ if and only if there are two positive constants C', L such that the inequality

$$|w^{(k)}(x)| \le 2^{-5} C' L^k (k!)^\lambda (1 + k)^{-2}$$

[1)] In [**4**] only linear differential equations are considered. However, the coefficients of the equations in [**4**] are assumed to be in some wider class of functions than the Gevrey class, i.e., the ultradifferentiable class.

holds for all $x \in I$ and all $k \in \boldsymbol{Z}_+$. So we write, according to Yamanaka [**6**],

$$\Gamma_\lambda(k) = 2^{-5}(k!)^\lambda(1+k)^{-2}$$

for $k = 0, 1, 2, \ldots$ and define

$$|w|_L = \sup\left\{\frac{|w^{(k)}(x)|}{L^k \Gamma_\lambda(k)}; x \in I, k \in \boldsymbol{Z}_+\right\}$$

for each C^∞ function $w : I \to \boldsymbol{R}$. We denote by $\gamma_L(I)$ the family of all C^∞ functions $w : I \to \boldsymbol{R}$ such that $|w|_L < \infty$.

Besides the family $\gamma_L(I)$ we need another type of Gevrey family. If $w : I \to \boldsymbol{R}$ is a C^∞ function, we write

$$\text{(2.1)} \qquad |\!|w|\!| = \sup_{x \in I} |w(x)|, \quad \|w\|_L = \max\{2^6 |\!|w|\!|, 2^3 L^{-1} |w'|_L\}$$

and define

$$\mathscr{G}_L(I) = \left\{w : I \xrightarrow{C^\infty} \boldsymbol{R}; \|w\|_L < \infty\right\}.$$

Between the two types of Gevrey families $\gamma_L(I)$ and $\mathscr{G}_L(I)$ there is the following relation.

Proposition 2.1. *If* $0 < L < M$, *then* $\gamma_L(I) \subset \mathscr{G}_M(I) \subset \gamma_M(I)$ *and the inclusion maps in these inclusion relations are linear and bounded.*

Proof. See Lemma 5.2 of Yamanaka [**7**]. □

The norm $\|\cdot\|_L$ has the following useful property.

Proposition 2.2. *If* v *and* w *are in* $\mathscr{G}_L(I)$, *then the product* vw *is again in* $\mathscr{G}_L(I)$ *and the inequality* $\|vw\|_L \le \|v\|_L \|w\|_L$ *holds.*

Proof. See Theorem 5.4 of Yamanaka [**7**]. □

As for the result of differentiation of a function belonging to the family $\gamma_L(I)$ there is the following fact.

Proposition 2.3. *Let* L *be a positive constant,* α *a constant greater than* 1 *and* q *a positive integer. Assume that* $w \in \gamma_L(I)$. *Then the* q*th derivative* $w^{(q)}$ *of* w *is in the family* $\gamma_{\alpha L}(I)$ *and the inequality*

$$|w^{(q)}|_{\alpha L} \le (\alpha L)^q \left(\frac{\lambda q}{\log \alpha}\right)^{\lambda q} |w|_L$$

holds.

Proof.

$$\frac{|w^{(k+q)}(x)|}{(\alpha L)^k \Gamma_\lambda(k)} \leq \frac{|w|_L L^{k+q} \Gamma_\lambda(k+q)}{(\alpha L)^k \Gamma_\lambda(k)} \leq |w|_L (\alpha L)^q \alpha^{-(k+q)} (k+q)^{\lambda q}$$

$$\leq |w|_L (\alpha L)^q \sup_{t \geq 0} \alpha^{-t} t^{\lambda q} = |w|_L (\alpha L)^q \left(\frac{\lambda q}{e \log \alpha}\right)^{\lambda q} \leq |w|_L (\alpha L)^q \left(\frac{\lambda q}{\log \alpha}\right)^{\lambda q}. \quad \square$$

For us the following modification of the above proposition is useful.

Proposition 2.4. *Let L and M be positive constants such that $L < M$. Assume that w is in $\gamma_L(I)$ and q is a positive integer. Then $w^{(q)}$ is in $\gamma_M(I)$ and the inequality*

$$|w^{(q)}|_M \leq M^{(1+\lambda)q} \left(\frac{\lambda q}{M - L}\right)^{\lambda q} |w|_L \tag{2.2}$$

holds.

Proof. Write $\alpha = M/L$. Then we see by Proposition 2.3 that $w^{(q)}$ is in $\gamma_{\alpha L}(I) = \gamma_M(I)$ and the inequality

$$|w^{(q)}|_M \leq M^q \left(\frac{\lambda q}{\log \alpha}\right)^{\lambda q} |w|_L \tag{2.3}$$

holds. Further we have

$$\log \alpha = \int_1^\alpha \frac{1}{t} dt > \frac{\alpha - 1}{\alpha} = \frac{M - L}{M}.$$

Substituting this relation into (2.3), we obtain (2.2). $\square$

As for the result of composition of two Gevrey type functions there is the following fact.

Proposition 2.5. *Let I, J be open intervals and L, M be positive constants. Assume that $w : J \to \boldsymbol{R}$ is a C^∞ function such that $w' \in \gamma_L(J)$ and $v : I \to J$ is a C^∞ function such that $v' \in \gamma_M(I)$. Assume further that the inequality*

$$|v'|_M \leq L^{-1} M \tag{2.4}$$

holds. Then the derivative $(w \circ v)'$ of the composite function $w \circ v : I \to \boldsymbol{R}$ belongs to the family $\gamma_M(I)$ and the inequality

$$|(w \circ v)'|_M \leq L^{-1} M |w'|_L$$

holds.

Proof. See Theorem 3.1 of Yamanaka [**6**]. □

The Proposition 2.5 is modified as follows.

Proposition 2.6. *Let I, J be open intervals and L, M be positive constants. Assume that $w : J \to \boldsymbol{R}$ is in the family $\mathscr{G}_L(J)$ and $v : I \to J$ is in the family $\mathscr{G}_M(I)$. Assume further that the inequality* (2.4) *holds. Then the composite function $w \circ v : I \to \boldsymbol{R}$ belongs to the family $\mathscr{G}_M(I)$ and the inequality*

$$\|w \circ v\|_M \le \|w\|_L$$

holds.

Proof. See Theorem 5.3 of Yamanaka [**7**]. □

2.2. Gevrey functions of several variables.

In this subsection we consider Gevrey functions of several variables. For a function of m variables we denote by ∂_j the partial differentiation with respect to the jth variable and write $\partial = (\partial_1, \dots, \partial_m)$. Further, if $k = (k_1, \dots, k_m)$ is an element of $\boldsymbol{Z}_+^m$, then we write $\partial^k = \partial_1^{k_1} \cdots \partial_m^{k_m}$. Let U be an open set of $\boldsymbol{R}^m$. If $f : U \to \boldsymbol{R}$ is a C^∞ function and there are positive constants C, M such that the inequality

$$|\partial^k f(x)| \le C M^{|k|} (k!)^\lambda, \quad \text{where } |k| = k_1 + \cdots + k_m, \ k! = k_1! \cdots k_m!,$$

holds everywhere in U for any m dimensional index $k = (k_1 \dots, k_m)$, then f is called a Gevrey function on U of order λ. A C^∞ function $f : U \to \boldsymbol{R}$ is a Gevrey function of order λ, if and only if there are positive constants C', L such that the inequality

$$|\partial^k f(x)| \le C' L^{|k|} \Gamma_\lambda(|k|)$$

holds everywhere in U for any m dimensional index k. For this reason we write

$$|f|_L = \sup_{x,k} \frac{|\partial^k f(x)|}{L^{|k|} \Gamma_\lambda(|k|)}$$

for any C^∞ function $f : U \to \boldsymbol{R}$ and define

$$\gamma_L(U) = \left\{ f : U \overset{C^\infty}{\longrightarrow} \boldsymbol{R}; |f|_L < \infty \right\}.$$

Further we write, like (2.1),

$$|\!|w|\!| = \sup_{x \in U} |w(x)|, \quad \|w\|_L = \max\left\{ 2^6 |\!|w|\!|, 2^3 L^{-1} \max_i |\partial_i w|_L \right\}$$

and define

$$\mathscr{G}_L(U) = \left\{ w : U \overset{C^\infty}{\longrightarrow} \boldsymbol{R}; \|w\|_L < \infty \right\}.$$

It is necessary for us to know what comes out when m Gevrey functions $g_1(x), \dots, g_m(x)$ of one variable x are substituted for the first m variables $y_1, \dots, y_m$ in a Gevrey function $f(y_1, \dots, y_m, x)$ of $m+1$ variables $y_1, \dots, y_m, x$.

Proposition 2.7. *Let $J_1, \dots, J_m$ and I be open intervals and L, M be positive constants. Write $U = J_1 \times \cdots \times J_m \times I$. Let f be an element of the family $\mathscr{G}_L(U)$ and $g_i : I \to J_i$, $i = 1, \dots, m$, be in the family $\mathscr{G}_M(I)$. Assume that*

$$M \geq L\left(1 + \max_i |g_i'|_M\right). \tag{2.5}$$

Put

$$\varphi(x) = f(g_1(x), \dots, g_m(x), x)$$

for $x \in I$. Then φ is in $\mathscr{G}_M(I)$ and the inequality

$$\|\varphi\|_M \leq \|f\|_L$$

holds.

Proof. See Lemma 8.1 of Yamanaka [**6**]. □

2.3. Partial Gevrey functions.

It is necessary for us to consider functions of $m+1$ variables which are in a Gevrey class with respect to the first m variables only. We call them partial Gevrey functions. For a function $f(y_1, \dots, y_m, t)$ of $m+1$ variables $y_1, \dots, y_m, t$ we write $\tilde{\partial} = (\partial_1, \dots, \partial_m)$. For a non-negative integer j_0 we write $\boldsymbol{Z}_+(j_0) = \{j \in \boldsymbol{Z}_+; j \leq j_0\}$. Let U be an open set of $\boldsymbol{R}^m$, I a real open interval and j_0 a non-negative integer. Then we denote by $C^\infty(U) \otimes C^{j_0}(I)$ the set of all functions $f : U \times I \to \boldsymbol{R}$ such that the partial derivative $\tilde{\partial}^k \partial_{m+1}^j f : U \times I \to \boldsymbol{R}$ exists and continuous for each $(k, j) \in \boldsymbol{Z}_+^m \times \boldsymbol{Z}_+(j_0)$. Further, if $h(t)$ is a positive valued function of $t \in I$, we write

$$\gamma_h(U) \otimes C^{j_0}(I) = \left\{ f \in C^\infty(U) \otimes C^{j_0}(I); \sup_{t \in I} |f(\cdot, \dots, \cdot, t)|_{h(t)} < \infty \right\},$$

$$\mathscr{G}_h(U) \otimes C^{j_0}(I) = \left\{ f \in C^\infty(U) \otimes C^{j_0}(I); \sup_{t \in I} \|f(\cdot, \dots, \cdot, t)\|_{h(t)} < \infty \right\}.$$

If $h(t)$ is identically equal to a positive constant L, then we write $\gamma_L(U) \otimes C^{j_0}(I)$ and $\mathscr{G}_L(U) \otimes C^{j_0}(I)$ instead of $\gamma_h(U) \otimes C^{j_0}(I)$ and $\mathscr{G}_h(U) \otimes C^{j_0}(I)$, respectively.

3. The Cauchy problem with shrinking at the time variable.

In this section we want to solve the Cauchy problem (1.1):

$$\partial_2 u(x,t) = f(u(x,t), \partial_1^p u(x, \alpha(t)t), x, t), \quad u(x,0) = 0.$$

In the first subsection we prove the existence of a solution to the problem.

3.1. Existence of a solution.

The purpose of this subsection is to prove the following theorem.

THEOREM 3.1. *Let R, T_0, L, M, m, q and r_0 be positive constants. Assume that $M > \max\{1, L\}$, $m < 1$ and $\lambda pq < 1$, where p is the integer in the equation (1.1). Write $U = \{(v,w,x) \in \boldsymbol{R}^3; |v| < R, |w| < R, |x| < r_0\}$ and assume that f is a member of the family $\mathscr{G}_L(U) \otimes C^0(-T_0, T_0)$. Assume further that $\alpha(t)$ is a real valued continuous function of $t \in (-T_0, T_0)$ satisfying the inequality $0 < \alpha(t) \le m$. Write $h(t) = M(1+|t|^q)$. Then there is a positive number T_1 such that the Cauchy problem (1.1) has a solution $u \in \mathscr{G}_h(-r_0, r_0) \otimes C^1(-T_1, T_1)$.*

PROOF. Solving the Cauchy problem (1.1) is clearly equivalent to solving the integral equation

$$v(x,t) = f\left(\int_0^t v(x,\tau)\,d\tau, \int_0^{\alpha(t)t} \partial_1^p v(x,\tau)\,d\tau, x, t\right). \tag{3.1}$$

Now, from the assumption that f is in $\mathscr{G}_L(U) \otimes C^0(-T_0, T_0)$ it immediately follows that

$$A_f := \sup_{(v,w,x)\in U, |t|<T_0} |f(v,w,x,t)| < \infty \tag{3.2}$$

and

$$B_f := \sup\{|\partial_i f(\cdot,\cdot,\cdot,t)|_L; |t| < T_0, 1 \le i \le 3\} < \infty. \tag{3.3}$$

For any two positive constants K and T we define

$$\mathscr{F}(K,T) = \left\{ v \in \mathscr{G}_h(-r_0, r_0) \otimes C^0(-T,T); \sup_{|t|<T} \|v(\cdot,t)\|_{h(t)} \le K \right\}.$$

Note first that, if $v \in \mathscr{F}(K,T)$, then

$$|\!|\!| v(\cdot,t) |\!|\!| = \sup_{|x|<r_0} |v(x,t)| \le 2^{-6}\|v(\cdot,t)\|_{h(t)} \le 2^{-6}K, \quad |t| < T, \tag{3.4}$$

and

$$|\partial_1 v(\cdot,t)|_{h(t)} \le 2^{-3}h(t)\|v(\cdot,t)\|_{h(t)} \le 2^{-3}h(t)K, \quad |t| < T. \tag{3.5}$$

Let us next show that, for any given $K > 0$, there is a number $T_2(K)$ such that, if $v \in \mathscr{F}(K, T_2(K))$, then the right hand side of (3.1) makes sense for all $(x, t) \in (-r_0, r_0) \times (-T_2(K), T_2(K))$. To this end take for the moment a $T > 0$ arbitrarily and let v be in $\mathscr{F}(K, T)$. We see by (3.4) that

$$\sup_{|x|<r_0}\left|\int_0^t v(x,\tau)\,d\tau\right| < 2^{-6}KT, \quad |t| < T.$$

It follows that, if T satisfies

$$T \le 2^6 K^{-1} R \tag{3.6}$$

and $|t| < T$, then we can substitute $\int_0^t v(x,\tau)\,d\tau$ for v in $f(v, w, x, t)$. It follows from (3.5) that, if $v \in \mathscr{F}(K, T)$ and $|t| < T$, then

$$|\partial_1^p v(x,t)| \le 2^{-3}Kh(t)^p \Gamma_\lambda(p-1)$$

and

$$\begin{aligned}\left|\int_0^{\alpha(t)t} \partial_1^p v(x,\tau)\,d\tau\right| &\le 2^{-3}K\Gamma_\lambda(p-1)\int_0^{\alpha(t)|t|} h(\tau)^p\,d\tau \\ &< 2^{-3}K\Gamma_\lambda(p-1)M^p(1+T^q)^p mT \\ &< 2^{-3}K\Gamma_\lambda(p-1)M^p(1+T)^{pq}T.\end{aligned}$$

Therefore, if T satisfies the inequality

$$2^{-3}K\Gamma_\lambda(p-1)M^p(1+T)^{pq}T \le R \tag{3.7}$$

and $|t| < T$, then we can substitute $\int_0^{\alpha(t)t} \partial_1^p v(x,\tau)\,d\tau$ for w in $f(v, w, x, t)$.

We denote by $T_2(K)$ the maximum value of T that satisfies (3.6) and (3.7). We know from the above arguments that, if $v \in \mathscr{F}(K, T_2(K))$ and $|t| < T_2(K)$, then the right member of (3.1) is well defined for all $(x, t) \in (-r_0, r_0) \times (-T_2(K), T_2(K))$.

It follows that, if $v \in \mathscr{F}(K, T_2(K))$, we can define

$$\Phi(v)(x,t) = f\left(\int_0^t v(x,\tau)\,d\tau, \int_0^{\alpha(t)t} \partial_1^p v(x,\tau)\,d\tau, x, t\right)$$

for all $(x, t) \in (-r_0, r_0) \times (-T_2(K), T_2(K))$.

Let us next show that there is a number $T_3(K) \in (0, T_2(K)]$ such that, if v is in $\mathscr{F}(K, T_3(K))$, then $\Phi(v)$ is in $\mathscr{G}_h(-r_0, r_0) \otimes C^0(-T_3(K), T_3(K))$. To this end take for the moment a $T \in (0, T_2(K)]$ arbitrarily and let v be in $\mathscr{F}(K, T)$. Write

$$\varphi(x,t) = \int_0^t v(x,\tau)\,d\tau \tag{3.8}$$

and

$$\psi(x,t) = \int_0^{\alpha(t)t} \partial_1^p v(x,\tau)\,d\tau. \tag{3.9}$$

We want to apply Proposition 2.7 to the composite function $f(\varphi(x,t),\psi(x,t),x,t)$. By (3.5) we have, for each $(k,\tau) \in \boldsymbol{Z}_+ \times (-T,T)$,

$$|\partial_1^{k+1} v(x,\tau)| \le |\partial_1 v(\cdot\,,\tau)|_{h(\tau)} h(\tau)^k \Gamma_\lambda(k) \le 2^{-3} K h(\tau)^{k+1} \Gamma_\lambda(k).$$

Therefore we have

$$\begin{aligned} |\partial_1^{k+1}\varphi(x,t)| &\le \left|\int_0^t |\partial_1^{k+1} v(x,\tau)|\,d\tau\right| \le \int_0^{|t|} 2^{-3} K h(\tau)^{k+1} \Gamma_\lambda(k)\,d\tau \\ &\le 2^{-3} K h(t)^{k+1} \Gamma_\lambda(k) T \end{aligned}$$

and

$$|\partial_1 \varphi(\cdot\,,t)|_{h(t)} \le 2^{-3} K h(t) T. \tag{3.10}$$

Now suppose that T satisfies the inequality

$$T \le 2^3 K^{-1}\left(\frac{1}{L} - \frac{1}{M}\right). \tag{3.11}$$

Then we have

$$2^{-3} K h(t) T \le \frac{h(t)}{L} - \frac{h(t)}{M} \le \frac{h(t)}{L} - 1.$$

Therefore we see by (3.10) that the following assertion is correct.

Assertion 3.1. *If T satisfies $0 < T \le T_2(K)$ and (3.11) and v is in $\mathscr{F}(K,T)$, then the function φ defined by (3.8) satisfies the inequality*

$$|\partial_1 \varphi(\cdot\,,t)|_{h(t)} \le \frac{h(t)}{L} - 1 \tag{3.12}$$

for $|t| < T$.

As for $\psi(x,t)$ we have, since $0 < \alpha(t) \le m < 1$,

$$\begin{aligned} |\partial_1^{k+1}\psi(x,t)| &\le \left|\int_0^{\alpha(t)t} |\partial_1^{p+k+1} v(x,\tau)|\,d\tau\right| \\ &\le \left|\int_0^{mt} |\partial_1^{p+k+1} v(x,\tau)|\,d\tau\right| \\ &= m\left|\int_0^{t} |\partial_1^{p+k+1} v(x,ms)|\,ds\right| \end{aligned} \tag{3.13}$$

$$\leq \left| \int_0^t |\partial_1^{p+1} v(\cdot, ms)|_{h(s)} h(s)^k \Gamma_\lambda(k)\, ds \right|$$

$$\leq h(t)^k \Gamma_\lambda(k) \left| \int_0^t |\partial_1^{p+1} v(\cdot, ms)|_{h(s)}\, ds \right|.$$

We use here Proposition 2.4 and (3.5). As a result we obtain the following estimation of $|\partial_1^{p+1} v(\cdot, ms)|_{h(s)}$.

$$|\partial_1^{p+1} v(\cdot, ms)|_{h(s)} \leq h(s)^{(1+\lambda)p} \left(\frac{\lambda p}{h(s) - h(ms)} \right)^{\lambda p} |\partial_1 v(\cdot, ms)|_{h(ms)}$$

$$\leq h(s)^{2\lambda p} \left(\frac{\lambda p}{M(1 - m^q)|s|^q} \right)^{\lambda p} 2^{-3} K h(ms)$$

$$\leq h(T)^{2\lambda p} M^{-\lambda p} \left(\frac{\lambda p}{1 - m^q} \right)^{\lambda p} 2^{-3} K h(t) |s|^{-\lambda pq}.$$

It follows that

$$\left| \int_0^t |\partial_1^{p+1} v(\cdot, ms)|_{h(s)}\, ds \right| \leq \frac{h(T)^{2\lambda p}}{M^{\lambda p}} \left(\frac{\lambda p}{1 - m^q} \right)^{\lambda p} \frac{K h(t)}{2^3} \frac{T^{1-\lambda pq}}{1 - \lambda pq}. \tag{3.14}$$

Substituting (3.14) in (3.13), we obtain

$$|\partial_1^{k+1} \psi(x, t)| \leq \left(\frac{\lambda p}{1 - m^q} \right)^{\lambda p} \frac{M^{\lambda p} 2^{-3} K}{1 - \lambda pq} (1 + T^q)^{2\lambda p} T^{1-\lambda pq} h(t)^{k+1} \Gamma_\lambda(k)$$

and

$$|\partial_1 \psi(\cdot, t)|_{h(t)} \leq \left(\frac{\lambda p}{1 - m^q} \right)^{\lambda p} \frac{M^{\lambda p} 2^{-3} K}{1 - \lambda pq} (1 + T^q)^{2\lambda p} T^{1-\lambda pq} h(t). \tag{3.15}$$

Now suppose that T satisfies the inequality

$$\left(\frac{\lambda p}{1 - m^q} \right)^{\lambda p} \frac{M^{\lambda p}}{1 - \lambda pq} (1 + T^q)^{2\lambda p} T^{1-\lambda pq} \leq 2^3 K^{-1} \left(\frac{1}{L} - \frac{1}{M} \right) \tag{3.16}$$

and $|t| < T$. Then we have, in virtue of (3.15),

$$|\partial_1 \psi(\cdot, t)|_{h(t)} \leq \frac{h(t)}{L} - \frac{h(t)}{M} \leq \frac{h(t)}{L} - 1.$$

It follows that the following assertion is correct.

Assertion 3.2. *If T satisfies $0 < T \le T_2(K)$ and* (3.16) *and v is in $\mathscr{F}(K,T)$, then the function ψ defined by* (3.9) *satisfies the inequality*

$$|\partial_1\psi(\cdot\,,t)|_{h(t)} \le \frac{h(t)}{L} - 1 \tag{3.17}$$

for $|t| < T$.

We denote by $T_3(K)$ the maximum value of $T \in (0, T_2(K)]$ that satisfies (3.11) and (3.16). Then, by Assertion 3.1 and 3.2, we see that, if $0 < T \le T_3(K)$ and v is in $\mathscr{F}(K,T)$, then the inequality

$$h(t) \ge L(1 + \max\{|\partial_1\varphi(\cdot\,,t)|_{h(t)}, |\partial_1\psi(\cdot\,,t)|_{h(t)}\}) \tag{3.18}$$

holds for $|t| < T$. The condition (3.18) is of the same type as (2.5) in Proposition 2.7. Therefore we can use it here and conclude that, if $0 < T \le T_3(K)$ and v is in $\mathscr{F}(K,T)$, then $\Phi(v)$ is in the family $\mathscr{G}_h(-r_0, r_0) \otimes C^0(-T,T)$ and the norm $\|\Phi(v)(\cdot\,,t)\|_{h(t)}$ is estimated as follows.

$$\begin{aligned} \|\Phi(v)(\cdot\,,t)\|_{h(t)} &\le \|f(\cdot\,,\cdot\,,\cdot\,,t)\|_L \\ &= \max\left\{2^6|f(\cdot\,,\cdot\,,\cdot\,,t)|, 2^3L^{-1}\max_{1\le i\le 3}|\partial_i f(\cdot\,,\cdot\,,\cdot\,,t)|_L\right\} \\ &\le \max\{2^6A_f, 2^3L^{-1}B_f\}. \end{aligned} \tag{3.19}$$

We want $\Phi(v)$ to be in the family $\mathscr{F}(K,T)$. In view of (3.19) we see that, in order for $\Phi(v)$ to be in this family, it is enough for K to satisfy the inequality $K \ge \max\{2^6A_f, 2^3L^{-1}B_f\}$. So we define

$$K_0 := \max\{2^6A_f, 2^3L^{-1}B_f\}.$$

We have proved that, if $0 < T \le T_3(K_0)$, then Φ maps $\mathscr{F}(K_0,T)$ into itself.

Our next task is to estimate the difference $\Phi(v_1) - \Phi(v_0)$, where v_1 and v_0 are two elements of the family $\mathscr{F}(K_0, T_3(K_0))$. Take two elements v_0, v_1 of this family arbitrarily and set $v_\theta = \theta v_1 + (1-\theta)v_0$ for $0 < \theta < 1$. Further define

$$\varphi_\theta(x,t) = \int_0^t v_\theta(x,\tau)\,d\tau \tag{3.20}$$

and

$$\psi_\theta(x,t) = \int_0^{\alpha(t)t} \partial_1^p v_\theta(x,\tau)\,d\tau. \tag{3.21}$$

Write

$$A_i(x,t) = \int_0^1 \partial_i f(\varphi_\theta(x,t), \psi_\theta(x,t), x, t)\, d\theta \tag{3.22}$$

for $i = 1, 2$. Then the difference $\Phi(v_1) - \Phi(v_0)$ is expressed as follows.

$$\begin{aligned} \Phi(v_1)(x,t) - \Phi(v_0)(x,t) &= f(\varphi_1(x,t), \psi_1(x,t), x, t) - f(\varphi_0(x,t), \psi_0(x,t), x, t) \\ &= \int_0^1 \frac{\partial}{\partial\theta} f(\varphi_\theta(x,t), \psi_\theta(x,t), x, t)\, d\theta \\ &= A_1(x,t)(\varphi_1(x,t) - \varphi_0(x,t)) + A_2(x,t)(\psi_1(x,t) - \psi_0(x,t)). \end{aligned} \tag{3.23}$$

Using this relation and Proposition 2.2, we obtain

$$\begin{aligned} \|\Phi(v_1)(\cdot\,,t) - \Phi(v_0)(\cdot\,,t)\|_{h(t)} \le{}& \|A_1(\cdot\,,t)\|_{h(t)} \|\varphi_1(\cdot\,,t) - \varphi_0(\cdot\,,t)\|_{h(t)} \\ &+ \|A_2(\cdot\,,t)\|_{h(t)} \|\psi_1(\cdot\,,t) - \psi_0(\cdot\,,t)\|_{h(t)}. \end{aligned} \tag{3.24}$$

In order to estimate the right member of this inequality we write

$$g_{i,\theta,t}(x) = \partial_i f(\varphi_\theta(x,t), \psi_\theta(x,t), x, t)$$

and define $L_1 = (L+M)/2$. Since $f \in \mathscr{G}_L(U) \otimes C^0(-T_0, T_0)$ and $L_1 > L$, we see that

$$D_f := \sup\{\|\partial_i f(\cdot\,,\cdot\,,\cdot\,,t)\|_{L_1}; |t| < T_0, 1 \le i \le 3\} < \infty. \tag{3.25}$$

Therefore we see again by Proposition 2.7 that, if, instead of (3.12) and (3.17), the conditions

$$|\partial_1 \varphi_\theta(\cdot\,,t)|_{h(t)} \le \frac{h(t)}{L_1} - 1, \quad |t| < T \tag{3.26}$$

and

$$|\partial_1 \psi_\theta(\cdot\,,t)|_{h(t)} \le \frac{h(t)}{L_1} - 1, \quad |t| < T \tag{3.27}$$

are satisfied, then $g_{i,\theta,t} \in \mathscr{G}_h(-r_0, r_0)$ and $\|g_{i,\theta,t}\|_{h(t)} \le \|\partial_i f(\cdot\,,\cdot\,,\cdot\,,t)\|_{L_1} \le D_f$. From the last inequality we obtain

$$\|A_i(\cdot\,,t)\|_{h(t)} \le \sup_{0\le\theta\le 1} \|g_{i,\theta,t}\|_{h(t)} \le D_f. \tag{3.28}$$

In order that the condition (3.26) is satisfied it is enough that T satisfies, instead of (3.11),

$$T \le 2^3 K_0^{-1}\left(\frac{1}{L_1} - \frac{1}{M}\right). \tag{3.29}$$

In order that the condition (3.27) is satisfied it is enough that T satisfies, instead of (3.16),

$$\left(\frac{\lambda p}{1-m^q}\right)^{\lambda p} \frac{M^{\lambda p}}{1-\lambda pq}(1+T^q)^{2\lambda p} T^{1-\lambda pq} \le 2^3 K_0^{-1}\left(\frac{1}{L_1} - \frac{1}{M}\right). \tag{3.30}$$

We need further to estimate $\|\varphi_1(\cdot\,,t) - \varphi_0(\cdot\,,t)\|_{h(t)}$ and $\|\psi_1(\cdot\,,t) - \psi_0(\cdot\,,t)\|_{h(t)}$ in terms of $\sup_\tau \|v_1(\cdot\,,\tau) - v_0(\cdot\,,\tau)\|_{h(\tau)}$. By (3.20) we clearly have

$$\|\varphi_1(\cdot\,,t) - \varphi_0(\cdot\,,t)\|_{h(t)} \le T \sup_{|\tau|<T} \|v_1(\cdot\,,\tau) - v_0(\cdot\,,\tau)\|_{h(\tau)}. \tag{3.31}$$

From (3.21) we obtain

$$\begin{aligned}
&|\psi_1(x,t) - \psi_0(x,t)| \\
&\quad \le \left| \int_0^{\alpha(t)t} |\partial_1^p v_1(x,\tau) - \partial_1^p v_0(x,\tau)|\, d\tau \right| \\
&\quad \le \left| \int_0^{\alpha(t)t} |\partial_1 v_1(\cdot\,,\tau) - \partial_1 v_0(\cdot\,,\tau)|_{h(\tau)} h(\tau)^{p-1} \Gamma_\lambda(p-1)\, d\tau \right| \\
&\quad \le \left| \int_0^{\alpha(t)t} 2^{-3} h(\tau) \|v_1(\cdot\,,\tau) - v_0(\cdot\,,\tau)\|_{h(\tau)} h(\tau)^{p-1} \Gamma_\lambda(p-1)\, d\tau \right| \\
&\quad \le 2^{-3} \Gamma_\lambda(p-1) h(T)^p T \sup_{|\tau|<T} \|v_1(\cdot\,,\tau) - v_0(\cdot\,,\tau)\|_{h(\tau)}
\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}
&|\partial_1^{k+1}\psi_1(x,t) - \partial_1^{k+1}\psi_0(x,t)| \\
&\quad \le \left| \int_0^{\alpha(t)t} |\partial_1^{p+k+1} v_1(x,\tau) - \partial_1^{p+k+1} v_0(x,\tau)|\, d\tau \right| \\
&\quad \le \left| \int_0^{mt} |\partial_1^{p+k+1} v_1(x,\tau) - \partial_1^{p+k+1} v_0(x,\tau)|\, d\tau \right| \\
&\quad \le m \left| \int_0^{t} |\partial_1^{p+k+1} v_1(x,ms) - \partial_1^{p+k+1} v_0(x,ms)|\, ds \right|
\end{aligned} \tag{3.33}$$

$$\leq \left| \int_0^t |\partial_1^{p+1} v_1(\cdot, ms) - \partial_1^{p+1} v_0(\cdot, ms)|_{h(s)} h(s)^k \Gamma_\lambda(k)\, ds \right|$$

$$\leq \left| \int_0^t h(s)^{(1+\lambda)p} \left(\frac{\lambda p}{h(s) - h(ms)} \right)^{\lambda p} |\partial_1 v_1(\cdot, ms) - \partial_1 v_0(\cdot, ms)|_{h(ms)} h(s)^k \Gamma_\lambda(k)\, ds \right|$$

$$\leq h(t)^{2\lambda p + k} \Gamma_\lambda(k) M^{-\lambda p} \left(\frac{\lambda p}{1 - m^q} \right)^{\lambda p} \frac{T^{1-\lambda pq}}{1 - \lambda pq} \sup_{|\tau|<T} |\partial_1 v_1(\cdot, \tau) - \partial_1 v_0(\cdot, \tau)|_{h(\tau)}$$

$$\leq h(t)^{2\lambda p + k} \frac{\Gamma_\lambda(k)}{M^{\lambda p}} \left(\frac{\lambda p}{1 - m^q} \right)^{\lambda p} \frac{T^{1-\lambda pq}}{1 - \lambda pq} \sup_{|\tau|<T} \left\{ \frac{h(\tau)}{2^3} \|v_1(\cdot, \tau) - v_0(\cdot, \tau)\|_{h(\tau)} \right\}$$

$$\leq 2^{-3} h(t)^{2\lambda p + k} \frac{\Gamma_\lambda(k)}{M^{\lambda p}} h(T) \left(\frac{\lambda p}{1 - m^q} \right)^{\lambda p} \frac{T^{1-\lambda pq}}{1 - \lambda pq} \sup_{|\tau|<T} \|v_1(\cdot, \tau) - v_0(\cdot, \tau)\|_{h(\tau).}$$

It follows from (3.32) and (3.33) that

$$\begin{aligned} (3.34)\quad & \|\psi_1(\cdot, t) - \psi_0(\cdot, t)\|_{h(t)} \\ &= \max\left\{ 2^6 \sup_x |\psi_1(x, t) - \psi_0(x, t)|, 2^3 h(t)^{-1} |\partial_1 \psi_1(\cdot, t) - \partial_1 \psi_0(\cdot, t)|_{h(t)} \right\} \\ &\leq \max\left\{ 2^3 \Gamma_\lambda(p-1) h(T)^p T, \frac{h(T)^{2\lambda p}}{M^{\lambda p}} \left(\frac{\lambda p}{1 - m^q} \right)^{\lambda p} \frac{T^{1-\lambda pq}}{1 - \lambda pq} \right\} \\ &\qquad \times \sup_{|\tau|<T} \|v_1(\cdot, \tau) - v_0(\cdot, \tau)\|_{h(\tau)}. \end{aligned}$$

If we define

$$(3.35)\qquad C_h(T) = \max\left\{ 2^3 \Gamma_\lambda(p-1) h(T)^p T, \frac{h(T)^{2\lambda p}}{M^{\lambda p}} \left(\frac{\lambda p}{1 - m^q} \right)^{\lambda p} \frac{T^{1-\lambda pq}}{1 - \lambda pq} \right\},$$

then (3.34) is rewritten as

$$(3.36)\qquad \|\psi_1(\cdot, t) - \psi_0(\cdot, t)\|_{h(t)} \leq C_h(T) \sup_{|\tau|<T} \|v_1(\cdot, \tau) - v_0(\cdot, \tau)\|_{h(\tau)}.$$

In view of (3.24), (3.28), (3.31) and (3.36) we see that, if T satisfies the inequalities $0 < T \leq T_3(K_0)$, (3.29) and (3.30), then the inequality

$$\|\Phi(v_1)(\cdot, t) - \Phi(v_0)(\cdot, t)\|_{h(t)} \leq D_f(T + C_h(T)) \sup_{|\tau|<T} \|v_1(\cdot, \tau) - v_0(\cdot, \tau)\|_{h(\tau)}$$

holds for $|t| < T$. If T satisfies one more inequality

$$(3.37)\qquad D_f(T + C_h(T)) \leq 2^{-1},$$

then the inequality

$$\|\Phi(v_1)(\cdot\,,t) - \Phi(v_0)(\cdot\,,t)\|_{h(t)} \le 2^{-1} \sup_{|\tau|<T} \|v_1(\cdot\,,\tau) - v_0(\cdot\,,\tau)\|_{h(\tau)}$$

holds for $|t| < T$.

We denote by T_1 the maximum value of $T \in (0, T_3(K_0)]$ that satisfies the inequalities (3.29), (3.30) and (3.37). Then Φ maps $\mathscr{F}(K_0, T_1)$ into itself and the inequality

$$\sup_{|t|<T_1} \|\Phi(v_1)(\cdot\,,t) - \Phi(v_0)(\cdot\,,t)\|_{h(t)} \le 2^{-1} \sup_{|t|<T_1} \|v_1(\cdot\,,t) - v_0(\cdot\,,t)\|_{h(t)}$$

holds for every pair $(v_1, v_0) \in \mathscr{F}(K_0, T_1) \times \mathscr{F}(K_0, T_1)$. It follows that there is a unique element $v \in \mathscr{F}(K_0, T_1)$ such that

$$\Phi(v) = v.$$

This element $v \in \mathscr{F}(K_0, T_1)$ satisfies the integral equation (3.1) for $|x| < r_0$ and $|t| < T_1$ and

$$u(x,t) := \int_0^t v(x,\tau)\,d\tau \tag{3.38}$$

becomes a solution of the Cauchy problem (1.1). Since v is in the family $\mathscr{G}_h(-r_0, r_0) \otimes C^0(-T_1, T_1)$, the function u defined by (3.38) enters in the family $\mathscr{G}_h(-r_0, r_0) \otimes C^1(-T_1, T_1)$. □

3.2. Uniqueness of the solution.

The solution of the Cauchy problem (1.1) is unique in the following sense.

THEOREM 3.2. *Let R, T_0, T_1, L, M, r_0 and m be positive constants. Assume that $T_1 \le T_0$ and $m < 1$. Write $U = \{(v, w, x) \in \boldsymbol{R}^3; |v| < R, |w| < R, |x| < r_0\}$. In the differential equation* (1.1) *assume that f is an element of the family $\mathscr{G}_L(U) \otimes C^0(-T_0, T_0)$ and $\alpha(t)$ is a real valued continuous function of $t \in (-T_0, T_0)$ satisfying the inequality $0 < \alpha(t) \le m$. Suppose that there are two solutions $u_i \in \mathscr{G}_M(-r_0, r_0) \otimes C^1(-T_1, T_1)$, $i = 0, 1$, such that*

$$|u_i(x,t)| < R \quad \text{and} \quad |\partial_1^p u_i(x,t)| < R \quad \text{for } |x| < r_0, |t| < T_1.$$

Then $u_1(x,t) = u_0(x,t)$ for all $(x,t) \in (-r_0, r_0) \times (-T_1, T_1)$.

PROOF. Write $v_i(x,t) = \partial_2 u_i(x,t)$ for $i = 0, 1$. Then each v_i is a solution of the integral equation (3.1) and is in $\mathscr{G}_M(-r_0, r_0) \otimes C^0(-T_1, T_1)$. Further, by the assumption of the theorem, the quantity

$$\max_i \sup_{|t|<T_1} |\partial_1 v_i(\cdot\,,t)|_M$$

is finite. We denote this quantity by S_v. Next take a number $M^* > \max\{M, L\}$. Then we have of course

$$|\partial_1 v_i(\cdot\,, t)|_{M^*} \le |\partial_1 v_i(\cdot\,, t)|_M \le S_v.$$

Moreover, since $M^* > M$, we see that there is another positive number S_v' such that $S_v' \ge S_v$ and

$$|\partial_1^{p+1} v_i(\cdot\,, t)|_{M^*} \le S_v'.$$

Next let us see that the difference $w(x,t) := v_1(x,t) - v_0(x,t)$ becomes a solution of a linear integral equation. For this purpose set $v_\theta = \theta v_1 + (1-\theta)v_0$ for $0 < \theta < 1$ and define $\varphi_\theta(x,t)$, $\psi_\theta(x,t)$ and $A_i(x,t)$ by (3.20), (3.21) and (3.22), respectively. Then we have (3.23). In the present case, however, we have $\Phi(v_0) = v_0$ and $\Phi(v_1) = v_1$. So we have

$$v_1(x,t) - v_0(x,t) = A_1(x,t)(\varphi_1(x,t) - \varphi_0(x,t)) + A_2(x,t)(\psi_1(x,t) - \psi_0(x,t)).$$

This means that $w(x,t)$ is a solution of the linear integral equation

$$w(x,t) = A_1(x,t)\int_0^t w(x,\tau)\,d\tau + A_2(x,t)\int_0^{\alpha(t)t} \partial_1^p w(x,\tau)\,d\tau. \tag{3.39}$$

Our purpose here is to show that the equation (3.39) has only the zero solution. For this purpose we have first to confirm that $A_i(x,t)$ is a Gevrey function of x. This is seen as follows. Since $|\partial_1 v_\theta(\cdot\,,t)|_{M^*} \le S_v \le S_v'$, we have $|\partial_1 \varphi_\theta(\cdot\,,t)|_{M^*} \le S_v' T_1$. Similarly we have $|\partial_1 \psi_\theta(\cdot\,,t)|_{M^*} \le S_v' T_1$. In view of these facts we take a positive number $q < (\lambda p)^{-1}$ and write

$$L_1 = \frac{L + M^*}{2}, \quad M^\dagger = \max\{M^*, L_1(1 + S_v' T_1)\}, \quad h(t) = M^\dagger(1 + |t|^q).$$

Then we have

$$\begin{aligned} &L_1(1 + \max\{|\partial_1\varphi_\theta(\cdot\,,t)|_{h(t)}, |\partial_1\psi_\theta(\cdot\,,t)|_{h(t)}\}) \\ &\quad \le L_1(1 + \max\{|\partial_1\varphi_\theta(\cdot\,,t)|_{M^\dagger}, |\partial_1\psi_\theta(\cdot\,,t)|_{M^\dagger}\}) \\ &\quad \le L_1(1 + S_v' T_1) \le h(t). \end{aligned}$$

Note also that, since $f \in \mathscr{G}_L(U) \otimes C^0(-T_0, T_0)$ and $L_1 > L$, the finite constant D_f is defined by (3.25). In view of these facts we can use Proposition 2.7 again and conclude that A_i is in $\mathscr{G}_h(-r_0, r_0) \otimes C^0(-T_1, T_1)$ and

$$\sup_{|t|<T_1} \|A_i(\cdot\,,t)\|_{h(t)} \le D_f. \tag{3.40}$$

Next we estimate the two integrals in the right member of (3.39) in terms of

$\|w(\cdot\,,t)\|_{h(t)}$. Take a positive number $T \le T_1$ arbitrarily. It is clear that the inequality

$$\sup_{|t|<T}\left\|\int_0^t w(\cdot\,,\tau)\,d\tau\right\|_{h(t)} \le T \sup_{|t|<T}\|w(\cdot\,,t)\|_{h(t)} \tag{3.41}$$

holds. Moreover, by the same argument for obtaining (3.34), we obtain

$$\sup_{|t|<T}\left\|\int_0^{\alpha(t)t} \partial_1^p w(\cdot\,,\tau)\,d\tau\right\|_{h(t)} \le C_h(T) \sup_{|t|<T}\|w(\cdot\,,t)\|_{h(t)}, \tag{3.42}$$

where $C_h(T)$ is the constant defined by (3.35).

From (3.39), (3.40), (3.41) and (3.42) we obtain

$$\sup_{|t|<T}\|w(\cdot\,,t)\|_{h(t)} \le D_f(T + C_h(T)) \sup_{|t|<T}\|w(\cdot\,,t)\|_{h(t)}.$$

It follows that, if T is sufficiently small, then

$$\sup_{|t|<T}\|w(\cdot\,,t)\|_{h(t)} = 0$$

and

$$|t| < T \Rightarrow \sup_{|x|<r_0}|w(x,t)| = 0.$$

So we write now

$$T_2 = \sup\left\{T \in (0, T_1]; 0 \le t < T \Rightarrow \sup_{|x|<r_0}|w(x,t)| = 0\right\},$$

$$T_3 = \sup\left\{T \in (0, T_1]; 0 \ge t > -T \Rightarrow \sup_{|x|<r_0}|w(x,t)| = 0\right\}.$$

We want to show that $T_1 = T_2 = T_3$. For this purpose suppose $T_2 < T_1$. Then, if $T_2 \le t < \min\{T_2/m, T_1\}$, then

$$\int_0^{\alpha(t)t} \partial_1^p w(x,\tau)\,d\tau = 0.$$

It follows that the integral equation (3.39) reduces to

$$w(x,t) = A_1(x,t)\int_{T_2}^t w(x,\tau)\,d\tau \tag{3.43}$$

in the interval $T_2 \le t < \min\{T_2/m, T_1\}$. From (3.43) we obtain

$$
\begin{aligned}
|w(x,t)| &\le \sup_{|s|<T_1, |\xi|<r_0} |A_1(\xi,s)| \int_{T_2}^{t} |w(x,\tau)|\, d\tau \\
&\le 2^{-6} \sup_{|s|<T_1} \|A(\cdot\,,s)\|_{h(s)} \int_{T_2}^{t} |w(x,\tau)|\, d\tau \\
&\le 2^{-6} D_f \int_{T_2}^{t} |w(x,\tau)|\, d\tau.
\end{aligned}
$$

It follows that $w(t,x) = 0$ for all (x,t) such that $T_2 \le t < \min\{T_2/m, T_1\}$ and $|x| < r_0$. This contradicts the definition of T_2. There arises a similar contradiction, if we assume that $T_3 < T_1$. Thus the proof of the fact that $T_1 = T_2 = T_3$ and the proof of the theorem itself is now complete. □

4. Gevrey functions, 2.

This section is a preparation for solving the Cauchy problem (1.2) in the next section, §5. Here we investigate the properties of the composite function $w \circ v$ of a Gevrey function w and a real analytic function v. First let us make some notational agreement. If ρ is a positive constant, we write $D(\rho) = \{z \in \boldsymbol{C}; |z| < \rho\}$ and denote by $\mathscr{A}(\rho)$ the set of all holomorphic functions from $D(\rho)$ into $\boldsymbol{C}$ whose restrictions on the real interval $(-\rho, \rho)$ are real valued. Further, if ρ and m are positive constants, we write

$$
\mathscr{B}(\rho, m) = \{v \in \mathscr{A}(\rho); |v(z)| \le m \text{ for all } z \in D(\rho)\}.
$$

Let us now prove the following proposition.

Proposition 4.1. *Let L, m, r_0 and ρ_0 be positive constants. Write $I_0 = (-r_0, r_0)$. Assume that $m < 1$ and*

$$
\rho_0 \ge \frac{r_0 + m^{-3/4} L^{-1}}{1 - m^{1/4}}. \tag{4.1}
$$

Let β be an element of $\mathscr{B}(\rho_0, m)$ and denote by α the restriction to the real interval I_0 of β. Let w be a member of the Gevrey family $\mathscr{G}_L(I_0)$. Write $v(x) = \alpha(x)x$. Then the composite function $w \circ v : I_0 \to \boldsymbol{R}$ is in the family $\mathscr{G}_{\sqrt{m}L}(I_0)$ and the inequality

$$
|(w \circ v)'|_{\sqrt{m}L} \le C(m, \lambda) |w'|_L \tag{4.2}
$$

holds, where $C(m, \lambda)$ is a constant ≥ 1 that depends only on m and λ.

Proof. We have to estimate the successive derivatives of the composite function $w \circ v$. To this end we use the higher order chain rule given by Yamanaka [**6**]. By the formula given in Theorem 2.1 of [**6**] we have

$$(4.3)\qquad (w \circ v)^{(n)}(x) = \sum_{j=1}^{n} \binom{n}{j} w^{(j)}(v(x)) \left[\frac{\partial^{n-j}}{\partial h^{n-j}} \left\{ \int_0^1 v'(x+\theta h)\, d\theta \right\}^j \right]_{h=0} = \sum_{j=1}^{n} \binom{n}{j} w^{(j)}(v(x)) \left[\frac{\partial^{n-j}}{\partial h^{n-j}} \left\{ \int_0^1 v_0'(x+\theta h)\, d\theta \right\}^j \right]_{h=0},$$

where $v_0(z) = \beta(z)z$. If we write

$$\varphi_x(h) = \int_0^1 v_0'(x+\theta h)\, d\theta,$$

$$\psi_{j,x}(h) = \{\varphi_x(h)\}^j,$$

then the formula (4.3) is rewritten as

$$(4.4)\qquad (w \circ v)^{(n)}(x) = \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} w^{(j)}(v(x)) \psi_{j,x}^{(n-j)}(0).$$

Let us estimate the magnitude of $\psi_{j,x}^{(n-j)}(0)$. For this purpose we write $r_1 = m^{-3/4}L^{-1}$ and $r_2 = \rho_0 - r_0 - r_1$. Note that the condition (4.1) implies that

$$r_2 = \rho_0 - (r_0 + m^{-3/4}L^{-1}) \geq \rho_0 - \rho_0(1 - m^{1/4}) = \rho_0 m^{1/4}.$$

Using this fact and the fact that the inequality $|v_0(z)| \leq \rho_0 m$ holds for $|z| \leq \rho_0$, we see that, if $|z| < r_0 + r_1$, then

$$|v_0'(z)| \leq \frac{\rho_0 m}{r_2} \leq \frac{\rho_0 m}{\rho_0 m^{1/4}} = m^{3/4}.$$

Therefore we see that, if $x \in I_0$, $h \in D(r_1)$, then we have

$$|\varphi_x(h)| \leq m^{3/4} \quad \text{and} \quad |\psi_{j,x}(h)| \leq m^{3j/4}.$$

It follows that the inequality

$$(4.5)\qquad |\psi_{j,x}^{(k)}(0)| \leq m^{3j/4} \frac{k!}{r_1^k}$$

holds for all $x \in I_0$ and all $k \in \boldsymbol{Z}_+$.

Now take an element w of $\mathscr{G}_L(I_0)$. Using (4.4) and (4.5), we obtain the following estimation of the derivative $(w \circ v)^{(n+1)}(x)$.

$$(4.6) \qquad |(w\circ v)^{(n+1)}(x)| \le \sum_{j=1}^{n+1}\frac{(n+1)!}{j!(n+1-j)!}|w^{(j)}(v_0(x))|\cdot|\psi_{j,x}^{(n+1-j)}(0)|$$

$$= \sum_{j=0}^{n}\frac{(n+1)!}{(j+1)!(n-j)!}|w^{(j+1)}(v_0(x))|\cdot|\psi_{j+1,x}^{(n-j)}(0)|$$

$$\le |w'|_L \sum_{j=0}^{n}\frac{(n+1)!}{(j+1)!(n-j)!}L^j\Gamma_\lambda(j)m^{3(j+1)/4}\frac{(n-j)!}{r_1^{n-j}}$$

$$= 2^{-5}|w'|_L(n+1)!\sum_{j=0}^{n}\frac{L^j j!^\lambda}{(j+1)!(1+j)^2}\frac{m^{3(j+1)/4}}{r_1^{n-j}}$$

$$\le 2^{-5}|w'|_L(n+1)!(n+1)!^{\lambda-1}(m^{3/4}L)^n\sum_{j=0}^{n}\frac{(m^{3/4}Lr_1)^{j-n}}{(1+j)^2}$$

$$\le 2^{-5}|w'|_L(m^{3/4}L)^n(n+1)!^\lambda\sum_{j=0}^{\infty}\frac{1}{(1+j)^2}$$

$$\le 2^{-4}(n+1)^\lambda|w'|_L(m^{3/4}L)^n n!^\lambda.$$

Note here that

$$(4.7) \qquad C(m,\lambda) := \sup_n 2(n+1)^{\lambda+2}m^{n/4} < \infty.$$

It follows from (4.6) and the definition (4.7) of the constant $C(m,\lambda)$ that the inequality

$$|(w\circ v)^{(n+1)}(x)| \le C(m,\lambda)2^{-5}|w'|_L(\sqrt{m}L)^n n!^\lambda(1+n)^{-2}$$

holds for all n and all $x\in I_0$. This means that $(w\circ v)'$ is in $\gamma_{\sqrt{m}L}(I_0)$ and the inequality (4.2) holds. It is clear that $C(m,\lambda)\ge 1$. □

5. The Cauchy problem with shrinking at the space variable.

In this section we want to solve the Cauchy problem (1.2):

$$\partial_2 u(x,t) = f(u(x,t), \partial_1^p u(\alpha(x,t)x,t), x, t), \quad u(x,0) = 0.$$

In the first subsection we prove the existence of a solution to the problem.

5.1. Existence of a solution.

The purpose of this subsection is to prove the following theorem. In the theorem we use the symbol ⊗ in the same meaning as in the subsection 2.3 of §2.

THEOREM 5.1. *Let* R, T_0, L, M *and* r_0 *be positive constants. Assume that* $M > \max\{1, L\}$. *Write* $U = \{(v, w, x) \in \boldsymbol{R}^3; |v| < R, |w| < R, |x| < r_0\}$ *and assume that* f *is a member of the family* $\mathscr{G}_L(U) \otimes C^0(-T_0, T_0)$. *Assume that* $\alpha(x, t)$ *is a real valued continuous function of* $(x, t) \in (-r_0, r_0) \times (-T_0, T_0)$ *such that the following condition* [C1] *is satisfied*:

[C1]: *For each* $t \in (-T_0, T_0)$ *the function* $\alpha(\cdot, t) : (-r_0, r_0) \to \boldsymbol{R}$ *is the restriction to the real interval* $(-r_0, r_0)$ *of a member of the family* $\mathscr{B}(\rho_0, m)$, *where* m *is a positive constant less than* 1 *and* ρ_0 *is another positive constant satisfying the condition* (4.1) *in Proposition* 4.1.

Then there is a positive number T_1 *such that the Cauchy problem* (1.2) *has a solution* $u \in \mathscr{G}_M(-r_0, r_0) \otimes C^1(-T_1, T_1)$.

PROOF. Solving the initial value problem (1.2) is clearly equivalent to solving the integral equation

$$v(x, t) = f\left(\int_0^t v(x, \tau)\, d\tau, \int_0^t \partial_1^p v(\alpha(x, t)x, \tau)\, d\tau, x, t\right). \tag{5.1}$$

Now, from the assumption that f is in $\mathscr{G}_L(U) \otimes C^0(-T_0, T_0)$ it follows that the finite constants A_f and B_f are defined by (3.2) and (3.3), respectively. For any two positive constants K and T we define

$$\mathscr{F}(K, T) = \left\{ v \in \mathscr{G}_M(-r_0, r_0) \otimes C^0(-T, T); \sup_{|t|<T} \|v(\cdot, t)\|_M \le K \right\}.$$

Note first that, if $v \in \mathscr{F}(K, T)$, then

$$|\!|v(\cdot, t)|\!| = \sup_{|x|<r_0} |v(x, t)| \le 2^{-6}\|v(\cdot, t)\|_M \le 2^{-6}K, \quad |t| < T, \tag{5.2}$$

and

$$|\partial_1 v(\cdot, t)|_M \le 2^{-3}M\|v(\cdot, t)\|_M \le 2^{-3}MK, \quad |t| < T. \tag{5.3}$$

Let us next show that, for any given $K > 0$, there is a number $T_2(K)$ such that, if $v \in \mathscr{F}(K, T_2(K))$, then the right-hand side of (5.1) makes sense for all $(x, t) \in (-r_0, r_0) \times (-T_2(K), T_2(K))$. To this end take for the moment a $T > 0$ arbitrarily and let v be in $\mathscr{F}(K, T)$. We see by (5.2) that

$$\sup_{|x|<r_0} \left| \int_0^t v(x, \tau)\, d\tau \right| < 2^{-6}KT, \quad |t| < T.$$

It follows that, if T satisfies

$$T \le 2^6 K^{-1} R$$

and $|t| < T$, then we can substitute $\int_0^t v(x,\tau)\,d\tau$ for v in $f(v,w,x,t)$. It follows from (5.3) that, if $v \in \mathscr{F}(K,T)$ and $|t| < T$, then

$$|\partial_1^p v(x,t)| \leq 2^{-3} K M^p \Gamma_\lambda(p-1)$$

and

$$\left|\int_0^t \partial_1^p v(\alpha(x,t)x,\tau)\,d\tau\right| < 2^{-3} K M^p \Gamma_\lambda(p-1) T.$$

Therefore, if T satisfies the inequality

$$T \leq \frac{2^3 R}{K \Gamma_\lambda(p-1) M^p}$$

and $|t| < T$, then we can substitute $\int_0^t \partial_1^p v(\alpha(x,t)x,\tau)\,d\tau$ for w in $f(v,w,x,t)$.

So we write

$$T_2(K) = \min\left\{T_0, 2^6 K^{-1} R, \frac{2^3 R}{K \Gamma_\lambda(p-1) M^p}\right\}.$$

From the above arguments we know that, if $v \in \mathscr{F}(K, T_2(K))$ and $|t| < T_2(K)$, then the right member of (5.1) is well-defined for all $(x,t) \in (-r_0, r_0) \times (-T_2(K), T_2(K))$. It follows that, if $v \in \mathscr{F}(K, T_2(K))$, we can define

$$\Phi(v)(x,t) = f\left(\int_0^t v(x,\tau)\,d\tau, \int_0^t \partial_1^p v(\alpha(x,t)x,\tau)\,d\tau, x, t\right)$$

for all $(x,t) \in (-r_0, r_0) \times (-T_2(K), T_2(K))$.

Let us next show that there is a number $T_3(K) \in (0, T_2(K)]$ such that, if v is in $\mathscr{F}(K, T_3(K))$, then $\Phi(v)$ is in $\mathscr{G}_M(-r_0, r_0) \otimes C^0(-T_3(K), T_3(K))$. To this end take for the moment a $T \in (0, T_2(K)]$ arbitrarily and let v be in $\mathscr{F}(K,T)$. Write

$$\varphi(x,t) = \int_0^t v(x,\tau)\,d\tau,$$

$$\omega_t(x,\tau) = \partial_1^p v(\alpha(x,t)x,\tau)$$

and

$$\psi(x,t) = \int_0^t \omega_t(x,\tau)\,d\tau.$$

We want to apply Proposition 2.7 to the composite function $(-r_0, r_0) \ni x \mapsto f(\varphi(x,t), \psi(x,t), x, t)$. As for $\varphi(x,t)$ it immediately follows from (5.3) that the following inequality holds.

$$\sup_{|t|<T} |\partial_1 \varphi(\cdot\,,t)|_M \leq 2^{-3} MKT. \tag{5.4}$$

In order to consider $\psi(x,t)$ we write $a = 1/\sqrt{m}$. Then, since $a > 1$ and $\partial_1 v(\cdot,t) \in \gamma_M(-r_0,r_0)$, we see that $\partial_1\partial_1^p v(\cdot,t) = \partial_1^{p+1}v(\cdot,t)$ is in $\gamma_{aM}(-r_0,r_0)$ and, in virtue of Proposition 2.3 and (5.3), we have the following estimation of $|\partial_1^{p+1}v(\cdot,t)|_{aM}$.

$$\begin{aligned}(5.5)\qquad |\partial_1\partial_1^p v(\cdot,t)|_{aM} &\le |\partial_1 v(\cdot,t)|_M (aM)^p\left(\frac{\lambda p}{\log a}\right)^{\lambda p}\\ &\le 2^{-3}M\|v(\cdot,t)\|_M M^p\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\\ &= 2^{-3}\left(\frac{a\lambda p}{\log a}\right)^{\lambda p} M^{p+1}\|v(\cdot,t)\|_M.\end{aligned}$$

Since $\partial_1\partial_1^p v(\cdot,t)$ is in $\gamma_{aM}(-r_0,r_0)$ and α satisfies the condition [C1] in the theorem, we can use here Proposition 4.1 and conclude that $\partial_1\omega_t(\cdot,\tau)$ is in the family $\gamma_M(-r_0,r_0)$ and satisfies the inequality

$$\begin{aligned}(5.6)\qquad |\partial_1\omega_t(\cdot,\tau)|_M &\le C(m,\lambda)|\partial_1\partial_1^p v(\cdot,\tau)|_{aM}\\ &\le 2^{-3}M^{p+1}C(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\|v(\cdot,\tau)\|_M.\end{aligned}$$

It follows from (5.6) and the condition $\|v(\cdot,t)\|_M \le K$ that

$$(5.7)\qquad |\partial_1\psi(\cdot,t)|_M \le 2^{-3}C(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p} M^{p+1}KT.$$

Examining the two inequalities (5.4) and (5.7), we know that the inequality

$$(5.8)\qquad \sup_{|t|<T}\max\{|\partial_1\varphi(\cdot,t)|_M, |\partial_1\psi(\cdot,t)|_M\} \le \frac{M}{L} - 1$$

holds, if T satisfies the inequality

$$\begin{aligned}(5.9)\qquad T &\le \left(\frac{M}{L}-1\right)\frac{2^3}{MK}\min\left\{1, \frac{1}{C(m,\lambda)M^p}\left(\frac{\log a}{a\lambda p}\right)^{\lambda p}\right\}\\ &= \left(\frac{M}{L}-1\right)\frac{2^3}{C(m,\lambda)M^{p+1}K}\left(\frac{\log a}{a\lambda p}\right)^{\lambda p}.\end{aligned}$$

Therefore, if $T \in (0, T_2(K)]$ satisfies (5.9), then we can use Proposition 2.7 and conclude that the function $x \mapsto \Phi(v)(x,t) = f(\varphi(x,t),\psi(x,t),x,t)$ is in $\mathscr{G}_M(-r_0,r_0)$ and the inequality $\|\Phi(v)(\cdot,t)\|_M \le \|f(\cdot,\cdot,\cdot,t)\|_L$ holds for $|t| < T$. So we put

$$T_3(K) = \min\left\{T_2(K), \left(\frac{M}{L}-1\right)\frac{2^3}{C(m,\lambda)M^{p+1}K}\left(\frac{\log a}{a\lambda p}\right)^{\lambda p}\right\}.$$

We now know that, if $0 < T \le T_3(K)$ and $v \in \mathscr{F}(K,T)$, then $\Phi(v)$ is in $\mathscr{G}_M(-r_0, r_0) \otimes C^0(-T,T)$ and

$$\begin{aligned}(5.10)\qquad \sup_{|t|<T} \|\Phi(v)(\cdot\,,t)\|_M &\le \sup_{|t|<T} \|f(\cdot\,,\cdot\,,\cdot\,,t)\|_L \\ &= \sup_{|t|<T} \max\left\{ 2^6 |f(\cdot\,,\cdot\,,\cdot\,,t)|, 2^3 L^{-1} \max_{1\le i\le 3} |\partial_i f(\cdot\,,\cdot\,,\cdot\,,t)|_L \right\} \\ &\le \max\{2^6 A_f, 2^3 L^{-1} B_f\}.\end{aligned}$$

We want $\Phi(v)$ to be in the family $\mathscr{F}(K,T)$. In view of (5.10) we see that, in order for $\Phi(v)$ to be in this family, it is enough for K to satisfy the inequality $K \ge \max\{2^6 A_f, 2^3 L^{-1} B_f\}$. So we define

$$K_0 := \max\{2^6 A_f, 2^3 L^{-1} B_f\}.$$

We have proved that, if $0 < T \le T_3(K_0)$, then Φ maps $\mathscr{F}(K_0, T)$ into itself.

Our next task is to estimate the difference $\Phi(v_1) - \Phi(v_0)$, where v_1 and v_0 are two elements of the family $\mathscr{F}(K_0, T_3(K_0))$. Take two elements v_0, v_1 of this family arbitrarily and set $v_\theta = \theta v_1 + (1-\theta) v_0$ for $0 < \theta < 1$. Further define

$$(5.11)\qquad \varphi_\theta(x,t) = \int_0^t v_\theta(x,\tau)\,d\tau,$$

$$(5.12)\qquad \omega_{\theta,t}(x,\tau) = \partial_1^p v_\theta(\alpha(x,t)x,\tau)$$

and

$$(5.13)\qquad \psi_\theta(x,t) = \int_0^t \omega_{\theta,t}(x,\tau)\,d\tau.$$

Write

$$(5.14)\qquad A_i(x,t) = \int_0^1 \partial_i f(\varphi_\theta(x,t), \psi_\theta(x,t), x, t)\,d\theta$$

for $i = 1,2$. Then

$$\begin{aligned}&\Phi(v_1)(x,t) - \Phi(v_0)(x,t) \\ &\qquad = A_1(x,t)(\varphi_1(x,t) - \varphi_0(x,t)) + A_2(x,t)(\psi_1(x,t) - \psi_0(x,t))\end{aligned}$$

and

$$\begin{aligned}(5.15)\qquad \|\Phi(v_1)(\cdot\,,t) - \Phi(v_0)(\cdot\,,t)\|_M &\le \|A_1(\cdot\,,t)\|_M \|\varphi_1(\cdot\,,t) - \varphi_0(\cdot\,,t)\|_M \\ &\quad + \|A_2(\cdot\,,t)\|_M \|\psi_1(\cdot\,,t) - \psi_0(\cdot\,,t)\|_M.\end{aligned}$$

In order to estimate the right member of this inequality we write

$$g_{i,\theta,t}(x) = \partial_i f(\varphi_\theta(x,t), \psi_\theta(x,t), x, t)$$

and define $L_1 = (L+M)/2$. Since $f \in \mathscr{G}_L(U) \otimes C^0(-T_0, T_0)$ and $L_1 > L$, we see that

$$D_f := \sup\{\|\partial_i f(\cdot,\cdot,\cdot,t)\|_{L_1}; |t| < T_0, 1 \le i \le 3\} < \infty. \tag{5.16}$$

Therefore we see again by Proposition 2.7 that, if, instead of (5.8), the conditions

$$|\partial_1 \varphi_\theta(\cdot,t)|_M \le \frac{M}{L_1} - 1, \quad |t| < T \tag{5.17}$$

and

$$|\partial_1 \psi_\theta(\cdot,t)|_M \le \frac{M}{L_1} - 1, \quad |t| < T \tag{5.18}$$

are satisfied, then $g_{i,\theta,t} \in \mathscr{G}_M(-r_0, r_0)$ and $\|g_{i,\theta,t}\|_M \le \|\partial_i f(\cdot,\cdot,\cdot,t)\|_{L_1} \le D_f$ holds. From the last inequality we obtain

$$\|A_i(\cdot,t)\|_M \le \sup_{0 \le \theta \le 1} \|g_{i,\theta,t}\|_M \le D_f. \tag{5.19}$$

The condition (5.17) is satisfied, if T satisfies

$$T \le \frac{2^3}{K_0}\left(\frac{1}{L_1} - \frac{1}{M}\right), \tag{5.20}$$

since we have, like (5.4), $|\partial_1 \varphi_\theta(\cdot,t)|_M \le 2^{-3} M K_0 T$ for $|t| < T$. The condition (5.18) is satisfied, if T satisfies

$$T \le \frac{2^3}{K_0 M^p C(m,\lambda)} \left(\frac{\log a}{a\lambda p}\right)^{\lambda p} \left(\frac{1}{L_1} - \frac{1}{M}\right). \tag{5.21}$$

We need further to estimate $\|\varphi_1(\cdot,t) - \varphi_0(\cdot,t)\|_M$ and $\|\psi_1(\cdot,t) - \psi_0(\cdot,t)\|_M$ in terms of $\sup_\tau \|v_1(\cdot,\tau) - v_0(\cdot,\tau)\|_M$. By (5.11) we clearly have

$$\|\varphi_1(\cdot,t) - \varphi_0(\cdot,t)\|_M \le T \sup_{|\tau|<T} \|v_1(\cdot,\tau) - v_0(\cdot,\tau)\|_M. \tag{5.22}$$

As for $\|\psi_1(\cdot,t) - \psi_0(\cdot,t)\|_M$ we note first that we have, like (5.5),

$$|\partial_1^{p+1} v_1(\cdot,t) - \partial_1^{p+1} v_0(\cdot,t)|_{aM} \le \left(\frac{a\lambda p}{\log a}\right)^{\lambda p} M^p |\partial_1 v_1(\cdot,t) - \partial_1 v_0(\cdot,t)|_M.$$

From this we obtain, like (5.6),

$$\begin{aligned} |\partial_1(\omega_{1,t} - \omega_{0,t})(\cdot,\tau)|_M &\le C(m,\lambda) |\partial_1^{p+1} v_1(\cdot,\tau) - \partial_1^{p+1} v_0(\cdot,\tau)|_{aM} \\ &\le M^p C(m,\lambda) \left(\frac{a\lambda p}{\log a}\right)^{\lambda p} |\partial_1(v_1 - v_0)(\cdot,\tau)|_M. \end{aligned} \tag{5.23}$$

Moreover we have

$$(5.24)\qquad |(\omega_{1,t}-\omega_{0,t})(\cdot,\tau)| \le |\partial_1^p(v_1-v_0)(\cdot,\tau)| \le |\partial_1(v_1-v_0)(\cdot,\tau)|_M M^{p-1}\Gamma_\lambda(p-1) \le 2^{-5}M^{p-1}p^{\lambda p}|\partial_1(v_1-v_0)(\cdot,\tau)|_M.$$

From (5.23) and (5.24) we obtain

$$(5.25)\qquad \|(\omega_{1,t}-\omega_{0,t})(\cdot,\tau)\|_M = \max\{2^6|(\omega_{1,t}-\omega_{0,t})(\cdot,\tau)|, 2^3M^{-1}|\partial_1(\omega_{1,t}-\omega_{0,t})(\cdot,\tau)|_M\}$$
$$\le \max\left\{2M^{p-1}p^{\lambda p}, 2^3M^{p-1}C(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\right\}|\partial_1 v_1(\cdot,\tau)-\partial_1 v_0(\cdot,\tau)|_M$$
$$= 2^3M^{p-1}C(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}|\partial_1 v_1(\cdot,\tau)-\partial_1 v_0(\cdot,\tau)|_M$$
$$\le M^pC(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\|v_1(\cdot,t)-v_0(\cdot,t)\|_M.$$

It follows from (5.25) that

$$(5.26)\qquad \|(\psi_1-\psi_0)(\cdot,t)\|_M \le T\sup_{|\tau|<T}\|(\omega_{1,t}-\omega_{0,t})(\cdot,\tau)\|_M \le TM^pC(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\sup_{|\tau|<T}\|v_1(\cdot,\tau)-v_0(\cdot,\tau)\|_M.$$

From (5.15), (5.19), (5.22) and (5.26) it follows that, if T satisfies the inequalities $0<T\le T_3(K_0)$, (5.20) and (5.21), then the inequality

$$\|\Phi(v_1)(\cdot,t)-\Phi(v_0)(\cdot,t)\|_M \le D_fT\left\{1+C(m,\lambda)M^p\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\right\}\sup_{|t|<T}\|v_1(\cdot,t)-v_0(\cdot,t)\|_M$$
$$\le 2D_fTM^pC(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\sup_{|t|<T}\|v_1(\cdot,t)-v_0(\cdot,t)\|_M$$

holds for $|t|<T$. If T satisfies one more inequality

$$(5.27)\qquad T \le \frac{1}{4D_fM^pC(m,\lambda)}\left(\frac{\log a}{a\lambda p}\right)^{\lambda p},$$

then the inequality

$$\|\Phi(v_1)(\cdot,t) - \Phi(v_0)(\cdot,t)\|_M \le 2^{-1}\|v_1(\cdot,t) - v_0(\cdot,t)\|_M$$

holds for $|t| < T$.

We denote by T_1 the maximum value of $T \in (0, T_3(K_0)]$ that satisfies the inequalities (5.20), (5.21) and (5.27). Then Φ maps $\mathscr{F}(K_0, T_1)$ into itself and the inequality

$$\sup_{|t|<T_1} \|\Phi(v_1)(\cdot,t) - \Phi(v_0)(\cdot,t)\|_M \le 2^{-1} \sup_{|t|<T_1} \|v_1(\cdot,t) - v_0(\cdot,t)\|_M$$

holds for every pair $(v_1, v_0) \in \mathscr{F}(K_0, T_1) \times \mathscr{F}(K_0, T_1)$.

It follows that there is a solution $v \in \mathscr{F}(K_0, T_1)$ of the integral equation (5.1). It further follows that there is a solution in $\mathscr{G}_M(-r_0, r_0) \otimes C^1(-T_1, T_1)$ of the Cauchy problem (1.2). □

5.2. Uniqueness of the solution.

The solution of the Cauchy problem (1.2) is unique in the following sense.

THEOREM 5.2. *Let R, T_0, T_1, L, M and r_0 be positive constants. Assume that $T_1 \le T_0$. Write $U = \{(v, w, x) \in \boldsymbol{R}^3; |v| < R, |w| < R, |x| < r_0\}$. In the differential equation* (1.2) *assume that f is an element of the family $\mathscr{G}_L(U) \otimes C^0(-T_0, T_0)$. Assume further that $\alpha(x,t)$ is a real valued continuous function of $(x,t) \in (-r_0, r_0) \times (-T_0, T_0)$ such that the condition* [C1] *in* Theorem 5.1 *is satisfied. Suppose that there are two solutions $u_i \in \mathscr{G}_M(-r_0, r_0) \otimes C^1(-T_1, T_1)$, $i = 0, 1$, such that*

$$|u_i(x,t)| < R \quad \text{and} \quad |\partial_1^p u_i(x,t)| < R \quad \text{for } |x| < r_0, \ |t| < T_1.$$

Then $u_1(x,t) = u_0(x,t)$ for all $(x,t) \in (-r_0, r_0) \times (-T_1, T_1)$.

PROOF. Write $v_i(x,t) = \partial_2 u_i(x,t)$ for $i = 0, 1$. Then each v_i is a solution of the integral equation (5.1) and is in $\mathscr{G}_M(-r_0, r_0) \otimes C^0(-T_1, T_1)$. Further, by the assumption of the theorem, the quantity

$$\max_i \sup_{|t|<T_1} |\partial_1 v_i(\cdot,t)|_M$$

is finite. We denote this quantity by S_v. Next take a number $M^* > \max\{M, L\}$. Then we have of course

$$|\partial_1 v_i(\cdot,t)|_{M^*} \le |\partial_1 v_i(\cdot,t)|_M \le S_v.$$

Moreover, since $M^* > M$, we see that there is another positive number S_v' such that $S_v' \ge S_v$ and

$$|\partial_1^p v_i(\cdot,t)|_{M^*} \le S_v'.$$

Next write $w(x,t) = v_1(x,t) - v_0(x,t)$. As was done in subsection 3.2, we can construct a linear integral equation to which $w(x,t)$ is a solution. In fact set $v_\theta = \theta v_1 + (1-\theta)v_0$ for $0 < \theta < 1$ and define $\varphi_\theta(x,t)$, $\omega_{\theta,t}(x,\tau)$, $\psi_\theta(x,t)$ and $A_i(x,t)$ by (5.11), (5.12), (5.13) and (5.14), respectively. We define further

$$\eta_t(x,\tau) = \partial_1^p w(\alpha(x,t)x,\tau).$$

Then $w(x,t)$ becomes a solution of the integral equation

$$w(x,t) = A_1(x,t)\int_0^t w(x,\tau)\,d\tau + A_2(x,t)\int_0^t \eta_t(x,\tau)\,d\tau. \tag{5.28}$$

Let us see that there is a positive number $M_* \geq M^*$ such that $\|A_i(\cdot\,,t)\|_{M_*} < \infty$ for $|t| < T_1$. Since $|\partial_1 v_\theta(\cdot\,,t)|_{M^*} \leq S_v \leq S_v'$, we have $|\partial_1\varphi_\theta(\cdot\,,t)|_{M^*} \leq S_v'T_1$. Similarly we have $|\partial_1\psi_\theta(\cdot\,,t)|_{M^*} \leq S_v'T_1$. In view of these facts we write

$$L_1 = \frac{L+M^*}{2}, \quad M_* = \max\{M^*, L_1(1+S_v'T_1)\}.$$

Then we have

$$L_1(1+\max\{|\partial_1\varphi_\theta(\cdot\,,t)|_{M_*}, |\partial_1\psi_\theta(\cdot\,,t)|_{M_*}\}) \leq L_1(1+S_v'T_1) \leq M_*.$$

Note also that, since $f \in \mathscr{G}_L(U) \otimes C^0(-T_0,T_0)$ and $L_1 > L$, the finite constant D_f is defined by (5.16). In view of these facts we can use Proposition 2.7 again and conclude that A_i is in $\mathscr{G}_{M_*}(-r_0,r_0) \otimes C^0(-T_1,T_1)$ and

$$\sup_{|t|<T_1} \|A_i(\cdot\,,t)\|_{M_*} \leq D_f. \tag{5.29}$$

Next we estimate the two integrals in the right member of (5.28). It is clear that the inequality

$$\left\|\int_0^t w(\cdot\,,\tau)\,d\tau\right\|_{M_*} \leq \left|\int_0^t \|w(\cdot\,,\tau)\|_{M_*}\,d\tau\right| \tag{5.30}$$

holds. Moreover, just like (5.6), we have

$$|\partial_1\eta_t(\cdot\,,\tau)|_{M_*} \leq 2^{-3}M_*^{p+1}C(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\|w(\cdot\,,\tau)\|_{M_*} \tag{5.31}$$

and

$$\begin{aligned} |\eta_t(\cdot\,,\tau)| &= \sup_x |\eta_t(x,\tau)| \leq \Gamma_\lambda(0)|\eta_t(\cdot\,,\tau)|_{M_*} \\ &\leq \Gamma_\lambda(0)2^{-3}M_*^pC(m,\lambda)\left(\frac{a\lambda(p-1)}{\log a}\right)^{\lambda(p-1)}\|w(\cdot\,,\tau)\|_{M_*} \\ &\leq 2^{-8}M_*^pC(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\|w(\cdot\,,\tau)\|_{M_*}. \end{aligned} \tag{5.32}$$

It follows from (5.31) and (5.32) that the inequality

$$(5.33)\qquad \|\eta_t(\cdot,\tau)\|_{M_*} = \max\{2^6|\eta_t(\cdot,\tau)|, 2^3 M_*^{-1}|\partial_1\eta_t(\cdot,\tau)|_{M_*}\} \le M_*^p C(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\|w(\cdot,\tau)\|_{M_*}$$

holds. From (5.28), (5.29), (5.30) and (5.33) we obtain the inequality

$$\|w(\cdot,t)\|_{M_*} \le \|A_1(\cdot,t)\|_{M_*}\left\|\int_0^t w(\cdot,\tau)\,d\tau\right\|_{M_*} + \|A_2(\cdot,t)\|_{M_*}\left\|\int_0^t \eta_t(\cdot,\tau)\,d\tau\right\|_{M_*} \le D_f\left\{1 + M_*^p C(m,\lambda)\left(\frac{a\lambda p}{\log a}\right)^{\lambda p}\right\}\left|\int_0^t \|w(\cdot,\tau)\|_{M_*}\,d\tau\right|$$

that holds for $|t| < T_1$. It follows that $\|w(\cdot,t)\|_{M_*} = 0$ for $|t| < T_1$. This means that $w(x,t) = 0$ and $v_0(x,t) = v_1(x,t)$ for all $(x,t) \in (-r_0, r_0) \times (-T_1, T_1)$. □

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