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Two criterions of Wiener type for minimally thin sets and rarefied sets in a cone

Dedicated to Professor Masayuki Itô on his 60th birthday

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Abstract. We shall give two criterions of Wiener type which characterize minimally thin sets and rarefied sets in a cone. We shall also show that a positive superharmonic function on a cone behaves regularly outside a rarefied set in a cone. These facts are known for a half space which is a special cone.

1. Introduction.

Let $\boldsymbol{R}$ and $\boldsymbol{R}_+$ be the set of all real numbers and all positive real numbers, respectively. We denote by $\boldsymbol{R}^n$ $(n \geq 2)$ the n-dimensional Euclidean space. A point in $\boldsymbol{R}^n$ is denoted by $P = (X, y)$, $X = (x_1, x_2, \dots, x_{n-1})$. The boundary and the closure of a set S in $\boldsymbol{R}^n$ are denoted by ∂S and $\bar{S}$, respectively. We introduce the system of the spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in $\boldsymbol{R}^n$ which are related to the cartesian coordinates $(x_1, x_2, \dots, x_{n-1}, y)$ by

$$x_1 = r(\Pi_{j=1}^{n-1} \sin\theta_j) \quad (n \geq 2), \quad y = r\cos\theta_1,$$

and if $n \geq 3$, then

$$x_{n+1-k} = r(\Pi_{j=1}^{k-1} \sin\theta_j)\cos\theta_k \quad (2 \leq k \leq n-1),$$

where $0 \leq r < +\infty$, $-(1/2)\pi \leq \theta_{n-1} < (3/2)\pi$, and if $n \geq 3$, then $0 \leq \theta_j \leq \pi$ $(1 \leq j \leq n-2)$. The unit sphere and the upper half unit sphere are denoted by $\boldsymbol{S}^{n-1}$ and $\boldsymbol{S}_+^{n-1}$, respectively. For simplicity, a point $(1, \Theta)$ on $\boldsymbol{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set $\Omega, \Omega \subset \boldsymbol{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Lambda \subset \boldsymbol{R}_+$ and $\Omega \subset \boldsymbol{S}^{n-1}$, the set

$$\{(r, \Theta) \in \boldsymbol{R}^n; r \in \Lambda, (1, \Theta) \in \Omega\}$$

in $\boldsymbol{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half-space

$$\boldsymbol{R}_+ \times \boldsymbol{S}_+^{n-1} = \{(X, y) \in \boldsymbol{R}^n; y > 0\}$$

will be denoted by $\boldsymbol{T}_n$.

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Lelong-Ferrand [**15**] investigated the regularity of value distribution of a positive superharmonic function on $\boldsymbol{T}_n$. Let $G(P,Q)$ $(P \in \boldsymbol{T}_n, Q \in \boldsymbol{T}_n)$ be the Green function of $\boldsymbol{T}_n$. The regularized reduced function $\hat{R}_h^E(P)$ of $h(X,y) = y$ $((X,y) \in \boldsymbol{T}_n)$ relative to a bounded set E of $\boldsymbol{T}_n$ has a unique positive measure λ_E on $\boldsymbol{T}_n$ such that

$$\hat{R}_h^E(P) = G\lambda_E(P)$$

for any $P \in \boldsymbol{T}_n$, where $G\lambda_E(P)$ $(P \in \boldsymbol{T}_n)$ is the Green potential of λ_E. By $\lambda(E)$, we denote the total mass $\lambda_E(\boldsymbol{T}_n)$ of λ_E which was called *charge extérieure de E* by Lelong-Ferrand [**15**, p. 129]. We denote the (Green) energy

$$\int_{\boldsymbol{T}_n} (G\lambda_E)(P)\, d\lambda_E(P)$$

of λ_E by $\gamma(E)$, which was originally introduced by Lelong-Ferrand [**15**] and called *puissance extérieure de E*. Let E be a subset of $\boldsymbol{T}_n$ and set

$$E_k = E \cap I_k,$$

where

$$I_k = \{(r,\Theta) \in \boldsymbol{T}_n; 2^k \le r < 2^{k+1}\} \quad (k = 0, 1, 2, \dots).$$

Lelong-Ferrand said that E is *effilé* at ∞ with respect to $\boldsymbol{T}_n$, if

$$\sum_{k=0}^{\infty} \gamma(E_k) 2^{-kn} < +\infty, \tag{1.1}$$

and proved the following fact.

Theorem A (Lelong-Ferrand [**15**, Théorème 1c]). *Let $v(P) = v(r,\Theta)$ be a positive superharmonic function on $\boldsymbol{T}_n$ and put*

$$c = \inf_{P=(X,y) \in \boldsymbol{T}_n} \frac{v(P)}{y}.$$

Then there is a subset $E, E \subset \boldsymbol{T}_n$ effilé at ∞ with respect to $\boldsymbol{T}_n$ such that

$$\frac{v(P)}{y} \quad (P = (X,y) = (r,\Theta) \in \boldsymbol{T}_n)$$

uniformly converges to c on $\boldsymbol{T}_n - E$ as $r \to +\infty$.

Essén and Jackson [7, Remark 3.1] observed that a subset E of $\boldsymbol{T}_n$ is effilé at ∞ with respect to $\boldsymbol{T}_n$ if and only if E is minimally thin at ∞ with respect to $\boldsymbol{T}_n$. To state the definition of minimally thin sets at ∞ with respect to $\boldsymbol{T}_n$, which is based on minimal thinness at a Martin boundary point (Brelot [**5**, p. 122], Doob

[**6**, p. 208]), observe that the Martin boundary of $\boldsymbol{T}_n$ is $\partial\boldsymbol{T}_n \cup \{\infty\}$ and ∞ is a minimal Martin boundary point of $\boldsymbol{T}_n$. Let $K(P,Q)$ $(P \in \boldsymbol{T}_n, Q \in \partial\boldsymbol{T}_n)$ be the Martin kernel with the reference point $(0,0,\dots,0,1) \in \boldsymbol{T}_n$. Then $K(P,\infty) = y$ for any $P = (X,y) \in \boldsymbol{T}_n$. A subset E of $\boldsymbol{T}_n$ is said to be *minimally thin* at ∞ with respect to $\boldsymbol{T}_n$, if there exists a point $P = (X,y)$ such that

$$\hat{R}^E_{K(\cdot,\infty)}(P) \neq y,$$

where $\hat{R}^E_{K(\cdot,\infty)}$ is the regularized reduced function of $K(P,\infty) = y$ $(P = (X,y) \in \boldsymbol{T}_n)$ relative E.

In connection with Ahlfors and Heins [**1**], Hayman [**11**], Ušakova [**16**] and Azarin [**4**], Essén and Jackson [**7**] introduced the following notion, similar to the minimal thinness. A subset E of $\boldsymbol{T}_n$ is said to be *rarefied* at ∞ with respect to $\boldsymbol{T}_n$, if

$$\sum_{k=0}^{\infty} \lambda(E_k) 2^{-k(n-1)} < +\infty \tag{1.2}$$

(Essén and Jackson [**7**, p. 244]). Since a rarefied set at ∞ with respect to $\boldsymbol{T}_n$ is also a minimally thin set at ∞ with respect to $\boldsymbol{T}_n$ (Essén and Jackson [**7**, Remark 3.2]), we can expect much stronger conclusion than the conclusion of Theorem A. In fact, the following theorem was proved.

Theorem B (Essén and Jackson [**7**, Theorem 4.6 and Remark 4.2]). *Let $v(P) = v(r,\Theta)$ be a positive superharmonic function on $\boldsymbol{T}_n$ and put*

$$c^*(v) = \inf_{P=(X,y)\in\boldsymbol{T}_n} \frac{v(P)}{y}.$$

Then there is a rarefied set E at ∞ with respect to $\boldsymbol{T}_n$ such that $v(P)r^{-1}$ uniformly converges to $c^*(v)\cos\theta_1$ on $\boldsymbol{T}_n - E$ as $r \to +\infty$ $(P = (r,\Theta) \in \boldsymbol{T}_n)$.

There is also another definition of rarefied sets at ∞ with respect to $\boldsymbol{T}_n$. A subset E of $\boldsymbol{T}_n$ is said to be rarefied at ∞ with respect to $\boldsymbol{T}_n$, if there exists a positive superharmonic function v in $\boldsymbol{T}_n$ such that

$$\inf_{P=(X,y)\in\boldsymbol{T}_n} \frac{v(P)}{y} = 0$$

and

$$v(P) \geq r$$

for any $P = (r,\Theta) \in E$ (Essén and Jackson [**7**, Remark 4.4], Aikawa and Essén [**3**, Difinition 12.4, p. 74] and Hayman [**12**, p. 474]). It was proved that a subset E

of $\boldsymbol{T}_n$ is rarefied at ∞ with respect to $\boldsymbol{T}_n$ according to this definition if and only if (1.2) holds (Aikawa and Essén [**3**, Theorem 13.1]).

By $C_n(\Omega)$, we denote the set $\boldsymbol{R}_+ \times \Omega$ in $\boldsymbol{R}^n$ with a domain Ω on $\boldsymbol{S}^{n-1} (n \geq 2)$ having smooth boundary. We call it a cone. Then $\boldsymbol{T}_n$ is the special cone obtained by putting $\Omega = \boldsymbol{S}_+^{n-1}$. When we transform $\boldsymbol{T}_n$ or $C_n(\Omega)$ into a bounded domain by a Kelvin transformation, we know that $\boldsymbol{T}_n$ is a typical domain having a smooth boundary at ∞, but $C_n(\Omega)$ except $\boldsymbol{T}_n$ is an example of domains which have corners at ∞.

In this paper, we shall give two criterions of Wiener type for minimally thin sets and rarefied sets at ∞ with respect to $C_n(\Omega)$, which extend (1.1) and (1.2) (Theorems 1 and 2). These criterions will be useful to make coverings over these exceptional sets in $C_n(\Omega)$ by a sequence of disks, which were exemplified in the case of the half space $\boldsymbol{T}_n$ by Essén and Jackson [7] and Essén, Jackson and Rippon [**8**]. By using one of them, we shall generalize Theorem B for positive superharmonic functions on $C_n(\Omega)$ (Theorem 3), while the generalization of Theorem A can be regarded as the special case of the Fatou boundary limit theorem for the Martin space (Remark 2). Finally these criterions will give some connection between both exceptional sets (Theorem 4).

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2. Statement of results.

Let Ω be a domain on $\boldsymbol{S}^{n-1}$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. Consider the Dirichlet problem

$$(\Lambda_n + \lambda)f = 0 \quad \text{on } \Omega$$

$$f = 0 \quad \text{on } \partial\Omega,$$

where Λ_n is the spherical part of the Laplace operator Δ_n

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_n.$$

We denote the least positive eigenvalue of this boundary value problem by λ_Ω and the normalized positive eigenfunction corresponding to λ_Ω by $f_\Omega(\Theta)$;

$$\int_\Omega f_\Omega^2(\Theta)\, d\sigma_\Theta = 1,$$

where $d\sigma_\Theta$ is the surface element on $\boldsymbol{S}^{n-1}$. We denote the solutions of the equation

$$t^2 + (n-2)t - \lambda_\Omega = 0$$

by $\alpha_\Omega, -\beta_\Omega$ $(\alpha_\Omega, \beta_\Omega > 0)$ and write δ_Ω for $\alpha_\Omega + \beta_\Omega$. If $\Omega = \boldsymbol{S}_+^{n-1}$, then $\alpha_\Omega = 1$, $\beta_\Omega = n-1$, and

$$f_\Omega(\Theta) = (2ns_n^{-1})^{1/2} \cos\theta_1,$$

where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\boldsymbol{S}^{n-1}$.

In the following, we put the strong assumption relative to Ω on $\boldsymbol{S}^{n-1}$: If $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\boldsymbol{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see Gilbarg and Trudinger [**10**, pp. 88–89] for the definition of $C^{2,\alpha}$-domain).

It is known that the Martin boundary Δ of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$ and each point of Δ is a minimal Martin boundary point. When we denote the Martin kernel by $K(P,Q)$ $(P \in C_n(\Omega), Q \in \partial C_n(\Omega) \cup \{\infty\})$ with respect to a reference point chosen suitably, we know

$$K(P,\infty) = r^{\alpha_\Omega} f_\Omega(\Theta), \quad K(P,O) = \kappa r^{-\beta_\Omega} f_\Omega(\Theta) \quad (P \in C_n(\Omega)),$$

where O denotes the origin of $\boldsymbol{R}^n$ and κ is a positive constant.

A subset E of $C_n(\Omega)$ is said to be *minimally thin* at $Q \in \Delta$ with respect to $C_n(\Omega)$ (Brelot [**5**, p. 122], Doob [**6**, p. 208]), if there exists a point $P \in C_n(\Omega)$ such that

$$\hat{R}^E_{K(\cdot,Q)}(P) \neq K(P,Q),$$

where $\hat{R}^E_{K(\cdot,Q)}(P)$ is the regularized reduced function of $K(\cdot,Q)$ relative to E (Helms [**13**, p. 134]).

Let E be a bounded subset of $C_n(\Omega)$. Then $\hat{R}^E_{K(\cdot,\infty)}$ is bounded on $C_n(\Omega)$ and hence the greatest harmonic minorant of $\hat{R}^E_{K(\cdot,\infty)}$ is zero (see Yoshida [**17**, Corollary 5.1]). When we denote by $G(P,Q)$ $(P \in C_n(\Omega), Q \in C_n(\Omega))$ the Green function of $C_n(\Omega)$, we see from the Riesz decomposition theorem that there exists a unique positive measure λ_E on $C_n(\Omega)$ such that

$$\hat{R}^E_{K(\cdot,\infty)}(P) = G\lambda_E(P)$$

for any $P \in C_n(\Omega)$ and λ_E is concentrated on B_E, where

$$B_E = \{P \in C_n(\Omega); E \text{ is not thin at } P\}$$

(see Brelot [**5**, Theorem VIII, 11] and Doob [**6**, XI, 14. Theorem (d)]). We denote the total mass $\lambda_E(C_n(\Omega))$ of λ_E by $\lambda_\Omega(E)$. The (Green) energy $\gamma_\Omega(E)$ of λ_E is defined by

$$\gamma_\Omega(E) = \int_{C_n(\Omega)} (G\lambda_E)\, d\lambda_E$$

(see Helms [**13**, p. 223]). Then $\lambda_{\Omega}(E)$ and $\gamma_{\Omega}(E)$ with $\Omega = \boldsymbol{S}_{+}^{n-1}$ are simply denoted by $\lambda(E)$ and $\gamma(E)$, respectively.

Let E be a subset of $C_n(\Omega)$ and $E_k = E \cap I_k(\Omega)$, where

$$I_k(\Omega) = \{(r, \Theta) \in C_n(\Omega); 2^k \le r < 2^{k+1}\}.$$

Theorem 1. *A subset E of $C_n(\Omega)$ is minimally thin at ∞ with respect to $C_n(\Omega)$ if and only if*

$$\sum_{k=0}^{\infty} \gamma_{\Omega}(E_k) 2^{-k\delta_{\Omega}} < +\infty. \tag{2.1}$$

When $\Omega = \boldsymbol{S}_{+}^{n-1}$, we immediately obtain

Corollary 1 (Aikawa and Essén [**3**, Theorem 11.3]). *A set $E, E \subset \boldsymbol{T}_n$, is minimally thin at ∞ with respect to $\boldsymbol{T}_n$ if and only if*

$$\sum_{k=0}^{\infty} \gamma(E_k) 2^{-kn} < +\infty.$$

A subset E of $C_n(\Omega)$ is said to be *rarefied* at ∞ with respect to $C_n(\Omega)$, if there exists a positive superharmonic function $v(P)$ in $C_n(\Omega)$ such that

$$\inf_{P \in C_n(\Omega)} \frac{v(P)}{K(P, \infty)} = 0$$

and

$$E \subset H_v,$$

where

$$H_v = \{P = (r, \Theta) \in C_n(\Omega); v(P) \ge r^{\alpha_{\Omega}}\}.$$

Theorem 2. *A subset E of $C_n(\Omega)$ is rarefied at ∞ with respect to $C_n(\Omega)$ if and only if*

$$\sum_{k=0}^{\infty} 2^{-k\beta_{\Omega}} \lambda_{\Omega}(E_k) < +\infty.$$

Corollary 2 (Aikawa and Essén [**3**, Theorem 13.1]). *A subset E of $\boldsymbol{T}_n$ is rarefied at ∞ with respect to $\boldsymbol{T}_n$ if and only if*

$$\sum_{k=0}^{\infty} 2^{-k(n-1)} \lambda(E_k) < +\infty.$$

In the following, we set

$$c(v) = \inf_{P \in C_n(\Omega)} \frac{v(P)}{K(P, \infty)}$$

for a positive superharmonic function $v(P)$ on $C_n(\Omega)$. We immediately see that $c(v) < +\infty$. In fact, let $u(P)$ be a subharmonic function on $C_n(\Omega)$ satisfying

$$\overline{\lim_{P \to Q, P \in C_n(\Omega)}} u(P) \le 0 \tag{2.2}$$

for any $Q \in \partial C_n(\Omega)$ and

$$\sup_{P=(r,\Theta) \in C_n(\Omega)} \frac{u(P)}{r^{\alpha_\Omega} f_\Omega(\Theta)} = \ell < +\infty. \tag{2.3}$$

Then we know $\ell > -\infty$ (e.g. see Yoshida [**17**, Lemma 6.1]). If we apply this fact to $u = -v$, then we obtain

$$c(v) < +\infty.$$

The following theorem 3 which is obtained from Theorem 2 generalizes Theorem B.

Theorem 3. *Let $v(P)$ be a positive superharmonic function on $C_n(\Omega)$. Then there exists a rarefied set E at ∞ with respect to $C_n(\Omega)$ such that $v(P)r^{-\alpha_\Omega}$ uniformly converges to $c(v)f_\Omega(\Theta)$ on $C_n(\Omega) - E$ as $r \to +\infty$ $(P = (r, \Theta) \in C_n(\Omega))$.*

Remark 1. We observe the following fact from the definition of rarefied sets. Given any rarefied set E at ∞ with respect to $C_n(\Omega)$, there exists a positive superharmonic function $v(P)$ on $C_n(\Omega)$ such that $v(P)r^{-\alpha_\Omega} \ge 1$ on E and $c(v) = 0$. Hence $v(P)r^{-\alpha_\Omega}$ does not converge to $c(v)f_\Omega(\Theta) = 0$ on E as $r \to +\infty$.

Let $u(P)$ be a subharmonic function on $C_n(\Omega)$ satisfying (2.2) and (2.3). Then

$$v(P) = \ell r^{\alpha_\Omega} f_\Omega(\Theta) - u(P) \quad (P = (r, \Theta) \in C_n(\Omega)),$$

is a positive superharmonic function on $C_n(\Omega)$ such that $c(v) = 0$. If we apply Theorem 3 to this $v(P)$, then we obtain the following Corollary which is a part of Azarin's result [**4**, Theorem 2].

Corollary 3. *Let $u(P)$ be a subharmonic function on $C_n(\Omega)$ satisfying* (2.2) *and* (2.3). *Then there exists a rarefied set E at ∞ with respect to $C_n(\Omega)$ such that $u(P)r^{-\alpha_\Omega}$ uniformly converges to $\ell f_\Omega(\Theta)$ on $C_n(\Omega) - E$ as $r \to +\infty$ $(P = (r, \Theta) \in C_n(\Omega))$.*

Remark 2. Without (2.1) the conical version of theorem A is immediately obtained by specializing the Fatou boundary limit theorem for the Martin space (e.g. see Doob [**6**, XII, 13. Theorem (a)]): for any positive superharmonic function $v(P)$,

$$\operatorname*{mf\,lim}_{P=(r,\Theta)\in C_n(\Omega),\,r\to+\infty} \frac{v(P)}{K(P,\infty)} = c(v),$$

where "mf limit" means minimal-fine limit i.e. there exists a minimally thin set E at ∞ with respect to $C_n(\Omega)$ such that $v(P)/K(P,\infty)$ uniformly converges to $c(v)$ on $C_n(\Omega) - E$ as $r \to +\infty$ $(P = (r,\Theta) \in C_n(\Omega))$, and for any minimally thin set E at ∞ with respect to $C_n(\Omega)$ there exists a positive superharmonic function $v(P)$ such that

$$\lim_{P=(r,\Theta)\in E,\,r\to+\infty} \frac{v(P)}{K(P,\infty)} = +\infty.$$

A cone $C_n(\Omega')$ is called a subcone of $C_n(\Omega)$, if $\overline{\Omega'} \subset \Omega$ ($\overline{\Omega'}$ is the closure of Ω' on $\boldsymbol{S}^{n-1}$). As in $\boldsymbol{T}_n$ (Essén and Jackson [**7**, Remark 3.2]), we have

Theorem 4. *Let E be a subset of $C_n(\Omega)$. If E is rarefied at ∞ with respect to $C_n(\Omega)$, then E is minimally thin at ∞ with respect to $C_n(\Omega)$. If E is contained in a subcone of $C_n(\Omega)$ and E is minimally thin at ∞ with respect to $C_n(\Omega)$, then E is also rarefied at ∞ with respect to $C_n(\Omega)$.*

3. Lemmas.

In the following we denote the sets $[a,b) \times \Omega$, $[a,b) \times \partial\Omega$ and $(0,b) \times \partial\Omega$ by $C_n(\Omega; a,b)$, $S_n(\Omega; a,b)$ and $S_n(\Omega; 0,b)$, respectively $(0 < a < b \le +\infty)$. Hence $S_n(\Omega; 0,+\infty)$ denoted simply by $S_n(\Omega)$ is $\partial C_n(\Omega) - \{O\}$.

First of all, we remark that

$$C_1 r^{\alpha_\Omega} t^{-\beta_\Omega} f_\Omega(\Theta) f_\Omega(\Phi) \le G(P,Q) \le C_2 r^{\alpha_\Omega} t^{-\beta_\Omega} f_\Omega(\Theta) f_\Omega(\Phi) \tag{3.1}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in C_n(\Omega)$ satisfying $0 < r/t \le 1/2$, where C_1 and C_2 are two positive constants (Azarin [**4**, Lemma 1], Essén and Lewis [**9**, Lemma 2]).

Lemma 1. *Let μ be a positive measure on $C_n(\Omega)$ such that there is a sequence of points $P_i = (r_i, \Theta_i) \in C_n(\Omega)$, $r_i \to +\infty$ $(i \to +\infty)$ satisfying*

$$G\mu(P_i) = \int_{C_n(\Omega)} G(P_i, Q)\, d\mu(t,\Phi) < +\infty \quad (i = 1,2,3,\cdots; Q = (t,\Phi) \in C_n(\Omega)).$$

Then for a positive number ℓ,

$$\int_{C_n(\Omega;\ell,+\infty)} t^{-\beta_\Omega} f_\Omega(\Phi)\, d\mu(t,\Phi) < +\infty \tag{3.2}$$

and

$$\lim_{L\to+\infty} L^{-\delta_\Omega}\int_{C_n(\Omega;0,L)} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi)=0. \tag{3.3}$$

Proof. Take a positive number ℓ satisfying $P_1=(r_1,\Theta_1)\in C_n(\Omega)$, $r_1\le \ell/2$. Then from (3.1), we have

$$C_1 r_1^{\alpha_\Omega} f_\Omega(\Theta_1)\int_{C_n(\Omega;\ell,+\infty)} t^{-\beta_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi)\le \int_{C_n(\Omega)} G(P_1,Q)\,d\mu(Q)<+\infty,$$

which gives (3.2). For any positive number ε, from (3.2) we can take a large number A such that

$$\int_{C_n(\Omega;A,+\infty)} t^{-\beta_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi)<\frac{\varepsilon}{2}.$$

If we take a point $P_i=(r_i,\Theta_i)\in C_n(\Omega), r_i\ge 2A$, then we have from (3.1)

$$C_1 r_i^{-\beta_\Omega} f_\Omega(\Theta_i)\int_{C_n(\Omega;0,A)} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi)\le \int_{C_n(\Omega)} G(P_i,Q)\,d\mu(Q)<+\infty.$$

If L $(L>A)$ is a sufficiently large, then

$$\begin{aligned}
&L^{-\delta_\Omega}\int_{C_n(\Omega;0,L)} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi)\\
&\quad\le L^{-\delta_\Omega}\int_{C_n(\Omega;0,A)} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi)+\int_{C_n(\Omega;A,L)} t^{-\delta_\Omega} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi)\\
&\quad\le L^{-\delta_\Omega}\int_{C_n(\Omega;0,A)} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi)+\int_{C_n(\Omega;A,+\infty)} t^{-\delta_\Omega} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi)<\varepsilon,
\end{aligned}$$

which gives (3.3). □

Lemma 2. *Let μ be a positive measure on $C_n(\Omega)$ for which $G\mu(P)$ is defined. Then for any positive number B the set*

$$\{(r,\Theta)\in C_n(\Omega); G\mu(r,\Theta)\ge B r^{\alpha_\Omega} f_\Omega(\Theta)\}$$

is minimally thin at ∞ with respect to $C_n(\Omega)$.

Proof. To the positive superharmonic function $G\mu$, apply a result in Doob [**6**, p. 213] which was stated in Remark 2. Then

$$\underset{P=(r,\Theta)\in C_n(\Omega),\, r\to+\infty}{\mathrm{mf}\lim} \frac{G\mu(P)}{K(P,\infty)}=\inf_{P\in C_n(\Omega)}\frac{G\mu(P)}{K(P,\infty)}=0,$$

because the greatest harmonic minorant of $G\mu$ is zero. This gives the conclusion. □

The following Lemma 3 is essentially due to Azarin [**4**]. Here we shall give a simple proof different from Azarin's proof.

Lemma 3 (Azarin [**4**, Theorem 1]). *Let $v(P)$ be a positive superharmonic function on $C_n(\Omega)$ and put*

$$c(v) = \inf_{P \in C_n(\Omega)} \frac{v(P)}{K(P,\infty)}, \quad c_0(v) = \inf_{P \in C_n(\Omega)} \frac{v(P)}{K(P,O)}. \tag{3.4}$$

Then there are a unique positive measure μ on $C_n(\Omega)$ and a unique positive measure ν on $S_n(\Omega)$ such that

$$v(P) = c(v)K(P,\infty) + c_0(v)K(P,O) + \int_{C_n(\Omega)} G(P,Q)\,d\mu(Q) + \int_{S_n(\Omega)} \frac{\partial G(P,Q)}{\partial n_Q}\,d\nu(Q),$$

where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$.

Proof. By the Riesz decomposition theorem, we have a unique measure μ on $C_n(\Omega)$ such that

$$v(P) = \int_{C_n(\Omega)} G(P,Q)\,d\mu(Q) + h(P) \quad (P \in C_n(\Omega)), \tag{3.5}$$

where h is the greatest harmonic minorant of v on $C_n(\Omega)$. Further by the Martin representation theorem we have another positive measure ν' on $\partial C_n(\Omega) \cup \{\infty\}$

$$\begin{aligned} h(P) &= \int_{\partial C_n(\Omega) \cup \{\infty\}} K(P,Q)\,d\nu'(Q) \\ &= K(P,\infty)\nu'(\{\infty\}) + K(P,O)\nu'(\{O\}) + \int_{S_n(\Omega)} K(P,Q)\,d\nu'(Q) \quad (P \in C_n(\Omega)). \end{aligned}$$

We see from (3.4) that $\nu'(\{\infty\}) = c(v)$ and $\nu'(\{O\}) = c_0(v)$ (see Yoshida [**17**, p. 292]). Since

$$K(P,Q) = \lim_{P_1 \to Q,\, P_1 \in C_n(\Omega)} \frac{G(P,P_1)}{G(P_0,P_1)} = \frac{\partial G(P,Q)}{\partial n_Q} \bigg/ \frac{\partial G(P_0,Q)}{\partial n_Q} \tag{3.6}$$

(P_0 is a fixed reference point of the Martin kernel), we also obtain

$$h(P) = c(v)K(P,\infty) + c_0(v)K(P,O) + \int_{S_n(\Omega)} \frac{\partial G(P,Q)}{\partial n_Q} dv(Q)$$

by taking

$$dv(Q) = \left\{\frac{\partial G(P_0,Q)}{\partial n_Q}\right\}^{-1} dv'(Q) \quad (Q \in S_n(\Omega)).$$

Finally this and (3.5) give the conclusion of this lemma. □

We remark the following inequality which follows from (3.1).

$$(3.7) \quad C_1 r^{\alpha_\Omega} t^{-\beta_\Omega - 1} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi) \le \frac{\partial G(P,Q)}{\partial n_Q} \le C_2 r^{\alpha_\Omega} t^{-\beta_\Omega - 1} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)$$

$$\left(\text{resp. } C_1 t^{\alpha_\Omega - 1} r^{-\beta_\Omega} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi) \le \frac{\partial G(P,Q)}{\partial n_Q} \le C_2 t^{\alpha_\Omega - 1} r^{-\beta_\Omega} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\right)$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and $Q = (t,\Phi) \in S_n(\Omega)$ satisfying $0 < r/t \le 1/2$ (resp. $0 < t/r \le 1/2$), where $\partial/\partial n_\Phi$ denotes the differentiation at $\Phi \in \partial\Omega$ along the inward normal into Ω (Azarin [**4**, Lemma 1]).

Lemma 4. *Let v be a positive measure on $S_n(\Omega)$ such that there is a sequence of points $P_i = (r_i,\Theta_i) \in C_n(\Omega), r_i \to +\infty \ (i \to +\infty)$ satisfying*

$$\int_{S_n(\Omega)} \frac{\partial G(P_i,Q)}{\partial n_Q} dv(Q) < +\infty \quad (i = 1,2,3,\dots).$$

Then for a positive number ℓ

$$\int_{S_n(\Omega;\ell,+\infty)} t^{-\beta_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\, dv(t,\Phi) < +\infty$$

and

$$\lim_{R\to+\infty} R^{-\delta_\Omega} \int_{S_n(\Omega;0,R)} t^{\alpha_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\, dv(t,\Phi) = 0.$$

Proof. If we use (3.7) in place of (3.1), we obtain this lemma in the completely paralleled way to the proof of Lemma 1. □

Lemma 5. *Let E be a bounded subset of $C_n(\Omega)$ and $u(P)$ be a positive superharmonic function on $C_n(\Omega)$ such that $u(P)$ is represented as*

$$(3.8) \quad u(P) = \int_{C_n(\Omega)} G(P,Q)\, d\mu_u(Q) + \int_{S_n(\Omega)} \frac{\partial}{\partial n_Q} G(P,Q)\, dv_u(Q) \quad (P \in C_n(\Omega))$$

with two positive measures μ_u and v_u on $C_n(\Omega)$ and $S_n(\Omega)$, respectively, and satisfies

$$u(P) \ge 1$$

for any $P \in E$. *Then*

$$\lambda_\Omega(E) \le \int_{C_n(\Omega)} t^{\alpha_\Omega} f_\Omega(\Phi)\, d\mu_u(t,\Phi) + \int_{S_n(\Omega)} t^{\alpha_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\, dv_u(t,\Phi). \tag{3.9}$$

When $u(P) = \hat{R}_1^E(P)$ $(P \in C_n(\Omega))$, *the equality holds in* (3.9).

Proof. Since λ_E is concentrated on B_E and $u(P) \ge 1$ for any $P \in B_E$, we see from (3.8) that

$$\begin{aligned} \lambda_\Omega(E) &= \int_{C_n(\Omega)} d\lambda_E \le \int_{C_n(\Omega)} u(P)\, d\lambda_E(P) \\ &= \int_{C_n(\Omega)} \hat{R}^E_{K(\cdot,\infty)}(Q)\, d\mu_u(Q) \\ &\quad + \int_{S_n(\Omega)} \left(\int_{C_n(\Omega)} \frac{\partial}{\partial n_Q} G(P,Q)\, d\lambda_E(P) \right) dv_u(Q). \end{aligned} \tag{3.10}$$

Now we have

$$\hat{R}^E_{K(\cdot,\infty)}(Q) \le K(Q,\infty) = t^{\alpha_\Omega} f_\Omega(\Phi) \quad (Q = (t,\Phi) \in C_n(\Omega)). \tag{3.11}$$

Since

$$\int_{C_n(\Omega)} \frac{\partial}{\partial n_Q} G(P,Q)\, d\lambda_E(P) \le \varliminf_{\rho \to 0} \frac{1}{\rho} \int_{C_n(\Omega)} G(P,P_\rho)\, d\lambda_E(P)$$

for any $Q \in S_n(\Omega)$ $(P_\rho = (r_\rho, \Theta_\rho) = Q + \rho n_Q \in C_n(\Omega), n_Q$ is the inward normal unit vector at $Q)$ and

$$\int_{C_n(\Omega)} G(P,P_\rho)\, d\lambda_E(P) = \hat{R}^E_{K(\cdot,\infty)}(P_\rho) \le K(P_\rho,\infty) = r_\rho^{\alpha_\Omega} f_\Omega(\Theta_\rho),$$

we have

$$\int_{C_n(\Omega)} \frac{\partial}{\partial n_Q} G(P,Q)\, d\lambda_E(P) \le t^{\alpha_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi) \tag{3.12}$$

for any $Q = (t,\Phi) \in S_n(\Omega)$. Thus from (3.10), (3.11) and (3.12) we obtain (3.9).

When $u(P) = \hat{R}_1^E(P)$, $u(P)$ has the expression (3.8) by Lemma 3, because $\hat{R}_1^E(P)$ is bounded on $C_n(\Omega)$. Then we easily have the equalities only in (3.10), because $\hat{R}_1^E(P) = 1$ for any $P \in B_E$ (see Brelot [**5**, p. 61] and Doob [**6**, p. 169]). Hence if we can see that

$$\mu_u(\{P \in C_n(\Omega); \hat{R}^E_{K(\cdot,\infty)}(P) < K(P,\infty)\}) = 0 \tag{3.13}$$

and

(3.14)

$$\nu_u\left(\left\{Q=(t,\Phi)\in S_n(\Omega);\int_{C_n(\Omega)}\frac{\partial}{\partial n_Q}G(P,Q)\,d\lambda_E(P)<t^{\alpha_\Omega-1}\frac{\partial}{\partial n_\Phi}f_\Omega(\Phi)\right\}\right)=0,$$

then we can prove the equality in (3.9).

To see (3.13), we remark that

$$\{P\in C_n(\Omega);\hat{R}^E_{K(\cdot,\infty)}(P)<K(P,\infty)\}\subset C_n(\Omega)-B_E$$

and

$$\mu_u(C_n(\Omega)-B_E)=0$$

(see Brelot [**5**, Theorem VIII, 11] and Doob [**6**, XI, 14. Theorem (d)]).

To prove (3.14), we set

$$B'_E=\{Q\in S_n(\Omega);E\ \text{is not minimally thin at}\ Q\} \tag{3.15}$$

and $e=\{P\in E;\hat{R}^E_{K(\cdot,\infty)}(P)<K(P,\infty)\}$. Then e is a polar set (see Doob [**6**, VI, 3. (b)]) and hence for any $Q\in S_n(\Omega)$

$$\hat{R}^E_{K(\cdot,Q)}=\hat{R}^{E-e}_{K(\cdot,Q)}$$

(see Doob [**6**, VI, 3. (c)]). Thus at any $Q\in B'_E$, $E-e$ is not also minimally thin at Q and hence

$$\int_{C_n(\Omega)}K(P,Q)\,d\eta(P)=\underline{\lim}_{P'\to Q,P'\in E-e}\int_{C_n(\Omega)}K(P,P')\,d\eta(P) \tag{3.16}$$

for any positive measure η on $C_n(\Omega)$, where

$$K(P,P')=\frac{G(P,P')}{G(P_0,P')}\quad(P\in C_n(\Omega),P'\in C_n(\Omega))$$

(see Brelot [**5**, Theorem XV, 6]). Now, take $\eta=\lambda_E$ in (3.16). Since

$$\lim_{P\to Q,P\in C_n(\Omega)}\frac{K(P,\infty)}{G(P_0,P)}=t^{\alpha_\Omega-1}\frac{\partial}{\partial n_\Phi}f_\Omega(\Phi)\left\{\frac{\partial G(P_0,Q)}{\partial n_Q}\right\}^{-1}\quad(Q=(t,\Phi)\in S_n(\Omega))$$

(for the existence of the limit in the left side, see Jerison and Kenig [**14**, (7.9) in p. 87]), we obtain from (3.6)

$$\int_{C_n(\Omega)}\frac{\partial G(P,Q)}{\partial n_Q}d\lambda_E(P)=t^{\alpha_\Omega-1}\frac{\partial}{\partial n_\Phi}f_\Omega(\Phi)\underline{\lim}_{P'\to Q,P'\in E-e}\int_{C_n(\Omega)}\frac{G(P,P')}{K(P',\infty)}d\lambda_E(P)$$

for any $Q=(t,\Phi)\in B'_E$. Since

$$\int_{C_n(\Omega)}\frac{G(P,P')}{K(P',\infty)}d\lambda_E(P)=\frac{1}{K(P',\infty)}\hat{R}^E_{K(\cdot,\infty)}(P')=1$$

for any $P'\in E-e$, we have

$$\int_{C_n(\Omega)}\frac{\partial G(P,Q)}{\partial n_Q}d\lambda_E(P)=t^{\alpha_\Omega-1}\frac{\partial}{\partial n_\Phi}f_\Omega(\Phi)$$

for any $Q=(t,\Phi)\in B'_E$, which shows

(3.17)

$$\left\{Q=(t,\Phi)\in S_n(\Omega);\int_{C_n(\Omega)}\frac{\partial}{\partial n_Q}G(P,Q)\,d\lambda_E(P)<t^{\alpha_\Omega-1}\frac{\partial}{\partial n_\Phi}f_\Omega(\Phi)\right\}\subset S_n(\Omega)-B'_E.$$

Let h be the greatest harmonic minorant of $u(P)=\hat{R}^E_1(P)$ and ν'_u be the Martin representing measure of h. If we can prove that

$$\hat{R}^E_h=h \tag{3.18}$$

on $C_n(\Omega)$, then $\nu'_u(S_n(\Omega)-B'_E)=0$ (see Essén and Jackson [**7**, pp. 240–241], Brelot [**5**, Theorem XV, 11] and, Aikawa and Essén [**3**, Part II, p. 188]). Since

$$d\nu'_u(Q)=\frac{\partial}{\partial n_Q}G(P_0,Q)\,d\nu_u(Q)\quad(Q\in S_n(\Omega))$$

from (3.6), we also have $\nu_u(S_n(\Omega)-B'_E)=0$, which gives (3.14) from (3.17).

To prove (3.18), set $u^*=\hat{R}^E_1-h$. Then

$$u^*+h=\hat{R}^E_1=\hat{R}^E_{u^*+h}\le\hat{R}^E_{u^*}+\hat{R}^E_h$$

(see Brelot [**5**, VI, 10. d)] and Helms [**13**, Theorem 7.12 (iv)]), and hence

$$\hat{R}^E_h-h\ge u^*-\hat{R}^E_{u^*}\ge 0,$$ □

from which (3.18) follows.

4. Proof of Theorem 1.

Apply the Riesz decomposition theorem to the superharmonic function $\hat{R}^E_{K(\cdot,\infty)}$ on $C_n(\Omega)$. Then we have a positive measure μ on $C_n(\Omega)$ satisfying

$$G\mu(P)<\infty$$

for any $P\in C_n(\Omega)$ and a non-negative greatest harmonic minorant H of $\hat{R}^E_{K(\cdot,\infty)}$ such that

$$\hat{R}^E_{K(\cdot,\infty)} = G\mu + H. \tag{4.1}$$

We remark that $K(P,\infty)$ $(P \in C_n(\Omega))$ is a minimal function at ∞. If E is minimally thin at ∞ with respect to $C_n(\Omega)$, then $\hat{R}^E_{K(\cdot,\infty)}$ is a potential (see Doob [**6**, p. 208]) and hence $H \equiv 0$ on $C_n(\Omega)$. Since

$$\hat{R}^E_{K(\cdot,\infty)}(P) = K(P,\infty) \tag{4.2}$$

for any $P \in B_E$ (Brelot [**5**, p. 61] and Doob [**6**, p. 169]), we see from (4.1)

$$G\mu(P) = K(P,\infty) \tag{4.3}$$

for any $P \in B_E$.

Take a sufficiently large L from Lemma 1 such that

$$C_2 2^{-\delta_\Omega} L^{-\delta_\Omega} \int_{C_n(\Omega;0,L)} t^{\alpha_\Omega} f_\Omega(\Phi)\, d\mu(t,\Phi) < \frac{1}{4}.$$

Then from (3.1)

$$\int_{C_n(\Omega;0,L)} G(P,Q)\, d\mu(Q) \le \frac{1}{4} K(P,\infty)$$

for any $P = (r,\Theta) \in C_n(\Omega)$, $r \ge 2L$, and hence from (4.3)

$$\int_{C_n(\Omega;L,+\infty)} G(P,Q)\, d\mu(Q) \ge \frac{3}{4} K(P,\infty) \tag{4.4}$$

for any $P = (r,\Theta) \in B_E$, $r \ge 2L$. Now, divide $G\mu$ into three parts:

$$G\mu(P) = A_1^{(k)}(P) + A_2^{(k)}(P) + A_3^{(k)}(P) \quad (P \in C_n(\Omega)), \tag{4.5}$$

where

$$A_1^{(k)}(P) = \int_{C_n(\Omega;2^{k-1},2^{k+2})} G(P,Q)\, d\mu(Q), \quad A_2^{(k)}(P) = \int_{C_n(\Omega;0,2^{k-1})} G(P,Q)\, d\mu(Q)$$

$$A_3^{(k)}(P) = \int_{C_n(\Omega;2^{k+2},+\infty)} G(P,Q)\, d\mu(Q).$$

Then we shall show that there exists an integer N such that

$$B_E \cap \overline{I_k(\Omega)} \subset \left\{ P = (r,\Theta) \in C_n(\Omega);\, A_1^{(k)}(P) \ge \frac{1}{4} r^{\alpha_\Omega} f_\Omega(\Theta) \right\} \quad (k \ge N). \tag{4.6}$$

Take any $P = (r,\Theta) \in \overline{I_k(\Omega)} \cap C_n(\Omega)$. When by Lemma 1 we choose a sufficiently large integer N_1 such that

$$2^{-k\delta_\Omega}\int_{C_n(\Omega;0,2^{k-1})} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi) \le \frac{1}{4C_2} \quad (k \ge N_1)$$

and

$$\int_{C_n(\Omega,2^{k+2},+\infty)} t^{-\beta_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi) \le \frac{1}{4C_2} \quad (k \ge N_1),$$

we have from (3.1) that

$$A_2^{(k)}(P) \le \frac{1}{4} r^{\alpha_\Omega} f_\Omega(\Theta) \quad (k \ge N_1) \tag{4.7}$$

and

$$A_3^{(k)}(P) \le \frac{1}{4} r^{\alpha_\Omega} f_\Omega(\Theta) \quad (k \ge N_1). \tag{4.8}$$

Put

$$N = \max\left\{N_1, \left[\frac{\log L}{\log 2}\right] + 2\right\}.$$

If $P = (r,\Theta) \in B_E \cap \overline{I_k(\Omega)}$ $(k \ge N)$, then we have from (4.4), (4.5), (4.7) and (4.8) that

$$A_1^{(k)}(P) \ge \int_{C_n(\Omega;L,+\infty)} G(P,Q)\,d\mu(Q) - A_2^{(k)}(P) - A_3^{(k)}(P) \ge \frac{1}{4} r^{\alpha_\Omega} f_\Omega(\Theta),$$

which shows (4.6).

Since the measure λ_{E_k} is concentrated on B_{E_k} and $B_{E_k} \subset B_E \cap \overline{I_k(\Omega)}$, we finally obtain by (4.6) that

$$\begin{aligned}
\gamma_\Omega(E_k) &= \int_{C_n(\Omega)} (G\lambda_{E_k})\,d\lambda_{E_k} \le \int_{B_{E_k}} r^{\alpha_\Omega} f_\Omega(\Theta)\,d\lambda_{E_k}(r,\Theta) \le 4\int_{B_{E_k}} A_1^{(k)}(P)\,d\lambda_{E_k}(P) \\
&= 4\int_{C_n(\Omega;2^{k-1},2^{k+2})} \left\{\int_{C_n(\Omega)} G(P,Q)\,d\lambda_{E_k}(P)\right\} d\mu(Q) \\
&\le 4\int_{C_n(\Omega;2^{k-1},2^{k+2})} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi) \quad (k \ge N)
\end{aligned}$$

and hence

$$\begin{aligned}
\sum_{k=N}^{\infty} \gamma_\Omega(E_k) 2^{-k\delta_\Omega} &\le 4^{1+\delta_\Omega} \sum_{k=N}^{\infty} \int_{C_n(\Omega;2^{k-1},2^{k+2})} t^{\alpha_\Omega-\delta_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi) \\
&\le 12 \times 4^{\delta_\Omega} \int_{C_n(\Omega;2^{N-1},+\infty)} t^{-\beta_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi) < +\infty
\end{aligned}$$

from Lemma 1. This gives (2.1).

Next we shall prove "if" part. Since

$$\hat{R}^{E_k}_{K(\cdot,\infty)}(Q) = K(Q,\infty)$$

for any $Q \in B_{E_k}$ as in (4.2), we have

$$\gamma_\Omega(E_k) = \int_{B_{E_k}} K(Q,\infty)\, d\lambda_{E_k}(Q) \geq 2^{k\alpha_\Omega} \int_{B_{E_k}} f_\Omega(\Phi)\, d\lambda_{E_k}(t,\Phi) \quad (Q = (t,\Phi) \in C_n(\Omega))$$

and hence from (3.1)

$$(4.9)\quad \hat{R}^{E_k}_{K(\cdot,\infty)}(P) \leq C_2 r^{\alpha_\Omega} f_\Omega(\Theta) \int_{B_{E_k}} t^{-\beta_\Omega} f_\Omega(\Phi)\, d\lambda_{E_k}(t,\Phi) \leq C_2 r^{\alpha_\Omega} f_\Omega(\Theta) 2^{-k\delta_\Omega} \gamma_\Omega(E_k)$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any integer k satisfying $2^k \geq 2r$. If we define a measure μ on $C_n(\Omega)$ by

$$d\mu(Q) = \begin{cases} \sum_{k=0}^{\infty} d\lambda_{E_k}(Q) & (Q \in C_n(\Omega;1,+\infty)) \\ 0 & (Q \in C_n(\Omega;0,1)), \end{cases}$$

then from (2.1) and (4.9)

$$G\mu(P) = \int_{C_n(\Omega)} G(P,Q)\, d\mu(Q) = \sum_{k=0}^{\infty} \hat{R}^{E_k}_{K(\cdot,\infty)}(P)$$

is a finite-valued superharmonic function on $C_n(\Omega)$ and

$$G\mu(P) \geq \int_{C_n(\Omega)} G(P,Q)\, d\lambda_{E_k}(Q) = \hat{R}^{E_k}_{K(\cdot,\infty)}(P) = r^{\alpha_\Omega} f_\Omega(\Theta)$$

for any $P = (r,\Theta) \in B_{E_k}$, and from (3.1)

$$G\mu(P) \geq C' r^{\alpha_\Omega} f_\Omega(\Theta)$$

for any $P = (r,\Theta) \in C_n(\Omega;0,1)$, where

$$C' = C_1 \int_{C_n(\Omega;2,+\infty)} t^{-\beta_\Omega} f_\Omega(\Phi)\, d\mu(t,\Phi).$$

If we set

$$E' = \bigcup_{k=0}^{\infty} B_{E_k}, \quad E_1 = E \cap C_n(\Omega;0,1) \quad \text{and} \quad C = \min(C',1),$$

then

$$E' \subset \{P = (r,\Theta) \in C_n(\Omega); G\mu(P) \geq C r^{\alpha_\Omega} f_\Omega(\Theta)\}.$$

Hence by Lemma 2, E' is minimally thin at ∞ with respect to $C_n(\Omega)$ i.e. there is a point $P' \in C_n(\Omega)$ such that

$$\hat{R}^{E'}_{K(\cdot,\infty)}(P') \neq K(P',\infty).$$

Since E' is equal to E except a polar set (see Brelot [**5**, p. 57] and Doob [**6**, p. 177]), we know that

$$\hat{R}^{E'}_{K(\cdot,\infty)}(P) = \hat{R}^{E}_{K(\cdot,\infty)}(P)$$

for any $P \in C_n(\Omega)$ (see Helms [**13**, Corollary 8.37]) and hence

$$\hat{R}^{E}_{K(\cdot,\infty)}(P') \neq K(P',\infty).$$

This shows that E is minimally thin at ∞ with respect to $C_n(\Omega)$. □

5. Proof of Theorem 2.

Let E be a rarefied set at ∞ with respect to $C_n(\Omega)$. Then there exists a positive superharmonic function $v(P)$ on $C_n(\Omega)$ such that $c(v) = 0$ and

$$E \subset H_v. \tag{5.1}$$

By Lemma 3, we can find two positive measures μ on $C_n(\Omega)$ and ν on $S_n(\Omega)$ such that

$$v(P) = c_0(v)K(P,O) + \int_{C_n(\Omega)} G(P,Q)\,d\mu(Q) + \int_{S_n(\Omega)} \frac{\partial G(P,Q)}{\partial n_Q} d\nu(Q) \quad (P \in C_n(\Omega)).$$

Now we write

$$v(P) = c_0(v)K(P,O) + B_1^{(k)}(P) + B_2^{(k)}(P) + B_3^{(k)}(P), \tag{5.2}$$

where

$$B_1^{(k)}(P) = \int_{C_n(\Omega;0,2^{k-1})} G(P,Q)\,d\mu(Q) + \int_{S_n(\Omega;0,2^{k-1})} \frac{\partial G(P,Q)}{\partial n_Q} d\nu(Q),$$

$$B_2^{(k)}(P) = \int_{C_n(\Omega;2^{k-1},2^{k+2})} G(P,Q)\,d\mu(Q) + \int_{S_n(\Omega;2^{k-1},2^{k+2})} \frac{\partial G(P,Q)}{\partial n_Q} d\nu(Q)$$

and

$$B_3^{(k)}(P) = \int_{C_n(\Omega;2^{k+2},+\infty)} G(P,Q)\,d\mu(Q)$$
$$+ \int_{S_n(\Omega;2^{k+2},+\infty)} \frac{\partial G(P,Q)}{\partial n_Q} d\nu(Q) \quad (P \in C_n(\Omega); k = 1,2,3,\dots).$$

First we shall show the existence of an integer N such that

$$H_v \cap I_k(\Omega) \subset \left\{ P = (r,\Theta) \in I_k(\Omega); B_2^{(k)}(P) \geq \frac{1}{2} r^{\alpha_\Omega} \right\} \tag{5.3}$$

for any integer $k, k \geq N$. Since $v(P)$ is finite almost everywhere on $C_n(\Omega)$, from Lemmas 1 and 4 applied to

$$\int_{C_n(\Omega)} G(P,Q)\,d\mu(Q) \quad \text{and} \quad \int_{S_n(\Omega)} \frac{\partial G(P,Q)}{\partial n_Q}\,dv(Q)$$

respectively, we can take an integer N such that for any $k, k \geq N$,

$$2^{-k\delta_\Omega} \int_{C_n(\Omega;0,2^{k-1})} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi) \leq \frac{1}{12C_2 J_\Omega}, \tag{5.4}$$

$$\int_{C_n(\Omega;2^{k+2},\infty)} t^{-\beta_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi) \leq \frac{1}{12C_2 J_\Omega}, \tag{5.5}$$

$$2^{-k\delta_\Omega} \int_{S_n(\Omega;0,2^{k-1})} t^{\alpha_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\,dv(t,\Phi) \leq \frac{1}{12C_2 J_\Omega} \tag{5.6}$$

and

$$\int_{S_n(\Omega;2^{k+2},\infty)} t^{-\beta_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\,dv(t,\Phi) \leq \frac{1}{12C_2 J_\Omega}, \tag{5.7}$$

where

$$J_\Omega = \sup_{\Theta \in \Omega} f_\Omega(\Theta).$$

Then for any $P = (r,\Theta) \in I_k(\Omega)$ $(k \geq N)$, we have

$$\begin{aligned} B_1^{(k)}(P) &\leq C_2 J_\Omega r^{-\beta_\Omega} \int_{C_n(\Omega;0,2^{k-1})} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi) \\ &\quad + C_2 J_\Omega r^{-\beta_\Omega} \int_{S_n(\Omega;0,2^{k-1})} t^{\alpha_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\,dv(t,\Phi) \\ &\leq \frac{1}{12} r^{\alpha_\Omega} + \frac{1}{12} r^{\alpha_\Omega} = \frac{1}{6} r^{\alpha_\Omega} \end{aligned}$$

from (3.1), (3.7), (5.4) and (5.6), and

$$\begin{aligned} B_3^{(k)}(P) &\leq C_2 J_\Omega r^{\alpha_\Omega} \int_{C_n(\Omega;2^{k+2},\infty)} t^{-\beta_\Omega} f_\Omega(\Phi)\,d\mu(t,\Phi) \\ &\quad + C_2 J_\Omega r^{\alpha_\Omega} \int_{S_n(\Omega;2^{k+2},\infty)} t^{-\beta_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\,dv(t,\Phi) \\ &\leq \frac{1}{12} r^{\alpha_\Omega} + \frac{1}{12} r^{\alpha_\Omega} = \frac{1}{6} r^{\alpha_\Omega} \end{aligned}$$

from (3.1), (3.7), (5.5) and (5.7). Further we can assume that

$$6\kappa c_0(v)J_\Omega \le r^{\delta_\Omega}$$

for any $P = (r, \Theta) \in I_k(\Omega)$ $(k \ge N)$, hence if $P = (r, \Theta) \in I_k(\Omega) \cap H_v$ $(k \ge N)$, then we obtain

$$B_2^{(k)}(P) \ge r^{\alpha_\Omega} - \frac{1}{6} r^{\alpha_\Omega} - \frac{1}{6} r^{\alpha_\Omega} - \frac{1}{6} r^{\alpha_\Omega} = \frac{1}{2} r^{\alpha_\Omega}$$

from (5.2), which gives (5.3).

Now we observe from (5.1) and (5.3) that

$$B_2^{(k)}(P) \ge 2^{k\alpha_\Omega - 1} \quad (k \ge N)$$

for any $P \in E_k$. If we define a function $u_k(P)$ on $C_n(\Omega)$ by

$$u_k(P) = 2^{1-k\alpha_\Omega} B_2^{(k)}(P),$$

then

$$u_k(P) \ge 1 \quad (P \in E_k, k \ge N)$$

and

$$u_k(P) = \int_{C_n(\Omega)} G(P, Q)\, d\mu_k(Q) + \int_{S_n(\Omega)} \frac{\partial G(P, Q)}{\partial n_Q} d\nu_k(Q)$$

with two measures

$$d\mu_k(Q) = \begin{cases} 2^{1-k\alpha_\Omega}\, d\mu(Q) & (Q \in C_n(\Omega; 2^{k-1}, 2^{k+2})) \\ 0 & (Q \in C_n(\Omega; 0, 2^{k-1}) \cup C_n(\Omega; 2^{k+2}, \infty)) \end{cases}$$

and

$$d\nu_k(Q) = \begin{cases} 2^{1-k\alpha_\Omega}\, d\nu(Q) & (Q \in S_n(\Omega; 2^{k-1}, 2^{k+2})) \\ 0 & (Q \in S_n(\Omega; 0, 2^{k-1}) \cup S_n(\Omega; 2^{k+2}, \infty)). \end{cases}$$

Hence by applying Lemma 5 to $u_k(P)$, we obtain

$$\lambda_\Omega(E_k) \le 2^{1-k\alpha_\Omega} \left\{ \int_{C_n(\Omega; 2^{k-1}, 2^{k+2})} t^{\alpha_\Omega} f_\Omega(\Phi)\, d\mu(t, \Phi) + \int_{S_n(\Omega; 2^{k-1}, 2^{k+2})} t^{\alpha_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\, d\nu(t, \Phi) \right\}$$

$(k \ge N)$. Finally we have

$$\sum_{k=N}^{\infty} 2^{-k\beta_\Omega} \lambda_\Omega(E_k) \le 6 \cdot 4^{\delta_\Omega} \left\{ \int_{C_n(\Omega; 2^{N-1}, +\infty)} t^{-\beta_\Omega} f_\Omega(\Phi)\, d\mu(t, \Phi) + \int_{S_n(\Omega; 2^{N-1}, +\infty)} t^{-\beta_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\, d\nu(t, \Phi) \right\}.$$

If we take a sufficiently large N, then the integrals of the right side are finite from Lemmas 1 and 4.

Suppose that a subset E of $C_n(\Omega)$ satisfies

$$\sum_{k=0}^{\infty} 2^{-k\beta_\Omega} \lambda_\Omega(E_k) < +\infty.$$

Then from the second part of Lemma 5 applied to E_k, we have

$$\sum_{k=1}^{\infty} 2^{-k\beta_\Omega} \left\{ \int_{C_n(\Omega)} t^{\alpha_\Omega} f_\Omega(\Phi)\, d\mu_k^*(t,\Phi) + \int_{S_n(\Omega)} t^{\alpha_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\, d\nu_k^*(t,\Phi) \right\} < \infty, \tag{5.8}$$

where μ_k^* and ν_k^* are two positive measures on $C_n(\Omega)$ and $S_n(\Omega)$, respectively, such that

$$\hat{R}_1^{E_k}(P) = \int_{C_n(\Omega)} G(P,Q)\, d\mu_k^*(Q) + \int_{S_n(\Omega)} \frac{\partial}{\partial n_Q} G(P,Q)\, d\nu_k^*(Q). \tag{5.9}$$

Consider a function $v_0(P)$ on $C_n(\Omega)$ defined by

$$v_0(P) = \sum_{k=-1}^{\infty} 2^{(k+1)\alpha_\Omega} \hat{R}_1^{E_k}(P) \quad (P \in C_n(\Omega)),$$

where

$$E_{-1} = E \cap \{P = (r,\Theta) \in C_n(\Omega); 0 < r < 1\}.$$

Then $v_0(P)$ is a superharmonic function on $C_n(\Omega)$ or identically $+\infty$ on $C_n(\Omega)$. Take any positive integer k_0 and represent $v_0(P)$ by

$$v_0(P) = v_1(P) + v_2(P) \quad (P \in C_n(\Omega)),$$

where

$$v_1(P) = \sum_{k=-1}^{k_0+1} 2^{(k+1)\alpha_\Omega} \hat{R}_1^{E_k}(P), \quad v_2(P) = \sum_{k=k_0+2}^{\infty} 2^{(k+1)\alpha_\Omega} \hat{R}_1^{E_k}(P).$$

Since μ_k^* and ν_k^* are concentrated on $B_{E_k} \subset \overline{E_k} \cap C_n(\Omega)$ and $B'_{E_k} \subset \overline{E_k} \cap S_n(\Omega)$ (see (3.15) for the notation B'_{E_k}), respectively (Brelot [**6**, Theorem XV, 11]), we have from (3.1) and (3.7) that

$$\begin{aligned} 2^{(k+1)\alpha_\Omega} \int_{C_n(\Omega)} G(P',Q)\, d\mu_k^*(Q) &\le C_2 2^{(k+1)\alpha_\Omega} (r')^{\alpha_\Omega} f_\Omega(\Theta') \int_{C_n(\Omega)} t^{-\beta_\Omega} f_\Omega(\Phi)\, d\mu_k^*(t,\Phi) \\ &\le C_2 2^{\alpha_\Omega} (r')^{\alpha_\Omega} f_\Omega(\Theta') 2^{-k\beta_\Omega} \int_{C_n(\Omega)} t^{\alpha_\Omega} f_\Omega(\Phi)\, d\mu_k^*(t,\Phi) \end{aligned}$$

and

$$2^{(k+1)\alpha_\Omega}\int_{S_n(\Omega)}\frac{\partial}{\partial n_Q}G(P',Q)\,d\nu_k^*(Q)$$
$$\le C_2 2^{(k+1)\alpha_\Omega}(r')^{\alpha_\Omega}f_\Omega(\Theta')2^{-k\beta_\Omega}\int_{S_n(\Omega)}t^{\alpha_\Omega-1}\frac{\partial}{\partial n_\Phi}f_\Omega(\Phi)\,d\nu_k^*(t,\Phi)$$

for a point $P'=(r',\Theta')\in C_n(\Omega)$, $r'\le 2^{k_0+1}$, and any integer $k\ge k_0+2$. Hence we know

$$v_2(P')\le C_2 2^{\alpha_\Omega}(r')^{\alpha_\Omega}f_\Omega(\Theta')\sum_{k=k_0+2}^{\infty}2^{-k\beta_\Omega}\left\{\int_{C_n(\Omega)}t^{\alpha_\Omega}f_\Omega(\Phi)\,d\mu_k^*(t,\Phi)\right.$$
$$\left.+\int_{S_n(\Omega)}t^{\alpha_\Omega-1}\frac{\partial}{\partial n_\Phi}f_\Omega(\Phi)\,d\nu_k^*(t,\Phi)\right\}. \tag{5.10}$$

This and (5.8) show that $v_2(P')$ is finite and hence $v_0(P)$ is a positive superharmonic function on $C_n(\Omega)$. To see

$$c(v_0)=\inf_{P\in C_n(\Omega)}\frac{v_0(P)}{K(P,\infty)}=0, \tag{5.11}$$

consider the representations of $v_0(P), v_1(P)$ and $v_2(P)$

$$v_0(P)=c(v_0)K(P,\infty)+c_0(v_0)K(P,O)+\int_{C_n(\Omega)}G(P,Q)\,d\mu_{(0)}(Q)$$
$$+\int_{S_n(\Omega)}\frac{\partial G(P,Q)}{\partial n_Q}\,d\nu_{(0)}(Q),$$

$$v_1(P)=c(v_1)K(P,\infty)+c_0(v_1)K(P,O)+\int_{C_n(\Omega)}G(P,Q)\,d\mu_{(1)}(Q)$$
$$+\int_{S_n(\Omega)}\frac{\partial G(P,Q)}{\partial n_Q}d\nu_{(1)}(Q),$$

and

$$v_2(P)=c(v_2)K(P,\infty)+c_0(v_2)K(P,O)+\int_{C_n(\Omega)}G(P,Q)\,d\mu_{(2)}(Q)$$
$$+\int_{S_n(\Omega)}\frac{\partial G(P,Q)}{\partial n_Q}d\nu_{(2)}(Q)$$

by Lemma 3. It is evident from (5.9) that $c(v_1)=0$ for any k_0. Since $c(v_0)=c(v_2)$ and

$$c(v_2) = \inf_{P \in C_n(\Omega)} \frac{v_2(P)}{K(P,\infty)} \le \frac{v_2(P')}{K(P,\infty)}$$

$$\le C_2 2^{\alpha_\Omega} \sum_{k_0+2}^{\infty} 2^{-k\beta_\Omega} \left\{ \int_{C_n(\Omega)} t^{\alpha_\Omega} f_\Omega(\Phi)\, d\mu_k^*(t,\Phi) \right.$$

$$\left. + \int_{S_n(\Omega)} t^{\alpha_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)\, d\nu_k^*(t,\Phi) \right\} \to 0 \quad (k_0 \to +\infty)$$

from (5.8) and (5.10), we know $c(v_0) = 0$, which is (5.11).

Since $\hat{R}_1^{E_k} = 1$ on $B_{E_k}, B_{E_k} \subset \overline{E_k} \cap C_n(\Omega)$ (Brelot [**5**, p. 61] and Doob [**6**, p. 169]), we see

$$v_0(P) \ge 2^{(k+1)\alpha_\Omega} \ge r^{\alpha_\Omega}$$

for any $P = (r,\Theta) \in B_{E_k}$ $(k = -1, 0, 1, 2, \ldots)$. If we set $E' = \bigcup_{k=-1}^{\infty} B_{E_k}$, then

$$E' \subset H_{v_0}. \tag{5.12}$$

Since E' is equal to E except a polar set S, we can take another positive superharmonic function v_3 on $C_n(\Omega)$ such that $v_3 = G\eta$ with a positive measure η on $C_n(\Omega)$ and v_3 is identically $+\infty$ on S (see Doob [**6**, p. 58]). Finally, define a positive superharmonic function v on $C_n(\Omega)$ by

$$v = v_0 + v_3.$$

Since $c(v_3) = 0$, it is easy to see from (5.11) that $c(v) = 0$. Also we see from (5.12) that $E \subset H_v$. Thus we complete to prove that E is a rarefied set at ∞ with respect to $C_n(\Omega)$. □

6. Proofs of Theorems 3 and 4.

PROOF OF THEOREM 3. By Lemma 3 we have

$$v(P) = c(v)K(P,\infty) + c_0(v)K(P,O) + \int_{C_n(\Omega)} G(P,Q)\, d\mu(Q) + \int_{S_n(\Omega)} \frac{\partial G(P,Q)}{\partial n_Q} d\nu(Q)$$

for two positive measures μ and ν on $C_n(\Omega)$ and $S_n(\Omega)$, respectively. Then

$$v_1(P) = v(P) - c(v)K(P,\infty) - c_0(v)K(P,O) \quad (P = (r,\Theta) \in C_n(\Omega))$$

also is a positive superharmonic function on $C_n(\Omega)$ such that

$$\inf_{P=(r,\Theta) \in C_n(\Omega)} \frac{v_1(P)}{K(P,\infty)} = 0.$$

We shall prove the existence of a rarefied set E at ∞ with respect to $C_n(\Omega)$ such that

$$v_1(P)r^{-\alpha_\Omega} \quad (P = (r, \Theta) \in C_n(\Omega))$$

uniformly converges to 0 on $C_n(\Omega) - E$ as $r \to +\infty$. Let $\{\varepsilon_i\}$ be a sequence of positive numbers ε_i satisfying $\varepsilon_i \to 0$ $(i \to +\infty)$. Put

$$E_i = \{P = (r, \Theta) \in C_n(\Omega); v_1(P) \geq \varepsilon_i r^{\alpha_\Omega}\} \quad (i = 1, 2, 3, \dots).$$

Then E_i $(i = 1, 2, 3, \dots)$ is rarefied at ∞ with respect to $C_n(\Omega)$ and hence

$$\sum_{k=0}^{\infty} 2^{-k\beta_\Omega} \lambda_\Omega((E_i)_k) < +\infty \quad (i = 1, 2, 3, \dots)$$

by Theorem 2. Take a sequence $\{q_i\}$ such that

$$\sum_{k=q_i}^{\infty} 2^{-k\beta_\Omega} \lambda_\Omega((E_i)_k) < \frac{1}{2^i} \quad (i = 1, 2, 3, \dots)$$

and set

$$E = \bigcup_{i=1}^{\infty} \bigcup_{k=q_i}^{\infty} (E_i)_k.$$

Then

$$\lambda_\Omega(E_m) \leq \sum_{i=1}^{\infty} \sum_{k=q_i}^{\infty} \lambda_\Omega(E_i \cap I_k \cap I_m) \quad (m = 1, 2, 3, \dots),$$

because λ_Ω is a countably sub-additive set function as in Aikawa [**2**, Lemma 2.4 (iii)] and Essén and Jackson [**7**, p. 241]. Since

$$\begin{aligned} \sum_{m=1}^{\infty} \lambda_\Omega(E_m) 2^{-m\beta_\Omega} &\leq \sum_{i=1}^{\infty} \sum_{k=q_i}^{\infty} \sum_{m=1}^{\infty} \lambda_\Omega(E_i \cap I_k \cap I_m) 2^{-m\beta_\Omega} \\ &= \sum_{i=1}^{\infty} \sum_{k=q_i}^{\infty} \lambda_\Omega((E_i)_k) 2^{-k\beta_\Omega} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1, \end{aligned}$$

we know by Theorem 2 that E is a rarefied set at ∞ with respect to $C_n(\Omega)$.

It is easy to see that

$$v_1(P)r^{-\alpha_\Omega} \quad (P = (r, \Theta) \in C_n(\Omega))$$

uniformly converges to 0 on $C_n(\Omega) - E$ as $r \to +\infty$. □

Proof of Theorem 4. Since λ_{E_k} is concentrated on $B_{E_k} \subset \overline{E_k} \cap C_n(\Omega)$, we see

$$\gamma_\Omega(E_k) = \int_{C_n(\Omega)} \hat{R}^{E_k}_{K(\cdot,\infty)}(P)\, d\lambda_{E_k}(P) \le \int_{C_n(\Omega)} K(P,\infty)\, d\lambda_{E_k}(P) \le J_\Omega 2^{(k+1)\alpha_\Omega} \lambda_\Omega(E_k)$$

and hence

$$\sum_{k=0}^{\infty} 2^{-k\delta_\Omega} \gamma_\Omega(E_k) \le J_\Omega 2^{\alpha_\Omega} \sum_{k=0}^{\infty} 2^{-k\beta_\Omega} \lambda_\Omega(E_k),$$

which gives the conclusion in the first part from Theorems 1 and 2.

To prove the second part, put $J'_\Omega = \min_{\Theta \in \overline{\Omega'}} f_\Omega(\Theta)$. Since

$$K(P,\infty) = r^{\alpha_\Omega} f_\Omega(\Theta) \ge J'_\Omega r^{\alpha_\Omega} \ge J'_\Omega 2^{k\alpha_\Omega} \quad (P = (r,\Theta) \in E_k),$$

and

$$\hat{R}^{E_k}_{K(\cdot,\infty)}(P) = K(P,\infty)$$

for any $P \in B_{E_k}$, we have

$$\gamma_\Omega(E_k) = \int_{C_n(\Omega)} \hat{R}^{E_k}_{K(\cdot,\infty)}(P)\, d\lambda_{E_k}(P) \ge J'_\Omega 2^{k\alpha_\Omega} \lambda_\Omega(E_k).$$

Since

$$J'_\Omega \sum_{k=0}^{\infty} 2^{-k\beta_\Omega} \lambda_\Omega(E_k) \le \sum_{k=0}^{\infty} 2^{-k\delta_\Omega} \gamma_\Omega(E_k) < +\infty$$

from Theorem 1, it follows from Theorem 2 that E is rarefied at ∞ with respect to $C_n(\Omega)$. □

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