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A linear operator and strongly starlike functions

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Abstract. Making use of an integral operator I^{α} which was defined and studied earlier by Srivastava et al., the author introduces two novel families of strongly starlike functions $ST_{\alpha}(\beta, \gamma)$ and $CV_{\alpha}(\beta, \gamma)$. Certain properties of these classes are discussed.

1. Introduction.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. A function $f(z)$ belonging to A is said to be starlike of order γ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in E) \tag{1.2}$$

for some γ $(0 \leq \gamma < 1)$. We denote by $S^*(\gamma)$ all of such functions. Also, a function in A is said to be convex of order γ if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \quad (z \in E) \tag{1.3}$$

for some γ $(0 \leq \gamma < 1)$. We denote by $C(\gamma)$ the subclass of A consisting of all functions which are convex of order γ in E. Clearly, $f(z) \in C(\gamma)$ if and only if $zf'(z) \in S^*(\gamma)$.

If $f(z) \in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta \quad (z \in E) \tag{1.4}$$

for some γ $(0 \leq \gamma < 1)$ and β $(0 < \beta \leq 1)$, then $f(z)$ is said to be strongly starlike of order β and type γ in E, and denoted by $f(z) \in S^*(\beta, \gamma)$. If $f(z) \in A$ satisfies

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$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right)\right|<\frac{\pi}{2}\beta \quad (z\in E) \tag{1.5}$$

for some γ $(0\le\gamma<1)$ and β $(0<\beta\le 1)$, then we say that $f(z)$ is strongly convex of order β and type γ in E, and we denote by $C(\beta,\gamma)$ the class of all such functions. It is obvious that $f(z)\in A$ belongs to $C(\beta,\gamma)$ if and only if $zf'(z)\in S^*(\beta,\gamma)$. Further, we note that $S^*(1,\gamma)=S^*(\gamma)$ and $C(1,\gamma)=C(\gamma)$.

For $c>-1$ and $f(z)\in A$, we recall here the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ as

$$L_c(f)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt. \tag{1.6}$$

The operator $L_c(f)$ when $c\in N=\{1,2,3,\dots\}$ was studied by Bernardi [**1**]. For $c=1$, $L_1(f)$ was investigated by Libera [**6**].

Recently, Jung, Kim and Srivastava [**4**] introduced the following one-parameter family of integral operator:

$$I^\alpha f(z)=\frac{2^\alpha}{z\Gamma(\alpha)}\int_0^z\left(\log\frac{z}{t}\right)^{\alpha-1}f(t)\,dt, \quad (\alpha>0, f(z)\in A). \tag{1.7}$$

They showed that

$$I^\alpha f(z)=z+\sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^\alpha a_n z^n. \tag{1.8}$$

The operator I^α is closely related to the multiplier transformations studied earlier by Flett [**2**]. It follows from (1.8) that one can define the operator I^α for any real number α. Certain properties of this operator have been studied by Srivastava et al. [**4**], Uralegaddi and Somanatha [**13**], Li [**5**] and the author [**7**].

Using the operator I^α, we now introduce the following classes:

$$ST_\alpha(\beta,\gamma)=\left\{f(z)\in A: I^\alpha f(z)\in S^*(\beta,\gamma), \frac{z(I^\alpha f(z))'}{I^\alpha f(z)}\ne\gamma \text{ for all } z\in E\right\} \tag{1.9}$$

and

$$CV_\alpha(\beta,\gamma)=\left\{f(z)\in A: I^\alpha f(z)\in C(\beta,\gamma), \frac{(z(I^\alpha f(z))')'}{(I^\alpha f(z))'}\ne\gamma \text{ for all } z\in E\right\}. \tag{1.10}$$

It is obvious that $f(z)\in CV_\alpha(\beta,\gamma)$ if and only if $zf'(z)\in ST_\alpha(\beta,\gamma)$.

In this note, we shall investigate some properties of the classes $ST_\alpha(\beta,\gamma)$ and $CV_\alpha(\beta,\gamma)$. The basic tool of our investigation is the following lemma which is due to Nunokawa [**11**].

Lemma. *Let a function* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *be analytic in* E *and* $p(z) \neq 0$ $(z \in E)$. *If there exists a point* $z_0 \in E$ *such that*

$$|\arg p(z)| < \frac{\pi}{2}\beta \quad (|z| < |z_0|) \quad \textit{and} \quad |\arg p(z_0)| = \frac{\pi}{2}\beta \quad (0 < \beta \leq 1),$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \qquad \left(\textit{when } \arg p(z_0) = \frac{\pi}{2}\beta\right),$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \left(\textit{when } \arg p(z_0) = -\frac{\pi}{2}\beta\right),$$

and $p(z_0)^{1/\beta} = \pm ia$ $(a > 0)$.

2. Main results.

Our first inclusion theorem is stated as

Theorem 1. *For any real number* α, $ST_\alpha(\beta, \gamma) \subset ST_{\alpha+1}(\beta, \gamma)$.

Proof. Let $f(z) \in ST_\alpha(\beta, \gamma)$. Define the function $p(z)$ by

$$\frac{z(I^{\alpha+1}f(z))'}{I^{\alpha+1}f(z)} = \gamma + (1-\gamma)p(z), \tag{2.1}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in E and $p(z) \neq 0$ for all $z \in E$. Using the identity (easy to verify)

$$z(I^{\alpha+1}f(z))' = 2I^\alpha f(z) - I^{\alpha+1}f(z). \tag{2.2}$$

(2.1) may be written as

$$\frac{I^\alpha f(z)}{I^{\alpha+1}f(z)} = \frac{1}{2}[(1+\gamma) + (1-\gamma)p(z)]. \tag{2.3}$$

Differentiating both sides of (2.3) logarithmically, we obtain

$$\frac{z(I^\alpha f(z))'}{I^\alpha f(z)} - \gamma = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{(1+\gamma) + (1-\gamma)p(z)}. \tag{2.4}$$

Suppose now that there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi}{2}\beta \quad (|z| < |z_0|) \quad \text{and} \quad |\arg p(z_0)| = \frac{\pi}{2}\beta. \tag{2.5}$$

Then, by applying Lemma, we can write that $z_0 p'(z_0)/p(z_0) = ik\beta$ and $(p(z_0))^{1/\beta} = \pm ia$ $(a > 0)$.

Therefore, if $\arg p(z_0) = -(\pi/2)\beta$, then

$$\begin{aligned}\frac{z_0(I^\alpha f(z_0))'}{I^\alpha f(z_0)} - \gamma &= (1-\gamma)p(z_0)\left[1 + \frac{z_0 p'(z_0)/p(z_0)}{(1+\gamma)+(1-\gamma)p(z_0)}\right] \\ &= (1-\gamma)a^\beta e^{-i\pi\beta/2}\left[1 + \frac{ik\beta}{(1+\gamma)+(1-\gamma)a^\beta e^{-i\pi\beta/2}}\right]. \end{aligned} \tag{2.6}$$

Thus we have

$$\arg\left\{\frac{z_0(I^\alpha f(z_0))'}{I^\alpha f(z_0)} - \gamma\right\} = -\frac{\pi}{2}\beta + \arg\left\{1 + \frac{ik\beta}{(1+\gamma)+(1-\gamma)a^\beta e^{-i\pi\beta/2}}\right\}$$

$$= -\frac{\pi}{2}\beta$$

$$+ \operatorname{Tan}^{-1}\left\{\frac{k\beta[(1+\gamma)+(1-\gamma)a^\beta\cos(\pi\beta/2)]}{(1+\gamma)^2 + 2(1-\gamma^2)a^\beta\cos(\pi\beta/2) + (1-\gamma)^2 a^{2\beta} - k\beta(1-\gamma)a^\beta\sin(\pi\beta/2)}\right\}$$

$$\le -\frac{\pi}{2}\beta \quad \left(\text{where } k \le -\frac{1}{2}\left(a + \frac{1}{a}\right) \le -1\right),$$

which contradicts the condition $f(z) \in ST_\alpha(\beta,\gamma)$.

Similarly, if $\arg p(z_0) = (\pi/2)\beta$, then we have

$$\arg\left\{\frac{z_0(I^\alpha f(z_0))'}{I^\alpha f(z_0)} - \gamma\right\} \ge \frac{\pi}{2}\beta,$$

which also contradicts the condition $f(z) \in ST_\alpha(\beta,\gamma)$.

Thus the function $p(z)$ has to satisfy $|\arg p(z)| < (\pi/2)\beta$ $(z \in E)$, which leads us to the following

$$\left|\arg\left\{\frac{z(I^\alpha f(z))'}{I^\alpha f(z)} - \gamma\right\}\right| < \frac{\pi}{2}\beta \quad (z \in E).$$

This evidently completes the proof of Theorem 1. □

We next state

Theorem 2. *For any real number* α, $CV_\alpha(\beta,\gamma) \subset CV_{\alpha+1}(\beta,\gamma)$.

Proof.

$$f(z) \in CV_\alpha(\beta,\gamma) \Leftrightarrow I^\alpha f(z) \in C(\beta,\gamma) \Leftrightarrow z(I^\alpha f(z))' \in S^*(\beta,\gamma)$$
$$\Leftrightarrow I^\alpha(zf'(z)) \in S^*(\beta,\gamma) \Leftrightarrow zf'(z) \in ST_\alpha(\beta,\gamma)$$
$$\Rightarrow zf'(z) \in ST_{\alpha+1}(\beta,\gamma) \Leftrightarrow I^{\alpha+1}(zf'(z)) \in S^*(\beta,\gamma)$$
$$\Leftrightarrow z(I^{\alpha+1}f(z))' \in S^*(\beta,\gamma) \Leftrightarrow I^{\alpha+1}f(z) \in C(\beta,\gamma) \Leftrightarrow f(z) \in CV_{\alpha+1}(\beta,\gamma). \quad \square$$

The following theorem deals with the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ defined by (1.6).

Theorem 3. *Let $c > -\gamma$ and $0 \le \gamma < 1$. If $f(z) \in A$ and $z(I^\alpha L_c f(z))'/I^\alpha L_c f(z) \neq \gamma$ for all $z \in E$, then $f(z) \in ST_\alpha(\beta,\gamma)$ implies that $L_c(f) \in ST_\alpha(\beta,\gamma)$.*

Proof. Let $f(z) \in ST_\alpha(\beta,\gamma)$. Put

$$\frac{z(I^\alpha L_c f(z))'}{I^\alpha L_c f(z)} = \gamma + (1-\gamma)p(z), \tag{2.7}$$

where $p(z)$ is analytic in E, $p(0) = 1$ and $p(z) \neq 0$ $(z \in E)$. From (1.6) we have

$$z(I^\alpha L_c f(z))' = (c+1)I^\alpha f(z) - cI^\alpha L_c f(z). \tag{2.8}$$

Using (2.7) and (2.8), we get

$$(c+1)\frac{I^\alpha f(z)}{I^\alpha L_c f(z)} = (c+\gamma) + (1-\gamma)p(z). \tag{2.9}$$

Differentiating (2.9) logarithmically, we obtain

$$\frac{z(I^\alpha f(z))'}{I^\alpha f(z)} - \gamma = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{(c+\gamma)+(1-\gamma)p(z)}. \tag{2.10}$$

Suppose that there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi}{2}\beta \quad (|z| < |z_0|) \quad \text{and} \quad |\arg p(z_0)| = \frac{\pi}{2}\beta.$$

Then, applying Lemma, we can write that $z_0 p'(z_0)/p(z_0) = ik\beta$ and $(p(z_0))^{1/\beta} = \pm ia$ $(a > 0)$.

If $\arg p(z_0) = (\pi/2)\beta$, then

$$\frac{z_0(I^\alpha f(z_0))'}{I^\alpha f(z_0)} - \gamma = (1-\gamma)p(z_0)\left[1 + \frac{z_0 p'(z_0)/p(z_0)}{(c+\gamma)+(1-\gamma)p(z_0)}\right]$$
$$= (1-\gamma)a^\beta e^{i\pi\beta/2}\left[1 + \frac{ik\beta}{(c+\gamma)+(1-\gamma)a^\beta e^{i\pi\beta/2}}\right].$$

This shows that

$$\arg\left\{\frac{z_0(I^\alpha f(z_0))'}{I^\alpha f(z_0)}-\gamma\right\}=\frac{\pi}{2}\beta+\arg\left\{1+\frac{ik\beta}{(c+\gamma)+(1-\gamma)a^\beta e^{i\pi\beta/2}}\right\}$$

$$=\frac{\pi}{2}\beta+\mathrm{Tan}^{-1}$$

$$\times\left\{\frac{k\beta[(c+\gamma)+(1-\gamma)a^\beta\cos(\pi\beta/2)]}{(c+\gamma)^2+2(c+\gamma)(1-\gamma)a^\beta\cos(\pi\beta/2)+(1-\gamma)^2a^{2\beta}+k\beta(1-\gamma)a^\beta\sin(\pi\beta/2)}\right\}$$

$$\geq\frac{\pi}{2}\beta\quad\left(\text{where } k\geq\frac{1}{2}\left(a+\frac{1}{a}\right)\geq 1\right),$$

which contradicts the condition $f(z)\in ST_\alpha(\beta,\gamma)$.

Similarly, we can prove the case $\arg p(z_0)=-(\pi/2)\beta$. Thus we conclude that the function $p(z)$ has to satisfy $|\arg p(z)|<(\pi/2)\beta$ for all $z\in E$. This shows that

$$\left|\arg\left\{\frac{z(I^\alpha L_c f(z))'}{I^\alpha L_c f(z)}-\gamma\right\}\right|<\frac{\pi}{2}\beta\quad(z\in E).$$

Now the proof is complete. □

Theorem 4. *Let $c>-\gamma$ and $0\leq\gamma<1$. If $f(z)\in A$ and $(z(I^\alpha L_c f(z))')'/(I^\alpha L_c f(z))'\neq\gamma$ for all $z\in E$, then $f(z)\in CV_\alpha(\beta,\gamma)$ implies that $L_c(f)\in CV_\alpha(\beta,\gamma)$.*

Proof.

$$\begin{aligned}f(z)\in CV_\alpha(\beta,\gamma)&\Leftrightarrow zf'(z)\in ST_\alpha(\beta,\gamma)\Rightarrow L_c(zf'(z))\in ST_\alpha(\beta,\gamma)\\&\Leftrightarrow z(L_c f(z))'\in ST_\alpha(\beta,\gamma)\Leftrightarrow L_c f(z)\in CV_\alpha(\beta,\gamma).\end{aligned}$$

□

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