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Shannon graphs, subshifts and lambda-graph systems

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Abstract. The relationship between presentations of subshifts by Shannon graphs and by λ-graph systems is studied. A class of presentations of subshifts by λ-graph systems is characterized. A notion of synchronization is introduced. A class of presentations by λ-graph systems, that are specifically associated to subshifts that fall under this notion, is characterized.

1. Introduction.

Let Σ be a finite alphabet. On $\Sigma^{\boldsymbol{Z}}$ one has the left shift that sends a point $(\sigma_i)_{i \in \boldsymbol{Z}} \in \Sigma^{\boldsymbol{Z}}$ into the point $(\sigma_{i+1})_{i \in \boldsymbol{Z}} \in \Sigma^{\boldsymbol{Z}}$. Subshifts are the dynamical systems that are obtained by restricting the shift to a closed shift-invariant set $X \subset \Sigma^{\boldsymbol{Z}}$. For an introduction to the theory of subshifts see [**Ki**] or [**LM**]. A subshift $X \subset \Sigma^{\boldsymbol{Z}}$ is uniquely determined by its set of admissible words, that is the words in the alphabet Σ that appear somewhere in a point $(x_i)_{i \in \boldsymbol{Z}} \in X$. We consider directed graphs whose edges are labeled by symbols in Σ. We define the admissible words of a labeled directed graph as the words that appear as label sequences of finite paths on the graph. We say that a labeled directed graph is Shannon if its labeling is such that for every symbol $\alpha \in \Sigma$ and every vertex v of the graph there is at most one edge that leaves v and that carries the label α. (Also different terminology is in use, e.g. directed graphs with such a labeling are referred to also as deterministic transition systems). The set of label sequences that are carried by the paths on the Shannon graph that leave a vertex v we call the forward context of v, and we say that a Shannon graph is forward separated if different vertices of the graph differ in their forward contexts. One says that a Shannon graph presents the subshift $X \subset \Sigma^{\boldsymbol{Z}}$ if its set of admissible words of the graph coincides with the set of admissible words of X. There is a theory of presenting subshifts by Shannon graphs, that began with Fischer's paper [**Fis**], where the case of sofic systems [**We**] was considered (cf. e.g. [**BK**], [**FF**]).

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In this paper we consider on the one hand presentations of subshifts by forward separated Shannon graphs that are such that every vertex has a predecessor and a successor, and if we speak of presenting forward separated Shannon graphs we mean such graphs. On the other hand we consider λ-graph systems as introduced in [**Ma**] that also present subshifts. A λ-graph system is a labeled directed Bratteli diagram [**Br**] with an additional structure. The vertex set of the λ-graph system we write as a disjoint union

$$V = \bigcup_{n \in \boldsymbol{Z}_+} V_{-n},$$

where $\boldsymbol{Z}_+ = \{0, 1, 2, \ldots\}$ and V_0 contains one vertex that we call the zero-vertex. (In [**Ma**] the components of the vertex sets of the λ-graph systems were indexed by $\boldsymbol{N}$. The reason for the change in indexation is that we give now preference to the other time direction. Both points of view are completely equivalent.) All edges of the λ-graph system start at some vertex of some V_{-n}, and end at some vertex of V_{-n+1}, $n \in \boldsymbol{N}$. Here it is required that every vertex has a predecessor and that every vertex except the zero-vertex has a successor. The additional structure that turns a labeled directed Bratteli diagram into a λ-graph system is a map

$$\iota : \bigcup_{n \in \boldsymbol{N}} V_{-n} \to V$$

such that

$$\iota(V_{-n}) = V_{-n+1}, \quad n \in \boldsymbol{N},$$

and such that a condition that expresses compatibility with the labeling is satisfied. In this paper, we consider only λ-graph systems that are forward separated Shannon graphs.

In section 2 of the paper we point out that there is a one-to-one correspondence between the class of compact presenting forward separated Shannon graphs and the class of forward separated Shannon λ-graph systems. Hence, presenting a subshift in the one way is equivalent to presenting it in the other way.

λ-graph systems are described algebraically by symbolic matrix systems ([**Ma**]). By means of these symbolic matrix systems it was possible to compute K-theoretic groups such as Bowen-Franks groups introduced in [**Ma**] for several subshifts (cf. [**Ma2**]). Also in [**Ma**] strong shift equivalence of λ-graph systems was formulated in terms of symbolic matrix systems. Labeled directed graphs are described algebraically by their symbolic adjacency matrices and in terms of these a translation of strong shift equivalence of forward separated Shannon λ-

graph systems into strong shift equivalence of the corresponding compact presenting forward separated Shannon graphs can be formulated (compare here [**BK**], [**Na**], [**Na3**]). A topological conjugacy between subshifts induces a natural one-to-one map between the set of presenting forward separated Shannon graphs of one subshift onto the set of presenting forward separated Shannon graphs of the other subshift, and, as we will see in section 3, compact presenting forward separated Shannon graphs are strong shift equivalent if and only if one graph is carried into the other by the natural map that is induced by a topological conjugacy between the presented subshifts.

Besides the maximal forward separated Shannon λ-graph system that presents a subshift every subshift is presented by another λ-graph system that was described in [**Ma**], and that was called the canonical λ-graph system of the subshift. In the canonical λ-graph system of a subshift V_{-n} is the set of future contexts up to time horizon $n \in \boldsymbol{N}$ of the infinite pasts in the subshift, a directed edge with label α indicates that the symbol α can be observed, while the time horizon diminishes by one, and applying the map ι means to lower the time horizon by one and simultaneously shifting once in the positive time direction. To the canonical λ-graph system of a subshift there corresponds the closure of the Shannon graph of future contexts of the infinite pasts in the subshifts (cf. [**Kr3**]). In section 4 we characterize the canonical λ-graph systems of subshifts and we point out that these are invariantly associated to the subshifts. Orbits under the map ι in a forward separated Shannon λ-graph system that describe the future context of an infinite past in the presented subshift, we call contextual. We prove in section 4 that the appearance of a non-contextual ι-orbit in the canonical λ-graph system of a subshift is an invariant of topological conjugacy.

An admissible word of a subshift is called synchronizing if every word in its past context is compatible with every word in its future context. A topologically transitive subshift that has a synchronizing word is called synchronizing. An admissible word of a subshift that has for all $n \in \boldsymbol{N}$ a transitive past that is compatible with all words of length n in its future context we call protosynchronizing. A protosynchronizing subshift is one with a protosynchronizing word. Examples of protosynchronizing subshifts are the semisynchronizing subshifts ([**Kr3**]). These are the subshifts that have a semisynchronizing word, that is an admissible word with a transitive past that is compatible with the entire future context of the word. Synchronizing and protosynchronizing subshifts allow presentations by specific forward separated Shannon λ-graph systems, and these need not coincide with their canonical λ-graph systems. In section 5 we give a characterization of these specific λ-graph systems and point out that these are invariantly associated to the subshifts. In section 6 we give examples. One of these examples is a protosynchronizing subshift that is not semisynchronizing.

2. Shannon graphs and λ-graph systems.

We introduce notation. For this consider directed graphs (V, E) where every edge in the edge set E has an initial vertex and a final vertex in the vertex set V. If there is an edge in the graph with initial vertex $v \in V$ and final vertex $v' \in V$, then we say that v is a predecessor of v', or that v' is a successor of v. There are finite paths $(e_i)_{j \le i \le k}$, $e_i \in E$, $j \le i \le k$, $j, k \in \boldsymbol{Z}$, $j \le k$ on the graph that start at the initial vertex of e_j and end at the final vertex of e_k, where the final vertex of e_i coincides with the initial vertex of e_{i+1}, $j \le i < k$. Similarly one has semi-infinite paths $(e_i)_{j \le i < \infty}$, $j \in \boldsymbol{Z}$ respectively $(e_i)_{-\infty < i \le k}$, $k \in \boldsymbol{Z}$, on the graph, and also bi-infinite paths $(e_i)_{i \in \boldsymbol{Z}}$. We say that a vertex $v' \in V$ is connected to a vertex $v \in V$ if there is a path that starts at v and ends at v', and we say that a directed graph is irreducible if every vertex is connected to every other vertex.

Let now (V, E) be a Shannon graph with labels in Σ. We say that a finite or left-infinite word c can precede an entry into the vertex $v \in V$, or that c can lead into v, if there is a path that carries c and that ends at v. We say that a vertex $v \in V$ accepts a finite or right-infinite word c if there is a path that carries c and starts at v. Given a Shannon graph S, we denote for a finite word c that is accepted by the vertex $v \in V$, the vertex at which the path ends that starts at v and carries c by $\Phi_S(v, c)$. The partial function Φ_S we call the transition function of the Shannon graph S.

We denote by $\Gamma_n^+(v)$ the set of words of length n that are accepted by the vertex $v \in V$, $n \in \boldsymbol{N}$, and we denote by $\Gamma^+(v)$ the set of right-infinite words that are accepted by v. The forward context of v is given by the union of the sets $\Gamma_n^+(v)$, $n \in \boldsymbol{N}$, or equivalently, by $\Gamma^+(v)$.

Shannon graphs S and $\tilde{S}$ with vertex sets V and $\tilde{V}$ with the same label alphabet Σ and with transition functions Φ and $\tilde{\Phi}$ are said to be isomorphic if there is a bijection $\phi : V \to \tilde{V}$ such that for all $v \in V$, $\phi(v)$ accepts an $\alpha \in \Sigma$ precisely if v accepts α, and if v accepts α, then

$$\phi(\Phi(v, \alpha)) = \tilde{\Phi}(\phi(v), \alpha).$$

To a finite alphabet Σ there is associated a Shannon graph. The vertex set $\mathscr{V}(\Sigma)$ of this Shannon graph is the set of non-empty closed subsets of $\Sigma^{\boldsymbol{N}}$. We denote by $\Gamma_n(v)$ the projection of a $v \in \mathscr{V}(\Sigma)$ onto $\Sigma^{[1,n]}$, $n \in \boldsymbol{N}$. Let d be a metric of $\Sigma^{\boldsymbol{N}}$, e.g.

$$d((\sigma_n)_{n \in \boldsymbol{N}}, (\sigma'_n)_{n \in \boldsymbol{N}}) = \sum_{n \in \boldsymbol{N}} 2^{-n} d(\sigma_n, \sigma'_n).$$

On $\mathscr{V}(\Sigma)$ we use the compact topology that is given by a Hausdorff metric d_H

$$d_H(v,v') = \max\left(\sup_{y\in v} d(y,v'), \sup_{y'\in v'} d(y',v)\right), \quad v,v' \in \mathscr{V}(\Sigma),$$

where $d(y,v') = \inf_{y'\in v'} d(y,y')$ and $d(y',v) = \inf_{y\in v} d(y',y)$. A metric D that is equivalent to d_H is, for instance, given by

$$D(v,v') = \sum_{n\in \mathbf{N}} 2^{-|\Sigma|^2} |\Gamma_n(v) \cap (\Sigma^{[1,n]} - \Gamma_n(v')) \cup (\Sigma^{[1,n]} - \Gamma_n(v')) \cap \Gamma_n(v')|,$$
$$v,v' \in \mathscr{V}(\Sigma).$$

$v \in \mathscr{V}(\Sigma)$ does not accept $\alpha \in \Sigma$ if there is no sequence in v that starts with α. If $v \in \mathscr{V}(\Sigma)$ accepts $\alpha \in \Sigma$, then

$$\Phi_{\mathscr{V}(\Sigma)}(v,\alpha) = \{(\sigma_{i+1})_{i\in \mathbf{N}} : (\sigma_i)_{i\in \mathbf{N}} \in v, \sigma_1 = \alpha\}.$$

The mapping $\Phi_{\mathscr{V}(\Sigma)}$ is continuous. With the transition function $\Phi_{\mathscr{V}(\Sigma)}$, $\mathscr{V}(\Sigma)$ is the vertex set of a Shannon graph such that every vertex has a successor. This Shannon graph is forward separated:

$$\Gamma^+(v) = v, \quad v \in \mathscr{V}(\Sigma).$$

It is the maximal presenting Shannon graph of the full shift over the alphabet Σ.

We say that a subset V of $\mathscr{V}(\Sigma)$ is transition complete if for all $v \in V$ and all α accepted by v, also $\Phi_{\mathscr{V}(\Sigma)}(v,\alpha) \in V$.

Lemma 2.1. *If $V \subset \mathscr{V}(\Sigma)$ is transition complete, then the closure of V is also transition complete.*

Proof. Let $\bar{V}$ be the closure of V, let $v \in \bar{V}$, and let v accept $\alpha \in \Sigma$. There are $v_i \in V$, $i \in \mathbf{N}$ that accept α, and such that $v = \lim_{i\to\infty} v_i$. The transition completeness of V implies that

$$\Phi_{\mathscr{V}(\Sigma)}(v_i,\alpha) \in V, \quad i \in \mathbf{N},$$

and one has

$$\Phi_{\mathscr{V}(\Sigma)}(v,\alpha) = \lim_{i\to\infty} \Phi_{\mathscr{V}(\Sigma)}(v_i,\alpha),$$

and therefore

$$\Phi_{\mathscr{V}(\Sigma)}(v,\alpha) \in \bar{V}. \qquad \square$$

Transition complete subsets of $\mathscr{V}(\Sigma)$ with the restriction of $\Phi_{\mathscr{V}(\Sigma)}$ as transition functions are forward separated Shannon graphs such that every vertex has a successor. On the other hand, given a forward separated Shannon graph with label alphabet Σ and vertex set V, such that every vertex has a successor, one has the one-to-one correspondence that assigns to a vertex $v \in V$ its forward context $\Gamma^+(v)$, and by means of this one-to-one correspondence one can identify the Shannon graph with the transition complete subgraph $\{\Gamma^+(v) : v \in V\}$ of $\mathscr{V}(\Sigma)$. Using this identification, one can define by means of Lemma 2.1 for every forward separated Shannon graph such that every vertex has a successor a Shannon graph as its closure. This closure also has the property that every vertex has a successor.

Proposition 2.2. *Let $V \subset \mathscr{V}(\Sigma)$ be a transition complete subset of $\mathscr{V}(\Sigma)$ such that every $v \in V$ has a predecessor in V. Then the closure $\overline{V}$ is a transition complete subset of $\mathscr{V}(\Sigma)$ such that every $v \in \overline{V}$ has a predecessor.*

Proof. Let $v \in \overline{V}$, and let $v_i \in V$, $i \in \boldsymbol{N}$ be such that $\lim_{i\to\infty} v_i = v$. There is an $\alpha \in \Sigma$ together with $u_i \in V$, $i \in \boldsymbol{N}$ such that $v_i = \Phi_{\mathscr{V}(\Sigma)}(u_i, \alpha)$. By compactness one can then select a subsequence u_{i_k}, $k \in \boldsymbol{N}$, such that one has a limit $u = \lim_{k\to\infty} u_{i_k}$ and one has then $\Phi_{\mathscr{V}(\Sigma)}(u, \alpha) = v$. By Lemma 2.1, $\overline{V}$ is transition complete. □

Consider now a λ-graph system [**Ma**] $\mathfrak{L}$ with vertex set

$$V = \bigcup_{n \in \boldsymbol{Z}_+} V_{-n},$$

that is labeled with symbols from a finite alphabet Σ. For the case that the λ-graph system $\mathfrak{L}$ is a forward separated Shannon graph, it is required that a $v_{-n} \in V_{-n}$ accepts an $\alpha \in \Sigma$ if and only if $\iota(v_{-n})$ accepts α, and if $v_{-n} \in V_{-n}$ accepts α then

$$\iota(\Phi_{\mathfrak{L}}(v_{-n}, \alpha)) = \Phi_{\mathfrak{L}}(\iota(v_{-n}), \alpha), \quad n \in \boldsymbol{N}.$$

The sequences $(v_{-n})_{n \in \boldsymbol{Z}_+}$, $v_{-n} \in V_{-n}$ such that $v_{-n+1} = \iota(v_{-n})$, $n \in \boldsymbol{N}$ are the ι-orbits. If here any of the v_{-n}, and therefore all of the v_{-n}, accept an $\alpha \in \Sigma$, then $(\Phi_{\mathfrak{L}}(v_{-n}, \alpha))_{n \in \boldsymbol{N}}$ is also an ι-orbit. Putting on the set

$$\prod_{n \in \boldsymbol{Z}_+} V_{-n}$$

the product of the discrete topologies, one has that the set of ι-orbits is a compact forward separated Shannon graph.

To the alphabet Σ, there is associated a forward separated and right-resolving λ-graph system $\mathfrak{L}_\Sigma$ as in the following way. For $n \in \boldsymbol{Z}_+$, the vertex set V_{-n} is the set of all non-empty subsets of the length n words Σ^n. For $v_{-n} \in V_{-n}$, we put $\iota(v_{-n}) = \{(\alpha_1, \dots, \alpha_{n-1}) \in \Sigma^{n-1} \mid (\alpha_1, \dots, \alpha_{n-1}, \alpha_n) \in v_{-n}\}$. Then $\iota(v_{-n})$ defines an element of V_{-n+1}. If a vertex $v_{-n} \in V_{-n}$ is written as

$$v_{-n} = \alpha_1 v^1_{-n+1} \cup \alpha_2 v^2_{-n+1} \cup \cdots \cup \alpha_k v^k_{-n+1}$$

for $\alpha_i \in \Sigma$ with $\alpha_i \neq \alpha_j$ $(i \neq j)$ and $v^i_{-n+1} \in V_{-n+1}$ as a disjoint union, then we define $\Phi_\Sigma(v_{-n}, \alpha_i) = v^i_{-n+1}$ for each $n \in \boldsymbol{N}$. The resulting labeled graph with map ι becomes a forward separated and right-resolving λ-graph system. We write it as $\mathfrak{L}_\Sigma$. The λ-graph system $\mathfrak{L}_\Sigma$ has a maximal property, that is, any forward separated and right-resolving λ-graph system over Σ is a sub λ-graph system of $\mathfrak{L}_\Sigma$. We also know that the compact forward separated Shannon graph $\mathscr{V}(\Sigma)$ is the Shannon graph of the set of ι-orbits of $\mathfrak{L}_\Sigma$.

We associate a λ-graph system $\mathfrak{L}(S)$ with a forward separated Shannon graph S. We assume that the Shannon graph is given by a transition complete set $V_0 \subset \mathscr{V}(\Sigma)$. We introduce for $n \in \boldsymbol{N}$ the equivalence relation $\approx_{(n)}$ into V_0. For $v, v' \in V_0$

$$v \approx_{(n)} v'$$

means that

$$\Gamma_n(v) = \Gamma_n(v').$$

V_{-n} is then the set of $\approx_{(n)}$-equivalence classes. The zero-vertex is V_0. The vertices in V_{-n} can be identified with the elements in the set $\{\Gamma_n(v) : v \in V_0\}$. Here $\Gamma_n(v)$ accepts $\alpha \in \Sigma$ precisely if v accepts α, and then

$$\Phi_{\mathfrak{L}(S)}(\Gamma_n(v), \alpha) = \Gamma_{n-1}(\Phi_S(v, \alpha)).$$

In particular, the associated λ-graph system of the Shannon graph $\mathscr{V}(\Sigma)$ is $\mathfrak{L}_\Sigma$.

Proposition 2.3. *Let $V \subset \mathscr{V}(\Sigma)$ be transition complete. Then the λ-graph system of V is isomorphic to the λ-graph system of the closure of V.*

Proof. The mapping that sends an n-equivalence class v of V into its closure, $n \in \boldsymbol{N}$ is an isomorphism of the λ-graph system of V onto the λ-graph system of the closure of V. □

Proposition 2.4. *Let $V \subset \mathscr{V}(\Sigma)$ be compact and transition complete. Then V as a Shannon graph is isomorphic to the Shannon graph of ι-orbits of the λ-graph system of V.*

Proof. The mapping that sends a $v \in V$ into the ι-orbit $(u_{-n})_{n \in \boldsymbol{Z}_+}$, that is given by

$$u_{-n} = \{u \in V : \Gamma_n(u) = \Gamma_n(v)\}$$

is an isomorphism of the Shannon graph V onto the Shannon graph of ι-orbits of the λ-graph system of V. □

To a transition complete subset $V \subset \mathscr{V}(\Sigma)$ there is associated the topological Markov chain $M(V)$ that contains all points $(v_i, x_i)_{i\in\boldsymbol{Z}} \in (V \times \Sigma)^{\boldsymbol{Z}}$ such that

$$v_{i+1} = \Phi_{\mathscr{V}(\Sigma)}(v_i, x_i), \quad i \in \boldsymbol{Z}.$$

By π_V we denote the projection that assigns to a point in $M(V)$ its $\Sigma^{\boldsymbol{Z}}$-coordinate. Given $V \subset \mathscr{V}(\Sigma)$ and $\tilde{V} \subset \mathscr{V}(\tilde{\Sigma})$ that present subshifts $X \subset \Sigma^{\boldsymbol{Z}}$ and $\tilde{X} \subset \tilde{\Sigma}^{\boldsymbol{Z}}$ and a topological conjugacy $\psi : X \to \tilde{X}$ we say that a shift-commuting map $\hat{\psi} : M(V) \to M(\tilde{V})$ is a lift of ψ if

$$\pi_{\tilde{V}} \circ \hat{\psi} = \psi \circ \pi_V.$$

Theorem 2.5. *Let $X \subset \Sigma^{\boldsymbol{Z}}$, $\tilde{X} \subset \tilde{\Sigma}^{\boldsymbol{Z}}$ be subshifts, and let $\psi : X \to \tilde{X}$ be a topological conjugacy. Let $V \subset \mathscr{V}(\Sigma)$ present the subshift X. Then there exists a unique $\tilde{V} \subset \mathscr{V}(\tilde{\Sigma})$ that presents $\tilde{X}$ such that ψ has a lift $\hat{\psi} : M(V) \to M(\tilde{V})$. The lift $\hat{\psi}$ is also uniquely determined.*

Proof. Let $L \in \boldsymbol{Z}_+$ be such that both ψ and ψ^{-1} are given by $(2L+1)$-block maps Ψ and $\tilde{\Psi}$, that is, Ψ maps the set of admissible words of X of length $2L+1$ into $\tilde{\Sigma}$, $\tilde{\Psi}$ maps the set of admissible words of $\tilde{X}$ of length $2L+1$ into Σ, ψ is given by

$$\psi(x) = (\Psi((x_{i+l})_{-L\le l\le L}))_{i\in\boldsymbol{Z}}, \quad x = (x_i)_{i\in\boldsymbol{Z}} \in X,$$

and ψ^{-1} is given by

$$\psi^{-1}(\tilde{x}) = (\tilde{\Psi}((\tilde{x}_{i+l})_{-L\le l\le L}))_{i\in\boldsymbol{Z}}, \quad \tilde{x} = (\tilde{x}_i)_{i\in\boldsymbol{Z}} \in \tilde{X},$$

(see e.g. [**LM**, Theorem 6.2.9]). We describe first $\tilde{V}$. For this, shifting $2L$ steps in the negative time direction, interpret V as a subset of the subsets of $\Sigma^{(-2L,\infty)}$. Then choose a $v \in V$ and a word w of length $3L+1$ that is accepted by v, and set $\tilde{w} = \Psi(w)$. Then define a set $\tilde{v} \subset \tilde{\Sigma}^{\boldsymbol{N}}$ as containing the points $\tilde{x} \in \tilde{X}_{[0,\infty)}$ such that

$$((w_i)_{-2L<i\le 0}, \tilde{\Psi}(\tilde{w}, \tilde{x})) \in v.$$

$\tilde{V}$ is then defined as containing all sets $\tilde{v}$ that can be obtained in this way from v.

We describe next the lift $\hat{\psi} : M(V) \to M(\tilde{V})$ of ψ. For this, let $(v_i, x_i)_{i\in\boldsymbol{Z}} \in M(V)$. Then setting

$$(\tilde{x}_i)_{i\in\boldsymbol{Z}} = \psi((x_i)_{i\in\boldsymbol{Z}}), \quad \hat{\psi}((v_i, x_i)_{i\in\boldsymbol{Z}}) = (\tilde{v}_i, \tilde{x}_i)_{i\in\boldsymbol{Z}},$$

one has that $\tilde{v}_i$ is given by

$$\tilde{v}_i = \{(\tilde{y}_j)_{i\le j<\infty} \in \tilde{X}_{[i,\infty)} : \tilde{\Psi}((\tilde{x}_j)_{i-L\le j<i}, (\tilde{y}_j)_{i\le j<\infty}) \in v_i\}, \quad i \in \boldsymbol{Z}.$$

To confirm the uniqueness of $\tilde{V}$ and of $\hat{\psi}$, assume that $V, V' \in \mathscr{V}(\Sigma)$ present both a subshift $Y \subset \Sigma^{\boldsymbol{Z}}$ and assume that

$$\eta : M(V) \to M(V')$$

is a lift of the identity, and prove that η is also the identity. Let $(v_i, x_i)_{i \in \boldsymbol{Z}} \in M(V)$. Setting

$$\eta((v_i, x_i)_{i \in \boldsymbol{Z}}) = (v_i', x_i)_{i \in \boldsymbol{Z}},$$

one has that

$$v_i \subset v_i', \quad i \in \boldsymbol{Z},$$

and symmetrically one has

$$v_i' \subset v_i, \quad i \in \boldsymbol{Z},$$

and the proof of the theorem is complete. □

3. Strong shift equivalence of Shannon graphs and λ-graph systems.

In this section, we will discuss strong shift equivalence of Shannon graphs and λ-graph systems. We follow the development for sofic systems in [**Na**] and [**BK**]. A Shannon graph $S = (V, E)$ over an alphabet Σ is said to be bipartite if there exist disjoint alphabets $C, D \subset \Sigma$ with $\Sigma = C \cup D$ and there exist disjoint sets $U, W \subset V$ with $V = U \cup W$ such that for every edge $e \in E$ we have its initial vertex in W, and its final vertex in U if the label of e is in D, and its initial vertex in U, and its final vertex in W if the label of e is in C.

We denote by $\mathfrak{S}_\Sigma$ the set of all finite formal sums of elements of Σ. A matrix with entries in $\mathfrak{S}_\Sigma$ is called a symbolic matrix. Let $\mathscr{M}_S$ denote the symbolic adjacency matrix associated to the Shannon graph S, that is defined as a map

$$\mathscr{M}_S(\,,\,) : V \times V \to \mathfrak{S}_\Sigma$$

in a natural way. Conversely, a 1-right resolving symbolic adjacency matrix defines a Shannon graph, where a symbolic matrix is said to be 1-right resolving if each symbol appears in every row at most one time.

Hence a Shannon graph S is bipartite if and only if the associated symbolic matrix $\mathscr{M}_S$ is of the form: $\mathscr{M}_S = \begin{bmatrix} 0 & \mathscr{P} \\ \mathscr{Q} & 0 \end{bmatrix}$ where $\mathscr{P}$ is a 1-right resolving $W \times U$ matrix with entries in $\mathfrak{S}_C$ and $\mathscr{Q}$ is a 1-right resolving $U \times W$ matrix with entries in $\mathfrak{S}_D$.

For two symbolic matrices $\mathscr{A}$ over an alphabet Σ and $\mathscr{A}'$ over an alphabet Σ' and a bijection ϕ from a subset of Σ onto a subset of Σ', we say $\mathscr{A}$ and $\mathscr{A}'$ specified equivalent under specification ϕ if $\mathscr{A}'$ can be obtained from $\mathscr{A}$ by replacing every symbol a appearing in $\mathscr{A}$ by $\phi(a)$. We write this as $\mathscr{A} \overset{\phi}{\simeq} \mathscr{A}'$. We call ϕ a specification from Σ to Σ'.

Let $S = (V, E)$, $S' = (V', E')$ be Shannon graphs over alphabets Σ, Σ' respectively. The associated symbolic adjacency matrices $\mathcal{M}_S, \mathcal{M}_{S'}$ for S, S' are said to be strong shift equivalent in 1-step if there exist alphabets C, D and specifications

$$\varphi : \Sigma \to CD, \quad \phi : \Sigma' \to DC$$

together with a 1-right resolving $V \times V'$ matrix $\mathscr{P}$ over C and a 1-right resolving $V' \times V$ matrix $\mathscr{Q}$ over D satisfying the following equations

$$\mathcal{M}_S \overset{\varphi}{\simeq} \mathscr{P}\mathscr{Q}, \quad \mathcal{M}_{S'} \overset{\phi}{\simeq} \mathscr{Q}\mathscr{P},$$

where $\mathscr{P}\mathscr{Q}$ is a $V \times V$ matrix over CD defined by

$$\mathscr{P}\mathscr{Q}(u, v) = \sum_{u' \in V'} \mathscr{P}(u, u')\mathscr{Q}(u', v), \quad u, v \in V,$$

and $\mathscr{Q}\mathscr{P}$ is similarly defined.

One has that $\mathcal{M}_S$ and $\mathcal{M}_{S'}$ are strong shift equivalent in 1-step if and only if there exists a bipartite Shannon graph $\hat{S}$ over the alphabet $C \cup D$ such that the associated symbolic matrix $\mathcal{M}_{\hat{S}}$ is of the form

$$\mathcal{M}_{\hat{S}} = \begin{bmatrix} 0 & \mathscr{P} \\ \mathscr{Q} & 0 \end{bmatrix}$$

where $\mathscr{P}$ is a 1-right resolving $V \times V'$ matrix over C and $\mathscr{Q}$ is a 1-right resolving $V' \times V$ matrix over D satisfying the equations

$$\mathcal{M}_S \overset{\varphi}{\simeq} \mathscr{P}\mathscr{Q}, \quad \mathcal{M}_{S'} \overset{\phi}{\simeq} \mathscr{Q}\mathscr{P}$$

for some specifications

$$\varphi : \Sigma \to CD, \quad \phi : \Sigma' \to DC.$$

We write this situation as

$$\mathcal{M}_S \underset{1-st}{\approx} \mathcal{M}_{S'}.$$

Two symbolic matrices $\mathcal{M}_S, \mathcal{M}_{S'}$ are said to be strong shift equivalent in N-step if there exist Shannon graphs S_i over alphabets Σ_i, $i = 1, 2, \ldots, N-1$ such that

$$\mathcal{M}_S \underset{1-st}{\approx} \mathcal{M}_{S_1} \underset{1-st}{\approx} \mathcal{M}_{S_2} \underset{1-st}{\approx} \cdots \underset{1-st}{\approx} \mathcal{M}_{S_{N-1}} \underset{1-st}{\approx} \mathcal{M}_{S'}.$$

We denote this situation by

$$\mathcal{M}_S \underset{N-st}{\approx} \mathcal{M}_{S'}$$

and simply call it a strong shift equivalence.

We also recall from [**Ma**] the notion of symbolic matrix systems, bipartite symbolic matrix systems and their strong shift equivalence. For a λ-graph system $\mathfrak{L}$ with vertex set $V = \bigcup_{n \in \boldsymbol{Z}_+} V_{-n}$, let $\mathscr{M}_{-n,-n-1}$ be the symbolic adjacency matrix of the labeled graph $\mathfrak{L}$ between the vertices V_{-n-1} and V_{-n}. That is, $\mathscr{M}_{-n,-n-1}$ has a symbol α in the component (i,j), $i \in V_{-n}$, $j \in V_{-n-1}$ if there exists an edge from j to i with symbol α. The matrices $I_{-n,-n-1}$, $n \in \boldsymbol{Z}_+$ with entries in $\{0,1\}$ are defined by the mappings $\iota : V_{-n-1} \to V_{-n}$ in a natural way. Then the sequence of pairs $(\mathscr{M}_{-n,-n-1}, I_{-n,-n-1})$, $n \in \boldsymbol{Z}_+$ of matrices satisfies the equations:

$$I_{-n,-n-1}\mathscr{M}_{-n-1,-n-2} = \mathscr{M}_{-n,-n-1}I_{-n-1,-n-2} \quad n \in \boldsymbol{Z}_+.$$

We call such sequence of pairs of matrices $(\mathscr{M}_{-n,-n-1}, I_{-n,-n-1})$, $n \in \boldsymbol{Z}_+$ a symbolic matrix system and write it as $(\mathscr{M}, I)$.

Let $(\mathscr{M}, I)$ and $(\mathscr{M}', I')$ be symbolic matrix systems over alphabets Σ, Σ' respectively, where $\mathscr{M}_{-n,-n-1}, I_{-n,-n-1}$ are $m(n) \times m(n+1)$ matrices and $\mathscr{M}'_{-n,-n-1}, I'_{-n,-n-1}$ are $m'(n) \times m'(n+1)$ matrices. Then $(\mathscr{M}, I)$, $(\mathscr{M}', I)$ are said to be strong shift equivalent in 1-step, written as $(\mathscr{M}, I) \underset{1-st}{\approx} (\mathscr{M}', I')$ if there exist alphabets C, D and specifications $\varphi : \Sigma \to CD$ and $\phi : \Sigma' \to DC$ such that for each $n \in \boldsymbol{N}$, there exist an $m(n-1) \times m'(n)$ matrix $\mathscr{H}_{-n}$ over C and an $m'(n-1) \times m(n)$ matrix $\mathscr{K}_{-n}$ over D satisfying the following equations:

$$I_{-n+1,-n}\mathscr{M}_{-n,-n-1} \overset{\varphi}{\simeq} \mathscr{H}_{-n}\mathscr{K}_{-n-1}, \quad I'_{-n+1,-n}\mathscr{M}'_{-n,-n-1} \overset{\phi}{\simeq} \mathscr{K}_{-n}\mathscr{H}_{-n-1}$$

and

$$\mathscr{H}_{-n}I'_{-n,-n-1} = I_{-n+1,-n}\mathscr{H}_{-n-1}, \quad \mathscr{K}_{-n}I_{-n,-n-1} = I'_{-n+1,-n}\mathscr{K}_{-n-1}.$$

$(\mathscr{M}, I)$ and $(\mathscr{M}', I')$ are said to be strong shift equivalent in N-step, or to be strong shift equivalent, written $(\mathscr{M}, I) \underset{N-st}{\approx} (\mathscr{M}', I')$, if there exist symbolic matrix systems $(\mathscr{M}^{(i)}, I^{(i)})$, $i = 1, 2, \ldots, N-1$ such that

$$(\mathscr{M}, I) \underset{1-st}{\approx} (\mathscr{M}^{(1)}, I^{(1)}) \underset{1-st}{\approx} (\mathscr{M}^{(2)}, I^{(2)}) \underset{1-st}{\approx} \cdots \underset{1-st}{\approx} (\mathscr{M}^{(N-1)}, I^{(N-1)}) \underset{1-st}{\approx} (\mathscr{M}', I').$$

We remark that if λ-graph systems are right-resolving and forward separated, strong shift equivalence is equivalent to properly strong shift equivalence of their symbolic matrix systems ([**Ma**], [**Ma3**]).

Proposition 3.1. *Let $V \subset \mathscr{V}(\Sigma)$ and $\tilde{V} \subset \mathscr{V}(\tilde{\Sigma})$ be compact. Then the symbolic adjacency matrices of V and $\tilde{V}$ are strong shift equivalent if and only if the symbolic matrix systems of V and $\tilde{V}$ are strong shift equivalent.*

Proof. It is enough to consider 1-step strong shift equivalence, in which case the statement of the proposition is confirmed by inspection using bipartite Shannon graphs and bipartite symbolic matrix systems. □

Proposition 3.2. *Let $V \subset \mathscr{V}(\Sigma)$ and $\tilde{V} \subset \mathscr{V}(\tilde{\Sigma})$ present the subshifts $X \subset \Sigma^{\boldsymbol{Z}}$ and $\tilde{X} \subset \tilde{\Sigma}^{\boldsymbol{Z}}$. Then the symbolic adjacency matrices of V and $\tilde{V}$ are strong shift equivalent if and only if there exists a topological conjugacy $\psi : X \to \tilde{X}$ with a lift that carries $M(V)$ into $M(\tilde{V})$.*

Proof. Recall the result of Nasu [**Na**] and [**Na2**], who introduced the notion of bipartite subshifts. A subshift X over an alphabet Σ is said to be bipartite if there exist disjoint subsets $C, D \subset \Sigma$ such that in any $(x_i)_{i \in \boldsymbol{Z}} \in X$ either $x_i \in C$ and $x_{i+1} \in D$ for all $i \in \boldsymbol{Z}$, or $x_i \in D$ and $x_{i+1} \in C$ for all $i \in \boldsymbol{Z}$. Let $X^{(2)}$ be the 2-higher power shift of X. Put

$$X_{CD} = \{(c_i d_i)_{i \in \boldsymbol{Z}} \in X^{(2)} \mid c_i \in C, d_i \in D\},$$

$$X_{DC} = \{(d_i c_i)_{i \in \boldsymbol{Z}} \in X^{(2)} \mid c_i \in C, d_i \in D\}.$$

These are subshifts over CD and DC respectively. Hence $X^{(2)}$ is partitioned into the two subshifts X_{CD} and X_{DC}. Nasu in [**Na**], [**Na2**] also introduced the notion of bipartite conjugacy. The conjugacy from X_{CD} onto X_{DC} that maps $(c_i d_i)_{i \in \boldsymbol{Z}}$ to $(d_i c_{i+1})_{i \in \boldsymbol{Z}}$ is called the forward bipartite conjugacy. The conjugacy from X_{CD} onto X_{DC} that maps $(c_i d_i)_{i \in \boldsymbol{Z}}$ to $(d_{i-1} c_i)_{i \in \boldsymbol{Z}}$ is called the backward bipartite conjugacy. A topological conjugacy between subshifts is called a symbolic conjugacy if it is a 1-block map given by a bijection between the underlying alphabets of the subshifts. Nasu in [**Na**] proved that any topological conjugacy ϕ between subshifts is factorized into a composition of the form:

$$\phi = \kappa_n \zeta_n \kappa_{n-1} \zeta_{n-1} \cdots \kappa_1 \zeta_1 \kappa_0$$

where $\kappa_0, \dots, \kappa_n$ are symbolic conjugacies and $\zeta_1, \dots, \zeta_n$ are either forward or backward bipartite conjugacies.

It follows that it is enough to confirm the statement of the proposition in the case of bipartite codings, in which case this is done by inspection using bipartite Shannon graphs. □

4. Presentations of subshifts.

We consider subshifts $X \subset \Sigma^{\boldsymbol{Z}}$. For the admissible blocks of the subshift X we use notation like

$$x_{[i,k]} = (x_j)_{i \le j \le k}, \quad x = (x_i)_{i \in \boldsymbol{Z}} \in X,$$

$$X_{[i,k]} = \{x_{[i,k]} : x \in X\}.$$

By $\Gamma^+(X, y)$ we denote the set of all $x \in X_{[i,\infty)}$ that can follow $y \in X_{(-\infty,i)}$, $i \in \boldsymbol{Z}$. Similarly for an admissible finite block a $\Gamma^+(X, a)$ denotes the set of all right-infinite words that can follow a, $\Gamma^-(X,a)$ having the time symmetric meaning. By $\Gamma_n^+(X,y)$ we denote the set of all admissible words of length n that can follow $y \in X_{(-\infty,i)}$, $i \in \boldsymbol{Z}$.

For a subshift $X \subset \Sigma^{\boldsymbol{Z}}$, let

$$V_n^+(X) = \{\Gamma_n^+(X,y) : y \in X_{(-\infty,0]}\}, \quad n \in \boldsymbol{N}$$

and

$$V^+(X) = \{\Gamma^+(X,y) : y \in X_{(-\infty,0]}\}.$$

$V^+(X)$ is a transition complete subset of $\mathscr{V}(\Sigma)$. As a Shannon graph $V^+(X)$ presents X. We remark at this point that a subshift X has one maximal presenting Shannon graph whose vertex set is the set of all closed subsets of the set $\Gamma^+(X,y) : y \in X_{(-\infty,0]}$.

Set for $a \in X_{[-l,0]}$, $l \in \boldsymbol{Z}_+$,

$$V^+(X,a) = \{\Gamma^+(X,ya) : y \in \Gamma^-(X,a)\}.$$

Lemma 4.1. *Let $l \in \boldsymbol{Z}_+$. $V^+(X)$ is compact if and only if $V^+(X,a)$ is compact for all $a \in X_{[-l,0]}$.*

Proof. On the one hand side, one has

$$V^+(X) = \bigcup_{a \in X_{[-l,0]}} V^+(X,a),$$

and on the other hand, one has for all $y \in \Gamma^-(X,a)$, $\Gamma^+(X,ya)$ given precisely by the $x \in \Gamma^+(X,y)$ such that $a = x_{[-l,0]}$. □

Proposition 4.2. *For subshifts X, the compactness of $V^+(X)$ is an invariant of topological conjugacy.*

Proof. Let $X \subset \Sigma^{\boldsymbol{Z}}$, $\tilde{X} \subset \tilde{\Sigma}^{\boldsymbol{Z}}$ be topologically conjugate subshifts, and let there be a topological conjugacy of $\tilde{X}$ onto X given for some $L \in \boldsymbol{Z}_+$ by a $(2L+1)$-block map Φ with the inverse of the conjugacy given by a $(2L+1)$-block map $\tilde{\Phi}$. Let $V^+(\tilde{X})$ be compact. According to Lemma 4.1, we establish the proposition by showing for $a \in X_{[-2L,0]}$ that $V^+(X,a)$ is compact. For this, let there be given

$$y^{(k)} \in \Gamma^-(X,a), \quad k \in \boldsymbol{N},$$

such that there is a limit

$$v = \lim_{k\to\infty} \Gamma^+(X, y^{(k)}a).$$

We have to show that there is a $y \in \Gamma^-(X,a)$ such that

$$(1) \qquad v = \Gamma^{+}(X, ya).$$

Selecting, if necessary, a subsequence we can assume without loss of generality that there is a block $\tilde{b} \in \tilde{X}_{[-3L,-L]}$ and

$$\tilde{y}^{(k)} \in \Gamma^{-}(\tilde{X}, \tilde{b}), \quad k \in \boldsymbol{N},$$

such that

$$\tilde{y}^{(k)} b = \tilde{\Phi}(y^{(k)} a),$$

and moreover, such that there is a limit

$$\tilde{v} = \lim_{k \to \infty} \Gamma^{+}(\tilde{X}, \tilde{y}^{(k)} \tilde{b}).$$

By Lemma 4.1 $\tilde{V}^{+}(\tilde{X}, \tilde{b})$ is compact, and we have a $\tilde{y} \in \Gamma^{-}(\tilde{X}, \tilde{b})$ such that $\tilde{v} = \Gamma^{+}(\tilde{X}, \tilde{y}\tilde{b})$. We set $y = \Phi(\tilde{y}\tilde{b})$. By construction $y \in \Gamma^{-}(X, a)$. We show that (1) holds. For this we observe that here

$$x \in \Gamma^{+}(X, ya)$$

is equivalent to

$$\tilde{x} = \tilde{\Phi}(a_{(-2L,0]}, x) \in \Gamma^{+}(\tilde{X}, \tilde{y}\tilde{b}),$$

which is equivalent to having a $k_0 \in \boldsymbol{N}$ such that

$$\tilde{x} \in \Gamma^{+}(\tilde{X}, \tilde{y}^{(k)} \tilde{b}), \quad k \geq k_0,$$

which in turn is equivalent to

$$x \in \Gamma^{+}(X, y^{(k)} a), \quad k \geq k_0,$$

and therefore to

$$x \in v. \qquad \square$$

We introduce an additional notation. Given a forward separated Shannon λ-graph system

$$V = \bigcup_{n \in \boldsymbol{Z}_{+}} V_{-n},$$

we denote for $n \in \boldsymbol{N}$ by $\mathscr{F}_{-n}(V)$ the set of words c such that there exists a vertex in V_{-n} into which c leads, and that is maximal among the vertices in V_{-n} into which c leads, and for $c \in \mathscr{F}_{-n}(V)$ denote by $v_{-n}(c)$ the vertex with the stated property.

For an admissible left-infinite word $(\sigma_{-i})_{i \in \boldsymbol{Z}_{+}}$ of a forward separated Shannon λ-graph system $(V_{-n}, E_{-n})_{n \in \boldsymbol{Z}_{+}}$ we say that a vertex $v \in V_{-n}$, $n \in \boldsymbol{Z}_{+}$, is associated to $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$ if the following holds: $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$ can lead into v, and v is maximal among the vertices of V_{-n} into which $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$ can lead.

LEMMA 4.3. *Let for all $n\in \boldsymbol{Z}_+$, $v_{-n}\in V_{-n}$ be associated to $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$. Then $(v_{-n})_{n\in \boldsymbol{Z}_+}$ is an ι-orbit.*

PROOF. Observe first that a vertex $v\in V_{-n}$, $n\in \boldsymbol{Z}_+$, is associated to $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$ if and only if the following holds: $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$ precedes an entry into v and there is an $I\in \boldsymbol{N}$ such that $(\sigma_{-i})_{I\geq i\geq 0}\in \mathscr{F}_{-n}(V)$ and $v=v_{-n}((\sigma_{-i})_{I\geq i\geq 0})$. Let now $v\in V_{-n}$, $n\in \boldsymbol{N}$, be associated to $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$ and select an $I\in \boldsymbol{N}$ as described. We prove that $\iota(v)$ is also associated to $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$. Assume that there is not the case and let $u\in V_{-n+1}$ be such that $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$ precedes an entry into u, and that there is a word $b\in \Gamma^+(u)$ that is not accepted by $\iota(v)$. Then let $w\in V_{-n-I+1}$ be such that there is a path from w to u that carries the word $(\sigma_{-i})_{I\geq i\geq 0}=c$, and let $w'\in V_{-n-I}$ be such that $w=\iota(w')$, w' accepts the word $cb\sigma$ for some $\sigma\in \Sigma$. It follows that v accepts the word $b\sigma$, and therefore $\iota(v)$ accepts the word b, a contradiction. □

We say that the ι-orbit $(v_{-n})_{n\in \boldsymbol{Z}_+}$ where v_{-n} is associated to $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$ is associated to $(\sigma_{-i})_{i\in \boldsymbol{Z}_+}$. The ι-orbits that are associated to admissible left-infinite words we call contextual ι-orbits. The contextual ι-orbits form a sub-Shannon graph of the Shannon graph of ι-orbits. By construction the Shannon graph of contextual ι-orbits of the canonical λ-graph system of a subshift is isomorphic to the Shannon graph of forward contexts of the subshift.

PROPOSITION 4.4. *The existence of a non-contextual ι-orbit in the canonical λ-graph system of a subshift $X\subset \Sigma^{\boldsymbol{Z}}$ is an invariant of topological conjugacy.*

PROOF. Observe that all ι-orbits of the canonical λ-graph system are contextual if and only if $V^+(X)$ is compact. Apply Proposition 4.2. □

For every subshift $X\subset \Sigma^{\boldsymbol{Z}}$ one has the maximal forward separated Shannon λ-graph system that presents X and that corresponds to the maximal presenting Shannon graph of the subshift. In its vertex set $(V_{-n})_{n\in \boldsymbol{Z}_+}$, V_{-n} contains all subsets of the sets in $V_n^+(X)$, $n\in \boldsymbol{N}$, and the corresponding compact forward separated Shannon graph is given by the set of all compact subsets of the sets that are in the closure of $V^+(X)$. Every subshift $X\subset \Sigma^{\boldsymbol{Z}}$ has also a presentation by the canonical forward separated λ-graph system of [**Ma**]. Its vertex set $(V_{-n}(X))_{n\in \boldsymbol{Z}_+}$ is given by

$$V_{-n}=V_n^+(X)\quad n\in \boldsymbol{N},$$

and the corresponding compact forward separated Shannon graph is given by the closure of $V^+(X)$.

We want to characterize the canonical λ-graph system of a subshift $X \subset \Sigma^{\boldsymbol{Z}}$.

Lemma 4.5. *Let $X \subset \Sigma^{\boldsymbol{Z}}$ be a subshift. For all $y \in X_{(-\infty,0]}$ and $n \in \boldsymbol{N}$ there is a $k_0 \in \boldsymbol{N}$ such that*

$$\Gamma_n^+(X, y_{[-k_0,0]}) = \Gamma_n^+(X, y).$$

Proof. It is

$$\Gamma_n^+(X, y) = \bigcap_{k \in \boldsymbol{N}} \Gamma_n^+(X, y_{[-k,0]}). \qquad \square$$

Lemma 4.6. *Let V be a forward separated Shannon λ-graph system. Let $l, m, n \in \boldsymbol{N}$, and let $a \in \mathscr{F}_{-m-n}(V)$ have length l. Let c be an admissible word of V of length $l+m$ that begins with a such that $c = ab$. Then*

$$c \in \mathscr{F}_{-n}(V), \quad v_{-n}(c) = \Phi_V(v_{-n-m}(a), b). \tag{2}$$

Proof. Set

$$u = \Phi_V(v_{-n-m}(a), b),$$

let u' be any vertex in V_{-n} into which c can lead, and let then w be any vertex in V_{-n-m-l} such that $u' = \Phi_V(w, c)$. It is

$$v_{-n-m}(a) \supset \Phi_V(w, a),$$

and therefore $u' \supset u$ and (2) follows. $\square$

Lemma 4.7. *Let V be a forward separated Shannon λ-graph system. For all $n \in \boldsymbol{N}$*

$$\mathscr{F}_{-n-1}(V) \subset \mathscr{F}_{-n}(V),$$

and for $c \in \mathscr{F}_{-n-1}(V)$,

$$\iota(v_{-n-1}(c)) = v_{-n}(c).$$

Proof. The map ι respects the labeling. $\square$

Proposition 4.8. *In the canonical λ-graph system $V(X)$ of a subshift $X \subset \Sigma^{\boldsymbol{Z}}$ one has for all $n \in \boldsymbol{N}$ that*

(1) *for all $v \in V_{-n}(X)$ there is a $c \in \mathscr{F}_{-n}(V(X))$ such that*

$$v = v_{-n}(c),$$

(2) *every left infinite word $(x_{-n})_{n \in \boldsymbol{N}}$ that can lead into the zero-vertex ends in a word in $\mathscr{F}_{-n}(V(X))$.*

Proof. Apply Lemma 4.5. □

We have a converse of the proposition.

Proposition 4.9. *Let*

$$V = \bigcup_{n \in \boldsymbol{Z}_+} V_{-n}$$

be a forward separated Shannon λ*-graph system such that for all* $n \in \boldsymbol{N}$

(1) *for all* $v \in V_{-n}(X)$ *there is a* $c \in \mathscr{F}_{-n}(V(X))$ *such that*

$$v = v_{-n}(c),$$

(2) *every left infinite word* $(x_{-n})_{n \in \boldsymbol{N}}$ *that can lead into the zero-vertex ends in a word in* $\mathscr{F}_{-n}(V(X))$.

Then V *is isomorphic to the canonical* λ*-graph system of the subshift that it presents.*

Proof. We have to show that to every left infinite word that can lead into the zero-vertex there is associated an ι-orbit and that these contextual ι-orbits are dense in the space of ι-orbits. Let $(x_{-n})_{n \in \boldsymbol{N}}$ be a word that can lead into the zero-vertex. Then the words $(x_{-i})_{i > I}$, $I \in \boldsymbol{N}$, can also lead into the zero-vertex and one finds from Lemma 4.6 that there is a sequence $I_n \in \boldsymbol{N}$ such that $I_{n+1} > I_n$, $n \in \boldsymbol{N}$, and such that the words $b_n = (x_{-i})_{-I_i \le i \le 0}$ are in $\mathscr{F}_{-n}(V)$, $n \in \boldsymbol{N}$. By means of Lemma 4.7, one has a selection argument that shows that there is an ι-orbit $(u_{-n})_{n \in \boldsymbol{Z}_+}$ such that for all $n \in \boldsymbol{N}$ there is a $K_n \in \boldsymbol{N}$ such that

$$v_{-n} = v_{-n}(b_k), \quad k > K_n.$$

It follows that the ι-orbit $(u_{-n})_{n \in \boldsymbol{Z}_+}$ is associated to the sequence $(x_{-n})_{n \in \boldsymbol{N}}$. To show the density of the contextual ι-orbits, let $n \in \boldsymbol{N}$ and $u \in V_{-n}$, and $c \in \mathscr{F}_{-n}(V)$, $u = v_{-n}(c)$. Write $c = (\gamma_{-k})_{0 \le k \le K}$ and let $(\gamma_{-k})_{k \in \boldsymbol{Z}_+}$ be a word that leads into u. Then the ι-orbit that is associated to $(\gamma_{-k})_{k \in \boldsymbol{Z}_+}$ contains u. □

Theorem 4.10. *Let* $X \subset \Sigma^{\boldsymbol{Z}}$, $\tilde{X} \subset \tilde{\Sigma}^{\boldsymbol{Z}}$ *be subshifts, and let* $\psi : X \to \tilde{X}$ *be a topological conjugacy. Then*

$$(\hat{\psi})(M(V(X))) = M(V(\tilde{X})). \tag{3}$$

Proof. It is (cf. [**Kr3**, Section 3]),

$$(\hat{\psi})(M(V^+(X))) = M(V^+(\tilde{X})).$$

Due to the uniform continuity of ψ with respect to the metrics on $V(\Sigma)$ and $V(\tilde{\Sigma})$ one has then also (3). □

5. Protosynchronization.

We call an admissible word c of a subshift $X \subset \Sigma^{\boldsymbol{Z}}$ protosynchronizing if for all $n \in \boldsymbol{N}$ there is a transitive $y \in \Gamma^{-}(X, c)$ such that

$$\Gamma_n^{+}(X, yc) = \Gamma_n^{+}(X, c).$$

Every admissible word of a subshift that begins with a protosynchronizing word is also protosynchronizing. For a protosynchronizing subshift $X \subset \Sigma^{\boldsymbol{Z}}$, that is a subshift with a protosynchronizing word, it is therefore possible to define the canonical protosynchronizing λ-graph system

$$V^{(ps)}(X) = \bigcup_{n \in \boldsymbol{Z}_+} V_{-n}^{(ps)}(X)$$

as the forward separated Shannon λ-graph system that has as elements of $V_{-n}^{(ps)}(X)$ the sets $\Gamma_n^{+}(X, c)$, c a protosynchronizing word of X, $n \in \boldsymbol{N}$.

For a forward separated Shannon λ-graph system V and a word

$$c \in \bigcap_{n \in \boldsymbol{N}} \mathscr{F}_{-n}(V)$$

we say that the ι-orbit $(v_{-n}(c))_{n \in \boldsymbol{Z}_+}$ is associated to c. We say that a forward separated Shannon λ-graph system V is a protosynchronizing λ-graph system, if

$$\bigcap_{n \in \boldsymbol{N}} \mathscr{F}_{-n}(V) \neq \emptyset,$$

and if also all ι-orbits of V that are associated to a word in $\bigcap_{n \in \boldsymbol{N}} \mathscr{F}_{-n}(V)$ contain for every vertex $u \in V$ a vertex that can be connected to u. The canonical protosynchronizing λ-graph system of a protosynchronizing subshift is protosynchronizing in this sense.

PROPOSITION 5.1. *Let $\tilde{X} \subset \tilde{\Sigma}^{\boldsymbol{Z}}$, $X \subset \Sigma^{\boldsymbol{Z}}$ be subshifts, X protosynchronizing, and let $\psi : \tilde{X} \to X$ be a topological conjugacy. Then $\tilde{X}$ is also protosynchronizing and*

$$\hat{\psi}(V^{(ps)}(\tilde{X})) = V^{(ps)}(X). \tag{1}$$

PROOF. Let $L \in \boldsymbol{Z}_+$ be such that both ψ and ψ^{-1} are given by $(2L+1)$-block maps. To prove that $\tilde{X}$ is protosynchronizing, let $n \in \boldsymbol{N}$, and consider the situation that one is given a left transitive point $x \in X$ such that for some $K \in \boldsymbol{Z}_+$, $(x_i)_{-2L-K \le i \le 0}$ is a protosynchronizing word for X, and such that

$$\Gamma_{2L+n}^{+}(X, x_{[-2L-K, 0]}) = \Gamma_{2L+n}^{+}(X, x_{(-\infty, 0]}).$$

Set $\tilde{x} = \psi^{-1}(x)$. Then

$$\Gamma_n^{+}(\tilde{X}, \tilde{x}_{[-L-K, L]}) = \Gamma_n^{+}(\tilde{X}, \tilde{x}_{(-\infty, L]}).$$

One concludes that $(\tilde{x}_i)_{-L-K\le i\le L}$ is a protosynchronizing word for $\tilde{X}$. (1) is seen by inspection. □

LEMMA 5.2. *Let $V = \bigcup_{n\in\boldsymbol{Z}_+} V_{-n}$ be a protosynchronizing λ-graph system, and let $m \in \boldsymbol{N}$, $u \in V_{-m}$. Then there exists a*

$$b \in \bigcap_{n\in\boldsymbol{N}} \mathscr{F}_{-n}(V) \tag{2}$$

such that

$$u = v_{-m}(b). \tag{3}$$

PROOF. Let

$$c \in \bigcap_{n\in\boldsymbol{N}} \mathscr{F}_{-n}(V).$$

There is for some $k \in \boldsymbol{N}$ a word a such that $v_{-m-k}(c)$ is connected to u by a. Set b equal to ca. By Lemma 4.6, (2) and (3) hold. □

LEMMA 5.3. *Let $V = \bigcup_{n\in\boldsymbol{Z}_+} V_{-n}$ be a protosynchronizing λ-graph system, and let a be an admissible word of V. Then there exists a word in $\bigcap_{n\in\boldsymbol{N}} \mathscr{F}_{-n}(V)$ that ends in a.*

PROOF. Let $m \in \boldsymbol{N}$, $u \in V_{-m}$ be such that u is connected by a to the zero-vertex. By Lemma 5.2, there is a $b \in \bigcap_{n\in\boldsymbol{N}} \mathscr{F}_{-n}(V)$ such that $u = v_{-m}(b)$. By Lemma 4.6 $ba \in \bigcap_{n\in\boldsymbol{N}} \mathscr{F}_{-n}(V)$. □

LEMMA 5.4. *For a protosynchronizing λ-graph system $V = \bigcup_{n\in\boldsymbol{Z}_+} V_{-n}$ every word in $\bigcap_{n\in\boldsymbol{N}} \mathscr{F}_{-n}(V)$ is protosynchronizing for the subshift X that is presented by V.*

PROOF. By Lemma 5.3, one has a sequence

$$b^{(k)} \in \bigcap_{n\in\boldsymbol{N}} \mathscr{F}_{-n}(V), \quad k \in \boldsymbol{N},$$

such that for every admissible word a of V infinitely many of the words $b^{(k)}$, $k \in \boldsymbol{N}$, end in a. Let $c \in \bigcap_{n\in\boldsymbol{N}} \mathscr{F}_{-n}(V)$ and let $n_0 \in \boldsymbol{N}$. One constructs inductively sequences $m(k) \in \boldsymbol{N}$, $k \in \boldsymbol{Z}_+$, and $l(k) \in \boldsymbol{N}$, $k \in \boldsymbol{N}$, such that $n_0 < m(0)$, and

$$m(k-1) < l(k) < m(k), \quad k \in \boldsymbol{N},$$

together with a sequence $u_k \in V_{-m(k)}$, $k \in \boldsymbol{Z}_+$ such that $v_{-n_0}(c) = \Phi_V(u, c)$ and such that for all $k \in \boldsymbol{N}$, $v_{-l(k)}(b^{(k)})$ can be connected to u_{k-1}, and $v_{-l(k)}(b^{(k)}) = \Phi_V(u_k, b^{(k)})$. Filling in the connections one obtains a transitive past y of c such that

$$v_{-n_0}(c) = \Gamma^+_{n_0}(X, yc). \qquad \square$$

Proposition 5.5. *A protosynchronizing λ-graph system V is isomorphic to the canonical protosynchronizing λ-graph system of the subshift X that it presents.*

Proof. In view of Lemma 5.4, we are left with proving that every protosynchronizing word b of X is in $\bigcap_{n\in \boldsymbol{N}} \mathscr{F}_{-n}(V)$. For this let $n_0 \in \boldsymbol{N}$, and let $c \in \bigcap_{n\in \boldsymbol{N}} \mathscr{F}_{-n}(V)$. Since b is protosynchronizing, there is an admissible word a of X such that cab is admissible, and such that

$$\Gamma^+_{n_0}(X, cab) = \Gamma^+_{n_0}(X, b).$$

Then

$$v_{-n_0}(cab) = \Gamma^+_{n_0}(X, b).$$

This means that $b \in \mathscr{F}_{-n_0}(V)$, and that $v_{-n_0}(b) = v_{-n_0}(cab)$. $\square$

Every admissible word of a subshift that contains a synchronizing word as a subword is also synchronizing. For a synchronizing subshift $X \subset \Sigma^{\boldsymbol{Z}}$ it is therefore possible to define the canonical synchronizing λ-graph system

$$V^{(s)}(X) = \bigcup_{n\in \boldsymbol{Z}_+} V^{(s)}_{-n}(X)$$

as the forward separated Shannon λ-graph system that has as elements of $V^{(s)}_{-n}(X)$ the sets $\Gamma^+_n(X, c)$, c a synchronizing word of X, $n \in \boldsymbol{N}$. For synchronizing subshifts $X \subset \Sigma^{\boldsymbol{Z}}$, $V^{(s)}(X)$ coincides with $V^{(ps)}(X)$.

6. Examples.

For sofic systems the canonical λ-graph system corresponds to the complete right-resolving extension of [**Kr**], [**Kr2**], and the canonical synchronizing λ-graph system of topologically transitive sofic systems corresponds to the Fisher automaton ([**Fis**]). There are topologically transitive sofic systems X such that $V(X)$ is equal to $V^{(s)}(X)$ ([**Fie**]).

We list some synchronizing non-sofic examples. These are coded systems $X(\mathscr{C})$ defined by codes $\mathscr{C}$ ([**BH**]).

1.

$$\Sigma = \{\gamma, 0, 1\},$$

$$\mathscr{C} = \{\gamma 0^l 1^l : l \in \boldsymbol{N}\}.$$

Here the canonical synchronizing λ-graph system is a proper subsystem of the canonical λ-graph system and all ι-orbits are contextual.

2. a)

$$\Sigma = \{\gamma, 0, 1\},$$
$$\mathscr{C} = \{\gamma 0^{kl} 1^{kl} : l \in \boldsymbol{N}\}, \quad k > 1.$$

b)

$$\Sigma = \{\alpha, \beta, 0, 1_\alpha, 1_\beta\},$$
$$\mathscr{C} = \{\alpha 0^{kl} 1_\alpha^{kl} : l \in \boldsymbol{N}\} \cup \{\beta 0^{kl} 1_\beta^{kl} : l \in \boldsymbol{N}\}, \quad k > 1.$$

Here in both cases the canonical synchronizing λ-graph system is a proper subsystem of the canonical λ-graph system and there are non-contextual ι-orbits.

One can obtain semisynchronizing nonsynchronizing examples by using these coded systems $X(\mathscr{C})$ and the Dyck shifts in a product construction.

We give an example of a protosynchronizing subshift that is not semisynchronizing. Let

$$\Sigma = \{\alpha, \beta, \gamma\},$$

and let S be the shift on $\Sigma^{\boldsymbol{Z}}$. We describe disjoint closed sets $E_1, E_2, E_3, E_4 \subset \Sigma^{\boldsymbol{Z}}$ such that the subshift

$$X = \bigcap_{i \in \boldsymbol{Z}} S^i(E_1 \cup E_2 \cup E_3 \cup E_4)$$

is protosynchronizing but not semisynchronizing. Here

$$E_1 = \{x \in \Sigma^{\boldsymbol{Z}} : x_1 \neq \alpha\} \cup \{x \in \Sigma^{\boldsymbol{Z}} : x_0 = x_1 = \alpha\},$$
$$E_2 = \{x \in \Sigma^{\boldsymbol{Z}} : x_0 \neq \alpha, x_1 = \alpha\}$$
$$\cap \bigcap_{i \in \boldsymbol{N}} (\{x \in \Sigma^{\boldsymbol{Z}} : x_{-i+1} \neq \alpha\} \cup \{x \in \Sigma^{\boldsymbol{Z}} : x_{-i} = x_{-i+1} = \alpha\}).$$

For

$$x \in \Sigma^{\boldsymbol{Z}} - (E_1 \cup E_2)$$

define $r(x)$ as the smallest integer $r > 1$ such that

$$x_{-r} \neq \alpha, \quad x_{-r+1} = \alpha.$$

Let D be the set of

$$x \in \Sigma^{\boldsymbol{Z}} - (E_1 \cup E_2)$$

such that the number of $i \in \boldsymbol{N}$, $1 \leq i \leq 2r(x)$, such that

$$x_{-i} = \beta$$

is greater than or equal to the number of $i \in \boldsymbol{N}$, $1 \le i \le 2r(x)$, such that

$$x_{-i} \neq \alpha, \quad x_{-i+1} = \alpha,$$

and for $x \in D$ define $h(x)$ as the smallest $h \in \boldsymbol{N}$, $1 \le i \le 2r(x)$, such that the number of $i \in \boldsymbol{N}$, $1 \le i \le h$, such that

$$x_{-i} = \beta$$

is equal to the number of $i \in \boldsymbol{N}$, $1 \le i \le h$, such that

$$x_{-i} \neq \alpha, \quad x_{-i+1} = \alpha.$$

Let

$$E_3 = \{x \in D : x_{-h(x)-q} \neq \gamma, q \in \boldsymbol{N}\},$$

and for $x \in D - E_3$ define $q(x)$ as the smallest $q \in \boldsymbol{N}$ such that

$$x_{-h(x)-q} = \gamma.$$

Then set

$$E_4 = (D - E_3) - \{x \in D - E_3 : x_i = \alpha, 1 < i \le 2q(x)\}.$$

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