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Local theta correspondence of depth zero representations and theta dichotomy

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Abstract. In this paper, we prove that depth zero representations are preserved by local theta correspondence for any type I reductive dual pairs over a p-adic field. Moreover, the minimal K-types of the paired depth zero irreducible admissible representations are paired by the theta correspondence for finite reductive dual pairs. As a consequence, we prove that the Iwahori-spherical representations are preserved by the local theta correspondence. Then we obtain some partial result of theta dichotomy for finite reductive dual pairs and p-adic reductive dual pairs of symplectic and orthogonal group, which is analogous to S. Kudla and S. Rallis' result for p-adic unitary groups.

0. Introduction.

Let F be a p-adic field with odd residual characteristic. Let D be a central division algebra over F with an involution, $\mathcal{O}$ be the ring of integers, $\mathfrak{p}$ be the prime ideal, f_D be the (finite) residue field and ϖ be a prime element. Let $\mathscr{V}$ (resp. $\mathscr{V}'$) be a finite-dimensional nondegenerate ε-Hermitian (resp. ε'-Hermitian) space over D where $\varepsilon, \varepsilon'$ are 1 or -1 and $\varepsilon\varepsilon' = -1$. Let $U(\mathscr{V})$ (resp. $U(\mathscr{V}')$) denote the group of isometries of $\mathscr{V}$ (resp. $\mathscr{V}'$). We can define a skew-symmetric F-bilinear form on the space $\mathscr{W} := \mathscr{V} \otimes_D \mathscr{V}'$. Then the pair $(U(\mathscr{V}), U(\mathscr{V}'))$ forms a *reductive dual pair* (over F) in the symplectic group $Sp(\mathscr{W})$ (*cf.* [**Hw2**], [**MVW**]). In particular, we have embeddings $\iota_{\mathscr{V}'} : U(\mathscr{V}) \to Sp(\mathscr{W})$ and $\iota_{\mathscr{V}} : U(\mathscr{V}') \to Sp(\mathscr{W})$. Let $\widetilde{Sp(\mathscr{W})}$ denote the metaplectic cover of $Sp(\mathscr{W})$. Let $\widetilde{U(\mathscr{V})}$ (resp. $\widetilde{U(\mathscr{V}')}$) denote the inverse image of $\iota_{\mathscr{V}'}(U(\mathscr{V}))$ (resp. $\iota_{\mathscr{V}}(U(\mathscr{V}'))$) in $\widetilde{Sp(\mathscr{W})}$. It is known that $\widetilde{U(\mathscr{V})}$ and $\widetilde{U(\mathscr{V}')}$ commute with each other in $\widetilde{Sp(\mathscr{W})}$. Therefore $\widetilde{U(\mathscr{V})} \cdot \widetilde{U(\mathscr{V}')}$ is a subgroup of $\widetilde{Sp(\mathscr{W})}$. By restricting the Weil representation of $\widetilde{Sp(\mathscr{W})}$ to the subgroup $\widetilde{U(\mathscr{V})} \cdot \widetilde{U(\mathscr{V}')}$ and pulling back to $\widetilde{U(\mathscr{V})} \times \widetilde{U(\mathscr{V}')}$ by the homomorphisms $\widetilde{U(\mathscr{V})} \times \widetilde{U(\mathscr{V}')} \to \widetilde{U(\mathscr{V})} \cdot \widetilde{U(\mathscr{V}')}$, there exists a one-to-one correspondence (called the *local theta correspondence*) between irreducible admissible representations of

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$\widetilde{U(\mathscr{V})}$ and irreducible admissible representations of $\widetilde{U(\mathscr{V}')}$ (*cf.* [**MVW**], [**Wp**]). To determine the explicit correspondence is extremely difficult. In fact very little is known except for few special cases.

Let $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair. Let L be a *good lattice* in $\mathscr{V}$ i.e., a lattice such that $L^*\varpi \subseteq L \subseteq L^*$ where $L^* := \{v \in \mathscr{V} \mid \boldsymbol{h}(v, l) \in \mathfrak{p}^\kappa \text{ for any } l \in L\}$ and κ is an integer which will be specified in subsection 2.1. Let G_L be the stabilizer of L in G and define

$$G_{L,0^+} := \{g \in G \mid (g-1).L^* \subseteq L, (g-1).L \subseteq L^*\varpi\}.$$

It is easy to see that $G_{L,0^+}$ is a normal subgroup of G_L. A *depth zero minimal K-type* for G is a pair (G_L, ζ) where ζ is an irreducible representation of the finite reductive quotient $G_L/G_{L,0^+}$ (the definition here is slightly different from the original definition in [**MP1**]). We say that an irreducible admissible representation (π, V) of G *contains* a minimal K-type (G_L, ζ) if $V^{G_{L,0^+}}$, the vectors fixed by $G_{L,0^+}$, is nonzero and $V^{G_{L,0^+}}$, as a representations of $G_L/G_{L,0^+}$, contains ζ. An irreducible admissible representation containing a depth zero minimal K-type is said to be of *depth zero* (*cf.* [**MP1**], [**MP2**]).

Fix an Iwahori subgroup I of G. Let $\tilde{I}$ denote the inverse image of $\iota_{\mathscr{V}'}(I)$ in $\tilde{G}$. By fixing a splitting $\beta_I : I \to \tilde{I}$, the concept of minimal K-types can be extended to the metaplectic covers $\tilde{G}$ (this was pointed out to me by Jiu-Kang Yu). In particular, an irreducible admissible representation (π, V) of $\tilde{G}$ is said to be *of depth zero* if $V^{\beta_I(G_{L,0^+})}$ is not trivial for some good lattice L such that $G_{L,0^+} \subseteq I$. The following is our first main result.

Theorem A. *Suppose that (π, V) and (π', V') are irreducible admissible representations of $\widetilde{U(\mathscr{V})}$ and $\widetilde{U(\mathscr{V}')}$ respectively and they are paired by the local theta correspondence. Then the depth of π is zero if and only if the depth of π' is zero.*

The concept of reductive dual pair can also be defined over a finite field. In particular, the theta correspondence for a finite reductive dual pair is also defined although this correspondence is not one-to-one in general. It is not difficult to see that if $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ is a reductive dual pair over F and L, L' are good lattices in $\mathscr{V}, \mathscr{V}'$ respectively, then $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$ is a reductive dual pair over the residue field of F. The following is our second main result.

Theorem B. *Let $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair. Let (π, V) (resp. (π', V')) be an irreducible admissible representation of $\widetilde{U(\mathscr{V})}$ (resp. $\widetilde{U(\mathscr{V}')}$) such that two representations are paired by the local theta correspondence. Suppose that π has a minimal K-type (G_L, ζ) for some good lattice L in $\mathscr{V}$ such that $G_{L,0^+} \subseteq I$. Then π' contains a minimal K-type $(G'_{L'}, \zeta')$ such that ζ and ζ' are paired by the theta correspondence for the dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$.*

The relation between the local theta correspondence of depth zero representations and the theta correspondence for finite reductive dual pairs has many nice applications. For example, it implies that an irreducible Iwahori-spherical representation (i.e., an irreducible admissible representation admitting nontrivial vectors fixed by an Iwahori subgroup) corresponds to an irreducible Iwahori-spherical representation. This result has been proved by A.-M. Aubert in [**Ab**] for those special cases called *unramified reductive dual pairs*. Another implications is that an irreducible unipotent representation (*cf.* [**Lt**]) corresponds to an irreducible unipotent representation under some restriction of residue characteristic of F.

Let $(U(\mathscr{V}), U(\mathscr{V}'))$ be a *split* reductive dual pair i.e., a reductive dual pair such that the two splittings $U(\mathscr{V}) \to \widetilde{U(\mathscr{V})}$ and $U(\mathscr{V}') \to \widetilde{U(\mathscr{V}')}$ exist. An explicit formula of the splittings with respect to a Schrödinger model of the Weil representation is given in [**Kl**]. If D is also commutative, the splittings $\tilde{\beta}^L : U(\mathscr{V}) \to \widetilde{U(\mathscr{V})}$ depending on a good lattice L in $\mathscr{V}$ is given in [**Pn1**]. The advantage of the splitting $\tilde{\beta}^L$ is that the depth of π is zero if and only if the depth of $\pi \circ \tilde{\beta}^L$ is zero where π is any irreducible admissible representation of $\widetilde{U(\mathscr{V})}$. Under the splitting $\tilde{\beta}^L$ we can describe the correspondence of depth zero supercuspidal representations for p-adic reductive dual pairs completely in terms of the correspondence of cuspidal representations for finite reductive dual pairs as follows. It is known that if ζ is an irreducible cuspidal representation of the finite classical group $G_L/G_{L,0^+}$, then the compactly induced representation $c\text{-Ind}_{G_L}^{G}\,\zeta$ is an irreducible supercuspidal representation of $G := U(\mathscr{V})$ (*cf.* [**MP2**]).

Theorem C. *Suppose that $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ is a split reductive dual pair such that D is commutative.*

(i) *Let (π, V) (resp. (π', V')) be an irreducible admissible representation of the group $\widetilde{U(\mathscr{V})}$ (resp. $\widetilde{U(\mathscr{V}')}$). Assume that π has a minimal K-type (G_L, ζ) for some good lattice L in $\mathscr{V}$. Suppose that $\pi \circ \tilde{\beta}$ is a supercuspidal representation of $U(\mathscr{V})$ for some splitting $\tilde{\beta} : U(\mathscr{V}) \to \widetilde{U(\mathscr{V})}$ and $\pi \otimes \pi'$ is a first occurrence in the theta correspondence. Then π' has a minimal K-type $(G'_{L'}, \zeta')$ such that $\zeta \otimes \zeta'$ is a first occurrence in the correspondence for the reductive dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$ and both ζ, ζ' are cuspidal representations.*

(ii) *Suppose that $\zeta \otimes \zeta'$ is a first occurrence of theta correspondence for the reductive dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$ where both ζ, ζ' are cuspidal. Then $\pi \otimes \pi'$ is a first occurrence for the reductive dual pair (G, G') where π is the irreducible admissible representation of $\tilde{G}$ such that $\pi \circ \tilde{\beta}^L \simeq c\text{-Ind}_{G_L}^{G}(\xi_L \otimes \zeta)$, π' is the irreducible admissible representation of $\tilde{G}'$ such that $\pi' \circ \tilde{\beta}^{L'} \simeq c\text{-Ind}_{G'_{L'}}^{G'}(\xi_{L'} \otimes \zeta')$, and ξ_L (resp. $\xi_{L'}$) is a character of G_L (resp. $G'_{L'}$) defined in subsection 8.2.*

A nice application of Theorem C is that we can extend some of S. Kudla and S. Rallis' result on theta dichotomy to other p-adic and finite reductive dual pairs. First we recall their result on theta dichotomy for p-adic unitary groups as follows. Let $\mathscr{V}$ be an ε-Hermitian space over a quadratic extension E of F. Let $\mathscr{V}'^+$ and $\mathscr{V}'^-$ be two ε'-Hermitian spaces over E defined in [**HKS**]. Let (π, V) be an irreducible supercuspidal representation of $U(\mathscr{V})$ and ℓ^+ (resp. ℓ^-) be the smallest dimension of $\mathscr{V}'^+$ (resp. $\mathscr{V}'^-$) such that π occurs in the theta correspondence for the pair $(U(\mathscr{V}), U(\mathscr{V}'^+))$ (resp. $(U(\mathscr{V}), U(\mathscr{V}'^-))$) with respect to the splittings given in [**Kl**]. Suppose that the dimensions of $\mathscr{V}$ and $\mathscr{V}'^\pm$ are all in the same parity. Then $\ell^+ + \ell^- = 2n + 2$ where n is the dimension of $\mathscr{V}$. It is interesting that the sum $\ell^+ + \ell^-$ does not depend on π although each of ℓ^+, ℓ^- does.

Let Z be an ε'-Hermitian space over a p-adic field or a finite field. Define $n_0(Z)$ as follows.

$$n_0(Z) := \begin{cases} 0, & \text{if } Z \text{ is symplectic;} \\ 1, & \text{if } Z \text{ is finite Hermitian;} \\ 2, & \text{if } Z \text{ is finite orthogonal or } p\text{-adic Hermitian;} \\ 4, & \text{if } Z \text{ is } p\text{-adic orthogonal.} \end{cases}$$

Define the sgn character of a classical group G as follows. If G is the trivial group or a symplectic group, let sgn be the trivial character of G. If G is a (nontrivial) orthogonal group (resp. unitary group), let sgn be the character of order two whose restriction to the special orthogonal group (resp. special unitary group) is trivial. The following two theorems are our results on theta dichotomy.

Theorem D. *Suppose that $(U(\boldsymbol{v}), U(\boldsymbol{v}'))$ and $(U(\boldsymbol{v}), U(\boldsymbol{v}''))$ are two related finite reductive dual pairs defined in subsection* 12.1. *Let ζ be an irreducible cuspidal representation of $U(\boldsymbol{v})$. Let ℓ_0' (resp. ℓ_0'') denote the smallest dimension of $\boldsymbol{v}'$ (resp. $\boldsymbol{v}''$) such that ζ (resp. $\zeta \otimes \mathrm{sgn}$) occurs in the theta correspondence for the dual pair $(U(\boldsymbol{v}), U(\boldsymbol{v}'))$ (resp. $(U(\boldsymbol{v}), U(\boldsymbol{v}''))$). Then*

$$\ell_0' + \ell_0'' = 2n + n_0(\boldsymbol{v}') = 2n + n_0(\boldsymbol{v}'')$$

where n is the dimension of $\boldsymbol{v}$.

Theorem E. *Let $(U(\mathscr{V}), U(\mathscr{V}'))$, $(U(\mathscr{V}), U(\mathscr{V}''))$ be two related p-adic reductive dual pair given in subsection* 13.1 *and* 13.2. *Suppose that π' is an irreducible depth zero supercuspidal representations of $\widetilde{U(\mathscr{V})}$ with a minimal K-type $(\widetilde{G_L}, \zeta)$. Let π'' be the irreducible depth zero supercuspidal representations of $\widetilde{U(\mathscr{V})}$ such that $\pi' \circ \tilde{\beta}^L = \mathrm{sgn} \otimes (\pi'' \circ \tilde{\beta}^L)$. Then*

$$\ell_0' + \ell_0'' = 2n + n_0(\mathscr{V}') = 2n + n_0(\mathscr{V}'')$$

where n is the dimension of $\mathscr{V}$ and ℓ_0' (resp. ℓ_0'') is the smallest dimension of $\mathscr{V}'$ (resp. $\mathscr{V}''$) such that π' (resp. π'') occurs in the theta correspondence for the reductive dual pair $(U(\mathscr{V}), U(\mathscr{V}'))$ (resp. $(U(\mathscr{V}), U(\mathscr{V}''))$).

Theorems D and E should be true for more general class of irreducible admissible representations but the author does not know how to tackle this generalization. An interesting consequence of Theorems D and E is that there exist a chain of irreducible (super)cuspidal representations such that any two successive representations are paired by the theta correspondence. A special case of the chains is the chain of unipotent cuspidal representations of finite classical groups. This special case is already studied by J. Adams and A. Moy in [**AM**]. Theorem D provides another proof of their result.

The content of this paper is as follows. In section 1, we introduce basic notation used in this paper and the concept of good lattices in an ε-Hermitian space. In this section, we also study the Schrödinger model and the generalized lattice model of the Weil representation of a p-adic symplectic group. In section 2, we introduce basic definitions of reductive dual pairs and theta correspondence. Depth zero minimal K-types for p-adic classical groups and their metaplectic covers are defined in section 3. In section 4, we prove our first main result, Theorem 4.2. From this theorem we conclude that depth zero representations are preserved by the local theta correspondence. We note here that the proof relies heavily on a deep result of J.-L. Waldspurger in [**Wp**]. In section 5, we prove that the depth zero minimal K-types of paired representations are paired by the theta correspondence for finite reductive dual pairs. Theorem 5.6 is our second main result. In sections 6, 7 we provide several consequences of Theorem 5.6. Iwahori-spherical representations and unipotent representations are studied in these two sections respectively. In section 8, we recall the splitting with respect to a generalized lattice model from [**Pn1**]. This splitting gives us the full advantage of studying theta correspondence for split reductive dual pairs by using minimal K-types. In section 9, we study the theta correspondence of irreducible depth zero supercuspidal representations for split reductive dual pairs. We also provide a few examples to illustrate the nice connection of theta correspondence of depth zero supercuspidal representations of p-adic groups and theta correspondence of cuspidal representation of finite groups. Section 10 is a remark on Shalika's and Tanaka's work [**Tn**] on constructing representation of modulo congruence group $SL_2(\boldsymbol{Z}/p^k\boldsymbol{Z})$ from the point of view of theta correspondence. In section 11, we recall Kudla and Rallis' work on theta dichotomy for p-adic unitary groups. Theorem 11.4 is the third main result in this paper, which reformulated Kudla and Rallis' result from our point of view. Sections 12 and 13 are consequence of Theorem 11.4. In section 12, we study the theta dichotomy for finite reductive dual pairs. In section 13, the theta dichotomy for p-adic reductive dual pairs is discussed.

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1. Preliminaries.

In this section, we provide the general setting of this work. Subsection 1.1 concerns the notation used throughout the paper. Materials in subsections 1.2 and 1.5 are from [**Wp**].

1.1. Notation.

Let F be a nonarchimedean local field, $\mathcal{O}_F$ be the ring of integers of F, $\mathfrak{p}_F$ be the prime ideal, ϖ_F be a uniformizer of $\mathcal{O}_F$, $\boldsymbol{f}_F := \mathcal{O}_F/\mathfrak{p}_F$ be the (finite) residue field, τ_F the identity automorphism of F. We assume that the characteristic of $\boldsymbol{f}_F$ is odd. Let q denote the cardinality of the finite field $\boldsymbol{f}_F$ and $\mathrm{ord} : F^\times \to \boldsymbol{Z}$ be the discrete valuation such that $\mathrm{ord}(\varpi_F) = 1$. We will fix a nontrivial (additive) character ψ of F.

Let E be a quadratic extension of F, $\mathcal{O}_E$ the ring of integers of E, ϖ_E a uniformizer of $\mathcal{O}_E$, $\boldsymbol{f}_E$ the residue field of E, τ_E the nontrivial automorphism of E over F. We make the choice such that $\varpi_E = \varpi_F$ if E is unramified, and $\tau_E(\varpi_E) = -\varpi_E$ if E is ramified. Let D' be a central quaternion algebra over F, $\varpi_{D'}$ be a uniformizer, $\tau_{D'}$ the canonical involution of D'. We assume that $\tau_{D'}(\varpi_{D'}) = -\varpi_{D'}$. We fix (D, ϖ, τ) to be one of the triples (F, ϖ_F, τ_F), (E, ϖ_E, τ_E) or $(D', \varpi_{D'}, \tau_{D'})$. Let $\mathcal{O}$ be the ring of integers, $\mathfrak{p}$ be the maximal ideal, $\boldsymbol{f}_D$ be the residues field of D. Let $\Pi_{\mathcal{O}} : \mathcal{O} \to \boldsymbol{f}_D$ be the usual quotient map.

We shall assume that D is commutative from section 8.

1.2. Good lattices in an ε-Hermitian space.

Let $(\mathscr{V}, \langle\,,\rangle)$ be a (finite-dimensional) nondegenerate right ε-Hermitian space over D where ε is 1 or -1. Let $U(\mathscr{V})$ be the group of isometries of $(\mathscr{V}, \langle\,,\rangle)$. Let L be a lattice in $\mathscr{V}$ i.e., a (free) right $\mathcal{O}$-module whose rank is equal to the dimension of $\mathscr{V}$. Fix an integer κ (more specific information about κ will be given in (2.1.b)). Define

$$L^* = \{v \in \mathscr{V} \mid \langle v, l \rangle \in \mathfrak{p}^\kappa \text{ for all } l \in L\}. \tag{1.2.a}$$

It is clear that L^* is also a lattice in $\mathscr{V}$. The lattice L^* is called the *dual lattice* (with respect to the integer κ) of L. The lattice L is said to be *self-dual* if $L^* = L$. The lattice L is called a *good lattice* if $L^*\varpi \subseteq L \subseteq L^*$. Let L be a lattice in $\mathscr{V}$. A decomposition $\mathscr{V} = \bigoplus_{i=1}^n \mathscr{V}_i$ of subspaces is called *L-admissible* if $L = \sum_{i=1}^n (L \cap \mathscr{V}_i)$. Define a *valuation* $\mathrm{ord}_L : \mathscr{V} \to \boldsymbol{Z}$ given by

$$\mathrm{ord}_L(v) := m \quad \text{if } v \in L\varpi^m - L\varpi^{m+1}. \tag{1.2.b}$$

Let L be a good lattice in $\mathscr{V}$. Then $\boldsymbol{l}^* := L^*/L$ and $\boldsymbol{l} := L/L^*\varpi$ are vector spaces over $\boldsymbol{f}_D$. Let $\Pi_{L^*} : L^* \to \boldsymbol{l}^*$, $\Pi_L : L \to \boldsymbol{l}$ be the quotient maps. We can define sesquilinear forms $\langle\,,\rangle_{\boldsymbol{l}^*}$, $\langle\,,\rangle_{\boldsymbol{l}}$ on $\boldsymbol{l}^*$ and $\boldsymbol{l}$ respectively by

$$\begin{aligned} \langle \Pi_{L^*}(w), \Pi_{L^*}(w') \rangle_{\boldsymbol{l}^*} &:= \Pi_{\mathcal{O}}(\langle w, w' \rangle \varpi^{1-\kappa}), \\ \langle \Pi_L(v), \Pi_L(v') \rangle_{\boldsymbol{l}} &:= \Pi_{\mathcal{O}}(\langle v, v' \rangle \varpi^{-\kappa}) \end{aligned} \tag{1.2.c}$$

where $w, w' \in L^*$, $v, v' \in L$. Note that the forms $\langle\,,\rangle_{\boldsymbol{l}^*}$ and $\langle\,,\rangle_{\boldsymbol{l}}$ are non-degenerate and depend on the choice of a prime element ϖ. The following table is from [**Wp**] lemme I.2.

	$\langle\,,\rangle_{\boldsymbol{l}^*}$	$\langle\,,\rangle_{\boldsymbol{l}}$
$D = F$	ε-symmetric over $\boldsymbol{f}_F$	ε-symmetric over $\boldsymbol{f}_F$
$D = E$, unramified	ε-Hermitian over $\boldsymbol{f}_D$	ε-Hermitian over $\boldsymbol{f}_D$
$D = E$, ramified	$(-1)^{\kappa+1}\varepsilon$-symmetric over $\boldsymbol{f}_F$	$(-1)^{\kappa}\varepsilon$-symmetric over $\boldsymbol{f}_F$
$D = D'$, κ even	$(-\varepsilon)$-symmetric over $\boldsymbol{f}_D$	ε-Hermitian over $\boldsymbol{f}_D$
$D = D'$, κ odd	ε-Hermitian over $\boldsymbol{f}_D$	$(-\varepsilon)$-symmetric over $\boldsymbol{f}_D$

A good lattice L in $\mathscr{V}$ is called *maximal* (resp. *minimal*) if it is a maximal element (resp. a minimal element) in the set of all good lattices in $\mathscr{V}$ with the partial order defined by inclusion. It is easy to see that L is maximal (resp. minimal) if and only if the space $(\boldsymbol{l}^*, \langle\,,\rangle_{\boldsymbol{l}^*})$ (resp. $(\boldsymbol{l}, \langle\,,\rangle_{\boldsymbol{l}})$) is anisotropic.

Two lattices L_1, L_2 in $\mathscr{V}$ are said to be *equivalent* if there is an element $g \in U(\mathscr{V})$ such that $g.L_1 = L_2$. The space $\mathscr{V}$ can be decomposed as an orthogonal direct sum $\mathscr{V}^\circ \oplus \mathscr{V}^1$ where $\mathscr{V}^\circ$ is anisotropic and $\mathscr{V}^1$ is a direct sum of hyperbolic planes. There is a unique good lattice A° in $\mathscr{V}^\circ$. Choose a complete polarization $\mathscr{V}^1 = X + Y$, a basis $x_1, \ldots, x_r$ of X, and the dual basis $y_1, \ldots, y_r$ of Y where r is the Witt index of $\mathscr{V}$. Then

$$B_i := x_1\mathcal{O}\varpi + \cdots + x_i\mathcal{O}\varpi + x_{i+1}\mathcal{O} + \cdots + x_r\mathcal{O} + y_r\mathcal{O} + \cdots + y_1\mathcal{O} \quad (1.2.d)$$

is a good lattice in $\mathscr{V}^1$ for each $0 \le i \le r$. Therefore $N_i := A^\circ + B_i$ is a good lattice in $\mathscr{V}$. It is not difficult to check that every good lattice in $\mathscr{V}$ is equivalent to one of N_i for $i = 0, \ldots, r$.

1.3. The "sgn" character.

Let G be a (finite or p-adic) symplectic, orthogonal or unitary group. We will use the notation "sgn" to denote the character of G defined as follows. If G is the trivial group or a symplectic group, let sgn be the trivial character of G. If G is a (nontrivial) orthogonal group (resp. unitary group), let sgn be the character of order two whose restriction to the special orthogonal group (resp. special unitary group) is trivial.

1.4. Weil representations and the metaplectic covers.

Let $(\mathscr{W}, \langle\langle\, , \rangle\rangle)$ be a symplectic space over F. Let $\mathrm{H}(\mathscr{W})$ be the *Heisenberg group* associated to $(\mathscr{W}, \langle\langle\, , \rangle\rangle)$. Let $(\rho_\psi, \mathscr{S})$ be the irreducible representation of $\mathrm{H}(\mathscr{W})$ with nontrivial central character ψ by the Stone-Von Neumann theorem. The symplectic group $Sp(\mathscr{W})$ acts on $\mathrm{H}(\mathscr{W})$. Define the *metaplectic cover* $\widetilde{Sp(\mathscr{W})}$ of $Sp(\mathscr{W})$ to be the topological subgroup of $Sp(\mathscr{W}) \times \mathrm{Aut}(\mathscr{S})$ consisting of the pairs $(g, M[g])$ where $M[g]$ satisfies the condition

$$M[g] \circ \rho_\psi(h) = \rho_\psi(g.h) \circ M[g] \quad (1.4.a)$$

for $g \in Sp(\mathscr{W})$, $M[g] \in \mathrm{Aut}(\mathscr{S})$ and any $h \in \mathrm{H}(\mathscr{W})$. The metaplectic group $\widetilde{Sp(\mathscr{W})}$ comes equipped with a representation ω_ψ on $\mathscr{S}$ given by

$$\omega_\psi(g, M[g]) := M[g]. \quad (1.4.b)$$

The representation $(\omega_\psi(g), \mathscr{S})$ of $\widetilde{Sp(\mathscr{W})}$ or the projective representation $(M[g], \mathscr{S})$ of $Sp(\mathscr{W})$ is called the *Weil representation* or *the oscillator representation*.

1.5. Generalized lattice model of the Weil representation.

Let $\mathfrak{p}_F^{\lambda_F}$ be the conductor of a character ψ of F i.e., λ_F is the smallest integer such that $\psi|_{\mathfrak{p}_F^{\lambda_F}}$ is trivial. Let B be a good lattice in $\mathscr{W}$ (with respect to λ_F). Let $\boldsymbol{b}^*$ be the quotient B^*/B. We know that $\boldsymbol{b}^*$ is a vector space over $\boldsymbol{f}_F$ with a nondegenerate skew-symmetric form $\langle\langle\, , \rangle\rangle_{\boldsymbol{b}^*}$ on $\boldsymbol{b}^*$ by (1.2.c). Let $\mathrm{H}(\boldsymbol{b}^*)$ be the Heisenberg group associated to the finite symplectic space $(\boldsymbol{b}^*, \langle\langle\, , \rangle\rangle_{\boldsymbol{b}^*})$. Let $\bar{\psi}$ denote the character of $\boldsymbol{f}_F$ defined by $\bar{\psi}(\Pi_{\mathcal{O}_F}(t)) := \psi(t\varpi_F^{\lambda_F - 1})$ where $t \in \mathcal{O}_F$. Let $(\overline{\omega}_{\bar{\psi}}, S)$ be the Schrödinger model of the Weil representation of the finite symplectic groups $Sp(\boldsymbol{b}^*)$ associated to the data $(\boldsymbol{b}^*, \langle\langle\, , \rangle\rangle_{\boldsymbol{b}^*}, \bar{\psi})$ (*cf.* [**Hw1**]). Although the skew-symmetric form $\langle\langle\, , \rangle\rangle_{\boldsymbol{b}^*}$ and character $\bar{\psi}$ depend on the choice of a prime element ϖ_F, the Weil representation $\overline{\omega}_{\bar{\psi}}$ does not. This can be seen easily from the Schrödinger model. Let $\bar{\rho}_{\bar{\psi}}$ denote the representation of $\mathrm{H}(\boldsymbol{b}^*)$ corresponding to the character $\bar{\psi}$ on the space S. Let $\mathrm{H}(B^*) := B^* \times \mathfrak{p}_F^{\lambda_F - 1}$. It is easy to check that $\mathrm{H}(B^*)$ is a subgroup of the Heisenberg group $\mathrm{H}(\mathscr{W})$. We have a reduction homomorphism

$$\Pi_{\mathrm{H}(B^*)} : \mathrm{H}(B^*) \to \mathrm{H}(\boldsymbol{b}^*) \quad \text{by} \quad (b, t) \mapsto (\Pi_{B^*}(b), \Pi_{\mathcal{O}_F}(t\varpi_F^{1-\lambda_F})),$$

where $\Pi_{B^*} : B^* \to \boldsymbol{b}^*$ and $\Pi_{\mathcal{O}_F} : \mathcal{O}_F \to \boldsymbol{f}_F$. Let K_B denote the stabilizer of B in $Sp(\mathscr{W})$, and

$$K_B' := \{g \in K_B \mid (g-1).B^* \subseteq B\}. \tag{1.5.a}$$

It is clear that K_B' is a normal subgroup of K_B and K_B/K_B' is isomorphic to $Sp(\boldsymbol{b}^*)$. Let $\tilde{\rho}_\psi$ denote the representation of $\mathrm{H}(B^*)$ inflated from $\bar{\rho}_{\bar{\psi}}$ by the projection $\Pi_{\mathrm{H}(B^*)}$, and $\tilde{\omega}_\psi$ be the representation of K_B inflated from $\overline{\omega}_{\bar{\psi}}$ by the projection $K_B \to K_B/K_B'$ and the isomorphism $K_B/K_B' \simeq Sp(\boldsymbol{b}^*)$. Let $\mathscr{S}(B)$ denote the space of locally constant, compactly supported maps $f : \mathscr{W} \to S$ such that

$$f(b + w) = \psi\left(\frac{1}{2}\langle\langle w, b\rangle\rangle\right)\tilde{\rho}_\psi(b).(f(w)), \tag{1.5.b}$$

for any $w \in \mathscr{W}, b \in B^*$. Define the action ρ_ψ^B of $\mathrm{H}(\mathscr{W})$ on $\mathscr{S}(B)$ by

$$(\rho_\psi^B(w, t).f)(w') = \psi(t)\psi\left(\frac{1}{2}\langle\langle w', w\rangle\rangle\right)f(w' + w)$$

for $w' \in \mathscr{W}$ and $(w, t) \in \mathrm{H}(\mathscr{W})$. We can define $M_B[g]$ in $\mathrm{Aut}(\mathscr{S}(B))$ such that $\rho_\psi^B(h)$ and $M_B[g]$ satisfy (1.4.a). Moreover we know that

$$(M_B[k].f)(w) = \tilde{\omega}_\psi(k).(f(k^{-1}.w)) \tag{1.5.c}$$

for $k \in K_B$ and $f \in \mathscr{S}(B)$. This realization of the Weil representation is known as a *generalized lattice model*. For any union of B^*-cosets R, define

$$\begin{aligned} \mathscr{S}(B)_R &:= \{f \in \mathscr{S}(B) \mid f \text{ has support in } R\}, \\ \mathscr{S}(B)_w &:= \mathscr{S}(B)_{B^*+w} \quad \text{for } w \in \mathscr{W}. \end{aligned} \tag{1.5.d}$$

If $B = A$ happens to be self-dual, the $(M_A[g], \mathscr{S}(A))$ is the well known *lattice model* of the Weil representation. In this case, it is clear that $\tilde{\omega}_\psi$ is trivial and $\mathscr{S}(A)_w$ is one-dimensional.

2. Reductive dual pairs and local theta correspondence.

In this section, we recall some basic definitions for reductive dual pairs and local theta correspondence. Of course, the material in this section is well known. Basic references are [**Hw2**], [**MVW**], or [**Rb**].

2.1. Reductive dual pairs.

Let D be as defined in subsection 1.1. Let $(\mathscr{V}, \langle\,,\rangle)$ (resp. $(\mathscr{V}', \langle\,,\rangle')$) be an ε-Hermitian (resp. ε'-Hermitian) space over D such that $\varepsilon\varepsilon' = -1$. Define $\mathscr{W} := \mathscr{V} \otimes_D \mathscr{V}'$, which will be denoted by $\mathscr{V} \otimes \mathscr{V}'$ latter for simplicity. Define a skew-symmetric F-bilinear from $\langle\!\langle\,,\rangle\!\rangle$ on $\mathscr{W}$ by

$$\langle\!\langle\,,\rangle\!\rangle := \mathrm{Trd}_{D/F}(\langle\,,\rangle \otimes \tau \circ \langle\,,\rangle') \tag{2.1.a}$$

where $\mathrm{Trd}_{D/F}$ denotes the reduced trace from D to F. Recall that ψ is a character of F with conductoral exponent λ_F. Define $\lambda := \lambda_F$ if D is F or an unramified quadratic extension of F, $\lambda := 2\lambda_F - 1$ otherwise. Let κ (resp. κ') be the integer used to define the dual lattices in $\mathscr{V}$ (resp. $\mathscr{V}'$) as in (1.2.a). We make the following assumption

$$\kappa + \kappa' = \lambda \tag{2.1.b}$$

throughout the paper. We also assume that the duality of lattices in $\mathscr{W}$ is defined with respect to the integer λ_F.

The pair $(U(\mathscr{V}), U(\mathscr{V}'))$ is called a (*type I*) *reductive dual pair* in $Sp(\mathscr{W})$. The reductive dual pair $(U(\mathscr{V}), U(\mathscr{V}'))$ is called *unramified* if it satisfies the following two conditions: (1) D is F itself or a unramified quadratic extension of F, (2) there exist self-dual lattices in both spaces $(\mathscr{V}, \langle\,,\rangle)$ and $(\mathscr{V}', \langle\,,\rangle')$.

2.2. Local theta correspondence.

From the definition of the form $\langle\!\langle\,,\rangle\!\rangle$, we know that there is an embedding $\iota_{\mathscr{V}'} : U(\mathscr{V}) \to Sp(\mathscr{W})$. Let $\widetilde{U(\mathscr{V})}$ be the inverse image of $\iota_{\mathscr{V}'}(U(\mathscr{V}))$ in $\widetilde{Sp(\mathscr{W})}$. The group $\widetilde{U(\mathscr{V})}$ is called the *metaplectic cover* of $U(\mathscr{V})$. Let $\widetilde{U(\mathscr{V}')}$ be defined similarly. One can check that $\widetilde{U(\mathscr{V})}$ and $\widetilde{U(\mathscr{V}')}$ commute with each other. Let $\widehat{U(\mathscr{V})}$ be the two-fold cover of $U(\mathscr{V})$ in $\widetilde{U(\mathscr{V})}$. We know that $\widehat{U(\mathscr{V})}$ is a totally disconnected group. A representations (π, V) of $\widetilde{U(\mathscr{V})}$ is called *admissible* if $\pi|_{\mathbf{C}^\times}(z)$ is multiplication by z and $\pi|_{\widehat{U(\mathscr{V})}}$ is an admissible representation of a totally disconnected group.

Let $(\omega_\psi, \mathscr{S})$ be the Weil representation of $\widetilde{Sp(\mathscr{W})}$ with respect to the character ψ of F. It is known that $(\omega_\psi, \mathscr{S})$ is an admissible representation of $\widetilde{Sp(\mathscr{W})}$. Then $(\omega_\psi, \mathscr{S})$ can be regarded as a representation of $\widetilde{U(\mathscr{V})} \times \widetilde{U(\mathscr{V}')}$ via the restriction to $\widetilde{U(\mathscr{V})} \cdot \widetilde{U(\mathscr{V}')}$ and the homomorphism $\widetilde{U(\mathscr{V})} \times \widetilde{U(\mathscr{V}')} \to \widetilde{U(\mathscr{V})} \cdot \widetilde{U(\mathscr{V}')}$. Let (π, V) (resp. (π', V')) be an irreducible admissible representation of the metaplectic group $\widetilde{U(\mathscr{V})}$ (resp. $\widetilde{U(\mathscr{V}')}$). The representation (π, V) is said to *correspond to* the representation (π', V') if there is a nontrivial $\widetilde{U(\mathscr{V})} \times \widetilde{U(\mathscr{V}')}$-map

$$\Pi : \mathscr{S} \to V \otimes_{\mathbf{C}} V'. \tag{2.2.a}$$

This establishes a correspondence, called the *local theta correspondence* or *Howe duality*, between some irreducible admissible representations of $\widetilde{U(\mathscr{V})}$ and some irreducible admissible representations of $\widetilde{U(\mathscr{V}')}$. It is proved by R. Howe (*cf.* [**MVW**], chapitre 5) and J.-L. Waldspurger (*cf.* [**Wp**]) that the local theta correspondence is one-to-one when the residue characteristic of F is odd.

2.3. Finite reductive dual pairs.

Reductive dual pairs can also be defined over a finite field. Let $\boldsymbol{f}$ be a finite field and $\boldsymbol{d}$ be $\boldsymbol{f}$ or a quadratic extension of $\boldsymbol{f}$. Let $\boldsymbol{v}$ (resp. $\boldsymbol{v}'$) be a ε-Hermitian (resp. ε'-Hermitian) spaces over $\boldsymbol{d}$ such that $\varepsilon\varepsilon' = -1$. Then $(U(\boldsymbol{v}), U(\boldsymbol{v}'))$ forms a (finite) *reductive dual pair* in the finite symplectic group $Sp(\boldsymbol{v} \otimes_{\boldsymbol{d}} \boldsymbol{v}')$. It is known that the theta correspondence for a finite reductive dual pair is in general not one-to-one. For convenience, we shall allow the dimension of $\boldsymbol{v}$ (or $\boldsymbol{v}'$) to be zero. Therefore we make the following conventions. We define the Weil representation of the symplectic group on a zero-dimensional space to be the trivial representation (of the trivial group). If one of the spaces $\boldsymbol{v}, \boldsymbol{v}'$ is trivial, then the theta correspondence for the dual pair $(U(\boldsymbol{v}), U(\boldsymbol{v}'))$ is just the trivial representation corresponds to the trivial representation.

3. Depth zero minimal K-types.

In this section, we recall the definition of depth zero minimal K-types for p-adic classical groups modified from [**MP1**] and [**MP2**].

3.1. Result of Moy and Prasad.

Let x be a point in the Bruhat-Tits building of a p-adic classical group $G := U(\mathscr{V})$. Let G_x denote the stabilizer of x, and $G_{x,0^+}$ be the pro-nilradical of G_x. It is known that $G_{x,0^+}$ is a normal subgroup of G_x. The quotient $G_{x,0}/G_{x,0^+}$ is a finite classical group. A *depth zero minimal K-type* of an irreducible admissible representation (π, V) of G is a pair (G_x, ζ) where ζ is an irreducible representation of the group $G_x/G_{x,0^+}$ such that $V^{G_{x,0^+}}$ is nontrivial and, as a representation of $G_x/G_{x,0^+}$, contains ζ. An irreducible admissible representations (π, V) containing a depth zero minimal K-type is said to be *of depth zero*. We should notice that the definition of a minimal K-type here is different from the original definition in [**MP1**] and [**MP2**]. Here we do not require ζ to be cuspidal. However, the notion of depth of an irreducible admissible representation π is still the same.

It is known that if x is a point in the building of G, then there exists a vertex v of a chamber such that $G_{v,0^+} \subseteq G_{x,0^+}$. Therefore an irreducible admissible representation of G is of depth zero if and only if it has nontrivial vectors fixed by $G_{v,0^+}$ for some vertex v.

3.2. Good lattices and vertices of the building.

Let G be the classical group $U(\mathscr{V})$ and L be a good lattice (*cf.* subsection 1.2) in $\mathscr{V}$. Define

$$G_L := \{g \in U(\mathscr{V}) \mid g.L = L\},$$
$$G_{L,0^+} := \{g \in U(\mathscr{V}) \mid (g-1).L^* \subseteq L, (g-1).L \subseteq L^*\varpi\}, \tag{3.2.a}$$
$$G_{L,1} := \{g \in U(\mathscr{V}) \mid (g-1).L \subseteq L\varpi\}.$$

It is clear that G_L, $G_{L,0^+}$ and $G_{L,1}$ are open compact subgroups of $U(\mathscr{V})$. In fact, G_L is a maximal open compact subgroup of $U(\mathscr{V})$. It is proved by H. Hijikata [**Hj**] that any maximal open compact subgroup of a classical group $U(\mathscr{V})$ is conjugate to one of G_{N_i} where N_i is a good lattice defined in subsection 1.2.

From [**BT**], it is not difficult to see that for a vertex v in the Bruhat-Tits building of G, there exists a good lattice L in $\mathscr{V}$ such that $G_v = G_L$ and $G_{v,0^+} = G_{L,0^+}$. And it is not difficult to figure that an irreducible admissible representation of G is depth zero if and only if it has nontrivial vectors fixed by $G_{L,0^+}$ for some good lattice L in $\mathscr{V}$.

3.3. Depth zero minimal K-types for metaplectic covers.

To extend the concept of depth zero minimal K-types to the metaplectic cover $\tilde{G}$, we need to fix a splitting $\beta : I \to \tilde{I}$ where I is a (fixed) Iwahori subgroup of G (this was pointed out to me by Jiu-Kang Yu). In this subsection, we want to discuss this issue.

Let $(\rho_\psi, \mathscr{S})$ be the irreducible representation of the Heisenberg group $\mathrm{H}(\mathscr{W})$ with respect to a nontrivial central character ψ. For $g \in Sp(\mathscr{W})$, let $M[g]$ be an element in $\mathrm{Aut}(\mathscr{S})$ satisfying (1.4.a). Define a cocycle $c : Sp(\mathscr{W}) \times Sp(\mathscr{W}) \to \boldsymbol{C}^\times$ with respect to the map $M : Sp(\mathscr{W}) \to \mathrm{Aut}(\mathscr{S})$ by

$$M[g] \circ M[g'] = c(g, g')M[gg']. \tag{3.3.a}$$

Let $(U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair in $Sp(\mathscr{W})$. Recall that the metaplectic cover $\widetilde{U(\mathscr{V})}$ is the group of elements of the form $(\iota_{\mathscr{V}'}(g), tM[\iota_{\mathscr{V}'}(g)])$ for $t \in \boldsymbol{C}^\times$, and the extension $\widetilde{U(\mathscr{V})} \to U(\mathscr{V})$ is given by $(\iota_{\mathscr{V}'}(g), tM[\iota_{\mathscr{V}'}(g)]) \mapsto g$. Let H be a subgroup of $U(\mathscr{V})$. A function $\beta : H \to \boldsymbol{C}^\times$ is called a *splitting* of the cocycle $c|_{\iota_{\mathscr{V}'}(H) \times \iota_{\mathscr{V}'}(H)}$ if it satisfies

$$c(\iota_{\mathscr{V}'}(g), \iota_{\mathscr{V}'}(g')) = \beta(gg')\beta(g)^{-1}\beta(g')^{-1} \tag{3.3.b}$$

for any $g, g' \in H$. If β is a splitting, the map $\tilde{\beta} : H \to \tilde{H}$ defined by $g \mapsto (\iota_{\mathscr{V}'}(g), \beta(g)M[\iota_{\mathscr{V}'}(g)])$ is a group homomorphism where $\tilde{H}$ is the inverse image of $\iota_{\mathscr{V}'}(H)$ in $\widetilde{U(\mathscr{V})}$. We will also called $\tilde{\beta}$ the *splitting* of the metaplectic cover $\tilde{H}$.

Let L be a good lattice in $\mathscr{V}$ such that $G_{L,0^+} \subseteq I$. Then I is contained in G_L. Let $(M_B[g], \mathscr{S}(B))$ be a generalized lattice model with respect to B defined by

$$B := B(L, L') := L^* \otimes L' \cap L \otimes L'^*.$$

for some good lattice L' in $\mathscr{V}'$. It is clear that B is a lattice in $\mathscr{W} := \mathscr{V} \otimes \mathscr{V}'$. Let $c_B(g, g)$ denote the cocycle defined in (3.3.a) with respect to $M_B[g]$. From (1.5.c), we see that $c_B|_{K_B \times K_B} = 1$. It is clear that $\iota_{\mathscr{V}'}(G_L) \subseteq K_B$. Hence the mapping $\beta_I : I \to \boldsymbol{C}^\times$ by $\beta_I(g) := 1$ for $g \in I$ is a splitting of $c_B|_{\iota_{\mathscr{V}'}(I) \times \iota_{\mathscr{V}'}(I)}$. Therefore the map $\tilde{\beta}_I : I \to \tilde{I}$ defined by $g \mapsto (\iota_{\mathscr{V}'}(g), M_B[\iota_{\mathscr{V}'}(g)])$ is an injective homomorphism. Although the lattice B depends on the good lattice L' in $\mathscr{V}'$, it turns out that the splitting $\tilde{\beta}_I$ only depends on the parity of the dimensions of L'^*/L' and $L'/L'^*\varpi$, which is the same for all good lattices in $\mathscr{V}'$ (*cf.* [**Pn1**]). Therefore $\tilde{\beta}_I$ depends only on $\mathscr{V}'$ but not on L'. We know that $I = G_{x_0}$ where x_0 is the barycenter of some Weyl chamber C_0 in the Bruhat-Tits building of G. It is known that $G_{x,0^+} \subseteq I$ if $x \in C_0$. Therefore we will identify $G_{x,0^+}$ with $\beta_I(G_{x,0^+})$ i.e., we regard $G_{x,0^+}$ as a subgroup of $\tilde{G}$ via the splitting β_I. For a general point y in the building, there exists an element $g \in G$ such that $g.y \in C_0$. Therefore $gG_{y,0^+}g^{-1} = G_{g.y,0^+} \subseteq I$. From now on, when we mention $G_{x,0^+}$ or $G_{L,0^+}$, we always restrict ourselves to the situation that $G_{x,0^+} \subseteq I$ or $G_{L,0^+} \subseteq I$ and regard $G_{x,0^+}$ and $G_{L,0^+}$ as subgroups of $\tilde{G}$.

An irreducible admissible representation (π, V) of $\tilde{G}$ is said to be *of depth zero* if $V^{G_{L,0^+}}$ is not trivial for some good lattice L in $\mathscr{V}$ (such that $G_{L,0^+} \subseteq I$). It is clear that this definition is independent of the choice of an Iwahori subgroup I. This can be seen as follows. Suppose that I_1, I_2 are two Iwahori subgroups of G. It is known that there is an element $g \in G$ such that $I_1 = gI_2g^{-1}$. Therefore $gG_{L,0^+}g^{-1} = G_{g.L,0^+} \subseteq I_2$ if $G_{L,0^+} \subseteq I_1$. Hence $V^{\beta_{I_1}(G_{L,0^+})}$ is nontrivial if and only if $V^{\beta_{I_2}(G_{g.L,0^+})}$ is nontrivial.

Suppose that $V^{G_{L,0^+}}$ is not trivial and $\tilde{G}$ is defined with respect to $M_B[g]$ where $B := B(L, L')$ for some L'. Then $\beta : G_L \to \tilde{G}_L$ by $g \mapsto (\iota_{\mathscr{V}'}(g), M_B[\iota_{\mathscr{V}'}(g)])$ extends β_I. So we can regard G_L as a subgroup of $\tilde{G}$. We say that π contains a *minimal K-type* (G_L, ζ) if $V^{G_{L,0^+}}$ is nontrivial and contains ζ.

4. Preservation of depth zero representations.

In this section we prove our first main result, which indicates that the depth zero representations are preserved by the local theta correspondence.

4.1.

Let $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair as usual. Suppose that L (resp. L') is a good lattice in $\mathscr{V}$ (resp. $\mathscr{V}'$). Define

$$B := B(L, L') := L^* \otimes L' \cap L \otimes L'^*. \tag{4.1.a}$$

It is clear that B is a lattice in $\mathscr{W} := \mathscr{V} \otimes \mathscr{V}'$ and is equal to $L \otimes L' +$

$L^*\varpi \otimes L'^*$. Hence we have $B(L,L')^* = L^* \otimes L' + L \otimes L'^* = L\varpi^{-1} \otimes L' \cap L^* \otimes L'^*$. It is easy to check that $B(L,L')$ is a good lattice in $\mathscr{W}$ i.e.,

$$B(L,L')^*\varpi_F \subseteq B(L,L') \subseteq B(L,L')^*.$$

Recall that K_B is the stabilizer of B in $Sp(\mathscr{W})$ and K'_B is the subgroup of elements g such that $(g-1).B^* \subseteq B$. It is easy to see that $G_{L,0^+}$ is a subgroup of K'_B. So is $G'_{L',0^+}$. Recall that we fix Iwahori subgroups I, I' of G, G' respectively and require that $G_{L,0^+} \subseteq I$, $G'_{L',0^+} \subseteq I'$. We regard $G_{L,0^+}$, $G'_{L',0^+}$ as subgroups of $\widetilde{Sp(\mathscr{W})}$ via identifying $G_{L,0^+}$, $G'_{L',0^+}$ with $\beta_I(G_{L,0^+})$, $\beta_{I'}(G'_{L',0^+})$ respectively.

PROPOSITION. *Let A be a good lattice in $\mathscr{W}$ such that $A^* \subseteq B(L,L')^*$. Then the subspace $\mathscr{S}(A)_{B(L,L')^*}$ of $\mathscr{S}(A)$ is fixed pointwise by $G_{L,0^+}$ and $G'_{L',0^+}$.*

The proof of this proposition is postponed to subsection 4.10.

4.2.

The following is our main result of this section, whose proof will be postponed to subsection 4.12.

THEOREM. *Let $(G,G') := (U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair and ψ be a nontrivial character of F. Suppose that L is a good lattice in $\mathscr{V}$. Then*

$$\mathscr{S}^{G_{L,0^+}} = \omega_\psi(\mathscr{H}').\left(\sum_{L' \in \mathscr{Q}(\Gamma'_{\mathrm{M}})} \mathscr{S}^{B(L,L')}\right) \tag{4.2.a}$$

where $\mathscr{Q}(\Gamma'_{\mathrm{M}})$ denotes the set of good lattices contained in a fixed maximal good lattice Γ'_{M} in $\mathscr{V}'$, $\mathscr{H}'$ is the Hecke algebra of $U(\mathscr{V}')$ and $(\omega_\psi, \mathscr{S})$ is the Weil representation of $\widetilde{Sp(\mathscr{W})}$.

4.3.

COROLLARY. *Let $(G,G') := (U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair. Let (π, V) (resp. (π', V')) be an irreducible admissible representation of $\widetilde{U(\mathscr{V})}$ (resp. $U(\mathscr{V}')$) such that two representations are paired by the local theta correspondence. Suppose that $V^{G_{L,0^+}}$ is nontrivial for some good lattice L in $\mathscr{V}$. Then there exists a good lattice L' in $\mathscr{V}'$ such that $V'^{G'_{L',0^+}}$ is nontrivial.*

PROOF. Suppose that $V^{G_{L,0^+}}$ is nontrivial for some good lattice L in $\mathscr{V}$. Let Π be the projection $\mathscr{S} \to V \otimes_C V'$ where $\mathscr{S}$ is the Weil representation. We have a nontrivial surjective map $\mathscr{S}^{G_{L,0^+}} \to V^{G_{L,0^+}} \otimes_C V'$. Because we assume that the space $V^{G_{L,0^+}}$ is nontrivial, by Theorem 4.2 there is an element $f \in \mathscr{S}^{B(L,L')}$ for some good lattice L' in $\mathscr{Q}(\Gamma'_{\mathrm{M}})$ such that $\Pi(f)$ is not zero. But $\Pi(f)$ is fixed by $G_{L,0^+}$ and $G'_{L',0^+}$ by Proposition 4.1. Therefore $\Pi(f)$ belongs to $V^{G_{L,0^+}} \otimes_C V'^{G'_{L',0^+}}$. Hence $V'^{G'_{L',0^+}}$ is also nontrivial. $\square$

4.4.

Corollary. *Let $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair. Let (π, V) (resp. (π', V')) be an irreducible admissible representation of $\widetilde{U(\mathscr{V})}$ (resp. $\widetilde{U(\mathscr{V}')}$) such that two representations are paired by the local theta correspondence. Then the depth of π is zero if and only if the depth of π' is zero.*

Proof. As in subsection 3.3 we know an irreducible admissible representation (π, V) of $\widetilde{U(\mathscr{V})}$ is of depth zero if and only if $V^{G_{L,0^+}}$ is nontrivial for some good lattice L in $\mathscr{V}$. Hence this corollary follows immediately from Corollary 4.3. $\square$

4.5.

Now we start the preparation for the proofs of Proposition 4.1 and Theorem 4.2. First we need to introduce the Cayley transforms. Recall that $(\mathscr{V}, \langle\,,\rangle)$ is a non-degenerate ε-Hermitian space. Let $\mathfrak{u}(\mathscr{V})$ be the space of elements $c \in \mathrm{End}_D(\mathscr{V})$ such that $\langle c.v, v'\rangle + \langle v, c.v'\rangle = 0$ for all $v, v' \in \mathscr{V}$. If c is an element in $\mathfrak{u}(\mathscr{V})$ and $1 + c$ is invertible, we define $u(c) := (1 - c)(1 + c)^{-1}$. It is easy to check that $u(c)$ is an element in $U(\mathscr{V})$ when it is defined. If $u \in U(\mathscr{V})$ and $1 + u$ is invertible, we define $c(u) := (1 - u)(1 + u)^{-1}$. It is also clear that $c(u)$ is an element in $\mathfrak{u}(\mathscr{V})$. For any two elements x, y in $\mathscr{V}$, we define $c_{x,y} : \mathscr{V} \to \mathscr{V}$ by

$$c_{x,y}.v := x\langle y, v\rangle - \varepsilon y\langle x, v\rangle. \tag{4.5.a}$$

It is easy to check that $c_{x,y}$ belongs to $\mathfrak{u}(\mathscr{V})$ for $x, y \in \mathscr{V}$. If $1 + c_{x,y}$ is invertible, define $u_{x,y} := u(c_{x,y})$ i.e., $u_{x,y} = (1 - c_{x,y})(1 + c_{x,y})^{-1}$. Then it is easy to check that $1 + u_{x,y}$ is invertible and $c(u_{x,y}) = c_{x,y}$. The following lemma is from [**Wp**].

Lemma. *Suppose that L is a good lattice in $\mathscr{V}$ and x, y are elements in $\mathscr{V}$.*

(i) *If $\mathrm{ord}_L(x) + \mathrm{ord}_L(y) \geq -\kappa$ and $\mathrm{ord}_{L^*}(x) + \mathrm{ord}_{L^*}(y) \geq 1 - \kappa$, then $u_{x,y}$ is defined and belongs to $G_{L,0^+}$.*

(ii) *If $\mathrm{ord}_L(x) + \mathrm{ord}_{L^*}(y) \geq 1 - \kappa$ and $\mathrm{ord}_L(y) + \mathrm{ord}_{L^*}(x) \geq 1 - \kappa$, then $u_{x,y}$ is defined and belongs to $G_{L,1}$.*

Proof. Part (i) is lemme I.17 of [**Wp**]. Part (ii) can be proved similarly. $\square$

4.6.

We fix a maximal good lattice Γ'_{M} and a minimal good lattice Γ'_{m} in $\mathscr{V}'$ such that $\Gamma'_{\mathrm{m}} \subseteq \Gamma'_{\mathrm{M}}$.

Lemma. *Let L be a good lattice in $\mathscr{V}$. Then $L^* \otimes \Gamma'_{\mathrm{M}} \cap L \otimes \Gamma'^*_{\mathrm{m}}$ is a good lattice in $\mathscr{W}$.*

PROOF. From I.15 in [**Wp**], we know that there exists a decomposition $\mathscr{V} = X_1 \oplus X_2$ such that $L = (L \cap X_1) \oplus (L \cap X_2)$ and $L^* = (L \cap X_1) \oplus (L \cap X_2)\varpi^{-1}$. There also exists a decomposition $\mathscr{V}' = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4$ such that

$$\begin{aligned}
\Gamma'_{\mathrm{m}} &= (\Gamma'_{\mathrm{M}} \cap Y_1)\varpi \oplus (\Gamma'_{\mathrm{M}} \cap Y_2) \oplus (\Gamma'_{\mathrm{M}} \cap Y_3) \oplus (\Gamma'_{\mathrm{M}} \cap Y_4)\\
\Gamma'_{\mathrm{M}} &= (\Gamma'_{\mathrm{M}} \cap Y_1) \oplus (\Gamma'_{\mathrm{M}} \cap Y_2) \oplus (\Gamma'_{\mathrm{M}} \cap Y_3) \oplus (\Gamma'_{\mathrm{M}} \cap Y_4)\\
\Gamma'^*_{\mathrm{M}} &= (\Gamma'_{\mathrm{M}} \cap Y_1) \oplus (\Gamma'_{\mathrm{M}} \cap Y_2)\varpi^{-1} \oplus (\Gamma'_{\mathrm{M}} \cap Y_3) \oplus (\Gamma'_{\mathrm{M}} \cap Y_4)\\
\Gamma'^*_{\mathrm{m}} &= (\Gamma'_{\mathrm{M}} \cap Y_1) \oplus (\Gamma'_{\mathrm{M}} \cap Y_2)\varpi^{-1} \oplus (\Gamma'_{\mathrm{M}} \cap Y_3)\varpi^{-1} \oplus (\Gamma'_{\mathrm{M}} \cap Y_4).
\end{aligned}$$

From the above decompositions it is easy to check that

$$L \otimes \Gamma'^*_{\mathrm{m}} \cap L^* \otimes \Gamma'_{\mathrm{M}} \subseteq L^* \otimes \Gamma'_{\mathrm{m}} + L \otimes \Gamma'^*_{\mathrm{M}} = (L \otimes \Gamma'^*_{\mathrm{m}} \cap L^* \otimes \Gamma'_{\mathrm{M}})^*.$$

Moreover we have

$$\begin{aligned}
L^* \otimes \Gamma'_{\mathrm{m}} + L \otimes \Gamma'^*_{\mathrm{M}} \subseteq L^* \otimes \Gamma'_{\mathrm{M}} + L \otimes \Gamma'^*_{\mathrm{M}} &= L\varpi^{-1} \otimes \Gamma'_{\mathrm{M}} \cap L^* \otimes \Gamma'^*_{\mathrm{M}}\\
&\subseteq (L \otimes \Gamma'^*_{\mathrm{m}} \cap L^* \otimes \Gamma'_{\mathrm{M}})\varpi^{-1}.
\end{aligned}$$

Hence $L^* \otimes \Gamma'_{\mathrm{M}} \cap L \otimes \Gamma'^*_{\mathrm{m}}$ is a good lattice in $\mathscr{W}$. □

4.7.

LEMMA. *Let Γ'_{M} be a fixed maximal good lattice in $\mathscr{V}'$.*

(i) *Suppose that M_1, M_2 are two $\mathcal{O}$-modules in $\mathscr{V}'$ such that $M_1 \subseteq M_2 \subseteq M_1\varpi^{-1}$, $M_1 \subseteq \Gamma'_{\mathrm{M}}$, and $\langle M_1, M_2\rangle' \subseteq \mathfrak{p}^{\kappa'}$. Then there exists a good lattice L' in $\mathscr{V}'$ such that $L' \subseteq \Gamma'_{\mathrm{M}}$, $M_1 \subseteq L'$ and $M_2 \subseteq L'^*$.*

(ii) *Let Γ'_{m} be a minimal good lattice in $\mathscr{V}'$ such that $\Gamma'_{\mathrm{m}} \subseteq \Gamma'_{\mathrm{M}}$. Suppose that M_1, M_2 are two $\mathcal{O}$-modules in $\mathscr{V}'$ such that $M_1 \subseteq M_2 \subseteq M_1\varpi^{-1}$, $M_1 \subseteq \Gamma'_{\mathrm{M}}$, $M_2 \subseteq \Gamma'^*_{\mathrm{m}}$, and $\langle M_1, M_2\rangle' \subseteq \mathfrak{p}^{\kappa'}$. Then there exists a good lattice L' in $\mathscr{V}'$ such that $\Gamma'_{\mathrm{m}} \subseteq L' \subseteq \Gamma'_{\mathrm{M}}$, $M_1 \subseteq L'$ and $M_2 \subseteq L'^*$.*

(iii) *Suppose that M is a $\mathcal{O}$-module in $\mathscr{V}'$ such that $\langle M, M\rangle' \subseteq \mathfrak{p}^{\kappa'}$ and $M \subseteq \Gamma'^*_{\mathrm{M}}$. Then M is contained in Γ'_{M}.*

PROOF. Without loss of generality, we may assume that M_1, M_2 are lattices in $\mathscr{V}'$. From the assumption we have $M_2 + \Gamma'^*_{\mathrm{M}} \subseteq M_1\varpi^{-1} + \Gamma'^*_{\mathrm{M}} \subseteq \Gamma'_{\mathrm{M}}\varpi^{-1}$. So we may regard $M_2/(M_2 \cap \Gamma'^*_{\mathrm{M}}) \simeq (M_2 + \Gamma'^*_{\mathrm{M}})/\Gamma'^*_{\mathrm{M}}$ as an $\boldsymbol{f}_D$-subspace of $\Gamma'_{\mathrm{M}}\varpi^{-1}/\Gamma'^*_{\mathrm{M}}$. Let $\{z_i\}_{i\in J}$ (for some finite index set J) be a subset of M_2 such that the images of these z_i in $M_2/(M_2 \cap \Gamma'^*_{\mathrm{M}})$ are linearly independent over $\boldsymbol{f}_D$. From the assumption we have $\langle M_2, M_2\rangle' \subseteq \mathfrak{p}^{\kappa'-1}$. Therefore $\langle z_i\varpi, z_j\varpi\rangle' \subseteq \mathfrak{p}^{\kappa'+1}$ for any $i, j \in J$. Therefore the set $\{z_i\varpi\}_{i\in J}$ as subset of Γ'_{M} satisfies the condition in [**Wp**] corollaire I.8 with $n = 1$. Thus there exists a Γ'_{M}-admissible decomposition $\mathscr{V}' = X \oplus \mathscr{V}'^{\circ} \oplus Y$ where X, Y are totally isotropic and in duality, and a basis $\{v_i\}_{i \in J}$ of X such that $v_i - z_i\varpi \in \Gamma'_{\mathrm{M}}\varpi$ for every $i \in J$. Note that a decomposition of $\mathscr{V}'$ is Γ'_{M}-admissible if and only if it is Γ'^{*}_{M}-admissible. Therefore from the choice of the set $\{z_i\}_{i \in J}$ we have

$$M_2 + \Gamma'^{*}_{\mathrm{M}} = (\Gamma'^{*}_{\mathrm{M}} \cap X)\varpi^{-1} \oplus (\Gamma'^{*}_{\mathrm{M}} \cap \mathscr{V}'^{\circ}) \oplus (\Gamma'^{*}_{\mathrm{M}} \cap Y). \tag{4.7.a}$$

From (4.7.a), we see that $(M_2 + \Gamma'^{*}_{\mathrm{M}})^{*}$ is a good lattice in $\mathscr{V}'$. Define $L' := (M_2 + \Gamma'^{*}_{\mathrm{M}})^{*} = M_2^{*} \cap \Gamma'_{\mathrm{M}}$. Therefore $L'^{*} = M_2 + \Gamma'^{*}_{\mathrm{M}}$. It is clear that $L' \subseteq \Gamma'_{\mathrm{M}}$ and $L' \subseteq M_2^{*}$. Hence $M_2 \subseteq L'^{*}$. Since $M_1 \subseteq M_2^{*}$ and $M_1 \subseteq \Gamma'_{\mathrm{M}}$, we have $M_1 \subseteq L'$. Then L' satisfies all requirements.

For (ii), let L' be given as in the proof of (i). We only need to check that $\Gamma'_{\mathrm{m}} \subseteq L'$. As in the previous paragraph, we have $L'^{*} = M_2 + \Gamma'^{*}_{\mathrm{M}}$. Hence $L'^{*} \subseteq \Gamma'^{*}_{\mathrm{m}}$ because $M_2 \subseteq \Gamma'^{*}_{\mathrm{m}}$ and $\Gamma'^{*}_{\mathrm{M}} \subseteq \Gamma'^{*}_{\mathrm{m}}$. Therefore $\Gamma'_{\mathrm{m}} \subseteq L'$.

Now we prove part (iii). Let $\{x_i\}_{i \in J}$ be a set of vectors in M such that their images $\{\bar{x}_i\}_{i \in J}$ in the quotient $(M + \Gamma'_{\mathrm{M}})/\Gamma'_{\mathrm{M}} \subseteq \Gamma'^{*}_{\mathrm{M}}/\Gamma'_{\mathrm{M}}$ are linearly independent for some finite index set J. Let $\langle\, ,\rangle'^{*}$ denote the form on the space $\Gamma'^{*}_{\mathrm{M}}/\Gamma'_{\mathrm{M}}$ by (1.2.c). Because we assume that $\langle M, M\rangle' \subseteq \mathfrak{p}^{\kappa'}$, $\langle \bar{x}_i, \bar{x}_i\rangle'^{*}$ must be zero for all $i \in J$. But we know the form $\langle\, ,\rangle'^{*}$ is anisotropic because Γ'_{M} is a maximal good lattice. Hence all $\bar{x}_i$ must be zero i.e., M is contained in Γ'_{M}. □

4.8.

Lemma. *Let A be a good lattice in $\mathscr{W}$ and w be an element in $\mathscr{W}$. Let K be the subgroup of K'_A of elements g such that $(g-1).w$ belongs to A. Then the map $\psi_w : K \to \boldsymbol{C}^{\times}$ defined by $g \mapsto \psi((1/2)\langle\!\langle (g-1).w, w\rangle\!\rangle)$ is a character of K.*

Proof. Let g_1, g_2 be two elements in K. Then both $(g_1^{-1} - 1).w$ and $(g_2 - 1).w$ belong to A. It is clear that $\langle\!\langle g.w, w\rangle\!\rangle = \langle\!\langle (g-1).w, w\rangle\!\rangle$ because $\langle\!\langle w, w\rangle\!\rangle = 0$. Now $g_1g_2 - 1$ is equal to $(g_1 - 1)(g_2 - 1) + (g_1 - 1) + (g_2 - 1)$. Therefore

$$\begin{aligned}
&\psi_w(g_1g_2)\\
&\quad= \psi\left(\frac{1}{2}\langle\!\langle (g_1g_2 - 1).w, w\rangle\!\rangle\right)\\
&\quad= \psi\left(\frac{1}{2}\langle\!\langle (g_1 - 1)(g_2 - 1).w, w\rangle\!\rangle\right)\psi\left(\frac{1}{2}\langle\!\langle (g_1 - 1).w, w\rangle\!\rangle\right)\psi\left(\frac{1}{2}\langle\!\langle (g_2 - 1).w, w\rangle\!\rangle\right).
\end{aligned}$$

It is easy to compute that $\langle\!\langle (g_1 - 1)(g_2 - 1).w, w\rangle\!\rangle = \langle\!\langle (g_2 - 1).w, (g_1^{-1} - 1).w\rangle\!\rangle$. Hence $\psi((1/2)\langle\!\langle (g_1 - 1)(g_2 - 1).w, w\rangle\!\rangle) = 1$ because both elements $(g_2 - 1).w$, $(g_1^{-1} - 1).w$ are in A and A is a good lattice. Therefore we have

$$\psi\left(\frac{1}{2}\langle\langle(g_1g_2-1).w,w\rangle\rangle\right)=\psi\left(\frac{1}{2}\langle\langle(g_1-1).w,w\rangle\rangle\right)\psi\left(\frac{1}{2}\langle\langle(g_2-1).w,w\rangle\rangle\right),$$

that is, $\psi_w(g_1g_2)=\psi_w(g_1)\psi_w(g_2)$. It is clear that the map ψ_w is continuous. Hence ψ_w is a character of K. $\square$

4.9.

Let K be a compact subgroup of $Sp(\mathscr{W})$ contained in K'_A for some good lattice A in $\mathscr{W}$. If w is an element in $\mathscr{W}$ and f is an element of $\mathscr{S}(A)_w$, then we define

$$f[w,K]:=\int_K \omega_\psi(k).f\,dk \tag{4.9.a}$$

where dk is a Haar measure on K. Then it is clear that $f[w,K]$ belongs to $\mathscr{S}(A)_{A^*+K.w}$. If $f[w,K]$ is not the zero vector, then $f[w,K]$ is fixed by K. Moreover those $f[w,K]$ when f runs over a basis of $\mathscr{S}(A)_w$ span the subspace of $\mathscr{S}(A)^K$ of functions with support in $A^*+K.w$, that is, we have

$$(\mathscr{S}(A)_{A^*+K.w})^K=\sum_{f\in\mathscr{S}(A)_w}\boldsymbol{C}f[w,K] \tag{4.9.b}$$

(*cf.* [**MVW**], chapitre 5, section III.1). The following lemma, which is from [**MVW**] chapitre 5 section III.3 plays an important role in the proofs of the main results in subsection 4.13.

LEMMA. *Let w be an element in $\mathscr{W}$ and K be a compact subgroup of K'_A. Suppose that f is a nonzero vector in $\mathscr{S}(A)_w$. Then $f[w,K]$ is nonzero if and only if f is fixed by the subgroup $K_1:=\{g\in K\mid g^{-1}.w\in A+w\}$.*

PROOF. First we prove that $f[w,K]$ is nonzero if and only if $f[w,K](w)$ is nonzero. Now $f[w,K]$ is a mapping with support in $A^*+K.w$. Suppose that $f[w,K]$ is nonzero, that is, $f[w,K](w')\neq 0$ for some element w' in $A^*+K.w$. Write $w'=a+k.w$ for some $a\in A^*$, $k\in K$. Now we have

$$f[w,K](a+k.w)=\psi\left(\frac{1}{2}\langle\langle k.w,a\rangle\rangle\right)\tilde{\rho}_\psi(a).(f[w,K](k.w))$$

by (1.5.b). Hence $f[w,K](w')\neq 0$ if and only if $f[w,K](k.w)\neq 0$. Moreover from (4.9.a) it is clear that $f[w,K](k.w)=f[w,K](w)$. Hence $f[w,K]$ is nonzero if and only if $f[w,K](w)$ is nonzero.

Now $f[w,K](w)=\int_K(\omega_\psi(k).f)(w)\,dk$ is equal to $\int_K f(k^{-1}.w)\,dk$ because K is contained in K'_A. Thus

$$\int_K f(k^{-1}.w)\,dk=\int_{K_1} f(k^{-1}.w)\,dk$$

because f is supported in $A^* + w$. Now $f(k^{-1}.w) = \psi((1/2)\langle\!\langle (k-1).w, w\rangle\!\rangle) \cdot f(w)$ for $k \in K_1$ because $(k-1).w$ belongs to A. Therefore we conclude that

$$f[w,K](w) = \int_{K_1} \psi_w(k) f(w)\, dk = \left(\int_{K_1} \psi_w(k)\, dk\right) f(w) \tag{4.9.c}$$

where ψ_w is defined in Lemma 4.8. By Lemma 4.8 we know that ψ_w is a character of K_1. The integral $\int_{K_1} \psi_w(k)\, dk$ in (4.9.c) is essentially equal to the sum of values of a character over all elements of a finite group. Therefore the last integral in (4.9.c) is nonzero if and only if ψ_w is trivial on K_1. Hence $f[w,K] \neq 0$ if and only if ψ_w is trivial on K_1. □

4.10.

We identify the space $\mathscr{W}$ with $\mathrm{Hom}_D(\mathscr{V}, \mathscr{V}')$. Hence an element $w \in \mathscr{W}$ can be regarded as a homomorphism from $\mathscr{V}$ to $\mathscr{V}'$. We use the notation $w.x$ to denote the image of x under the map w for $x \in \mathscr{V}$. The following lemma is [**Wp**] lemme II.4.

Lemma. *Let A be a good lattice in $\mathscr{W}$, g be an element in K'_A, w be an element in $\mathscr{W}$ and c denote the element $c(g)$ in $\mathfrak{sp}(\mathscr{W})$. Suppose that $c.w$ belongs to A^*.*

(i) *For $f \in \mathscr{S}(A)$ we have $(M_A[g].f)(w) = \psi(\langle\!\langle w, c.w\rangle\!\rangle)\tilde{\rho}_\psi(2c.w).f(w)$.*
(ii) *Suppose that $c = c_{x,y}$ for some $x, y \in \mathscr{V}$. Then $\langle\!\langle w, c_{x,y}.w\rangle\!\rangle = -2\,\mathrm{Trd}_{D/F}\langle w.x, w.y\rangle'$.*

Let $(\rho_\psi, \mathscr{S})$ be an irreducible smooth representation of the Heisenberg group $\mathrm{H}(\mathscr{W})$ associated to the character ψ of F of conductor $\mathfrak{p}_F^{\lambda_F}$. Suppose that Q is a lattice in $\mathscr{W}$ such that $Q \subseteq Q^*$. Then we have $\langle\!\langle b, b'\rangle\!\rangle \in \mathfrak{p}_F^{\lambda_F}$ for any $b, b' \in Q$. Therefore $Q \times \mathfrak{p}_F^{\lambda_F}$ is a subgroup of $\mathrm{H}(\mathscr{W})$. The representation $(\rho_\psi|_{Q\times\mathfrak{p}_F^{\lambda_F}}, \mathscr{S})$ factors through the projection $Q \times \mathfrak{p}_F^{\lambda_F} \to Q$ because $\mathfrak{p}_F^{\lambda_F}$ is in the kernel of ψ. The action of Q on the space $\mathscr{S}$ will be also denoted by ρ_ψ. If Q is self-dual, then it is not difficult to see that the representation ρ_ψ of the additive group Q on $\mathscr{S}$ is equivalent to its regular representation.

Suppose Q is a lattice in $\mathscr{W}$ and contained in a good lattice A. Then we have $Q \subseteq Q^*$. Hence Q acts on $\mathscr{S}(A)$ as in the previous paragraph. From (i) of Lemma 4.10 it is not difficult to check that

$$\mathscr{S}(A)^Q = \mathscr{S}(A)_{Q^*} \tag{4.10.a}$$

(a proof can be found in [**Pn2**]). Hence corollaire III.2 of [**Wp**] for $G = G_{L,0^+}$, θ trivial and A as defined in II.2 can be rewritten as

$$\mathscr{S}(A)^{G_{L,0^+}} = \omega_\psi(\mathscr{H}').(\mathscr{S}(A)^{B^*_{M,N}})^{G_{L,0^+}} \tag{4.10.b}$$

because $B^*_{M,N} \subseteq A \subseteq B_{M,N}$ where $B_{M,N}$ is as defined in [**Wp**]. Clearly, the lattice A in above expression is not essential. Hence we have

$$\mathscr{S}^{G_{L,0^+}} = \omega_\psi(\mathscr{H}').(\mathscr{S}^{B^*_{M,N}})^{G_{L,0^+}} \tag{4.10.c}$$

where $\mathscr{S}$ is any model of the Weil representation.

4.11.

PROOF OF PROPOSITION 4.1. Let B denote $B(L,L')$. Suppose $f \in \mathscr{S}(B)_{B^*}$ and $g \in G_{L,0^+}$. We know that g is in K'_B. We have $(\omega_\psi(g).f)(x) = (M_B[g].f)(x) = \tilde{\omega}_\psi(g).(f(g^{-1}.x))$ from (1.5.c) for any $x \in \mathscr{W}$. Now $\tilde{\omega}_\psi(g)$ is trivial because g is in K'_B. Hence $(\omega_\psi(g).f)(x)$ is not zero only if $g^{-1}.x$ belongs to B^*. But B^* is stable by g, so $\omega_\psi(g).f$ belongs to $\mathscr{S}(B)_{B^*}$. Now suppose w is in B^*. Hence $(g^{-1}-1).w$ is in B. Then

$$(\omega_\psi(g).f)(w) = f((g^{-1}-1).w + w) = \psi\left(\frac{1}{2}\langle\!\langle w, (g^{-1}-1).w\rangle\!\rangle\right)\tilde{\rho}((g^{-1}-1).w).f(w).$$

Because $(g^{-1}-1).w$ is in B, $\tilde{\rho}((g^{-1}-1).w)$ becomes trivial. Moreover $w \in B^*$ and $(g^{-1}-1).w \in B$ imply that $\psi((1/2)\langle\!\langle (g-1).w, w\rangle\!\rangle) = 1$. Hence $(\omega_\psi(g).f)(w) = f(w)$ for any $w \in B^*$. Therefore $\mathscr{S}(B)_{B^*}$ is fixed pointwise by $G_{L,0^+}$. By the remark in subsection 4.10 we conclude that $\mathscr{S}(A_0)_{B(L,L')^*}$ is fixed pointwise by $G_{L,0^+}$. By symmetry, $\mathscr{S}(A_0)_{B(L,L')^*}$ is also fixed pointwise by $G'_{L',0^+}$. □

4.12.

PROPOSITION. *Let L be a good lattice in $\mathscr{V}$. Suppose that w is in $L^* \otimes \Gamma'^*_{\mathrm{m}} \cap L\varpi^{-1} \otimes \Gamma'_{\mathrm{M}}$ and $f[w, G_{L,0^+}]$ is nonzero for some $f \in \mathscr{S}(A)_w$ where $A := L^* \otimes \Gamma'_{\mathrm{M}} \cap L \otimes \Gamma'^*_{\mathrm{m}}$ in $\mathscr{W}$. Then w belongs to $B(L,L')^*$ for some good lattice L' such that $\Gamma'_{\mathrm{m}} \subseteq L' \subseteq \Gamma'_{\mathrm{M}}$.*

PROOF. We know that A is a good lattice in $\mathscr{W}$ by Lemma 4.6. Identify $L^* \otimes \Gamma'^*_{\mathrm{m}} \cap L\varpi^{-1} \otimes \Gamma'_{\mathrm{M}}$ with $\mathrm{Hom}_{\mathcal{O}}(L\varpi^{-\kappa}, \Gamma'^*_{\mathrm{m}}) \cap \mathrm{Hom}_{\mathcal{O}}(L^*\varpi^{1-\kappa}, \Gamma'_{\mathrm{M}})$. Let x be an element in L, y be an element in $L^*\varpi^{1-\kappa}$. Then $\mathrm{ord}_{L^*}(x) \geq \mathrm{ord}_L(x) \geq 0$, $\mathrm{ord}_L(y) \geq -\kappa$, and $\mathrm{ord}_{L^*}(y) \geq 1-\kappa$. Then x, y satisfy the condition in Lemma 4.5(i). Hence $u_{x,y}$ is defined and belongs to $G_{L,0^+}$. Now $c_{x,y}.w \subseteq A$, so

$$(\omega_\psi(u_{x,y}).f)(w) = \psi(\langle\!\langle w, c_{x,y}.w\rangle\!\rangle)f(w)$$

by Lemma 4.10(i). Because we have $(u_{x,y}^{-1}-1).w \in A$ and we assume that $f[w, G_{L,0^+}]$ is nonzero, we have $\psi(\langle\!\langle w, c_{x,y}.w\rangle\!\rangle) = 1$ by Lemma 4.9. Hence we have $\langle\!\langle w, c_{x,y}.w\rangle\!\rangle \in \mathfrak{p}_F^{\lambda_F}$. Now $\mathrm{Trd}_{D/F}(\langle w.x, w.y\rangle')$ is in $\mathfrak{p}_F^{\lambda_F}$ by Lemma 4.10(ii). Therefore $\langle w.x, w.y\rangle'$ is in $\mathfrak{p}^\lambda = \mathfrak{p}^{\kappa+\kappa'}$ from the definition in subsection 2.1.

Therefore $\langle w.x\varpi^{-\kappa}, w.y\rangle' \in \mathfrak{p}^{\kappa'}$. Because x (resp. y) is arbitrary in L (resp. $L^*\varpi^{1-\kappa}$), we have

$$\langle w.L\varpi^{-\kappa}, w.L^*\varpi^{1-\kappa}\rangle' \subseteq \mathfrak{p}^{\kappa'}.$$

Now $w.L^*\varpi^{1-\kappa} \subseteq w.L\varpi^{-\kappa} \subseteq w.L^*\varpi^{-\kappa}$ because L is a good lattice. Because w is in $L^* \otimes \Gamma'^*_{\mathrm{m}} \cap L\varpi^{-1} \otimes \Gamma'_{\mathrm{M}}$, we have $w.L^*\varpi^{1-\kappa} \subseteq \Gamma'_{\mathrm{M}}$ and $w.L\varpi^{-\kappa} \subseteq \Gamma'^*_{\mathrm{m}}$. Hence by Lemma 4.7(ii) there exists a good lattice L' in $\mathscr{V}'$ such that $\Gamma'_{\mathrm{m}} \subseteq L' \subseteq \Gamma'_{\mathrm{M}}$, $w.L^*\varpi^{1-\kappa} \subseteq L'$ and $w.L\varpi^{-\kappa} \subseteq L'^*$. Therefore w is in $\mathrm{Hom}_{\mathcal{O}}(L\varpi^{-\kappa}, L'^*) \cap \mathrm{Hom}_{\mathcal{O}}(L^*\varpi^{1-\kappa}, L')$ which is exactly $L^* \otimes L'^* \cap L \otimes L'\varpi^{-1} = B(L, L')^*$. □

4.13.

Proof of Theorem 4.2. Now we begin to prove Theorem 4.2. The inclusion

$$\omega_\psi(\mathscr{H}').\left(\sum_{L' \subseteq \Gamma'_{\mathrm{M}}} \mathscr{S}(A)_{B(L,L')^*}\right) \subseteq \mathscr{S}(A)^{G_{L,0^+}}$$

is an easy consequence of Proposition 4.1 and the fact that $\widetilde{U(\mathscr{V})}$ and $\widetilde{U(\mathscr{V}')}$ commute with each other in $\widetilde{Sp(\mathscr{W})}$. We shall prove the opposite inclusion by discussion according to the following three separate cases: (1) $L = L^*$; (2) $L = L^*\varpi$; (3) $L^*\varpi \neq L \neq L^*$.

First suppose that we are in the first case, that is, L is self-dual. Hence L is a maximal good lattice in $\mathscr{V}$. Then $G_{L,0^+} = \{g \in G \mid (g-1).L \subseteq L\varpi\}$. Let $M := L\varpi$, $N := L$ and $\mathscr{R}(L), B_{M,N}, K_{M,N}$ be as defined in [**Wp**] I.15, I.16, and II.6. Clearly the pair (M, N) belongs to $\mathscr{R}(L)$. So $B_{M,N} = L\varpi^{-1} \otimes \Gamma'_{\mathrm{M}} + L \otimes \Gamma'^*_{\mathrm{M}} = L\varpi^{-1} \otimes \Gamma'_{\mathrm{M}}$ for a fixed maximal good lattice Γ'_{M} in $\mathscr{V}'$. It is easy to check that $K_{M,N} = G_{L,0^+}$ in this case. Now we have $(g-1).B_{M,N} \subseteq L \otimes \Gamma'_{\mathrm{M}}$ for all $g \in G_{L,0^+}$. Clearly $A := L \otimes \Gamma'_{\mathrm{M}} = B(L, \Gamma'_{\mathrm{M}})$ is a good lattice in $\mathscr{W}$, and $G_{L,0^+}$ is a subgroup of K'_A. Let x be an element in L, y be in $L\varpi^{1-\kappa}$. Then we have $\mathrm{ord}_L(x) \geq 0$ and $\mathrm{ord}_L(y) \geq 1-\kappa$. Therefore x, y satisfy the condition in Lemma 4.5(i). Hence $u_{x,y}$ is defined and belongs to $G_{L,0^+}$. Let w be an element in $B_{M,N}$. Then $c_{x,y}.w$ belongs to A. Therefore $\omega_\psi(u_{x,y}).f = \psi(\langle\!\langle w, c_{x,y}.w\rangle\!\rangle)f$ by Lemma 4.10(i) for $f \in \mathscr{S}(A)$. Suppose that $f[w, G_{L,0^+}]$ is nonzero. Then $\psi(\langle\!\langle w, c_{x,y}.w\rangle\!\rangle) = 1$ by Lemma 4.9. Hence we have $\langle\!\langle w, c_{x,y}.w\rangle\!\rangle \in \mathfrak{p}_F^{\lambda_F}$. Now $\mathrm{Trd}_{D/F}(\langle w.x, w.y\rangle') \in \mathfrak{p}_F^{\lambda_F}$ by Lemma 4.10(ii). From subsection 2.1 we know that $\langle w.x, w.y\rangle' \in \mathfrak{p}^\lambda = \mathfrak{p}^{\kappa+\kappa'}$. Hence we have $\langle w.x\varpi^{-\kappa}, w.y\rangle' \in \mathfrak{p}^{\kappa'}$. Therefore $\langle w.L\varpi^{-\kappa}, w.L\varpi^{1-\kappa}\rangle'$ is contained in $\mathfrak{p}^{\kappa'}$. Since w is in $L\varpi^{-1} \otimes \Gamma'_{\mathrm{M}}$, we have $w.L\varpi^{1-\kappa} \subseteq \Gamma'_{\mathrm{M}}$. Hence by Lemma 4.7(i) we know that there exists a good lattice L' in $\mathscr{V}'$ such that $L' \subseteq \Gamma'_{\mathrm{M}}$ and $w.L\varpi^{-\kappa} \subseteq L'^*$. Hence $w \in L \otimes L'^* = B(L, L')^*$ because $L = L^*$ in this case. Therefore we have proved that if w is in $B_{M,N}$ and

$f[w, G_{L,0^+}]$ is nonzero, then w must be in $B(L, L')^*$ for some good lattice $L' \subseteq \Gamma'_{\mathrm{M}}$. So we conclude

$$(\mathscr{S}(A)_{B_{M,N}})^{G_{L,0^+}} = \sum_{w \in B_{M,N}} \sum_{f \in \mathscr{S}(A)_w} f[w, G_{L,0^+}] \subseteq \sum_{L' \subseteq \Gamma'_{\mathrm{M}}} \mathscr{S}(A)_{B(L,L')^*}.$$

Therefore by (4.10.c) and [**Wp**] corollaire III.2 we have

$$\mathscr{S}^{G_{L,0^+}} = \omega_\psi(\mathscr{H}').((\mathscr{S}^{B^*_{M,N}})^{G_{L,0^+}}) \subseteq \omega_\psi(\mathscr{H}').\left(\sum_{L' \subseteq \Gamma'_{\mathrm{M}}} \mathscr{S}^{B(L,L')}\right).$$

Secondly, suppose that $L = L^*\varpi$. The proof for this case is very similar to the proof for the first case. Now again $G_{L,0^+} = \{g \in G \mid (g-1).L \subseteq L\varpi\}$ in this case. Let $M = N := L$. We have $(M, N) \in \mathscr{R}(\Gamma_{\mathrm{M}})$ where Γ_{M} is a fixed maximal good lattice in $\mathscr{V}$ containing L. Now we have $B_{M,N} = L^* \otimes \Gamma'_{\mathrm{M}} + L^* \otimes \Gamma'^*_{\mathrm{M}} = L^* \otimes \Gamma'^*_{\mathrm{M}}$ for a fixed maximal good lattice Γ'_{M} in $\mathscr{V}'$. And we also have $K_{M,N} = G_{L,0^+}$. Clearly $A := L \otimes \Gamma'^*_{\mathrm{M}}$ is a good lattice in $\mathscr{W}$. Then $G_{L,0^+}$ is a subgroup of K'_A. Let x be an element in L, y be an element in $L\varpi^{-\kappa}$. Then x, y satisfy the conditions in Lemma 4.5(i). Hence $u_{x,y}$ is defined and belongs to $G_{L,0^+}$. By the same argument in the first case we can prove that $\langle w.L\varpi^{-\kappa}, w.L\varpi^{-\kappa}\rangle' \subseteq \mathfrak{p}^{\kappa'}$. Now $w.L\varpi^{-\kappa}$ is contained in Γ'^*_{M} because $w \in B_{M,N}$. Therefore we get $w.L\varpi^{-\kappa} \in \Gamma'_{\mathrm{M}}$ by Lemma 4.7(iii). Let $L' := \Gamma'_{\mathrm{M}}$, which is of course a good lattice in $\mathscr{V}'$. Hence w is in $L^* \otimes L' = B(L, L')^*$ because now $L^* = L\varpi^{-1}$ in this case. Therefore we have $\mathscr{S}^{G_{L,0^+}} \subseteq \omega_\psi(\mathscr{H}').(\sum_{L' \subseteq \Gamma'_{\mathrm{M}}} \mathscr{S}^{B(L,L')})$ by the same argument in the first case.

Finally we suppose that $L^*\varpi \neq L \neq L^*$. Now we let $M := L^*\varpi$, $N := L$. Then we know that $(M, N) \in \mathscr{R}(\Gamma_{\mathrm{M}})$ and

$$B_{M,N} = L\varpi^{-1} \otimes \Gamma'^*_{\mathrm{M}} \cap L^* \otimes \Gamma'_{\mathrm{M}}\varpi^{-1}. \tag{4.12.a}$$

Also we have $K_{M,N} = G_{L,1}$. Suppose that g is an element in $G_{L,1}$. Then from (4.12.a), we have $(g-1).B_{M,N} \subseteq L \otimes \Gamma'^*_{\mathrm{M}} \cap L^* \otimes \Gamma'_{\mathrm{M}}$. Let A be the good lattice $L \otimes \Gamma'^*_{\mathrm{M}} \cap L^* \otimes \Gamma'_{\mathrm{M}}$. Hence

$$(g-1).B_{M,N} \subseteq A \subseteq A^* \subseteq B_{M,N}$$

for all $g \in G_{L,1}$. Hence $G_{L,1}$ is a subgroup of K'_A. We identify $B_{M,N}$ with $\mathrm{Hom}_{\mathcal{O}}(L^*\varpi^{1-\kappa}, \Gamma'^*_{\mathrm{M}}) \cap \mathrm{Hom}_{\mathcal{O}}(L\varpi^{1-\kappa}, \Gamma'_{\mathrm{M}})$. Let x be an element in L, and y be an element in $L\varpi^{1-\kappa}$. Then x, y satisfy the condition in Lemma 4.5(ii). Hence $u_{x,y}$ is defined and belongs to $G_{L,1}$. By the same argument in the first case we can prove that $\langle w.L\varpi^{1-\kappa}, w.L\varpi^{-\kappa}\rangle' \in \mathfrak{p}^{\kappa'}$. We know that $w.L\varpi^{1-\kappa} \subseteq \Gamma'_{\mathrm{M}}$. Therefore by Lemma 4.7(i) we have $w.L\varpi^{-\kappa}$ is contained in L''^* for some good lattice L'' in $\mathscr{V}'$ such that $L'' \subseteq \Gamma'_{\mathrm{M}}$. Let Γ'_{m} be a minimal good lattice in $\mathscr{V}'$ such that $\Gamma'_{\mathrm{m}} \subseteq L''$. Hence $w.L\varpi^{-\kappa}$ is contained in Γ'^*_{m}. Let x be an element in

$L^*\varpi$ and y be an element in $L^*\varpi^{1-\kappa}$. Then again x, y satisfy the condition in Lemma 4.5(ii). Hence $u_{x,y}$ is defined and belongs to $G_{L,1}$. By the same argument in the first case we can prove that $\langle w.L^*\varpi^{1-\kappa}, w.L^*\varpi^{1-\kappa}\rangle' \subseteq \mathfrak{p}^{\kappa'}$. Since w in $B_{M,N}$, we have $w.L^*\varpi^{1-\kappa} \subseteq \Gamma'^*_{\mathrm{M}}$. Therefore we have $w.L^*\varpi^{1-\kappa} \subseteq \Gamma'_{\mathrm{M}}$ by Lemma 4.7(iii). Therefore we have proved that if $w \in B_{M,N}$ and $f[w, G_{L,1}] \in \mathscr{S}(A)$ is nonzero for some $f \in \mathscr{S}(A)_w$, then w must be in $L^* \otimes \Gamma'^*_{\mathrm{m}} \cap L \otimes \Gamma'_{\mathrm{M}}\varpi^{-1}$ for some minimal good lattice $\Gamma'_{\mathrm{m}} \subseteq \Gamma'_{\mathrm{M}}$. Therefore $(\mathscr{S}(A)_{B_{M,N}})^{G_{L,1}} \subseteq \mathscr{S}(A)_R$ where $R := L^* \otimes \Gamma'^*_{\mathrm{m}} \cap L \otimes \Gamma'_{\mathrm{M}}\varpi^{-1}$. Hence by (4.10.a) we have $(\mathscr{S}(A_1)^{B^*_{M,N}})^{G_{L,1}} \subseteq \mathscr{S}(A_1)^{R(L,\Gamma'_{\mathrm{m}})^*}$, or

$$\left(\mathscr{S}^{B^*_{M,N}}\right)^{G_{L,1}} \subseteq \mathscr{S}^{R(L,\Gamma'_{\mathrm{m}})^*}$$

where S is any model of the Weil representation of $\widetilde{Sp(\mathscr{W})}$. Now by Proposition 4.11 we know that if $w \in R$ and $f[w, G_{L,0^+}] \in \mathscr{S}(A_0)$ is nonzero for some $f \in \mathscr{S}(A)_w$, then $w \in B(L, L')^*$ for some good lattice L' in $\mathscr{V}'$ such that $L' \subseteq \Gamma'_{\mathrm{M}}$. Hence we have proved

$$\begin{aligned}(\mathscr{S}(A)_{B_{M,N}})^{G_{L,0^+}} = ((\mathscr{S}(A)_{B_{M,N}})^{G_{L,1}})^{G_{L,0^+}} &\subseteq \left(\sum_{\Gamma'_{\mathrm{m}} \subseteq \Gamma'_{\mathrm{M}}} \mathscr{S}(A)_R\right)^{G_{L,0^+}} \\ &\subseteq \sum_{L' \subseteq \Gamma'_{\mathrm{M}}} \mathscr{S}(A)_{B(L,L')^*}.\end{aligned}$$

Therefore by (4.10.b) and [**Wp**] corollaire III.2 we have

$$\mathscr{S}(A)^{G_{L,0^+}} \subseteq \omega_\psi(\mathscr{H}').(\mathscr{S}(A)_{B_{M,N}})^{G_{L,0^+}} \subseteq \omega_\psi(\mathscr{H}').\left(\sum_{L' \subseteq \Gamma'_{\mathrm{M}}} \mathscr{S}(A)^{B(L,L')}\right).$$

Hence the theorem is proved. □

5. Correspondence of depth zero minimal *K*-types.

In section 4 we proved that the depth zero representations are preserved by the local theta correspondence. In this section we want to investigate how the depth zero minimal *K*-types of the paired representations are related.

5.1.

Let L (resp. L') be a good lattice in $\mathscr{V}$ (resp. $\mathscr{V}'$). Recall that $B := B(L, L') = L^* \otimes L' \cap L \otimes L'^*$ and $B^* = L^* \otimes L' + L \otimes L'^*$ from subsection 4.1. Define

$$B\varpi^r := L^* \otimes L'\varpi^r \cap L \otimes L'^*\varpi^r$$

$$B^*\varpi^r := L^* \otimes L'\varpi^r + L \otimes L'^*\varpi^r$$

for any integer r. Clearly $B\varpi^r$ and $B^*\varpi^r$ are $\mathcal{O}_F$-lattices in $\mathscr{W}$. As usual let K_B denote the stabilizer of B in $Sp(\mathscr{W})$ and K_B' be as defined in (1.5.a). Define

$$K_B'' := \{g \in K_B \mid (g-1).B^*\varpi^{-1} \subseteq B^*\} = \{g \in K_B \mid (g-1).B \subseteq B\varpi\}. \quad (5.1.a)$$

It is easy to see that $K_B'' \subseteq K_B' \subseteq K_B$. We know that B^*/B is a nondegenerate symplectic space over $\boldsymbol{f}_F$. It is not difficult to check that the quotient K_B/K_B' is isomorphic to the finite symplectic group $Sp(B^*/B)$.

Define the maps ord_B and ord_{B^*} from $\mathscr{W}$ to $\boldsymbol{Z}$ as follows: $\mathrm{ord}_B(w) := m$ if $w \in B\varpi^m - B\varpi^{m+1}$, and $\mathrm{ord}_{B^*}(w) := m$ if $w \in B^*\varpi^m - B^*\varpi^{m+1}$.

LEMMA. *Let x, y be elements in $\mathscr{W}$ such that* $\mathrm{ord}_B(x) + \mathrm{ord}_{B^*}(y) \geq 1 - \lambda$ *and* $\mathrm{ord}_B(y) + \mathrm{ord}_{B^*}(x) \geq 1 - \lambda$ *where λ is defined in subsection* 2.1. *Then $u_{x,y}$ is defined and belongs K_B''.*

PROOF. From the condition in the lemma, it is clear that $1 + c_{x,y}$ is invertible. Hence $u_{x,y}$ is well-defined. Let $b \in B^*\varpi^{-1}$. Then $c_{x,y}.b = x\langle\langle y, b\rangle\rangle + y\langle\langle x, b\rangle\rangle$. It is clear that $c_{x,y}.B^* \subseteq B^*\varpi$ by the condition in the lemma. Therefore $(u_{x,y} - 1).B^*\varpi^{-1} = -2c_{x,y}(1 + c_{x,y})^{-1}.B^*\varpi^{-1} = -2c_{x,y}.B^*\varpi^{-1} \subseteq B^*$. Thus $u_{x,y}$ belongs to K_B''. □

5.2.

LEMMA. *Let ψ be a character of F with conductor $\mathfrak{p}_F^{\lambda_F}$. Suppose that $w \in B\varpi^{-1}$ and $\psi(\langle\langle w, c(k).w\rangle\rangle) = 1$ for all $k \in K_B''$. Then w belongs to B^*.*

PROOF. Here we consider a new dual pair $(U(\mathscr{V}_a), U(\mathscr{V}_a'))$ in $Sp(\mathscr{W})$ where $\mathscr{V}_a := \mathscr{W}$ with the form $\langle\,,\rangle_a := \langle\langle\,,\rangle\rangle$ and $\mathscr{V}_a' := F$ with the form $\langle\,,\rangle_a$ given by $\langle t_1, t_2\rangle_a' := t_1t_2$. Let $\kappa_a := \lambda$ and $\kappa_a' := 0$. It is clear that $\langle\langle\,,\rangle\rangle$ is equal to $\langle\,,\rangle_a \otimes \langle\,,\rangle_a'$. Identify $B\varpi^{-1}$ with $\mathrm{Hom}_{\mathcal{O}}(B^*\varpi^{1-\lambda}, \mathcal{O})$. Suppose that x_0 is any element in B. Let $x := x_0$ and $y := x_0\varpi^{1-\lambda}$. Then x, y satisfy the condition in Lemma 5.1. Therefore $u_{x,y}$ is defined and belongs to K_B''. Now $c(u_{x,y}) = c_{x,y}$ from subsection 4.5. Then $\psi(\langle\langle w, c_{x,y}.w\rangle\rangle) = 1$ by the assumption. Therefore $\langle\langle w, c_{x,y}.w\rangle\rangle \in \mathfrak{p}_F^{\lambda_F}$. Then we get $\langle w.x, w.y\rangle_a' \in \mathfrak{p}_F^{\lambda_F}$ by Lemma 4.10(ii). Hence $\langle w.x_0\varpi^{1-\lambda}, w.x_0\varpi^{1-\lambda}\rangle_a'$ is in $\mathfrak{p}_F^{\lambda_F}\varpi^{1-\lambda} = \mathfrak{p}_F$. So $(w.x_0\varpi^{1-\lambda})^2 \in \mathfrak{p}_F \subseteq \mathfrak{p}$. Now $w.x_0\varpi^{1-\lambda} \in \mathcal{O}$ by the assumption that $w \in B\varpi^{-1}$. Therefore $w.x_0\varpi^{1-\lambda}$ is in $\mathfrak{p}$ because $\mathfrak{p}$ is a prime ideal of the ring $\mathcal{O}$. Since x_0 is arbitrary in B, we conclude that w is in $\mathrm{Hom}_{\mathcal{O}}(B\varpi^{-\lambda}, \mathcal{O})$. Hence w belongs to B^*. □

5.3.

PROPOSITION. *Let $(U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair and L (resp. L') be a good lattice in $\mathscr{V}$ (resp. $\mathscr{V}'$).*

(i) *If* $w \in B\varpi^{-1}$, *then* K''_B *stabilizes the space* $\mathscr{S}(B)_w$.
(ii) *If* $w \in B\varpi^{-1}$ *and* K''_B *fixes an nonzero element in* $\mathscr{S}(B)_w$, *then* w *belongs to* B^*.
(iii) $\mathscr{S}(B)^{K'_B} = \mathscr{S}(B)^{K''_B} = \mathscr{S}(B)_{B^*}$.
(iv) K_B/K'_B *acts on* $\mathscr{S}(B)_{B^*}$ *and the action is isomorphic to* $\overline{\omega}_{\bar{\psi}}$ *where* $\overline{\omega}_{\bar{\psi}}$ *is the Weil representation of* $Sp(B^*/B)$.

Proof. If k belongs to K''_B and w belongs to $B\varpi^{-1}$, then $k^{-1}.w$ is in $B + w$ by the definition of K''_B. Let f be a nonzero element in $\mathscr{S}(B)_w$. Now we have $\omega_\psi(k).f = \psi(\langle\langle w, c(k).w\rangle\rangle)f$ from Lemma 4.10(i). Hence (i) is proved.

If K''_B fixes a nonzero element f in $\mathscr{S}(B)_w$, then from (i) we know that $\psi(\langle\langle w, c(k).w\rangle\rangle) = 1$ for all $k \in K''_B$. So (ii) follows from Lemma 5.2.

Now we prove (iii). It is known that $\mathscr{S}(B)^{K'_B}$ is contained in $\mathscr{S}(B)^{K''_B}$ because K''_B is a subgroup of K'_B. Let $(U(\mathscr{V}_a), U(\mathscr{V}_a'))$ be the reductive dual pair considered in the proof of Lemma 5.2. Then $U(\mathscr{V}_a')$ is just the group of two elements and is contained in the center of $Sp(\mathscr{W})$. Let $M := B^*\varpi$ and $N := B$. Then the group $K_{M,N}$ defined in [**Wp**] I.16 is equal to K''_B, and the lattice $B_{M,N}$ defined in [**Wp**] II.6 is equal to $B\varpi^{-1}$. Let $H := K_{M,N}$ and θ be the trivial representation of H. Then corollaire III.2 of [**Wp**] becomes $\mathscr{S}(B)^{K''_B} = \omega_\psi(\mathscr{H}_a').(\mathscr{S}(B)_{B\varpi^{-1}})^{K''_B} = (\mathscr{S}(B)_{B\varpi^{-1}})^{K''_B}$. Now $(\mathscr{S}(B)_{B\varpi^{-1}})^{K''_B} \subseteq \mathscr{S}(B)_{B^*}$ by part (ii). On the other hand, if k is in K'_B, w is in B^* and f is in $\mathscr{S}(B)_w$, then it is clear that $\omega_\psi(k).f = \psi((1/2)\langle\langle (k-1).w, w\rangle\rangle)f = f$ since $(k-1).w \in B$. Therefore $\mathscr{S}(B)_{B^*}$ is contained in $\mathscr{S}(B)^{K'_B}$. Thus we have proved that

$$\mathscr{S}(B)^{K'_B} \subseteq \mathscr{S}(B)^{K''_B} \subseteq \mathscr{S}(B)_{B^*} \subseteq \mathscr{S}(B)^{K'_B}. \tag{5.3.a}$$

Hence (iii) is proved.

From (1.5.c), it is easy to see that K_B acts on the space $\mathscr{S}(B)_{B^*}$. Let Ω be the map from $\mathscr{S}(B)_{B^*}$ to S given by $f \mapsto f(0)$ where S is defined in subsection 1.5. Obviously Ω is an isomorphism of vector spaces. Now we have $\Omega(\omega_\psi(k).f) = (\omega_\psi(k).f)(0) = \tilde{\omega}_\psi(k).(f(0)) = \tilde{\omega}_\psi(k).(\Omega(f))$ for any $k \in K_B$ by the definition of Ω and (1.5.c). Since K'_B acts trivially on $\mathscr{S}(B)_{B^*}$, we can regard $\mathscr{S}(B)_{B^*}$ as a representation of K_B/K'_B. It is clear that Ω is an isomorphism of K_B/K'_B-representations. $\square$

5.4.

Lemma. *Let* L (*resp.* L') *be a good lattice in* $\mathscr{V}$ (*resp.* $\mathscr{V}'$) *and* $B := B(L, L')$. *Then we have* $\iota_{\mathscr{V}'}(G_L) \cap K'_B = \iota_{\mathscr{V}'}(G_{L,0^+})$ *and* $\iota_{\mathscr{V}}(G'_{L'}) \cap K'_B = \iota_{\mathscr{V}}(G'_{L',0^+})$.

Proof. From the discussion in [**Wp**] I.15, we know that there exists a decomposition $\mathscr{V} = X \oplus Y$ such that $L = L^X \oplus L^Y$ and $L^* = L^X \oplus L^Y\varpi^{-1}$, where $L^X := L \cap X$, $L^Y := L \cap Y$. Let $\mathscr{V}' = X' \oplus Y'$ be the similar decomposition such that $L' = L'^{X'} \oplus L'^{Y'}$ and $L'^* = L'^{X'} \oplus L'^{Y'}\varpi^{-1}$ where $L'^{X'} := L' \cap X'$, $L'^{Y'} := L' \cap Y'$. Then we have

$$B^* = (L^X \otimes L'^{X'}) \oplus (L^X \otimes L'^{Y'})\varpi^{-1} \oplus (L^Y \otimes L'^{X'})\varpi^{-1} \oplus (L^Y \otimes L'^{Y'})\varpi^{-1},$$

$$B = (L^X \otimes L'^{X'}) \oplus (L^X \otimes L'^{Y'}) \oplus (L^Y \otimes L'^{X'}) \oplus (L^Y \otimes L'^{Y'})\varpi^{-1}.$$

It is easy to check that $\iota_{\mathscr{V}'}(G_{L,0^+}) \subseteq K'_B$. Hence $\iota_{\mathscr{V}'}(G_{L,0^+}) \subseteq \iota_{\mathscr{V}'}(G_L) \cap K'_B$. On the other hand, if $g \in G_L$ and $\iota_{\mathscr{V}'}(g) \in K'_B$, then we have $(g-1).L^X \subseteq L^X\varpi \oplus L^Y$ and $(g-1).L^Y \subseteq L^X\varpi \oplus L^Y\varpi$. Hence $g \in G_{L,0^+}$. Therefore $\iota_{\mathscr{V}'}(G_{L,0^+}) = \iota_{\mathscr{V}'}(G_L) \cap K'_B$. The proof for $\iota_{\mathscr{V}}(G'_{L'}) \cap K'_B = \iota_{\mathscr{V}}(G'_{L',0^+})$ is similar. $\square$

5.5.

It is easy to check that B^*/B is isomorphic to $(\boldsymbol{l}^* \otimes_{\boldsymbol{d}} \boldsymbol{l}') \times (\boldsymbol{l}'^* \otimes_{\boldsymbol{d}} \boldsymbol{l})$ where $\boldsymbol{l}, \boldsymbol{l}^*, \boldsymbol{l}', \boldsymbol{l}'^*$ are defined in subsection 1.2, and the quotient $G_L/G_{L,0^+}$ is isomorphic to $U(\boldsymbol{l}) \times U(\boldsymbol{l}^*)$, and $G'_{L'}/G'_{L',0^+}$ is isomorphic to $U(\boldsymbol{l}') \times U(\boldsymbol{l}'^*)$.

Proposition. *The pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$ is a (possibly reducible) reductive dual pair in the finite symplectic group $Sp(B^*/B)$.*

Proof. Now $\mathscr{V}$ (resp. $\mathscr{V}'$) is an ε-Hermitian (resp. ε'-Hermitian) space over D with $\varepsilon\varepsilon' = -1$. We know that $\boldsymbol{l}$ (resp. $\boldsymbol{l}^*$, $\boldsymbol{l}'$, $\boldsymbol{l}'^*$) is a (non-degenerate) μ (resp. μ^*, μ', μ'^*)-Hermitian space over $\boldsymbol{f}_D$ for some μ (resp. μ^*, μ', μ'^*) equal to 1 or -1. To prove this theorem, we need to check that $\mu\mu'^* = -1$, $\mu^*\mu' = -1$ and

$$\langle\!\langle\, , \rangle\!\rangle_{\boldsymbol{b}^*} = \mathrm{Trd}_{\boldsymbol{f}_D/\boldsymbol{f}_F}((\langle\, , \rangle_{\boldsymbol{l}^*} \otimes \bar{\tau} \circ \langle\, , \rangle'_{\boldsymbol{l}'}) \oplus (\langle\, , \rangle_{\boldsymbol{l}} \otimes \bar{\tau} \circ \langle\, , \rangle'_{\boldsymbol{l}'^*})) \tag{5.5.a}$$

where $\bar{\tau}$ denotes the involution of $\boldsymbol{f}_D$ over $\boldsymbol{f}_F$ induced from τ i.e., $\bar{\tau}$ is given by $\bar{\tau}(\Pi_{\mathcal{O}}(t)) := \Pi_{\mathcal{O}}(\tau(t))$ for $t \in \mathcal{O}$. Consider two elements $w_1 := l_1^* \otimes l_1' + l_1 \otimes l_1'^*$, $w_2 := l_2^* \otimes l_2' + l_2 \otimes l_2'^*$ in B^* where $l_i \in L$, $l_i^* \in L^*$, $l_i' \in L'$ and $l_i'^* \in L'^*$. Now

$$\begin{aligned}
&\langle\!\langle \Pi_{B^*}(w_1), \Pi_{B^*}(w_2) \rangle\!\rangle_{\boldsymbol{b}^*} \\
&\quad = \Pi_{\mathcal{O}_F}(\langle\!\langle w_1, w_2 \rangle\!\rangle \varpi_F^{1-\lambda_F}) \\
&\quad = \Pi_{\mathcal{O}_F}(\langle\!\langle l_1^* \otimes l_1', l_2^* \otimes l_2' \rangle\!\rangle \varpi_F^{1-\lambda_F}) + \Pi_{\mathcal{O}_F}(\langle\!\langle l_1 \otimes l_1'^*, l_2^* \otimes l_2' \rangle\!\rangle \varpi_F^{1-\lambda_F}) \\
&\qquad + \Pi_{\mathcal{O}_F}(\langle\!\langle l_1^* \otimes l_1', l_2 \otimes l_2'^* \rangle\!\rangle \varpi_F^{1-\lambda_F}) + \Pi_{\mathcal{O}_F}(\langle\!\langle l_1 \otimes l_1'^*, l_2 \otimes l_2'^* \rangle\!\rangle \varpi_F^{1-\lambda_F}) \\
&\quad = \Pi_{\mathcal{O}_F}(\langle\!\langle l_1^* \otimes l_1', l_2^* \otimes l_2' \rangle\!\rangle \varpi_F^{1-\lambda_F}) + \Pi_{\mathcal{O}_F}(\langle\!\langle l_1 \otimes l_1'^*, l_2 \otimes l_2'^* \rangle\!\rangle \varpi_F^{1-\lambda_F}).
\end{aligned}$$

The last equality is due to the facts $\Pi_{\mathcal{O}_F}(\langle\!\langle l_1 \otimes l_1'^*, l_2^* \otimes l_2' \rangle\!\rangle \varpi_F^{1-\lambda_F}) \in \mathfrak{p}_F^{\lambda_F}$ and $\Pi_{\mathcal{O}_F}(\langle\!\langle l_1^* \otimes l_1', l_2 \otimes l_2'^* \rangle\!\rangle \varpi_F^{1-\lambda_F}) \in \mathfrak{p}_F^{\lambda_F}$. We have

$$\begin{aligned}
&\Pi_{\mathcal{O}_F}(\langle\!\langle l_1^* \otimes l_1', l_2^* \otimes l_2' \rangle\!\rangle \varpi_F^{1-\lambda_F}) \\
&\quad = \Pi_{\mathcal{O}_F}(\mathrm{Trd}_{D/F}(\langle l_1^*, l_2^* \rangle \tau(\langle l_1', l_2' \rangle')) \varpi_F^{1-\lambda_F}) \\
&\quad = \mathrm{Trd}_{\boldsymbol{f}_D/\boldsymbol{f}_F}(\Pi_{\mathcal{O}}(\langle l_1^*, l_2^* \rangle \tau(\langle l_1', l_2' \rangle') \varpi^{1-\lambda})) \\
&\quad = \mathrm{Trd}_{\boldsymbol{f}_D/\boldsymbol{f}_F}(\Pi_{\mathcal{O}}(\langle l_1^*, l_2^* \rangle \varpi^{1-\kappa}) \bar{\tau}(\Pi_{\mathcal{O}}(\langle l_1', l_2' \rangle') \varpi^{-\kappa'}))) \\
&\quad = \mathrm{Trd}_{\boldsymbol{f}_D/\boldsymbol{f}_F}(\langle \Pi_{L^*}(l_1^*), \Pi_{L^*}(l_2^*) \rangle_{\boldsymbol{l}^*} \bar{\tau}(\langle \Pi_{L'}(l_1'), \Pi_{L'}(l_2') \rangle_{\boldsymbol{l}'}')).
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&\Pi_{\mathcal{O}_F}(\langle\!\langle l_1 \otimes l_1'^*, l_2 \otimes l_2'^* \rangle\!\rangle \varpi_F^{1-\lambda_F}) \\
&\quad = \mathrm{Trd}_{\boldsymbol{f}_D/\boldsymbol{f}_F}(\langle \Pi_L(l_1), \Pi_L(l_2) \rangle_{\boldsymbol{l}} \bar{\tau}(\langle \Pi_{L'^*}(l_1'^*), \Pi_{L'^*}(l_2'^*) \rangle_{\boldsymbol{l}'^*}')).
\end{aligned}$$

Hence (5.5.a) is true.

Now we check $\mu\mu'^* = -1$ and $\mu^*\mu' = -1$. All the information is in the table in subsection 1.2. First suppose that $D = F$. Then $\boldsymbol{f}_D = \boldsymbol{f}_F$, $\mu = \mu^* = \varepsilon$, $\mu' = \mu'^* = \varepsilon'$. Hence $\mu\mu'^* = \mu^*\mu' = \varepsilon\varepsilon' = -1$. If D is a unramified quadratic extension of F, then $\boldsymbol{f}_D$ is a quadratic extension of $\boldsymbol{f}_F$, $\mu = \mu^* = \varepsilon$, $\mu' = \mu'^* = \varepsilon'$. Again $\mu\mu'^* = \mu^*\mu' = \varepsilon\varepsilon' = -1$. If D is a ramified quadratic extension of F, then $\boldsymbol{f}_D = \boldsymbol{f}_F$, $\mu^* = (-1)^{\kappa+1}\varepsilon$, $\mu = (-1)^{\kappa}\varepsilon$, $\mu'^* = (-1)^{\kappa'+1}\varepsilon'$, and $\mu' = (-1)^{\kappa'}\varepsilon'$. Note that $\kappa + \kappa' = 2\lambda_F + 1$ is odd in this case. Thus $\kappa + \kappa' + 1$ is even. Hence $\mu\mu'^* = \mu^*\mu' = (-1)^{\kappa+\kappa'+1}\varepsilon\varepsilon' = -1$. If D is a quaternion algebra of F, then again $\kappa + \kappa' = 2\lambda_F + 1$. Without loss of generality, we assume that κ is even and κ' is odd. Then $\mu^* = -\varepsilon$, $\mu = \varepsilon$, $\mu'^* = \varepsilon'$, and $\mu' = -\varepsilon'$. Therefore $\mu\mu'^* = \varepsilon\varepsilon' = -1$ and $\mu^*\mu' = (-\varepsilon)(-\varepsilon') = -1$. □

Suppose $\mathscr{V}, \mathscr{V}''$ are two ε-Hermitian spaces in the same Witt series. Let L, L'' be good lattices in $\mathscr{V}$, $\mathscr{V}''$ respectively. Then it is clear that the spaces L^*/L and L''^*/L'' (resp. $L/L^*\varpi$ and $L''/L''^*\varpi$) are in the same Witt tower.

5.6.

In this subsection, we prove our second main result of this paper.

Theorem. *Let $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair. Let (π, V) (resp. (π', V')) be an irreducible admissible representation of $\widetilde{U(\mathscr{V})}$ (resp. $\widetilde{U(\mathscr{V}')}$) of depth zero such that two representations are paired by the local theta correspondence. Suppose that π has a minimal K-type (G_L, ζ). Then π' has a minimal K-type $(G'_{L'}, \zeta')$ such that ζ and ζ' are paired in the theta correspondence for the dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$ in $Sp(B^*/B)$ and the Weil representation on $\mathscr{S}^{K_B'}$ where $B := B(L, L')$ is defined in subsection 4.1.*

Proof. By Theorem 4.2 we know that

$$\mathscr{S}(B)^{G_{L,0^+}} = \omega_\psi(\mathscr{H}').\left(\sum_{L' \in \mathscr{Q}(\Gamma'_{\mathrm{M}})} \mathscr{S}(B)_{B(L,L')^*}\right)$$

where $\mathscr{Q}(\Gamma'_{\mathrm{M}})$ is the set of good lattices contained in Γ'_{M} and $\mathscr{S}(B)$ is the generalized lattice model of the Weil representation of $\widetilde{Sp(\mathscr{W})}$ with respect to the good lattice B. Let Π be the projection $\mathscr{S}(B) \to V \otimes_C V'$. Let $W \subseteq V$ denote the representation space of ζ. Because $V^{G_{L,0^+}}$ is nontrivial, we have a nontrivial surjective map $\mathscr{S}(B)^{G_{L,0^+}} \to V^{G_{L,0^+}} \otimes_C V'$. Then there is a nonzero element $f \in \mathscr{S}(B)_{B(L,L')^*}$ for some good lattice $L' \subseteq \Gamma'_{\mathrm{M}}$ such that $\Pi(f)$ is not zero and $\Pi(f)$ is in the space $W \otimes_C V'$. But we know that $\Pi(f)$ is fixed by $G'_{L',0^+}$ by Proposition 4.1. Therefore $\Pi(f)$ is in $V^{G_{L,0^+}} \otimes_C V'^{G'_{L',0^+}}$. Hence $V'^{G'_{L',0^+}}$ is not trivial. Let (ζ', W') be an irreducible representation of $G'_{L'}/G'_{L',0^+}$ such that $\Pi(f)$ has a nonzero image in $W \otimes_C W'$. We know $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$ forms a finite reductive dual pair in $Sp(B^*/B)$ from Proposition 5.5. By Proposition 5.3(iii) and 5.3(iv), we know that $\mathscr{S}(B)_{B^*} = \mathscr{S}(B)^{K'_B}$ and the action of K_B/K'_B on $\mathscr{S}(B)_{B^*}$ is the Weil representation of $Sp(B^*/B)$ with respect to the nontrivial character $\bar{\psi}$ of $\boldsymbol{f}_F$. Now we have a nontrivial map from $\mathscr{S}(B)_{B^*}$ to $W \otimes_C W'$. Hence the two representations ζ and ζ' are paired by the theta correspondence for the reductive dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$. □

6. Correspondence of Iwahori-spherical representations.

We begin to give a few applications of Theorem 5.6. An irreducible admissible representation of a p-adic reductive group is called *Iwahori-spherical* if it admits a nontrivial vector fixed by an Iwahori subgroup. The following theorem generalizes [**Ab**] corollaire 4.3, where the result for the cases of unramified reductive dual pairs is proved.

Theorem. *Let $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair. Let I (resp. I') be an Iwahori subgroup of $U(\mathscr{V})$ (resp. $U(\mathscr{V}')$), and (π, V) (resp. (π', V')) be an irreducible admissible representation of $\widetilde{U(\mathscr{V})}$ (resp. $\widetilde{U(\mathscr{V}')}$) such that π, π' are paired by local theta correspondence. Then V^I is nontrivial if and only if $V'^{I'}$ is nontrivial i.e., an irreducible Iwahori-spherical representations corresponds to an irreducible Iwahori-spherical representation.*

Proof. First we assume that V^I is nontrivial. Therefore there is a good lattice L in $\mathscr{V}$ such that $V^{G_{L,0^+}}$ is nontrivial. Let ζ be an irreducible subrepresentation of $G_L/G_{L,0^+}$ occurring in $V^{G_{L,0^+}}$. By Theorem 5.6, there exists a good lattice L' in $\mathscr{V}'$ such that $V'^{G_{L',0^+}}$ is nontrivial and there exists an irreducible subrepresentation ζ' of $G'_{L'}/G'_{L',0^+}$ on $V'^{G'_{L',0^+}}$ such that ζ, ζ' are paired by the theta correspondence for the finite reductive dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$. Because V^I is nontrivial, ζ, as a representation of $G_L/G_{L,0^+}$, can be chosen to be an irreducible subrepresentation occurring in the induced representation $\mathrm{Ind}_T^{G_L/G_{L,0^+}}$ triv where T is the Levi factor of the Borel subgroup $I/G_{L,0^+}$ of $G_L/G_{L,0^+}$ and "triv" denote the trivial character of T. By the induction principle of the theta correspondence for finite reductive dual pairs (*cf.* [**AMR**], théorèm 3.7), the representation ζ' must be a subquotient of $\mathrm{Ind}_{T'}^{G'_{L'}/G'_{L',0^+}}$ triv for the trivial character of the Levi factor T' of the Borel subgroup of $I'/G'_{L',0^+}$ of $G'_{L'}/G'_{L',0^+}$. By Frobenius reciprocity, the representation ζ' restricting to the Borel subgroup $I'/G'_{L',0^+}$ contains a trivial representation. Therefore $V'^{G'_{L',0^+}}$, as a representation of $I'/G'_{L',0^+}$, contains the trivial representation. Hence $V'^{I'}$ is nontrivial. Therefore we have proved one direction of the theorem. By symmetry the other direction is also true. □

7. Correspondence of unipotent representations.

Our next consequence concerns the unipotent representations of p-adic reductive groups defined by G. Lusztig. First we recall some definitions for representations of finite reductive groups. Let H be a finite reductive group, T be a maximal torus in H, and θ be a character of T. P. Deligne and G. Lusztig define in [**DL**] a virtual character $R_T^H(\theta)$ of H. An irreducible representation of H is called *unipotent* if its character occurs as a constituent of $R_T^H(\mathrm{triv})$ for some T. Let $U(\mathscr{V})$ be a reductive p-adic group occurring as a member of a reductive dual pair. An irreducible admissible representation (π, V) of $\widetilde{U(\mathscr{V})}$ is called *unipotent* if $V^{G_{L,0^+}}$ is nontrivial for some good lattice L in $\mathscr{V}$ and $V^{G_{L,0^+}}$ contains a unipotent representation of the reductive finite group $G_L/G_{L,0+}$ (*cf.* [**Lt**], 1.5).

Theorem. *Let $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair. Let (π, V) (resp. (π', V')) be an irreducible admissible representation of $\widetilde{U(\mathscr{V})}$ (resp. $\widetilde{U(\mathscr{V}')}$) such that π and π' are paired by the local theta correspondence. For simplicity, we assume that $\varepsilon' = 1$. Suppose that the dual pair $(U(\mathscr{V}), U(\mathscr{V}'))$ satisfies one of the following conditions*:

(i) *D is F, the characteristic of $\boldsymbol{f}_F$ is large (see the proof), and $U(\mathscr{V}')$ is of type* D_m, ${}^2\mathrm{D}_m$, *or* ${}^2\mathrm{D}'_m$ (*cf.* [**Tt**], *pp.* 60–66) *for some m,*

(ii) *D is an unramified quadratic extension of F, no other restriction,*

(iii) *D is a ramified quadratic extension of F, the characteristic of $\boldsymbol{f}_F$ is large, and the dimensions of $\mathscr{V}, \mathscr{V}'$ are even,*

(iv) *D is a central quaternion algebra, the characteristic of $\boldsymbol{f}_F$ is large, and $U(\mathscr{V})$ is of type ${}^2\mathrm{D}''_m$ for some integer m.*

Then π is unipotent if and only if π' is unipotent.

Proof. Suppose that π is unipotent representation. So $V^{G_{L,0^+}}$ is nontrivial for some good lattice L in $\mathscr{V}$. Let ζ be a unipotent subrepresentation of $G_L/G_{L,0^+}$ on the space $V^{G_{L,0^+}}$. By Theorem 5.6, we know that there exists a good lattice L' in $\mathscr{V}'$ such that $V'^{G'_{L',0^+}}$ is nontrivial and there is a subrepresentation ζ' of $G'_{L'}/G'_{L',0^+}$ on $V'^{G'_{L',0^+}}$ such that ζ, ζ' are paired by the theta correspondence for the finite reductive dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$. J. Adams and A. Moy ([**AM**], theorem 3.5) prove that a unipotent representation corresponds to a unipotent representation for the following finite reductive dual pairs:

(1) $(Sp(\boldsymbol{v}), O(\boldsymbol{v}'))$ where the dimension of $\boldsymbol{v}'$ is even and the characteristic of $\boldsymbol{f}_F$ is large enough such that every maximal torus T in $(Sp(\boldsymbol{v})$ or $O(\boldsymbol{v}'))$ satisfies the condition that T/Z has at least two regular orbits under the Weyl group action where Z is the center of the group,

(2) $(U(\boldsymbol{v}), U(\boldsymbol{v}'))$, a dual pair of finite unitary groups.

Therefore we only need to check that the dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$ under the restrictions in (i), (ii), (iii), (iv) are a sum of irreducible reductive dual pairs of the above two types. First suppose now we are in the case (i) i.e., $U(\mathscr{V}')$ is of type D_m, ${}^2\mathrm{D}_m$, or ${}^2\mathrm{D}'_m$ for some m. Then it is clear that both $U(\boldsymbol{l}')$ and $U(\boldsymbol{l}'^*)$ are orthogonal groups from the table in subsection 1.2, and both $\boldsymbol{l}$ and $\boldsymbol{l}^*$ are even-dimensional for any good lattice L in $\mathscr{V}'$. Therefore $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$ is a dual pair of a sum of two irreducible dual pairs satisfying (1).

To check the cases (ii), (iii), (iv) is analogous. So we just sketch the proof. If we are in case (ii), then all groups $U(\boldsymbol{l})$, $U(\boldsymbol{l}^*)$, $U(\boldsymbol{l}')$, $U(\boldsymbol{l}'^*)$ are unitary groups. If we are in case (iii), then exact one of $U(\boldsymbol{l})$, $U(\boldsymbol{l}^*)$ (resp. $U(\boldsymbol{l}')$, $U(\boldsymbol{l}'^*)$) is an orthogonal group. And this orthogonal group is of even variables if and only if the dimension of $\mathscr{V}$ (resp. $\mathscr{V}'$) is even. For case (iv), $U(\boldsymbol{l})$ is an orthogonal group. This orthogonal group is of even variables if and only if $U(\mathscr{V})$ is of type ${}^2\mathrm{D}''_m$ for some m. Hence this theorem is proved. $\square$

8. Admissible splitting of metaplectic covers.

From this section to the end of this paper we will assume that D is commutative. It is known that most reductive dual pairs are split. In this section, we want to discuss the issue of splitting of the metaplectic covers of reductive dual pairs in this section.

8.1. Ranga-Rao cocycle and its splitting.

Let $(U(\mathscr{V}), U(\mathscr{V}'))$ be a reductive dual pair such that D is commutative and $\mathscr{W} := \mathscr{V} \otimes \mathscr{V}'$. Let $\mathscr{W} = X + Y$ be a complete polarization. Let $M_Y[g]$ be the action of the Schrödinger model. The cocycle $c_Y(g, g')$ defined as in (3.3.a) with respect to $M_Y[g]$ is called the *Ranga-Rao cocycle* and it is computed explicitly in [**RR**]. An explicit admissible splitting (if it exists) of $\widetilde{U(\mathscr{V})}$ with respect to the

Ranga-Rao cocycle is given in [**Kl**] theorem 3.1 by S. Kudla. We will use the notation $\beta^Y(g)$ to denote this splitting.

8.2. Splitting with respect to a generalized lattice model.

Let L (resp. L') be a good lattice in $\mathscr{V}$ (resp. $\mathscr{V}'$). Define $B := B(L, L')$ as in subsection 4.1. Let $c_B(g, g')$ denote the cocycle with respect to the generalized lattice model $(M_B[g], \mathscr{S}(B))$ of the Weil representation of $Sp(\mathscr{W})$. An admissible splitting β^L of the cocycle $c_B|_{\iota_{\mathscr{V}'}(U(\mathscr{V}))\times\iota_{\mathscr{V}'}(U(\mathscr{V}))}$ is given in [**Pn1**]. We shall normalize β^L such that $\beta^L|_{U(\mathscr{V})_L} = \xi_L$ where ξ_L is a character of $U(\mathscr{V})_L$ inflated from the character of $\xi \otimes \xi^*$ of $U(\boldsymbol{l}) \times U(\boldsymbol{l}^*)$ via the homomorphism $U(\mathscr{V})_L \to U(\boldsymbol{l}) \times U(\boldsymbol{l}^*)$ and ξ, ξ^* are defined as follows.

(i) If $D = F$, let ξ (resp. ξ^*) be the trivial character of $U(\boldsymbol{l})$ (resp. $U(\boldsymbol{l}^*)$).

(ii) Suppose that D is an unramified extension of F. Let ξ be the trivial character of $U(\boldsymbol{l})$ if $\boldsymbol{l}'^*$ is even-dimensional, be the sgn character of $U(\boldsymbol{l})$ if $\boldsymbol{l}'^*$ is odd-dimensional. Let ξ^* be the trivial character of $U(\boldsymbol{l}^*)$ if $\boldsymbol{l}'$ is even-dimensional, be the sgn character of $U(\boldsymbol{l}^*)$ if $\boldsymbol{l}'$ is odd-dimensional.

(iii) Suppose that D is a ramified extension of F. Then one of $\boldsymbol{l}$, $\boldsymbol{l}^*$ is a quadratic space and the other is a symplectic space. When $SU(\boldsymbol{l})/[U(\boldsymbol{l}), U(\boldsymbol{l})]$ (resp. $SU(\boldsymbol{l}^*)/[U(\boldsymbol{l}^*), U(\boldsymbol{l}^*)]$) is nontrivial, let ξ (resp. ξ^*) be a character of order two of $U(\boldsymbol{l})$ (resp. $U(\boldsymbol{l}^*)$) whose restriction to $SU(\boldsymbol{l})$ (resp. $SU(\boldsymbol{l}^*)$) is also nontrivial. Let ξ, ξ^* be the trivial characters otherwise.

From the definition, it is clear that $\xi = \xi^{-1}$, $\xi^* = (\xi^*)^{-1}$ and $\xi_L = \xi_L^{-1}$. We also note here that the character ξ_L depends only on the parity of the dimension of $\mathscr{V}'$.

8.3. Depth and splitting.

Let L be a good lattice in $\mathscr{V}$. The map $\tilde{\beta}^L : U(\mathscr{V}) \to \widetilde{U(\mathscr{V})}$ given by

$$\tilde{\beta}^L(g) := (\iota_{\mathscr{V}'}(g), \beta^L(g) M_B[\iota_{\mathscr{V}'}(g)]) \tag{8.3.a}$$

is a group homomorphism. Therefore, if π is an irreducible admissible representation of $\widetilde{U(\mathscr{V})}$, then $\pi \circ \tilde{\beta}^L$ is an irreducible admissible representation of $U(\mathscr{V})$. By (1.4.b), (1.5.c) and (8.3.a), we have

$$(\omega_\psi(\tilde{\beta}^L(k)).f)(w) = \xi_L(k)\tilde{\omega}_\psi(\iota_{\mathscr{V}'}(k)).(f(\iota_{\mathscr{V}'}(k)^{-1}.w)) \tag{8.3.b}$$

for $k \in U(\mathscr{V})_L$ and $f \in \mathscr{S}(B)$.

Proposition. *Let $G := U(\mathscr{V})$. Suppose that the extension $\tilde{G} \to G$ splits. Let π be an irreducible admissible representation of $\tilde{G}$. Then π is of depth zero if and only if $\pi \circ \tilde{\beta}^L$ is of depth zero where L is a good lattice in $\mathscr{V}$ such that $G_{L,0^+} \subseteq I$ and I is a fixed Iwahori subgroup of G.*

Proof. Suppose that π is of depth zero. Then π has nontrivial vectors fixed by $\tilde{\beta}_I(G_{L_1,0^+})$ for some good lattice L_1 in $\mathscr{V}$ such that $G_{L_1,0^+} \subseteq I$ for a fixed Iwahori subgroup I where β_I is defined in subsection 3.3. Compare β_I and β^L, we see that we only need to prove that the restriction $\beta^L|_{G_{L_1,0^+}}$ is trivial. From subsection 8.2, we know that $\beta^L|_{G_L} = \xi_L$. Suppose that $I = G_{x_0}$ for some point x_0 in the affine building of G. From the definition in subsection 8.2, it is clear that $\xi_L|_{G_{x_0,0^+}}$ is trivial. Since $G_{L_1,0^+} \subseteq G_{x_0,0^+}$, we see that $\beta^L|_{G_{L_1,0^+}}$ is trivial. Hence π has nontrivial vectors fixed by $\beta_I(G_{L_1,0^+})$ if and only if $\pi \circ \tilde{\beta}^L$ has nontrivial vectors fixed by $G_{L_1,0^+}$. Hence π is of depth zero if and only if $\pi \circ \tilde{\beta}^L$ is of depth zero. $\square$

8.4.

Corollary. *Suppose that $(U(\mathscr{V}), U(\mathscr{V}'))$ is a split reductive dual pair and D is commutative. Suppose that an irreducible admissible representation π of $U(\mathscr{V})$ corresponds to an irreducible admissible representation π' of $U(\mathscr{V}')$ with respect to the splittings β^L and $\beta^{L'}$ for good lattices L, L' in $\mathscr{V}$ and $\mathscr{V}'$ respectively. Then π is of depth zero if and only if π' is of depth zero.*

Proof. The corollary follows from Corollary 4.4 and Proposition 8.3 immediately. $\square$

9. Correspondence of depth zero supercuspidal representations.

In this section, we consider the local theta correspondence of depth zero supercuspidal representations of split reductive dual pairs.

9.1. Depth zero supercuspidal representations of classical groups. Let (π, V) be an irreducible depth zero supercuspidal representation of $G := U(\mathscr{V})$ with a minimal K-types (G_L, ζ) for some good lattice L in $\mathscr{V}$ and an irreducible representation ζ of $G_L/G_{L,0^+}$. We will use the notation $\tilde{\zeta}$ to denote the representation of G_L inflated from ζ of $G_L/G_{L,0^+}$. Then we know that ζ must be cuspidal and $\pi \simeq c\text{-Ind}_{G_L}^G \tilde{\zeta}$ where "c-Ind" denotes the *compact induction*. On the other hand, any irreducible cuspidal representation of $G_L/G_{L,0^+}$ induces an irreducible supercuspidal representation of G (*cf.* [**MP2**], corollary 6.8).

Lemma. *Suppose an irreducible supercuspidal representation (π, V) of G has a minimal K-type (G_L, ζ) for some good lattice L in $\mathscr{V}$. Then the lattice L is unique up to equivalence.*

Proof. Recall that two lattices L_1 and L_2 in $\mathscr{V}$ are said to be equivalent if there exists an element $g \in G$ such that $L_1 = g.L_2$. Note that G_L is a maximal open compact subgroup of G. Suppose that π has another minimal K-type (G_M, ζ') where ζ' is an irreducible cuspidal representation. Moreover, by the associativity of the minimal K-types $G_M/G_{M,0^+}$ must be isomorphic to $G_L/G_{L,0^+}$. Then there exists an element $g \in G$ such that the vertex corresponding to $g.M$ and the vertex corresponding to L are in the same chamber. But $G_L \cap G_{g.M}$ surjects onto $G_L/G_{L,0^+}$ and $G_M/G_{M,0^+}$ by the associativity of minimal K-types defined in [**MP1**] 5.1. Then L must be equal to $g.M$. Therefore L and M are equivalent. □

9.2. Depth zero supercuspidal representations of metaplectic covers.

Let $(U(\mathscr{V}), U(\mathscr{V}'))$ be a split reductive dual pair such that D is commutative. An irreducible admissible representation π of $\widetilde{U(\mathscr{V})}$ is called a *supercuspidal representation* if $\pi \circ \tilde{\beta}$ is a supercuspidal representation of $U(\mathscr{V})$ for some splitting $\tilde{\beta} : U(\mathscr{V}) \to \widetilde{U(\mathscr{V})}$. Since for different splittings $\tilde{\beta}_1$, $\tilde{\beta}_2$ the representations $\pi \circ \tilde{\beta}_1$, $\pi \circ \tilde{\beta}_2$ are different up to a character of $U(\mathscr{V})$, we see the definition is independent of the choice of the splitting $\tilde{\beta}$. As in subsection 9.1, an irreducible depth zero supercuspidal representation of $G := U(\mathscr{V})$ with a minimal K-type (G_L, ζ) is isomorphic to $c\text{-Ind}_{G_L}^{G} \tilde{\zeta}$. Therefore it is not difficult to see that if π is an irreducible depth zero supercuspidal representation of $\tilde{G}$ with minimal K-type (G_L, ζ), then $\pi \circ \tilde{\beta}^L$ is isomorphic to $c\text{-Ind}_{G_L}^{G}(\xi_L \otimes \tilde{\zeta})$ where ξ_L is defined in subsection 8.2.

9.3.

Theorem. *Suppose that $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ is a split reductive dual pair and D is commutative. Suppose that (π, V) (resp. (π', V')) is an irreducible admissible representation of $\widetilde{U(\mathscr{V})}$ (resp. $\widetilde{U(\mathscr{V}')}$) such that the representation $\pi \otimes \pi'$ is a first occurrence in the correspondence. Suppose that π is supercuspidal with a minimal K-type (G_L, ζ). Then π' has a minimal K-type $(G'_{L'}, \zeta')$ such that $\zeta \otimes \zeta'$ is a first occurrence in the correspondence for the finite reductive dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$.*

Proof. By Theorem 5.6, we know that π' has a minimal K-type $(G'_{L'}, \zeta')$ such that ζ and ζ' are paired by the theta correspondence for the finite reductive dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$. Suppose that $\zeta \otimes \zeta'$ is not a first occurrence. Then ζ' is not cuspidal by induction principle. Therefore the $G'_{L'}/G'_{L',0^+}$-representation $V'^{G'_{L',0^+}}$ contains a non-cuspidal component. Then $\pi' \circ \tilde{\beta}^{L'}$ can not be an irreducible supercuspidal representation of $U(\mathscr{V}')$ by [**MP2**] corollary 6.8. But we know that $\pi' \circ \tilde{\beta}^{L'}$ must be an irreducible supercuspidal representation of $U(\mathscr{V}')$ by the induction principle of the local theta correspondence. We get a contradiction. □

9.4.

Next we want to consider the converse of Theorem 9.3. Suppose that we have two finite reductive dual pairs $(U(\boldsymbol{v}), U(\boldsymbol{v}'^{*}))$ and $(U(\boldsymbol{v}^{*}), U(\boldsymbol{v}'))$ satisfying one of the following conditions:

(i) $\boldsymbol{v}$ and $\boldsymbol{v}^{*}$ are symplectic spaces, $\boldsymbol{v}'$ and $\boldsymbol{v}'^{*}$ are orthogonal spaces,

(ii) all $\boldsymbol{v}$, $\boldsymbol{v}^{*}$, $\boldsymbol{v}'$, $\boldsymbol{v}'^{*}$ are Hermitian spaces with respect to a quadratic extension of $\boldsymbol{f}_F$,

(iii) $\boldsymbol{v}$ and $\boldsymbol{v}'$ are symplectic spaces, $\boldsymbol{v}^{*}$ and $\boldsymbol{v}'^{*}$ are orthogonal spaces.

Then it is clear that there exists a field D over F and two spaces $\mathscr{V}$, $\mathscr{V}'$, integers κ, κ', and good lattice L and L' (with respect to the integers κ, κ') in $\mathscr{V}$, $\mathscr{V}'$ respectively such that

$$L/L^{*}\varpi \simeq \boldsymbol{v}, \quad L^{*}/L \simeq \boldsymbol{v}^{*}, \quad L'/L'^{*}\varpi \simeq \boldsymbol{v}', \quad L'^{*}/L' \simeq \boldsymbol{v}'^{*},$$

as Hermitian spaces over $\boldsymbol{f}_D$. This is just an easy consequence of the table in subsection 1.2. Therefore, we have $G_L/G_{L,0^+} \simeq U(\boldsymbol{v}) \times U(\boldsymbol{v}^{*})$ and $G'_{L'}/G'_{L',0^+} \simeq U(\boldsymbol{v}') \times U(\boldsymbol{v}'^{*})$ where $G := U(\mathscr{V})$ and $G' := U(\mathscr{V}')$.

Given a character ϕ of $\boldsymbol{f}$, clearly there exists a character ψ with conductoral exponent $\mathfrak{p}_F^{\lambda_F}$ such that $\bar{\psi} = \phi$ where $\bar{\psi}$ is defined in subsection 1.5. Of course, the character ψ is not unique.

9.5.

The following theorem is the converse of Theorem 9.3.

Theorem. *Suppose that $(G, G') := (U(\mathscr{V}), U(\mathscr{V}'))$ is a split reductive dual pair and D is commutative. Suppose that ζ, ζ' are irreducible cuspidal representations and $\zeta \otimes \zeta'$ is a first occurrence of theta correspondence for the finite reductive dual pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$. Then $\pi \otimes \pi'$ is a first occurrence for the reductive dual pair (G, G') where π is the irreducible admissible representation of $\tilde{G}$ such that $\pi \circ \tilde{\beta}^L \simeq c\text{-}\mathrm{Ind}_{G_L}^{G}(\xi_L \otimes \tilde{\zeta})$ and π' is the irreducible admissible representation of $\tilde{G}'$ such that $\pi' \circ \tilde{\beta}^{L'} \simeq c\text{-}\mathrm{Ind}_{G'_{L'}}^{G'}(\xi_{L'} \otimes \tilde{\zeta}')$ where ξ_L, $\xi_{L'}$ are defined in subsection* 8.2.

Proof. Because ζ and ζ' are irreducible cuspidal representations, $\xi_L \otimes \tilde{\zeta}$ and $\xi_{L'} \otimes \tilde{\zeta}'$ are representations inflated from irreducible cuspidal representations of $G_L/G_{L,0^+}$ and $G'_{L'}/G'_{L',0^+}$ respectively. By proposition 6.5 in [**MP2**], we know that both $c\text{-}\mathrm{Ind}_{G_L}^{G}(\xi_L \otimes \tilde{\zeta})$ and $c\text{-}\mathrm{Ind}_{G'_{L'}}^{G'}(\xi_{L'} \otimes \tilde{\zeta}')$ are irreducible supercuspidal representations. Moreover it is clear that π and π' are uniquely determined by the condition in the theorem. We know that π has the minimal K-type (G_L, ζ). Fix the group G and let G'' be a group in the Witt tower of G'. Suppose that $\pi \otimes \pi''$ is a first occurrence for some irreducible admissible representation π'' of $\tilde{G}''$. We know that π'' must be of depth zero by Corollary 4.4. Therefore by Theorem 9.3, π'' has a minimal K-type $(G''_{L''}, \zeta'')$ for some good lattice L'' in $\mathscr{V}''$ where $U(\mathscr{V}'') := G''$ such that $\zeta \otimes \zeta''$ occurs in the theta correspondence for the dual pair $(G_L/G_{L,0^+}, G''_{L''}/G''_{L'',0^+})$. Moreover we know that $\zeta \otimes \zeta''$ is a first occurrence. Write $G'_{L'}/G'_{L',0^+} \simeq U(\boldsymbol{l}') \times U(\boldsymbol{l}'^*)$ and $G''_{L''}/G''_{L'',0^+} \simeq U(\boldsymbol{l}'') \times U(\boldsymbol{l}''^*)$. Then $U(\boldsymbol{l}')$ and $U(\boldsymbol{l}'')$ (resp. $U(\boldsymbol{l}'^*)$ and $U(\boldsymbol{l}''^*)$) are in the same Witt tower. Therefore $\dim(\boldsymbol{l}') = \dim(\boldsymbol{l}'')$, $\dim(\boldsymbol{l}'^*) = \dim(\boldsymbol{l}''^*)$ and $\zeta' \simeq \zeta''$ because now both $\zeta \otimes \zeta'$ and $\zeta \otimes \zeta''$ are first occurrences. Hence $G' = G''$ and $\pi' = \pi''$. Now π' is an irreducible supercuspidal representation of $\tilde{G}'$ having a minimal K-type $(G_{L'}, \zeta')$. Hence $\pi'' \circ \tilde{\beta}^{L'} \simeq c\text{-Ind}_{G'_{L'}}^{G'}(\xi_{L'} \otimes \tilde{\zeta}')$ as in subsection 9.2. $\square$

Theorem 9.3 and Theorem 9.5 indicate that the theta correspondence of depth zero supercuspidal representations for p-adic reductive dual pairs can be completely described by the theta correspondence of cuspidal representations for finite reductive dual pairs. Here we note that the theta correspondence of cuspidal representations for finite reductive dual pair is one-to-one although the general correspondence for finite reductive dual pair is not. In the following subsections, we give a few examples to illustrate the nice relation of theta correspondence of p-adic and finite reductive dual pairs. These examples may not be entirely new but here we provide more precise information. All three examples are pairs of orthogonal and symplectic groups. The characters ξ_L, $\xi_{L'}$ are trivial in these situations, so we will just ignore them.

9.6.

Example. This example is originally from [**As**] theorem 8.3. Assume that ψ is a character of F with conductoral exponent $\mathfrak{p}_F^{\lambda_F}$ for some integer λ_F. Consider the dual pair $(Sp_4(F), O(\mathscr{V}'))$ where $\mathscr{V}'$ is a two-dimensional anisotropic quadratic space over F such that the corresponding quadratic form is split over a unramified quadratic extension of F. This pair is unramified and split. The sgn representation of $O(\mathscr{V}')$ is supercuspidal and first occurs in the correspondence for this dual pair. Assume it corresponds to the representation (π, V) of $Sp_4(F)$ (under our choice of splitting). Let L' be the unique good lattice in $\mathscr{V}'$. We know L' is self-dual and $G'_{L'}/G'_{L',0^+} \simeq O_2^-(\boldsymbol{f}_F)$, the anisotropic finite orthogonal group in two variables. The sgn representation factors through the first congruence subgroup $G'_{L',0^+}$ and becomes a representation of the finite orthogonal group $O_2^-(\boldsymbol{f}_F)$, still denoted by sgn. For the dual pair $(Sp_4(\boldsymbol{f}_F), O_2^-(\boldsymbol{f}_F))$ the sgn representation corresponds to the representation θ_{10} of $Sp_4(\boldsymbol{f}_F)$, in the notation of [**Sr**]. By Corollary 4.3, there exists a good lattice L in $\mathscr{V}$ such that $V^{G_{L,0^+}}$ is nontrivial. We know that $L^* = L\varpi^m$ where m is 0 or 1 depending on the parity of λ_F. By Theorem 9.5, we have $\pi \simeq c\text{-Ind}_{G_L}^{Sp_4(F)} \tilde{\theta}_{10}$. Note that G_L is conjugate to $Sp_4(\mathcal{O}_F)$ when λ_F is odd. This is the situation in [**As**]. But G_L is not conjugate to $Sp_4(\mathcal{O}_F)$ when λ_F is even.

9.7.

Example. Our next example is also originally from [**As**] but here we are able to provide more information. Assume the character ψ is of conductoral exponent $\mathfrak{p}_F^{\lambda_F}$. Consider the dual pair $(Sp_4(F), O(\mathscr{V}'))$ where $\mathscr{V}'$ is a two-dimensional anisotropic quadratic space over F such that the corresponding quadratic form is split over a ramified quadratic extension of F. We can identify $\mathscr{V}'$ with a ramified quadratic extension E of F with the quadratic form induced from the norm of E over F. We know we have two ramified quadratic extensions of F, namely $F(\sqrt{\varpi})$ and $F(\sqrt{\varepsilon\varpi})$ where ε is a nonsquare unit in $\mathcal{O}_F$. We know this pair is split but not unramified. Again the sgn representation of $O(\mathscr{V}')$ is supercuspidal and first occurs in the correspondence with $Sp_4(F)$. Assume it corresponds to the representation (π, V) of $Sp_4(F)$. Let L' be the unique good lattice in $\mathscr{V}'$. We know that $G'_{L'}/G'_{L',0^+} \simeq O_1 \times O_1$ where O_1 denote the orthogonal group in one variable i.e., the group of two elements. Now the sgn representation factors through the subgroup $G'_{L',0^+}$ and is equivalent to the representation $\mathrm{sgn} \otimes \mathrm{sgn}$ of the finite group $O_1 \times O_1$. We know that $SL_2(\boldsymbol{f}_F)$ has two irreducible $(q-1)/2$-dimensional representations ζ_1 and ζ_2, which are cuspidal. Now consider the theta correspondence for the dual pair $(SL_2(\boldsymbol{f}_F), O_1)$ over the finite field $\boldsymbol{f}_F$ with respect to a character ϕ. Note that the character ϕ depends on the choice of ϖ. We know that the sgn representation of O_1 corresponds to ζ_i where i is 1 or 2 depending on the character ϕ. By Theorem 9.3, $V^{G_{L,0^+}}$ is nontrivial for the unique (up to equivalence) good lattice L in $\mathscr{V}$ such that $L^*\varpi \subsetneq L \subsetneq L^*$. We know that $G_L/G_{L,0^+} \simeq SL_2(\boldsymbol{f}_F) \times SL_2(\boldsymbol{f}_F)$. Then the representation $\mathrm{sgn} \otimes \mathrm{sgn}$ of $G'_{L'}/G'_{L',0^+}$ corresponds to the representation $\zeta_i \otimes \zeta_j$ of $G_L/G_{L,0^+}$ where i, j are 1 or 2. And we know that $i = j$ if $E \simeq F(\sqrt{\varpi})$, and $i \neq j$ if $E \simeq F(\sqrt{\varepsilon\varpi})$. By Theorem 9.5, π is isomorphic to $c\text{-}\mathrm{Ind}_{G_L}^{Sp_4(F)} \widetilde{\zeta_i \otimes \zeta_j}$.

9.8.

Example. Let $\mathscr{V}$ be a four dimensional anisotropic quadratic space over F. Let $G := O_4(F)$ denote the anisotropic orthogonal group on $\mathscr{V}$. Now we want to determine the first occurrence of the sgn representation of $O_4(F)$. There is a unique good lattice L in $\mathscr{V}$ since $\mathscr{V}$ is anisotropic. And $G_L/G_{L,0^+} \simeq O_2^-(\boldsymbol{f}_F) \times O_2^-(\boldsymbol{f}_F)$. The sgn representation of $O_4(F)$ is of depth zero and, as a representation of $G_L/G_{L,0^+} = U(\boldsymbol{l}) \times U(\boldsymbol{l}^*)$, isomorphic to $\mathrm{sgn} \otimes \mathrm{sgn}$. We know that the first occurrence of sgn representation of $O_2^-(\boldsymbol{f}_F)$ is paired with θ_{10} of $Sp_4(\boldsymbol{f}_F)$. Therefore by Theorem 9.5, the first occurrence of sgn representation of $O_4(F)$ is paired with $c\text{-}\mathrm{Ind}_{G'_{L'}}^{G'} \widetilde{\theta_{10} \otimes \theta_{10}}$ where G' is the symplectic group $Sp_8(F)$ over an eight-dimensional space $\mathscr{V}'$ over F, L' is a good lattice on $\mathscr{V}'$ such that both L'^*/L' and $L'/L'^*\varpi$ are four-dimensional spaces over $\boldsymbol{f}_F$. The representation $c\text{-}\mathrm{Ind}_{G'_{L'}}^{G'} \widetilde{\theta_{10} \otimes \theta_{10}}$ is irreducible supercuspidal. We remark here that the first occurrence of the sgn character is already considered by S. Rallis in [**Rl**].

10. "Dual pairs" of modular congruence groups.

10.1. Modular congruence groups.

So far we have seen that the local theta correspondence of depth zero representations is closely connected with the theta correspondence for finite reductive dual pairs. Similar results should hold for representations of positive depths. Suppose that $(U(\mathscr{V}), U(\mathscr{V}'))$ is an unramified reductive dual pair in $Sp(\mathscr{W})$. Let L (resp. L') be a lattice in $\mathscr{V}$ (resp. $\mathscr{V}'$) such that $L^* = L\varpi^m$ (resp. $L'^* = L'\varpi^{m'}$). Define $A := L \otimes L'$. Then A is a lattice in $\mathscr{W}$ such that $A^* = A\varpi^{m+m'}$. Fix a positive integer k. Then we can define ε (resp. ε')-Hermitian form $\langle\,,\rangle_{\boldsymbol{l}_k}$ (resp. $\langle\,,\rangle_{\boldsymbol{l}_k}$) on the free module $\boldsymbol{l}_k := L/L\varpi^k$ (resp. $\boldsymbol{l}_k' := L'/L'\varpi^k$) over the (finite) commutative ring $\mathcal{O}/\mathfrak{p}^k$. We can also define a skew-symmetric form $\langle\!\langle\,,\rangle\!\rangle_{\boldsymbol{a}_k}$ on the free module $\boldsymbol{a}_k := A/A\varpi_F^k$ over $\mathcal{O}_F/\mathfrak{p}_F^k$. It is not difficult to verify that both G_L, $G_{L'}$ are subgroups of K_A. Moreover both $G_L/G_{L,(k-1)^+}$ and $G'_{L'}/G'_{L',(k-1)^+}$ are subgroups of $K_A/K_{A,(k-1)^+}$ where

$$G_{L,(k-1)^+} := \{g \in U(\mathscr{V}) \mid (g-1).L^* \subseteq L\varpi^{k-1}, (g-1).L \subseteq L^*\varpi^k\}$$

and $K_{A,(k-1)^+}$ is defined similarly. Therefore we have a "*modular dual pair*"

$$(G_L/G_{L,(k-1)^+}, G'_{L'}/G'_{L',(k-1)^+}) \tag{10.1.a}$$

in $K_A/K_{A,(k-1)^+}$. We can also define the "Weil representation" of $K_A/K_{A,(k-1)^+}$ as we do for a finite symplectic group. By restricting the "Weil representation", we can define a correspondence between some irreducible representations of the modular congruence group $G_L/G_{L,(k-1)^+}$ and some irreducible representations of $G'_{L'}/G'_{L',(k-1)^+}$. Of course, we cannot expect the correspondence for a modular congruence dual pair to be one-to-one. However this correspondence still provides a way to classify irreducible representations of one group by the irreducible representations of the other group. In fact, J. Shalika and S. Tanaka [**Tn**] have implicitly used the pairs $(SL_2(\boldsymbol{Z}/p^k\boldsymbol{Z}), O_2^+(\boldsymbol{Z}/p^k\boldsymbol{Z}))$ and $(SL_2(\boldsymbol{Z}/p^k\boldsymbol{Z}), O_2^-(\boldsymbol{Z}/p^k\boldsymbol{Z}))$ to construct irreducible representations of the modular congruence group $SL_2(\boldsymbol{Z}/p^k\boldsymbol{Z})$ where $p \in \boldsymbol{Z}$ is a prime number. But from their work, we also know that these two pairs are not enough to construct all irreducible representations of $SL_2(\boldsymbol{Z}/p^k\boldsymbol{Z})$ unless $k = 1$.

10.2. Tanaka's result.

Now we want to consider the third "modular dual pair". It can be constructed from the p-adic reductive dual pair $(Sp(\mathscr{V}), O(\mathscr{V}'))$ where both spaces $\mathscr{V}$ and $\mathscr{V}'$ are two-dimensional and the quadratic form on $\mathscr{V}'$ is not split over F but split over a ramified quadratic extension of F. This dual pair is not unramified. We know that $\mathscr{V}'$ has a unique good lattice L'. Let L be a self-dual lattice in $\mathscr{V}$. Now $A := L \otimes L'$ is a good lattice in $\mathscr{W}$. Again

$(G_L/G_{L,(k-1)^+}, G'_{L'}/G'_{L',(k-1)^+})$ is a "dual pair" in $K_A/K_{A,(k-1)^+}$. The "Weil representation" of $K_A/K_{A,(k-1)^+}$ can be realized on the space $\mathscr{S}(A_0)_{A^*\varpi^{k-1}}$. If $k=1$, then we just get the finite dual pair $(SL_2(\boldsymbol{f}) \times SL_0(\boldsymbol{f}), O_1 \times O_1)$, which is not really interesting. However, if $k>1$, then $G_L/G_{L,(k-1)^+}$ is isomorphic to $SL_2(\mathcal{O}_F/\mathfrak{p}_F^k)$ and the "Schrödinger model of the Weil representation" of $K_A/K_{A,(k-1)^+}$ can be realized as the space of complex valued functions on the set $\mathcal{O}_F/\mathfrak{p}_F^k \times \mathcal{O}_F/\mathfrak{p}_F^{k-1}$. By decomposing the "Weil representation" according to the representations of $G'_{L'}/G'_{L',(k-1)^+}$, which is a group with an abelian subgroup of index 2, we can construct all irreducible representations of $SL_2(\mathcal{O}_F/\mathfrak{p}_F^k)$ that cannot be obtained from the two "modular dual pairs" considered in the previous subsection. This is exactly what S. Tanaka has done in [**Tn**] although not necessarily from this point of view.

11. Theta dichotomy for p-adic unitary groups.

Theta dichotomy is a very interesting phenomenon which concerns the first occurrences in local theta correspondences. A result of theta dichotomy of supercuspidal representations for reductive dual pairs of p-adic unitary groups has been established by M. Harris, S. Kudla, S. Rallis and W. Sweet (*cf.* [**HKS**]). The theta dichotomy depends on the splitting of the metaplectic covers. The splittings used in [**HKS**] are the splittings given in [**Kl**] i.e., the splittings with respect to a Schrödinger model of the Weil representation. In this section, we want to investigate the theta dichotomy for reductive dual pairs of unitary groups by using the splittings with respect to a generalized lattice model. Under this splitting, Theorems 9.3 and 9.5 can be applied and new results of theta dichotomy for finite reductive dual pairs and other p-adic reductive dual pairs are obtained in the next two sections. In this section, let $D=E$ be a quadratic extension of F.

11.1. Kudla and Rallis' result.

Let $E := F(\sqrt{\Delta})$ be a quadratic extension of F. Let $\varepsilon_{E/F}$ denote the quadratic character of $F^\times$ associated to the extension E/F i.e., $\varepsilon_{E/F}(t) := (\Delta, t)_F$ where $(,)_F$ denotes the Hilbert symbol of F. Fix an ε-Hermitian space $\mathscr{V}$ over E. Let $\mathscr{V}'$ be an m-dimensional ε'-Hermitian space such that $\varepsilon\varepsilon' = -1$. Denote the spaces $\mathscr{V}'^{\pm}$, so that

$$\varepsilon_{E/F}((-1)^{m(m-1)/2} \det(\mathscr{V}'^{\pm})) = \pm 1. \tag{11.1.a}$$

Let $\{\mathscr{V}'^{+}_{m_i^+}\}$, $\{\mathscr{V}'^{-}_{m_i^-}\}$ be two Witt towers where m_i^+ (resp. m_i^-) denotes the dimension of $\mathscr{V}'^{+}_{m_i^+}$ (resp. $\mathscr{V}'^{-}_{m_i^-}$). For the dual pair $(U(\mathscr{V}), U(\mathscr{V}'^{\pm}_{m_i^\pm}))$, if a character χ of $E^\times$ such that $\chi|_{F^\times} = \varepsilon_{E/F}^{m_i^\pm}$ is fixed, then a splitting $\tilde{\beta}^{Y^\pm}$: $U(\mathscr{V}) \to \widetilde{U(\mathscr{V})}$ is determined in [**Kl**]. Because $\varepsilon_{E/F}$ is a character of order two, we will fix χ such that $\chi|_{F^\times} = \varepsilon_{E/F}^{m_0^\pm}$ for the whole tower $\{\mathscr{V}'^{\pm}_{m_i^\pm}\}$. The following theorem is described in [**HKS**] p. 944.

Theorem (Kudla-Rallis). *Let $(U(\mathscr{V}), U(\mathscr{V}'^{\pm}))$ be a reductive dual pair of unitary groups. Suppose that π^+, π^- are irreducible supercuspidal representations of $\widetilde{U(\mathscr{V})}$ such that $\pi^+ \circ \tilde{\beta}^{Y^+} = \pi^- \circ \tilde{\beta}^{Y^-}$. Let $\ell_0^{\pm}(\chi)$ be the smallest number $m^{\pm}$, the dimension of $\mathscr{V}'^{\pm}$, such that $\pi^{\pm}$ occurs in the theta correspondence for the dual pair $(U(\mathscr{V}), U(\mathscr{V}'^{\pm}))$. Moreover, suppose that the parity of the dimension of $\mathscr{V}$ and the parity of the dimensions of $\mathscr{V}'^{\pm}$ are the same. Then*

$$\ell_0^+(\chi) + \ell_0^-(\chi) = 2n + 2$$

where n is the dimension of $\mathscr{V}$.

This theorem is called the *preservation principle*. A corollary of the theorem is that there exist sequences of dimensions $n = n_0 < n_1 < n_2 < \cdots < n_i < \cdots$ and irreducible supercuspidal representations $\pi = \pi_0, \pi_1, \pi_2, \ldots, \pi_i, \ldots$ of $U(\mathscr{V}_i^{\pm})$ such that the dimension of $\mathscr{V}_i^{\pm}$ is n_i and $\pi_i \otimes \pi_{i+1}$ occurs in the theta correspondence for the dual pair $(U(\mathscr{V}_i^{\pm}), U(\mathscr{V}_{i+1}^{\mp}))$ with respect to the splittings $\beta^{Y_i^{\pm}}$, $\beta^{Y_{i+1}^{\mp}}$. (Here the signs of these $\mathscr{V}_i^{\pm}$'s are alternating i.e., if $\mathscr{V}_i^{\pm} = \mathscr{V}_i^{+}$, then $\mathscr{V}_{i+1}^{\mp} = \mathscr{V}_{i+1}^{-}$.) If π does not come from a smaller unitary group via the theta correspondence, the sequence of dimensions $\{n_i\}$ will be

$$n,\ n+2,\ n+6,\ n+12,\ n+20, \ldots, n+i(i+1), \ldots. \tag{11.1.b}$$

When the dimension of $\mathscr{V}$ and the dimensions of $\mathscr{V}'^{\pm}$ are of opposite parity, the similar statement in Theorem 11.1 is conjectured in [**HKS**] speculation 7.6. In particular, the sequence of dimensions analogous to (11.1.b) should be

$$n,\ n+1,\ n+4,\ n+9,\ n+16, \ldots, n+i^2, \ldots. \tag{11.1.c}$$

11.2. Splittings of two related Witt towers.

Suppose that $\mathscr{V}'^{+}$ and $\mathscr{V}'^{-}$ are two ε'-Hermitian spaces satisfying the convention (11.1.a). Let m^+ (resp. m^-) denote the dimension of $\mathscr{V}'^{+}$ (resp. $\mathscr{V}'^{-}$). Let L be a good lattice in $\mathscr{V}$. Let $B^{\pm} := B(L, L'^{\pm})$ for some good lattice $L'^{\pm}$ in $\mathscr{V}'^{\pm}$. Let $\beta^L_{\mathscr{V}'^{\pm}}$ denote the splitting defined in [**Pn1**] of the extension $\widetilde{U(\mathscr{V})} \to U(\mathscr{V})$ with respect to the generalized lattice model $(M_{B^{\pm}}[g], \mathscr{S}(B^{\pm}))$ for the dual pair $(U(\mathscr{V}), U(\mathscr{V}'^{\pm}))$. It is known that the splitting $\beta^L_{\mathscr{V}'^{\pm}}$ does not depend on the lattice $L'^{\pm}$.

Define a character η_L of $U(\mathscr{V})_L$ as follows. We have a homomorphism $U(\mathscr{V})_L \to U(\boldsymbol{l}) \times U(\boldsymbol{l}^*)$. If E is an unramified extension of F, then both $U(\boldsymbol{l}), U(\boldsymbol{l}^*)$ are finite unitary group. Let η_1 (resp. η_2) be the character of order two of $U(\boldsymbol{l})$ (resp. $U(\boldsymbol{l}^*)$) if the group is not trivial, and the trivial character otherwise. If E is a ramified extension of F, then one of $U(\boldsymbol{l}), U(\boldsymbol{l}^*)$ is an orthogonal group and the other is a symplectic group. Let η_1 (resp. η_2) be the character of $U(\boldsymbol{l})$ (resp. $U(\boldsymbol{l}^*)$) of order two whose restriction to $SU(\boldsymbol{l})$ (resp. $SU(\boldsymbol{l}^*)$) is trivial if the group is a (nontrivial) orthogonal group, be the trivial character otherwise. For both cases, let η_L be the character of $U(\mathscr{V})_L$ lifted from $\eta_1 \otimes \eta_2$ by the above homomorphism. It is easy to see that the character η_L factors through the determinant map det: $U(\mathscr{V})_L \to E^{(1)} := \{t \in E \mid t\tau_E(t) = 1\}$. Moreover, we can check that the restriction of the sgn character to $U(\mathscr{V})_L$ is equal to η_L. In fact, sgn is the only character of $U(\mathscr{V})$ whose restriction to $U(\mathscr{V})_L$ is equal to η_L if we are not in the case that E is a ramified extension and L is the good lattice such that the component of orthogonal group of $U(\boldsymbol{l}) \times U(\boldsymbol{l}^*)$ is trivial. For this exceptional case, there are two such characters. One of them is the sgn character and the other is the trivial character.

Recall that the function $\alpha_{\mathscr{V}'} : U(\mathscr{V}) \to \boldsymbol{C}^\times$ for a dual pair $(U(\mathscr{V}), U(\mathscr{V}'))$ is defined by

$$M_Y[\iota_{\mathscr{V}'}(g)] \circ \Psi = \alpha_{\mathscr{V}'}(g)\Psi \circ M_B[\iota_{\mathscr{V}'}(g)] \tag{11.2.a}$$

where Ψ is an isomorphism from $\mathscr{S}(B)$ to $\mathscr{S}(Y)$ which intertwines the actions of the Heisenberg group. We know that $\alpha_{\mathscr{V}'}$ is independent of the choice of Ψ. Let $\beta^{Y^\pm}$ be the splitting $\beta_{\mathscr{V}'^\pm,\chi}$ in [**Kl**] for the dual pair $(U(\mathscr{V}), U(\mathscr{V}'^\pm))$. Then we have

$$\alpha_{\mathscr{V}'^+}(g)\beta^{Y^+}_{\mathscr{V}'^+}(g) = \eta_L(g)\alpha_{\mathscr{V}'^-}(g)\beta^{Y^-}_{\mathscr{V}'^-}(g) \tag{11.2.b}$$

for $g \in U(\mathscr{V})_L$ from [**Pn1**]. Since we know that $\mathrm{sgn}|_{U(\mathscr{V})_L} = \eta_L$, we can and will normalize the generalized model so that we have

$$\alpha_{\mathscr{V}'^+}(g)\beta^{Y^+}_{\mathscr{V}'^+}(g) = \mathrm{sgn}(g)\alpha_{\mathscr{V}'^-}(g)\beta^{Y^-}_{\mathscr{V}'^-}(g) \tag{11.2.c}$$

for $g \in U(\mathscr{V})$.

11.3.

LEMMA. *Let $\beta^L_{\mathscr{V}'^\pm}$ denote the β^L for the dual pair $(U(\mathscr{V}), U(\mathscr{V}'^\pm))$. There exists a character σ of $U(\mathscr{V})$ such that*

$$\beta^L_{\mathscr{V}'^+}(g) = \sigma(g)\alpha_{\mathscr{V}'^+}(g)\beta^{Y^+}_{\mathscr{V}'^+}(g),$$

$$\beta^L_{\mathscr{V}'^-}(g) = \eta_L(g)\sigma(g)\alpha_{\mathscr{V}'^-}(g)\beta^{Y^-}_{\mathscr{V}'^-}(g).$$

for all $g \in U(\mathscr{V})_L$.

PROOF. From [**Pn1**] subsection 3.4, we know that there exist characters σ^+, σ^- of $U(\mathscr{V})$ such that $\beta^L_{\mathscr{V}'^+}(g) = \sigma^+(g)\alpha^+(g)\beta^{Y^+}_{\mathscr{V}'^+}(g)$, and $\beta^L_{\mathscr{V}'^-}(g) = \sigma^-(g)\alpha^-(g)\beta^{Y^-}_{\mathscr{V}'^-}(g)$. So we need to prove that

$$\eta_L\sigma^+|_{U(\mathscr{V})_L} = \sigma^-|_{U(\mathscr{V})_L}. \tag{11.3.a}$$

Let $\xi_L^\pm$ be the character ξ_L defined in subsection 8.2 for the dual pair $(U(\mathscr{V}), U(\mathscr{V}'^\pm))$. From subsection 8.2, we know that $\beta^L_{\mathscr{V}'^+}|_{U(\mathscr{V})_L} = \xi_L^+$ and $\beta^L_{\mathscr{V}'^-}|_{U(\mathscr{V})_L} = \xi_L^-$. And we know that the characters $\xi_L^\pm$ depends on only the parity of dimensions of $\mathscr{V}'^\pm$. The parity of the dimensions of $\mathscr{V}'^+$ and $\mathscr{V}'^-$ are the same, so $\xi_L^+ = \xi_L^-$. Hence $\beta^L_{\mathscr{V}'^+}|_{U(\mathscr{V})_L} = \beta^L_{\mathscr{V}'^-}|_{U(\mathscr{V})_L}$. Therefore to prove (11.3.a), we need only to prove that

$$\alpha_{\mathscr{V}^+}(g)\beta^{Y^+}_{\mathscr{V}'^+}(g) = \eta_L(g)\alpha_{\mathscr{V}^-}(g)\beta^{Y^-}_{\mathscr{V}'^-}(g)$$

for $g \in U(\mathscr{V})_L$. Hence this lemma follows from (11.2.b). □

In fact, under the normalization (11.2.c), we know that $\beta^L_{\mathscr{V}'^+} = \beta^L_{\mathscr{V}'^-}$.

11.4. Another version of theta dichotomy for *p*-adic unitary groups.
The following theorem is another form of Theorem 11.1.

Theorem. *Let $(U(\mathscr{V}), U(\mathscr{V}'^\pm))$ be a reductive dual pair of unitary groups and L be a good lattice in $\mathscr{V}$. Suppose that π^+, π^- are the irreducible supercuspidal representations of $\widetilde{U(\mathscr{V})}$ such that $\pi^+ \circ \tilde{\beta}^L = \mathrm{sgn} \otimes (\pi^- \circ \tilde{\beta}^L)$. Suppose that the parity of the dimension of $\mathscr{V}$ and the parity of the dimensions of $\mathscr{V}'^\pm$ are the same. Then*

$$\ell_0^+ + \ell_0^- = 2n + 2$$

where n is the dimension of $\mathscr{V}$, $\ell_0^\pm$ is the smallest dimension of $\mathscr{V}'^\pm$ such that $\pi^\pm$ occurs in the theta correspondence for the pair $(U(\mathscr{V}), U(\mathscr{V}'^\pm))$.

Proof. By Lemma 11.3, we see that $\pi^+ \circ \tilde{\beta}^Y_{\mathscr{V}'^+} = \sigma \otimes (\pi^+ \circ \tilde{\beta}^L_{\mathscr{V}'^+})$ and $\pi^- \circ \tilde{\beta}^Y_{\mathscr{V}'^-} = \mathrm{sgn} \otimes \sigma \otimes (\pi^- \circ \tilde{\beta}^L_{\mathscr{V}'^-})$. Therefore $\pi^+ \circ \tilde{\beta}^Y_{\mathscr{V}'^+} = \pi^- \circ \tilde{\beta}^Y_{\mathscr{V}'^-}$ from the assumption of the theorem. Therefore the theorem follows from Theorem 11.1 immediately. □

12. Theta dichotomy for finite reductive dual pairs.

The goal of this section is to establish certain results of theta dichotomy for finite reductive dual pairs from Theorems 9.3, 9.5 and 11.4. First we introduce related finite reductive dual pairs.

12.1.
Let $\boldsymbol{f}$ be a finite field of odd characteristic and $\boldsymbol{d}$ be either $\boldsymbol{f}$ itself or a quadratic extension of $\boldsymbol{f}$. Let ε be 1 or -1. We know that maximal dimension of an anisotropic ε-Hermitian space is one (resp. zero; two) when $\boldsymbol{d}$ is a quadratic extension of $\boldsymbol{f}$ (resp. $(\boldsymbol{d}, \varepsilon) = (\boldsymbol{f}, -1)$; $(\boldsymbol{d}, \varepsilon) = (\boldsymbol{f}, 1)$). Fix a nontrivial character ψ of $\boldsymbol{f}$. We shall consider the following *related* finite reductive dual pairs $(U(\boldsymbol{v}), U(\boldsymbol{v}'))$ and $(U(\boldsymbol{v}), U(\boldsymbol{v}''))$:

(i) $\boldsymbol{d}$ is a quadratic extension of $\boldsymbol{f}$ and the parities of the dimensions of $\boldsymbol{v}'$ and $\boldsymbol{v}''$ are different.
(ii) $\boldsymbol{v}$ is a symplectic space and $\boldsymbol{v}'$, $\boldsymbol{v}''$ are even-dimensional quadratic spaces with Witt indices equal to half of the dimension and half minus one of the dimension respectively.
(iii) $\boldsymbol{v}$ is a symplectic space and $\boldsymbol{v}'$, $\boldsymbol{v}''$ are two quadratic spaces defined as follows. Let $\boldsymbol{v}_1'$ be a one-dimensional quadratic space and $\boldsymbol{v}_1''$ be another one-dimensional quadratic space whose quadratic form is a multiple of the form on $\boldsymbol{v}_1'$ by a nonsquare element in $\boldsymbol{f}^\times$. Then $\boldsymbol{v}'$ (resp. $\boldsymbol{v}''$) is an odd dimensional quadratic space in the Witt towers of $\boldsymbol{v}_1'$ (resp. $\boldsymbol{v}_1''$).
(iv) $\boldsymbol{v}$ is a quadratic space and $\boldsymbol{v}'$, $\boldsymbol{v}''$ are symplectic spaces.

As usual, we allow some of the spaces $\boldsymbol{v}$, $\boldsymbol{v}'$, $\boldsymbol{v}''$ to be the trivial space $\{0\}$.

12.2. Trivial examples.

Although the examples that we are going to consider are trivial, they are important for the proof of our theorem in the next subsection. Let ζ be the trivial representation of a finite classical group $U(\boldsymbol{v})$. If $\boldsymbol{v}$ is the trivial space and $\zeta \otimes \zeta'$ is a first occurrence for the pair $(U(\boldsymbol{v}), U(\boldsymbol{v}'))$, then by convention ζ' is also the trivial representation. We know that ζ' is a cuspidal representation if and only if the Witt index of $\boldsymbol{v}'$ is zero. Another example is that $U(\boldsymbol{v})$ is an orthogonal group of one variable and $\mathrm{sgn} \otimes \zeta'$ for some ζ' is a first occurrence for the pair $(U(\boldsymbol{v}), U(\boldsymbol{v}'))$. Then we know that $\boldsymbol{v}'$ is a two-dimensional symplectic space.

12.3.

THEOREM. *Suppose that $(\boldsymbol{v}, \boldsymbol{v}')$ and $(\boldsymbol{v}, \boldsymbol{v}'')$ are related reductive dual pairs in subsection* 12.2. *Let ζ be an irreducible cuspidal representation of $U(\boldsymbol{v})$. Let ℓ_0' (resp. ℓ_0'') be the smallest dimension of $\boldsymbol{v}'$ (resp. $\boldsymbol{v}''$) such that ζ (resp. $\zeta \otimes \mathrm{sgn}$) occurs in the theta correspondence for the dual pair $(U(\boldsymbol{v}), U(\boldsymbol{v}'))$ (resp. $(U(\boldsymbol{v}), U(\boldsymbol{v}''))$) with respect to a character ϕ of $\boldsymbol{f}$. Then*

$$\ell_0' + \ell_0'' = \begin{cases} 2n+1, & \textit{for case (i)};\\ 2n+2, & \textit{for case (ii) and (iii)};\\ 2n, & \textit{for case (iv)}, \end{cases}$$

where n is the dimension of $\boldsymbol{v}$.

PROOF. Let F be a p-adic field such that the residue field $\boldsymbol{f}_F$ is isomorphic to $\boldsymbol{f}$. Suppose that we are in the first case in subsection 12.1 and n is even i.e., $\boldsymbol{v}$ is an even dimensional ε-Hermitian space over a quadratic extension $\boldsymbol{d}$ of a finite field $\boldsymbol{f}$. From the discussion in subsection 9.4, we know that there exists an n-

dimensional ε-Hermitian space $\mathscr{V}$ over an unramified quadratic extension E of F such that

(1) the Witt index of $\boldsymbol{v}$ is equal to the Witt index of $\mathscr{V}$, and

(2) there exists a good lattice L in $\mathscr{V}$ such that $\boldsymbol{l} \simeq \boldsymbol{v}$ and $\boldsymbol{l}^*$ is trivial i.e., $U(\mathscr{V})_L/U(\mathscr{V})_{L,0^+} \simeq U(\boldsymbol{v})$.

Let G denote the group $U(\mathscr{V})$. Let $\tilde{\zeta}$ be the representation of G_L inflated from ζ. We know that the representation $c\text{-Ind}_{G_L}^{G}(\xi_L \otimes \tilde{\zeta})$ of G is irreducible supercuspidal. Therefore the representation π^+ of $\tilde{G}$ such that $\pi^+ \circ \tilde{\beta}^L = c\text{-Ind}_{G_L}^{G}(\xi_L \otimes \tilde{\zeta})$ is irreducible supercuspidal from the definition in subsection 9.2. Let π^- be the representation of $\tilde{G}$ such that $\pi^+ \circ \tilde{\beta}^L = \operatorname{sgn} \otimes (\pi^- \circ \tilde{\beta}^L)$. It is clear that $\pi^- \circ \tilde{\beta}^L = c\text{-Ind}_{G_L}^{G}(\xi_L \otimes (\widetilde{\zeta \otimes \operatorname{sgn}}))$ where $\widetilde{\zeta \otimes \operatorname{sgn}}$ denotes the representation of G_L inflated from $\zeta \otimes \operatorname{sgn}$ of $U(\boldsymbol{v})$. Let $\mathscr{V}'^+$ (resp. $\mathscr{V}'^-$) be an ε'-Hermitian space of even dimension m^+ (resp. m^-) defined in (11.1.a) such that π^+ (resp. π^-) first occurs in the theta correspondence for the dual pair $(U(\mathscr{V}), U(\mathscr{V}'^+))$ (resp. $(U(\mathscr{V}), U(\mathscr{V}'^-))$ and is paired with the representation π'^+ (resp. π'^-) with respect to a nontrivial character ψ of F such that $\bar{\psi} = \phi$. By Theorem 11.4, we know that $m^+ + m^- = 2n + 2$. By Theorem 9.5, we know that π'^+ must be the representation $c\text{-Ind}_{G'^+_{L'^+}}^{G'^+}(\xi_{L'^+} \otimes \tilde{\zeta}'^+)$ for some good lattice L'^+ in $\mathscr{V}'^+$ where $G'^+ := U(\mathscr{V}'^+)$. Moreover we know that the restriction of $\tilde{\zeta}'^+$ to $G'^+_{L'^+,0^+}$ is trivial. So $\tilde{\zeta}'^+$ is inflated from a representation $\zeta_1'^+ \otimes \zeta_2'^+$ of $U(\boldsymbol{l}'^+) \times U((\boldsymbol{l}'^+)^*)$. We can regard $\zeta \simeq \zeta \otimes \operatorname{triv}$ as a representation of $U(\boldsymbol{l}) \times U(\boldsymbol{l}^*)$. By Theorem 9.3, we know that $\zeta \otimes \zeta_2'^+$ and $\operatorname{triv} \otimes \zeta_1'^+$ must be two first occurrences in the theta correspondence for the pairs $(U(\boldsymbol{l}), U((\boldsymbol{l}'^+)^*))$ and $(U(\boldsymbol{l}^*), U(\boldsymbol{l}'^+))$ respectively. Because $\mathscr{V}'^+$ is a split ε'-Hermitian space, we know that both $\boldsymbol{l}'^+$ and $(\boldsymbol{l}'^+)^*$ are even-dimensional. Hence $\boldsymbol{l}'^+$ must be trivial from subsection 12.2. By the same argument, we know that π'^- is the representation $c\text{-Ind}_{G'^-_{L'^-}}^{G'^-}(\xi_{L'^-} \otimes (\widetilde{\zeta_1'^- \otimes \zeta_2'^-}))$ where $G'^- := U(\mathscr{V}'^-)$ and L'^- is a good lattice in $\mathscr{V}'^-$ such that $G'^-_{L'^-}/G'^-_{L'^-,0^+} \simeq U(\boldsymbol{l}'^-) \times U((\boldsymbol{l}'^-)^*)$ and $\zeta_1'^-$ (resp. $\zeta_2'^-$) is a representation of $U(\boldsymbol{l}'^-)$ (resp. $U((\boldsymbol{l}'^-)^*)$). We can regard $\zeta \otimes \operatorname{sgn} \simeq (\zeta \otimes \operatorname{sgn}) \otimes \operatorname{triv}$ as a representation of $U(\boldsymbol{l}) \times U(\boldsymbol{l}^*)$. Therefore by Theorem 9.3, we know that $(\zeta \otimes \operatorname{sgn}) \otimes \zeta_2'^-$ and $\operatorname{triv} \otimes \zeta_1'^-$ must be two first occurrences in theta correspondence for the pairs $(U(\boldsymbol{l}), U((\boldsymbol{l}'^-)^*))$ and $(U(\boldsymbol{l}^*), U(\boldsymbol{l}'^-))$ respectively. Because $\mathscr{V}'^-$ is a non-split even dimensional ε'-Hermitian space, both $\boldsymbol{l}'^-$ and $(\boldsymbol{l}'^-)^*$ are odd-dimensional. Hence $\boldsymbol{l}'^-$ must be one-dimensional from subsection 12.2. Now the two finite reductive dual pairs $(U(\boldsymbol{v}), U((\boldsymbol{l}'^+)^*))$, $(U(\boldsymbol{v}), U((\boldsymbol{l}'^-)^*))$ are the related dual pairs case (i) in subsection 12.1 because the parities of dimensions of $(\boldsymbol{l}'^+)^*$ and of $(\boldsymbol{l}'^-)^*$ are different. Therefore $\ell_0' + \ell_0'' = m^+ + m^- - 1 = 2n + 1$. Hence the proof of the theorem is complete for case (i) such that the dimension of $\boldsymbol{v}$ is even. When the dimension of $\boldsymbol{v}$ is odd, the proof is almost the same. So we skip it.

Next we consider cases (ii). There exists an n-dimensional ε-Hermitian space $\mathscr{V}$ over a ramified quadratic extension E of F satisfying the condition (1) and (2) above i.e., $\dim(\boldsymbol{v}) = \dim(\mathscr{V})$ and $\boldsymbol{v} \simeq \boldsymbol{l}$ is a symplectic space. The proof for the first case can be applied to this case. Now $(\boldsymbol{l}'^{+})^{*}$ is a split orthogonal group of even number of variables, $(\boldsymbol{l}'^{-})^{*}$ is a non-split orthogonal group of even number of variables and both $\boldsymbol{l}'^{+}$, $\boldsymbol{l}'^{-}$ are trivial from subsection 12.2. Hence $\ell_0' + \ell_0'' = m^{+} + m^{-} = 2n + 2$.

For case (iii), let $\mathscr{V}$ be an $(n+1)$-dimensional ε-Hermitian space over a ramified quadratic extension E of F and L be a good lattice in $\mathscr{V}$ such that $\boldsymbol{l} \simeq \boldsymbol{v}$ and $\boldsymbol{l}^{*}$ is a one-dimensional quadratic space over $\boldsymbol{f}_E = \boldsymbol{f}_F$. It is clear that $\zeta \otimes \mathrm{triv}$ is an irreducible cuspidal representation of $U(\boldsymbol{l}) \times U(\boldsymbol{l}^{*})$ and the induced representation $c\text{-}\mathrm{Ind}_{G_L}^{G}(\xi_L \otimes \widetilde{(\zeta \otimes \mathrm{triv})})$ is an irreducible supercuspidal representation of G. Let $\mathscr{V}'^{+}$, $\mathscr{V}'^{-}$ be the spaces as above but they are assumed to be odd-dimensional. By Theorem 11.4, we have $m^{+} + m^{-} = 2(n+1) + 2 = 2n + 4$. Now we should notice that $(c\text{-}\mathrm{Ind}_{G_L}^{G}(\xi_L \otimes \widetilde{\zeta \otimes \mathrm{triv}})) \otimes \mathrm{sgn} = c\text{-}\mathrm{Ind}_{G_L}^{G}(\xi_L \otimes \widetilde{(\zeta \otimes \mathrm{sgn})})$. Similar to the previous two cases, we know that $U((\boldsymbol{l}'^{+})^{*})$, $U((\boldsymbol{l}'^{-})^{*})$ are two orthogonal groups in odd variables and they satisfy the condition in (iii) of subsection 12.1. We also know that $\mathrm{triv} \otimes \zeta_1'^{+}$ and $\mathrm{sgn} \otimes \zeta_1'^{-}$ are two first occurrences for the pairs $(U(\boldsymbol{l}^{*}), U(\boldsymbol{l}'^{+}))$, $(U(\boldsymbol{l}^{*}), U(\boldsymbol{l}'^{-}))$ respectively. Hence $\boldsymbol{l}'^{+}$ is trivial and $\boldsymbol{l}'^{-}$ is two-dimensional. Thus $\ell_0' + \ell_0'' = m^{+} + m^{-} - 2 = 2n + 2$.

For case (iv), let $\mathscr{V}$ be an n-dimensional ε-Hermitian space over a ramified quadratic extension E of F and L be a good lattice in $\mathscr{V}$ such that $\boldsymbol{l} \simeq \boldsymbol{v}$ and $\boldsymbol{l}^{*}$ is trivial. Let $\mathscr{V}'^{+}$, $\mathscr{V}'^{-}$ be as given in previous cases. So we have $m^{+} + m^{-} = 2n + 2$. Similar to the case (iii), we know that $U((\boldsymbol{l}'^{+})^{*})$, $U((\boldsymbol{l}'^{-})^{*})$ are two symplectic groups. We also know that both $\boldsymbol{l}'^{+}$, $\boldsymbol{l}'^{-}$ are odd-dimensional if $\mathscr{V}$ is odd-dimensional, $\boldsymbol{l}'^{+}$ is split even-dimensional and $\boldsymbol{l}'^{-}$ is non-split even-dimensional if $\mathscr{V}$ is even-dimensional. Since $\boldsymbol{l}^{*}$ is trivial, we know that either both $\boldsymbol{l}'^{+}$, $\boldsymbol{l}'^{-}$ are one-dimensional, or $\boldsymbol{l}'^{+}$ is trivial and $\boldsymbol{l}'^{-}$ is two-dimensional. Therefore $\ell_0' + \ell_0'' = m^{+} + m^{-} - 2 = 2n + 2 - 2 = 2n$. $\square$

12.4.

REMARK. In fact, if ζ is a unipotent cuspidal representation and we have some restriction on the characteristic of $\boldsymbol{f}$, Theorem 12.3 can be figured out from [**AM**] theorem 4.1 and theorem 5.2 as follows.

(i) Suppose ζ is an irreducible unipotent cuspidal representation of a unitary group $U(\boldsymbol{v})$. Let n be the dimension of $\boldsymbol{v}$. Then $n = i(i+1)/2$ for some integer i by G. Lusztig's computation. By [**AM**] theorem 4.1, $\{\ell_0', \ell_0''\}$ is $\{(i-1)i/2, (i+1)(i+2)/2\}$. Then

$$\ell_0' + \ell_0'' = \frac{(i-1)i}{2} + \frac{(i+1)(i+2)}{2} = i^2 + i + 1 = 2n + 1$$

as we expect. Here we should notice that the Weil representation of the finite symplectic group we use here is a little different from the Weil representation used in [**AM**]. Therefore the theta correspondence for the dual pairs of unitary groups is different by the sgn character (*cf.* [**AMR**] introduction).

(ii) Similarly, if ζ is an irreducible unipotent cuspidal representation of a symplectic group $U(\boldsymbol{v})$, then $n = 2i(i+1)$ for some i. If the characteristic of $\boldsymbol{f}$ is large enough and $\boldsymbol{v}'$, $\boldsymbol{v}''$ are even dimensional, from [**AM**] theorem 5.2, $\{\ell_0^+, \ell_0^-\}$ is $\{2i^2, 2(i+1)^2\}$. Hence

$$\ell_0^+ + \ell_0^- = 2i^2 + 2(i+1)^2 = 4i(i+1) + 2 = 2n + 2.$$

(iii) If ζ is an irreducible unipotent cuspidal representation of an orthogonal group $U(\boldsymbol{v})$, then $n = 2i^2$ for some i. Suppose that the characteristic of $\boldsymbol{f}$ is large enough. Then from [**AM**] theorem 5.2, $\{\ell_0^+, \ell_0^-\}$ is $\{2i(i-1), 2i(i+1)\}$. Hence

$$\ell_0^+ + \ell_0^- = 2i(i-1) + 2i(i+1) = 4i^2 = 2n.$$

If we accept the fact that an irreducible unipotent representation is paired with an irreducible unipotent representation (under some restriction for the symplectic-orthogonal pairs), then Theorem 12.3 provides another (independent) proofs of theorem 4.1 and theorem 5.2 in [**AM**].

12.5.

Remark. As in the remark after Theorem 11.1 for p-adic unitary groups, Theorem 12.3 suggests that there exist a sequence of dimensions $n = n_0 < n_1 < n_2 < \cdots < n_i < \cdots$ and irreducible cuspidal representations $\zeta = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_i, \ldots$ of $U(\boldsymbol{v}_i)$ such that $(\zeta_i \otimes \mathrm{sgn}) \otimes \zeta_{i+1}$ occurs in the theta correspondence for the dual pair $(U(\boldsymbol{v}_i), U(\boldsymbol{v}_{i+1}))$. Suppose that ζ does not come from a smaller group via the theta correspondence. We have the following cases.

(i) Suppose that $U(\boldsymbol{v}_0)$ is a unitary group. So all $U(\boldsymbol{v}_i)$ are unitary groups and the sequence of dimensions are

$$n,\ n+1,\ n+3,\ n+6,\ n+10, \ldots, n + \frac{i(i+1)}{2}, \ldots. \tag{12.5.a}$$

If $n = 0$, then ζ is the trivial character and these dimensions are exactly the dimensions of $\boldsymbol{v}$ where the finite unitary groups $U(\boldsymbol{v})$ contain unipotent cuspidal representations and ζ_i are the only irreducible unipotent cuspidal representations of the finite unitary groups.

(ii) Suppose that $U(\boldsymbol{v}_0)$ is an orthogonal group and n is even. Then $U(\boldsymbol{v}_i)$ are orthogonal groups and symplectic groups alternatively. The sequence of dimensions are

$$n,\ n,\ n+2,\ n+4,\ n+8,\ n+12,$$
$$n+18,\dots,n+\frac{i(i+1)}{2}-\left\lceil\frac{i}{2}\right\rceil,\dots. \tag{12.5.b}$$

If $n=0$, these are exactly the dimensions of $\boldsymbol{v}$ where the finite even orthogonal groups or symplectic groups $U(\boldsymbol{v})$ have unipotent cuspidal representations. Moreover ζ_i are the only irreducible unipotent cuspidal representations of finite symplectic groups and finite even orthogonal groups. Suppose that $U(\boldsymbol{v}_0)$ is a symplectic group. Then the sequence of dimensions are

$$n,\ n+2,\ n+4,\ n+8,\ n+12,\ n+18,$$
$$n+24,\dots,n+\frac{i(i+1)}{2}+\left\lceil\frac{i}{2}\right\rceil,\dots. \tag{12.5.c}$$

This sequence is just the sequence in (12.5.b) with the index shifted by one.

(iii) Suppose that $U(\boldsymbol{v}_0)$ is symplectic group and the other group is an odd orthogonal group. Then by Theorem 12.3, the sequence of dimensions are

$$n,\ n+1,\ n+2,\ n+5,\ n+8,$$
$$n+13,\dots,n+\frac{i(i+1)}{2}-\left\lfloor\frac{i}{2}\right\rfloor,\dots. \tag{12.5.d}$$

For $n=0$ and i even, these terms are exactly the dimensions of $\boldsymbol{v}$ where the finite odd orthogonal groups $U(\boldsymbol{v})$ have unipotent cuspidal representations. As we know in [**AM**], the unipotent representations are not preserved by the theta correspondence for the dual pairs of finite symplectic groups and odd orthogonal groups. However, according to the sequence of dimensions here, it should be reasonable to believe that those representations of odd orthogonal groups in the chain should be all unipotent cuspidal when $n=0$ in (12.5.d) although those representations of symplectic groups are not unipotent. Suppose that $U(\boldsymbol{v}_0)$ is an orthogonal group and n is odd. Then the sequence of dimensions are

$$n,\ n+1,\ n+4,\ n+7,\ n+12,$$
$$n+17,\dots,n+\frac{i(i+1)}{2}+\left\lfloor\frac{i}{2}\right\rfloor,\dots. \tag{12.5.e}$$

12.6.

REMARK. It might be interesting to compare the chains of irreducible depth zero supercuspidal representations of p-adic unitary groups described in subsection 11.1 with the chains of cuspidal representations of finite classical groups described in subsection 12.5. However, here we will regard the representation π_i in subsection 11.1 as a representation of $\widetilde{U(\mathscr{V}_i)}$ instead of $U(\mathscr{V}_i)$. Suppose that π_0 is an irreducible depth zero supercuspidal representation of $\tilde{G} := \widetilde{U(\mathscr{V}_0)}$ and does not come from a smaller unitary group via the theta correspondence. Let $\pi_0, \pi_1, \dots, \pi_i, \dots$ be the chain of irreducible supercuspidal representations as in subsection 11.1. Since each π_i is of depth zero, we know that there are a good lattice L_i in $\mathscr{V}_i$ and a cuspidal representation $\zeta_i' \otimes \zeta_i''$ of $G_{L_i}/G_{L_i,0^+} \simeq U(\boldsymbol{l}_i) \times U(\boldsymbol{l}_i^*)$ such that $\pi_i \circ \tilde{\beta}^{L_i} = c\text{-Ind}_{G_{L_i}}^{G}(\xi_{L_i} \otimes \zeta_i)$ where ζ_i is the representation of G_{L_i} inflated from the representation $\zeta_i' \otimes \zeta_i''$ of $G_{L_i}/G_{L_i,0^+}$. It is clear that the cuspidal representation ζ_0' (resp. ζ_0'') of $U(\boldsymbol{l}_0)$ (resp. $U(\boldsymbol{l}_0^*)$) does not come from a smaller group via the theta correspondence. Moreover we know that $\zeta_0', \zeta_1', \dots, \zeta_i', \dots$ (resp. $\zeta_0'', \zeta_1'', \dots, \zeta_i'', \dots$) is the chain of cuspidal representations as in subsection 12.5(i) starting from ζ_0' (resp. ζ_0''). Of course, we must have $\dim(\mathscr{V}_i) = \dim(\boldsymbol{l}_i) + \dim(\boldsymbol{l}_i^*)$. This can be easily verified according the following two cases when the the dimension of $\mathscr{V}_0$ and the the dimension of $\mathscr{V}_0'$ are of the same parity.

(i) Suppose that E is an unramified quadratic extension of F. Then $U(\boldsymbol{l}_i)$ and $U(\boldsymbol{l}_i^*)$ are all finite unitary group. Thus $\dim(\boldsymbol{l}_i) + \dim(\boldsymbol{l}_i^*) = \dim(\boldsymbol{l}_0) + i(i+1)/2 + \dim(\boldsymbol{l}_0^*) + i(i+1)/2$ from (12.5.a). On the other hand, $\dim(\mathscr{V}_i) = \dim(\mathscr{V}_0) + i(i+1)$ from (11.1.b). Since $\dim(\mathscr{V}_0) = \dim(\boldsymbol{l}_0) + \dim(\boldsymbol{l}_0^*)$, hence $\dim(\mathscr{V}_i) = \dim(\boldsymbol{l}_i) + \dim(\boldsymbol{l}_i^*)$ as we expect.

(ii) Suppose that E is a ramified quadratic extension of F. Then one of $U(\boldsymbol{l}_i)$, $U(\boldsymbol{l}_i^*)$ is a finite orthogonal group and the other is a finite symplectic group. If the dimension of $\mathscr{V}_0$ is even, then we have $\dim(\boldsymbol{l}_i) + \dim(\boldsymbol{l}_i^*) = \dim(\boldsymbol{l}_0) + i(i+1)/2 - \lceil i/2 \rceil + \dim(\boldsymbol{l}_0^*) + i(i+1)/2 + \lceil i/2 \rceil$ from (12.5.b) and (12.5.c). If the dimension of $\mathscr{V}_0$ is odd, then we have $\dim(\boldsymbol{l}_i) + \dim(\boldsymbol{l}_i^*) = \dim(\boldsymbol{l}_0) + i(i+1)/2 - \lfloor i/2 \rfloor + \dim(\boldsymbol{l}_0^*) + i(i+1)/2 + \lfloor i/2 \rfloor$ from (12.5.d) and (12.5.e). In any cases, we have $\dim(\mathscr{V}_i) = \dim(\boldsymbol{l}_i) + \dim(\boldsymbol{l}_i^*)$ as we expect again.

13. Theta dichotomy for p-adic reductive dual pairs.

An interesting phenomenon is that the theta dichotomy for irreducible depth zero supercuspidal representations of p-adic reductive dual pairs can be obtained from the theta dichotomy for irreducible cuspidal representations of finite reductive dual pairs via Theorems 9.3, 9.5 and 12.3. This is the subject of this section.

13.1.

First we consider theta dichotomy for reductive dual pairs $(U(\mathscr{V}), U(\mathscr{V}'^{\pm}))$ of unitary groups when the parity of the dimension of $\mathscr{V}$ and the parity of the dimension of $\mathscr{V}'^{\pm}$ are different.

THEOREM. *Let $(U(\mathscr{V}), U(\mathscr{V}'^{\pm}))$ be a reductive dual pair of unitary groups. Suppose that the parity of the dimension of $\mathscr{V}$ and the parity of the dimension of $\mathscr{V}'^{\pm}$ are different. Suppose that π^+ is an irreducible depth zero supercuspidal representations of $\widetilde{U(\mathscr{V})}$ with a minimal K-type $(\widetilde{G_L}, \zeta)$. Let π^- be the irreducible depth zero supercuspidal representations of $\widetilde{U(\mathscr{V})}$ such that $\pi^+ \circ \tilde{\beta}^L = \mathrm{sgn} \otimes (\pi^- \circ \tilde{\beta}^L)$. Then*

$$\ell_0^+ + \ell_0^- = 2n + 2$$

where n is the dimension of $\mathscr{V}$ and $\ell_0^{\pm}$ is the smallest dimension of $\mathscr{V}'^{\pm}$ such that $\pi^{\pm}$ occurs in the theta correspondence for the pair $(U(\mathscr{V}), U(\mathscr{V}'^{\pm}))$.

PROOF. Let $G := U(\mathscr{V})$ and $G'^{\pm} := U(\mathscr{V}'^{\pm})$. Because $\pi^+ \circ \tilde{\beta}^L = \mathrm{sgn} \otimes (\pi^- \circ \tilde{\beta}^L)$ and π^+ has a minimal K-type (G_L, ζ), it is clear that π^- has a minimal K-type $(G_L, \mathrm{sgn} \otimes \zeta)$. Because ζ is an irreducible cuspidal representation of $G_L/G_{L,0^+} \simeq U(\boldsymbol{l}) \times U(\boldsymbol{l}^*)$, we can write $\zeta \simeq \zeta_1 \otimes \zeta_2$ where ζ_1 (resp. ζ_2) is an irreducible cuspidal representation of $U(\boldsymbol{l})$ (resp. $U(\boldsymbol{l}^*)$). Hence clearly, $\mathrm{sgn} \otimes \zeta$ is isomorphic to $(\mathrm{sgn} \otimes \zeta_1) \otimes (\mathrm{sgn} \otimes \zeta_2)$ where the first sgn is a character of $U(\boldsymbol{l}) \times U(\boldsymbol{l}^*)$, the second sgn is a character of $U(\boldsymbol{l})$ and the third one is a character of $U(\boldsymbol{l}^*)$.

Suppose that π^+ (resp. π^-) corresponds to an irreducible supercuspidal representation π'^+ of $\widetilde{U(\mathscr{V}'^+)}$ (resp. π'^- of $\widetilde{U(\mathscr{V}'^-)}$) in the theta correspondence for the reductive dual pair $(U(\mathscr{V}), U(\mathscr{V}'^+))$ (resp. $(U(\mathscr{V}), U(\mathscr{V}'^-))$). Then by Theorem 9.3, we know that π'^+ has a minimal K-type $(G'^+_{L'^+}, \zeta'^+)$ such that $\zeta \otimes \zeta'^+$ is a first occurrence. This means that $\zeta'^+ \simeq \zeta_1'^+ \otimes \zeta_2'^+$ where $\zeta_1'^+$ (resp. $\zeta_2'^+$) is an irreducible cuspidal representation of $U(\boldsymbol{l}'^+)$ (resp. $U((\boldsymbol{l}'^+)^*)$), and $\zeta_1 \otimes \zeta_2'^+$ (resp. $\zeta_2 \otimes \zeta_1'^+$) is a first occurrence for the finite dual pair $(U(\boldsymbol{l}), U((\boldsymbol{l}'^+)^*))$ (resp. $(U(\boldsymbol{l}^*), U(\boldsymbol{l}'^+))$). By the same reason, we also know that π'^- has a minimal K-type $(G'^-_{L'^-}, \zeta_1'^- \otimes \zeta_2'^-)$ such that $\zeta_1'^-$ (resp. $\zeta_2'^-$) is an irreducible cuspidal representation of $U(\boldsymbol{l}'^-)$ (resp. $U((\boldsymbol{l}'^-)^*)$), and $(\mathrm{sgn} \otimes \zeta_1) \otimes \zeta_2'^-$ (resp. $(\mathrm{sgn} \otimes \zeta_2) \otimes \zeta_1'^-$) is a first occurrence for the finite dual pair $(U(\boldsymbol{l}), U((\boldsymbol{l}'^-)^*))$ (resp. $(U(\boldsymbol{l}^*), U(\boldsymbol{l}'^-))$).

If E is an unramified quadratic extension of F, the related reductive dual pairs

$$\{(U(\boldsymbol{l}), U((\boldsymbol{l}'^+)^*)), (U(\boldsymbol{l}^*), U(\boldsymbol{l}'^+))\} \quad \text{and} \quad \{(U(\boldsymbol{l}), U((\boldsymbol{l}'^-)^*)), (U(\boldsymbol{l}^*), U(\boldsymbol{l}'^-))\} \tag{13.1.a}$$

are case (i) of subsection 12.1. If E is a ramified quadratic extension of F, then one of the related reductive dual pairs in (13.1.a) is case (iv) of subsection 12.1 and the other is either case (ii) or case (iii) depending the parity of the dimension of $\mathscr{V}$. It clear that $n = \dim(\boldsymbol{l}) + \dim(\boldsymbol{l}^*)$, $\ell_0^+ = \dim(\boldsymbol{l}'^+) + \dim((\boldsymbol{l}'^+)^*)$ and $\ell_0^- = \dim(\boldsymbol{l}'^-) + \dim((\boldsymbol{l}'^-)^*)$. Therefore by Theorem 12.3, we have

$$\begin{aligned}\ell_0^+ + \ell_0^- &= \begin{cases} 2\dim(\boldsymbol{l}) + 1 + 2\dim(\boldsymbol{l}^*) + 1, & \text{if } E \text{ is an unramified quadratic extension;} \\ 2\dim(\boldsymbol{l}) + 2\dim(\boldsymbol{l}^*) + 2, & \text{if } E \text{ is a ramified quadratic extension} \end{cases} \\ &= 2n + 2. \end{aligned}$$

□

We have the following analogue of the result in subsection 12.6. Suppose that π_0 is an irreducible depth zero supercuspidal representation of $\widetilde{U(\mathscr{V}_0)}$ and does not come from a smaller unitary group via the theta correspondence. Let $\mathscr{V}_i, L_i, \boldsymbol{l}_i, \boldsymbol{l}_i^*, \zeta_i, \zeta_i', \zeta_i''$ be as in subsection 12.6. Now we have the following two cases.

(i) Suppose that E is an unramified quadratic extension of F. Then $U(\boldsymbol{l}_i)$ and $U(\boldsymbol{l}_i^*)$ are all finite unitary group. Because now the dimensions of $\mathscr{V}_0, \mathscr{V}_1$ are of opposite parity, we know that $\dim(\mathscr{V}_1) = \dim(\mathscr{V}_0) + 1$. Therefore either $\dim(\boldsymbol{l}_0) = \dim(\boldsymbol{l}_1^*)$ or $\dim(\boldsymbol{l}_0^*) = \dim(\boldsymbol{l}_1)$. If $\dim(\boldsymbol{l}_0) = \dim(\boldsymbol{l}_1^*)$, then $\zeta_1', \zeta_2', \ldots, \zeta_i', \ldots$ (resp. $\zeta_0'', \zeta_1'', \ldots, \zeta_i'', \ldots$) is the chain of cuspidal representations as in subsection 12.5(i) starting from ζ_1' (resp. ζ_0''). If $\dim(\boldsymbol{l}_0^*) = \dim(\boldsymbol{l}_1)$, then $\zeta_0', \zeta_1', \ldots, \zeta_i', \ldots$ (resp. $\zeta_1'', \zeta_2'', \ldots, \zeta_i'', \ldots$) is the chain of cuspidal representations as in subsection 12.5(i) starting from ζ_0' (resp. ζ_1''). In any case, we have $\dim(\boldsymbol{l}_i) + \dim(\boldsymbol{l}_i^*) = \dim(\boldsymbol{l}_0) + i(i+1)/2 + \dim(\boldsymbol{l}_0^*) + (i-1)i/2$ from (12.5.a). Hence $\dim(\mathscr{V}_i) = \dim(\mathscr{V}_0) + i^2$ as we expect.

(ii) Suppose that E is a ramified quadratic extension of F. Then one of $U(\boldsymbol{l}_i)$, $U(\boldsymbol{l}_i^*)$ is a finite orthogonal group and the other is a finite symplectic group. Define

$$\boldsymbol{l}_i' = \begin{cases} \boldsymbol{l}_i, & \text{if } i \text{ is even;} \\ \boldsymbol{l}_i^*, & \text{if } i \text{ is odd,} \end{cases} \quad \boldsymbol{l}_i'' = \begin{cases} \boldsymbol{l}_i^*, & \text{if } i \text{ is even;} \\ \boldsymbol{l}_i, & \text{if } i \text{ is odd.} \end{cases}$$

Suppose that $U(\boldsymbol{l}_0)$ is a finite symplectic group and $U(\boldsymbol{l}_0^*)$ is a finite orthogonal group.

(ii.a) If $\mathscr{V}_0$ is even-dimensional, then we have $\dim(\boldsymbol{l}_1') = \dim(\boldsymbol{l}_0) + 1$ and $\dim(\boldsymbol{l}_1'') = \dim(\boldsymbol{l}_0^*)$. More generally we have $\dim(\boldsymbol{l}_i') = \dim(\boldsymbol{l}_0) + i(i+1)/2 - \lfloor i/2 \rfloor$ from (12.5.d) and $\dim(\boldsymbol{l}_i'') = \dim(\boldsymbol{l}_0^*) + i(i+1)/2 - \lceil i/2 \rceil$ from (12.5.b). Hence $\dim(\mathscr{V}_i) = \dim(\boldsymbol{l}_i') + \dim(\boldsymbol{l}_i'') = \dim(\boldsymbol{l}_0) + \dim(\boldsymbol{l}_0^*) + i^2 = \dim(\mathscr{V}_0) + i^2$ as we expect.

(ii.b) If $\mathscr{V}_0$ is odd-dimensional, then we have $\dim(\boldsymbol{l}'_1) = \dim(\boldsymbol{l}_0)$ and $\dim(\boldsymbol{l}''_1) = \dim(\boldsymbol{l}^*_0) + 1$. More generally we have $\dim(\boldsymbol{l}'_i) = \dim(\boldsymbol{l}_0) + i(i-1)/2 + \lceil (i-1)/2 \rceil$ from (12.5.c) and $\dim(\boldsymbol{l}''_i) = \dim(\boldsymbol{l}^*_0) + i(i+1)/2 + \lfloor i/2 \rfloor$ from (12.5.e). Hence $\dim(\mathscr{V}_i) = \dim(\boldsymbol{l}'_i) + \dim(\boldsymbol{l}''_i) = \dim(\boldsymbol{l}_0) + \dim(\boldsymbol{l}^*_0) + i^2 = \dim(\mathscr{V}_0) + i^2$ as we expect again.

13.2.

In this section, we discuss theta dichotomy for p-adic reductive dual pairs of symplectic and orthogonal groups. Because we only consider split reductive dual pairs right now, we shall assume that the dimensions of the quadratic spaces are even. We have the following three types of related reductive dual pairs $(U(\mathscr{V}), U(\mathscr{V}'))$ and $(U(\mathscr{V}), U(\mathscr{V}''))$ of symplectic groups and orthogonal groups.

(i) $\mathscr{V}$ is a symplectic space and $\mathscr{V}'$ (resp. $\mathscr{V}''$) is an even-dimensional quadratic space whose Witt index is half (resp. half minus two) of its dimension.

(ii) $\mathscr{V}$ is a symplectic space, $\mathscr{V}'$ and $\mathscr{V}''$ is the quadratic spaces as follows. Let $\mathscr{V}'_0$ be a two-dimensional anisotropic quadratic space. Then we can associate a quadratic extension $F(\sqrt{\Delta})$ of F to $\mathscr{V}'_0$. Let $\mathscr{V}''_0$ be another two-dimensional anisotropic quadratic space with the quadratic form equal to the multiple of the form on $\mathscr{V}'_0$ by an element $a \in F^\times$ such that $(\Delta, a)_F = -1$. Let $\mathscr{V}'$ and $\mathscr{V}''$ be the quadratic spaces in the Witt towers of $\mathscr{V}'_0$ and $\mathscr{V}''_0$ respectively. We notice that the Witt indices of $\mathscr{V}'$ and $\mathscr{V}''$ are half minus one of their dimensions.

(iii) $\mathscr{V}$ is an even-dimensional quadratic space and $\mathscr{V}'$ is a symplectic space.

13.3.

THEOREM. *Suppose that $(U(\mathscr{V}), U(\mathscr{V}'))$ and $(U(\mathscr{V}), U(\mathscr{V}''))$ are one of the above three types of related reductive dual pairs. Suppose that π is an irreducible depth zero supercuspidal representation of $\widetilde{U(\mathscr{V})}$ having a minimal K-type (G_L, ζ). Let π^s be the irreducible depth zero supercuspidal representation of $\widetilde{U(\mathscr{V})}$ such that $\pi^s \circ \tilde{\beta}^L = \mathrm{sgn} \otimes (\pi \circ \tilde{\beta}^L)$. Let ℓ'_0 (resp. ℓ''_0) be the smallest dimension of $\mathscr{V}'$ (resp. $\mathscr{V}''$) such that π (resp. π^s) occurs in the theta correspondence for the pair $(U(\mathscr{V}), U(\mathscr{V}'))$. Then*

$$\ell'_0 + \ell''_0 = \begin{cases} 2n+4, & \text{for cases (i), (ii);} \\ 2n, & \text{for cases (iii),} \end{cases} \tag{13.3.a}$$

where n is the dimension of $\mathscr{V}$.

PROOF. Let $G := U(\mathscr{V})$, $G' := U(\mathscr{V}')$ and $G'' := U(\mathscr{V}'')$. First we consider cases (i) and (ii) in subsection 13.2. We know that $\pi \circ \tilde{\beta}^L$ is isomorphic to $c\text{-Ind}_{G_L}^G \tilde{\zeta}$ where $\tilde{\zeta}$ is the representation of G_L inflated from the representation ζ of $G_L/G_{L,0^+}$. We can write $\zeta = \zeta_1 \otimes \zeta_2$ where ζ_1 (resp. ζ_2) is an irreducible cuspidal representation of $U(\boldsymbol{l})$ (resp. $U(\boldsymbol{l}^*)$). Suppose that π (resp. π^s) corresponds to the irreducible supercuspidal representation π' of $\widetilde{U(\mathscr{V}')}$ (resp. π'' of $\widetilde{U(\mathscr{V}'')}$) in the theta correspondence for the dual pair $(U(\mathscr{V}), U(\mathscr{V}'))$ (resp. $(U(\mathscr{V}), U(\mathscr{V}'')))$. By Theorem 9.3, we know that π' must have a minimal K-type $(G'_{L'}, \zeta'_1 \otimes \zeta'_2)$ for some good lattice L' in $\mathscr{V}'$ where ζ'_1 (resp. ζ'_2) is an irreducible cuspidal representation of $U(\boldsymbol{l}')$ (resp. $U(\boldsymbol{l}'^*)$) such that both $\zeta_1 \otimes \zeta'_2$, $\zeta_2 \otimes \zeta'_1$ are first occurrences. By the same reason, π'' must have a minimal K-type $(G''_{L''}, \zeta''_1 \otimes \zeta''_2)$ for some good lattice L'' in $\mathscr{V}''$ where ζ''_1 (resp. ζ''_2) is an irreducible cuspidal representation of $U(\boldsymbol{l}'')$ (resp. $U(\boldsymbol{l}''^*)$) such that both $(\operatorname{sgn} \otimes \zeta_1) \otimes \zeta'_2$, $(\operatorname{sgn} \otimes \zeta_2) \otimes \zeta'_1$ are first occurrences. If we are in case (i) or in case (ii) with $F(\sqrt{\Delta})$ unramified, then the related dual pairs $(U(\boldsymbol{l}), U(\boldsymbol{l}'^*))$ and $(U(\boldsymbol{l}), U(\boldsymbol{l}''^*))$ are case (iii) in subsection 12.1. Similarly, the dual pairs $(U(\boldsymbol{l}^*), U(\boldsymbol{l}'))$ and $(U(\boldsymbol{l}^*), U(\boldsymbol{l}''))$ are also case (iii) in subsection 12.1. If we are in case (ii) and $F(\sqrt{\Delta})$ is a ramified quadratic extension of F, then the related dual pairs $(U(\boldsymbol{l}), U(\boldsymbol{l}'^*))$ and $(U(\boldsymbol{l}), U(\boldsymbol{l}''^*))$ are case (ii) in subsection 12.1. Similarly, the dual pairs $(U(\boldsymbol{l}^*), U(\boldsymbol{l}'))$ and $(U(\boldsymbol{l}^*), U(\boldsymbol{l}''))$ are also case (ii) in subsection 12.1. Therefore by Theorem 12.3, we have

$$\begin{aligned} \ell'_0 + \ell''_0 = \dim(\boldsymbol{l}') + \dim(\boldsymbol{l}'^*) + \dim(\boldsymbol{l}'') + \dim(\boldsymbol{l}''^*) &= 2\dim(\boldsymbol{l}) + 2 + 2\dim(\boldsymbol{l}^*) + 2 \\ &= 2n + 4. \end{aligned}$$

Hence the proof is complete for cases (i) and (ii). The proof of case (iii) is similar. $\square$

It might be reasonable to believe that the theorem should also be true for any irreducible supercuspidal representations or even for any irreducible admissible representations. In fact, it has been conjectured by S. Kudla and D. Prasad (*cf.* [**Rb**]) that (12.5.a) for type (iii) should be valid for any irreducible admissible representations of $U(\mathscr{V})$.

13.4.

Remark. Theorem 13.3 suggests that there exist a sequence of dimensions $n = n_0 < n_1 < n_2 < \cdots < n_i < \cdots$ and irreducible depth zero supercuspidal representations $\pi = \pi_0, \pi_1, \pi_2, \ldots, \pi_i, \ldots$ of $\widetilde{U(\mathscr{V}_i)}$ such that $\pi_i^s \otimes \pi_{i+1}$ occurs in the theta correspondence for the pair $(U(\mathscr{V}_i), U(\mathscr{V}_{i+1}))$ where $U(\mathscr{V}_i)$ are symplectic and orthogonal groups alternatively and π_i^s is the representation such that $\pi_i^s \circ \tilde{\beta}^{L_i} = \operatorname{sgn} \otimes (\pi_i \circ \tilde{\beta}^{L_i})$. Suppose that π_0 does not come from a smaller group via the theta correspondence and $U(\mathscr{V}_0)$ is an orthogonal group. The sequence of the dimensions are

$$n,\ n,\ n+4,\ n+8,\ n+16,\ n+24,\dots,n+i(i+1)+(-1)^i 2\lceil i/2\rceil,\dots. \tag{13.4.a}$$

We can write $\pi_0\circ\tilde{\beta}^{L_0}=c\text{-Ind}_{G_{L_0}}^{G}\zeta_0$ for some good lattice L_0 in $\mathscr{V}_0$ where ζ_0 is the representation of G_{L_0} inflated from the cuspidal representation $\zeta_0'\otimes\zeta_0''$ of $G_{L_0}/G_{L_0,0^+}\simeq U(\boldsymbol{l}_0)\times U(\boldsymbol{l}_0^*)$. Then it is clear that ζ_0' and ζ_0'' are cuspidal representations of finite classical groups and do not come from smaller groups via the theta correspondence. Then $\pi_i\circ\tilde{\beta}^{L_i}=c\text{-Ind}_{G_{L_i}}^{G_i}\zeta_i$ where ζ_i is the representation of G_{L_i} inflated from $\zeta_i'\otimes\zeta_i''$ where ζ_i' (resp. ζ_i'') is the i-th cuspidal representation in the chain in subsection 12.5(ii) or 12.5(iii) starting from ζ_0' (resp. ζ_0'').

13.5.

An immediate consequence of Theorem 13.3 is the following.

Corollary. *Suppose that $\mathscr{V}$ is a $2n$-dimensional symplectic space. All irreducible depth zero supercuspidal representations of $Sp(\mathscr{V})$ occur in the local theta correspondences with some $O(\mathscr{V}')$ where $\mathscr{V}'$ are $2n+2$ dimensional quadratic spaces of Witt indices $n+1$ or $n-1$.*

Proof. Let π be an irreducible admissible depth zero supercuspidal representation of $\widetilde{Sp(\mathscr{V})}$ and π^s be as defined in Theorem 13.3. Because the character sgn of $Sp(\mathscr{V})$ is trivial, we know that $\pi=\pi^s$. Suppose that L_1, L_2 are two good lattices in $\mathscr{V}$. We know that the splittings β^{L_1}, β^{L_2} are different up to a character of $Sp(\mathscr{V})$. Since $Sp(\mathscr{V})$ does not have any nontrivial character, we have $\beta^{L_1}=\beta^{L_2}$. Hence we can fix a splitting β^L for a good lattice in $\mathscr{V}$ and regard π as an irreducible depth zero supercuspidal representation of $Sp(\mathscr{V})$. Let $\mathscr{V}'$ (resp. $\mathscr{V}''$) be the quadratic space of even dimension m' (resp. m'') of Witt index $m'/2$ (resp. $m''/2-2$) such that π first occurs in the theta correspondence for the reductive dual pair $(Sp(\mathscr{V}),O(\mathscr{V}'))$ (resp. $(Sp(\mathscr{V}),O(\mathscr{V}''))$). By Theorem 13.3, we know that $m'+m''=4n+4$. Therefore at least one of m', m'' must be less than or equal to $2n+2$. □

D. Prasad has conjectured in [**Ps**] corollary 1 that all irreducible admissible representations of $Sp(\mathscr{V})$ should occur in the correspondence with some $O(\mathscr{V}')$ where $\mathscr{V}'$ is $2n+2$ dimensional. Corollary 13.5 provides some support for his conjecture.

References

[AM] J. Adams and A. Moy, Unipotent representations and reductive dual pairs over finite fields, Trans. Amer. Math. Soc., **340** (1993), 309–321.

[As] C. Asmuth, Weil representations of symplectic p-adic groups, Amer. J. Math., **101** (1979), 885–909.

[Ab] A.-M. Aubert, Correspondance de Howe et sous-groupes parahoriques, J. Reine Angew. Math., **392** (1988), 176–186.

[AMR] A.-M. Aubert, J. Michel and R. Rouquier, Correspondance de Howe pour les groupes finis, Duke Math. J., **83** (1996), 353–397.

[BT] F. Bruhat and J. Tits, Schémas en groupes et immeubles des groupes classiques sur un corps local, deuxième partie: groupes unitaires, Bull. Soc. Math. France, **115** (1987), 141–195.

[DL] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, Ann. of Math. (2), **103** (1976), 103–161.

[HKS] M. Harris, S. Kudla and W. Sweet, Theta dichotomy for unitary groups, J. Amer. Math. Soc., **9** (1996), 941–1004.

[Hj] H. Hijikata, Maximal compact subgroups of some p-adic classical groups, mimeographed notes, Yale University, 1964.

[Hw1] R. Howe, Invariant theory and duality for classical groups over finite fields with applications to their singular representation theory, preprint.

[Hw2] R. Howe, θ-series and invariant theory, In Automorphic forms, representations and L-functions, Proc. Sympos. Pure Math., **33** (1979), 275–285.

[Kl] S. Kudla, Splitting metaplectic covers of dual reductive pairs, Israel J. Math., **87** (1994), 361–401.

[Lt] G. Lusztig, Some examples of square integrable representations of semisimple p-adic groups, Trans. Amer. Math. Soc., **277** (1983), 623–653.

[MVW] C. Mœglin, M.-F. Vigneras and J.-L. Waldspurger, Correspondances de Howe sur un corps p-adiques, Lecture Notes in Math., **1291**, Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[Mn] C. Moen, The dual pair $(U(1), U(1))$ over a p-adic field, Pacific J. Math., **158** (1993), 365–386.

[MP1] A. Moy and G. Prasad, Unrefined minimal K-types for p-adic groups, Invent. Math., **116** (1994), 393–408.

[MP2] A. Moy and G. Prasad, Jacquet functors and unrefined minimal K-types, Comment. Math. Helv., **71** (1996), 98–121.

[Pn1] S.-Y. Pan, Splittings of the metaplectic covers of some reductive dual pairs, Pacific J. Math., **199** (2001), 163–227.

[Pn2] S.-Y. Pan, Depth preservation in local theta correspondence, Duke Math. J., **113** (2002), 531–592.

[Ps] D. Prasad, On the local Howe duality correspondence, Internat. Math. Res. Notices, **11** (1993), 279–287.

[Rl] S. Rallis, On the Howe duality conjecture, Compositio Math., **51** (1984), 333–399.

[RR] R. Ranga-Rao, On some explicit formulas in the theory of Weil representation, Pacific J. Math., **157** (1993), 335–371.

[Rb] B. Roberts, lecture notes, University of Maryland, 1994.

[Sr] B. Srinivasan, Weil representations of finite classical groups, Invent. Math., **51** (1979), 143–153.

[Tn] S. Tanaka, Irreducible representations of the binary modular congruence group mod p^{λ}, J. Math. Kyoto Univ., **7** (1967), 123–132.

[Tt] J. Tits, Reductive groups over local fields, In Automorphic forms, representations and L-functions, Proc. Sympos. Pure Math., **33** (1979), 29–71.

[Wp] J.-L. Waldspurger, Demonstration d'une conjecture de duality de Howe dans le case p-adiques, $p \neq 2$, In Festschrift in honor of I. Piatetski-Shapiro, Israel Math. Conf. Proc., **2** (1990), 267–324.

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