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Some extensions of the Marcinkiewicz interpolation theorem in terms of modular inequalities

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Abstract. Given a quasi-subaditive operator $T : L_0(\mu) \to L_0(\nu)$, we want to study mapping properties of interpolation type for which the following modular inequality holds

$$\int_{\mathscr{N}} P(|Tf(x)|)\, d\nu(x) \le \int_{\mathscr{M}} Q(|f(x)|)\, d\mu(x)$$

where P and Q are modular functions. These results generalize the classical Marcinkiewicz interpolation theorem.

§1. Introduction.

Let $(\mathscr{M}, \mu)$ and $(\mathscr{N}, \nu)$ be two σ-finite measure spaces and $L_0(\mu)$ and $L_0(\nu)$ the sets of measurable functions defined on $\mathscr{M}$ and $\mathscr{N}$ respectively. Given two operators $S : L_0(\mu) \to L_0(\nu)$ and $T : L_0(\mu) \to L_0(\nu)$, we say that T is S-subaditive if

$$|T(f+g)(x)| \le Sf(x) + Sg(x).$$

If $S = |T|$ then T is said to be subaditive. If f_μ^* is the decreasing rearrangement of f defined by $f_\mu^*(t) = \inf\{s > 0; \lambda_f^\mu(s) \le t\}$, where $\lambda_f^\mu(s) = \mu\{x; |f(x)| > s\}$ is the distribution function of f, then $Tf(t) = f_\mu^*(t)$ is S-subaditive, where $Sf(t) = f_\mu^*(t/2)$.

A function $Q : [0, \infty) \to [0, \infty)$ is called a modular function if Q is an increasing (non-decreasing) right-continuous function and $Q(0+) = 0$. Given a modular function Q, we set

$$L_Q(\mu) = L_Q = \left\{ f \in L_0(\mu); \|f\|_Q = \int_{\mathscr{M}} Q(|f(x)|)\, d\mu(x) < \infty \right\},$$

and given two modular functions P and Q, we say that T satisfies a (P, Q) modular inequality if the following inequality holds

$$\int_{\mathscr{N}} P(|Tf(x)|)\, d\nu(x) \le \int_{\mathscr{M}} Q(|f(x)|)\, d\mu(x). \tag{1}$$

Such modular inequalities have been studied previously by many authors (cf. [**KK**], [**CH**], [**L**]) and in a great number of contexts since many problems in Analysis deal with

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the question of determining the relation between the integrability property of a function and of its image Tf.

In particular, the theory of interpolation (see [**BS**], [**BL**]) is an extremely useful tool to deal with such kind of problem. Also the theory of Orlicz spaces (see [**KR**]) deals with this type of modular inequalities since an operator T, satisfying (1), satisfies that $T : L(Q) \to L(P)$ is bounded, where in this case $L(Q)$ and $L(P)$ represent the associate Orlicz spaces endowed with the Luxemburg norm. However, unlike the case we treat here, the functions P and Q are typically Young's or N-functions.

In [**CH**], the authors studied modular inequalities for the Hardy operator

$$Tf(t) = \frac{1}{t^{1/a}} \int_0^t f(s) s^{(1/a)-1}\, ds,$$

and also for the conjugate operator. In the case $a > 1$, they had to use some kind of interpolation result (see Theorem 3.5, [**CH**]) for which the concept of admissible function (see Definition 1.2 below) was introduced.

The purpose of this paper is to further investigate on these type of interpolation results by analyzing the minimum hypothesis needed on the operator T, the optimality of the obtained results and giving concrete examples.

The two following concepts are fundamental for our purpose:

Definition 1.1. A subset $D \subset L_0(\mu)$ is said to be upper-stable if

$$\forall f \in D, \forall \alpha > 0, f\chi_{\{|f|>\alpha\}} \in D,$$

and it is lower-stable if

$$\forall f \in D, \forall \alpha > 0, f\chi_{\{|f|\le\alpha\}} \in D.$$

Finally, we say that D is stable if it is upper-stable and lower-stable simultaneously.

Obviously the set D of decreasing (or increasing) functions in $L_0(\boldsymbol{R}^+)$ is upper-stable. The sets $D = \{f; \|f\|_\infty \le 1\}$ (or in general the unit ball of any lattice space), $D = \{\chi_E;\ E \text{ is } \mu\text{-measurable}\}$, $D = \{\text{radial functions in } L_0(\boldsymbol{R}^N)\}$, or D a lattice are stable sets. Finally, the set $D = \{g = f\chi_{(0,r)};\ f \text{ is increasing and } r > 0\}$ is lower stable.

Definition 1.2. A function $A : [0,\infty) \to [0,\infty)$ is said to be D-admissible for an operator T, if $A(0) = 0$, $A(t/y)$ is measurable on $[0,\infty) \times [0,\infty)$ and, for every $f \in D$,

$$\lambda_{Tf}^{\nu}(y) \le \int_{\mathcal{M}} A\left(\frac{|f(x)|}{y}\right) d\mu(x).$$

If $D = L_0(\mu)$, we simply say that A is admissible.

Examples.

(1) Obviously if T is any operator of weak type (p,p), then $A(t) = t^p$ is admissible.

(2) In [**BP**], the authors study a generalization of the Hardy-Littlewood maximal function as follows: let Φ be a Young function and let us define

$$M_{\Phi}f(x) = \sup_{x \in Q} \inf\left\{\lambda > 0; \int_Q \Phi(|f(y)|/\lambda)\, dy \le |Q|\right\},$$

where Q is a cube in $\boldsymbol{R}^N$. Then, they prove that Φ is an admissible function for this operator.

(3) Let $\tilde{T}$ be a subaditive operator such that $\tilde{T} : L^p(\mu) \to L^{q,\infty}(v)$, that is,

$$\lambda^v_{\tilde{T}f}(y)^{1/q} y \le \left(\int_{\mathcal{M}} |f(x)|^p\, d\mu(x)\right)^{1/p}.$$

Therefore, $\lambda^v_{\tilde{T}f}(y)^{p/q} \le \int_{\mathcal{M}} A(|f(x)|/y)\, d\mu(x)$ with $A(t) = t^p$. Now, simple computations show that $\lambda^v_{\tilde{T}f}(y)^{p/q} = \lambda_{Tf}(y)$, where $Tf(t) = (\tilde{T}f)^*_v(t^{q/p})$. Consequently, $A(t) = t^p$ is an admissible function for this operator T.

The paper is organized as follows: in Section 2, we develop our interpolation theorems for bounded operators on L^∞. In Section 3, we deal with distribution controlled operators, and Section 4 is devoted to study interpolation results when two admissible functions are known.

The notation used is standard: if f/g is bounded above and below by positive constants we write $f \approx g$ and we say that f and g are equivalent functions. Constants, denoted by C, are assumed to be positive and independent of the functions involved and may be different at different places. χ_E is the characteristic function of the set E.

Finally, inequalities, such as (1), are interpreted in the sense that if the right side is finite, so is the left side and the inequality holds and we assume that P and Q are modular functions.

§2. Modular interpolation results for bounded operators on L^∞.

The main result of this section is the following.

Theorem 2.1. *Let T be an S-subaditive operator and let $D \subset L_0(\mu)$ be an upper-stable set. Assume that $S : L^\infty(\mu) \to L^\infty(v)$ is bounded with constant M and that A is D-admissible for S. Then T satisfies a (P, Q) modular inequality on D, for every P and Q such that, for some $0 < c \le 1$,*

$$\int_0^{Mt/(1-c)} A\left(\frac{t}{cy}\right) dP(y) \le Q(t). \tag{2}$$

Proof. Let $f \in D$ and let us write $f = f_1 + f_2$, where

$$f_1 = f(x)\chi_{\{x;\, |f(x)| > (1-c)y/M\}},$$

with $0 < c \le 1$ as in (2) and $y > 0$ fixed. Then,

$$\lambda^v_{Tf}(y) \le \lambda^v_{Sf_1}(cy) + \lambda^v_{Sf_2}((1-c)y).$$

Now, since $\|Sf_2\|_\infty \le M\|f_2\|_\infty \le (1-c)y$, we obtain that $\lambda^v_{Sf_2}((1-c)y) = 0$ and hence, since D is upper-stable, $f_1 \in D$ and we get by (1) that

$$\begin{aligned}\int_{\mathcal{N}} P(|Tf(x)|)\,dv(x) &= \int_0^\infty \lambda^v_{Tf}(y)\,dP(y) \le \int_0^\infty \lambda^v_{Sf_1}(cy)\,dP(y)\\ &\le \int_0^\infty \int_{\mathcal{M}} A\left(\frac{|f_1(x)|}{cy}\right) d\mu(x)dP(y)\\ &\le \int_0^\infty \int_{\{x;\,|f(x)|>(1-c)y/M\}} A\left(\frac{|f(x)|}{cy}\right) d\mu(x)dP(y)\\ &= \int_{\mathcal{M}} \left[\int_0^{|f(x)|M/(1-c)} A\left(\frac{|f(x)|}{cy}\right) dP(y)\right] d\mu(x).\end{aligned}$$

Using (2), we get that the above expression can be majorized by $\int Q(|f(x)|)\,d\mu(x)$ and the result follows. $\square$

Remark 2.2.

(a) If we choose $P_\alpha(y) = \chi_{(\alpha,\infty)}$, we get that $dP_\alpha(y) = d\delta_\alpha$ and hence

$$\int_0^\infty A\left(\frac{t}{y}\right) d\delta_\alpha(y) = A(t/\alpha).$$

Consequently, if T satisfies a (P,Q) modular inequality on D, for every P and Q satisfying (2), we obtain that A is a D-admissible for T. That is, the previous result characterizes the admissible functions in terms of modular inequalities for subaditive operators $(S = T)$.

(b) From Theorem 2.1, one can easily see that the result remains true for operators T such that

$$|T(f+g)(x)| \le S_1 f(x) + S_2 g(x),$$

where $S_2 : L^\infty(\mu) \to L^\infty(v)$ is bounded with constant M and A is D-admissible for S_1. This can be useful in the following situation: let us consider $Tf(t) = (Sf)^*(t)$ for some appropriate subaditive operator S. Then $T(f+g) \le (Sf)^*(\alpha t) + (Sg)^*((1-\alpha)t)$ for every $0 \le \alpha \le 1$. Sometimes, the conclusion of Theorem 2.1 can be improved by choosing in a proper way this constant α.

(c) The result of Theorem 2.1 is optimal in the sense that, if

$$Q_{A,P}(t) := \int_0^{2Mt} A\left(\frac{2t}{y}\right) dP(y),$$

there is no R so that $R(t) \le Q_{A,P}(t)$ with $L_R \ne L_{Q_{A,P}}$ and such that every T satisfying the hypothesis of the theorem satisfies a (P,R) modular inequality. To see this, it is enough to consider the average $Tf(x) = (1/x)\int_0^x f$, $A(t) = t$ and $P(y) = y^p$.

(d) If $T : L^\infty(\mu) \to L^\infty(v)$ is bounded with norm M and $T : L^p(\mu) \to L^{q,\infty}(v)$ is bounded with norm 1, then, using Example (3), Theorem 2.1 and Remark 2.2-(b), we can conclude that, for every $r > p$, $T : L^r(\mu) \to L^{rq/p,r}(v)$ with norm less than or equal to

$$\left(\frac{r^{r+1}}{p^{p+1}}\,q\,\frac{M^{r-p}}{(r-p)^{r-p+1}}\right)^{1/r}.$$

Corollary 2.3. *Let T be an S-subaditive operator so that $T : L^\infty(\mu) \to L^\infty(\nu)$ is bounded with norm M and let us assume that, for some $\alpha \geq 0$, $A(t) = t^q(1 + (\log^+ t)^\alpha)$ is D-admissible for T, with D an upper-stable set. Then, T is bounded on $D \cap L^p$ for every $q < p < \infty$ with constant*

$$p2^p \left[\frac{M^{p-q}}{p-q} + \frac{1}{(p-q)^{\alpha+1}} \int_{(p-q)\log^+(1/M)}^{\infty} e^{-s} s^\alpha \, ds \right].$$

In many cases, it is also important to consider the weak-type spaces

$$L_{P,\infty}(\nu) = \left\{ f; \sup_{y>0} P(y) \lambda_f^\nu(y) < \infty \right\}.$$

We shall say that $T : D \cap L_Q(\mu) \to L_{P,\infty}(\nu)$ is bounded if the following inequality holds, for every $f \in D$,

$$\sup_{y>0} P(y) \lambda_{Tf}^\nu(y) \leq \int_{\mathcal{M}} Q(|f(x)|) \, d\mu(x).$$

The weak type version of Theorem 2.1 is the following:

Theorem 2.4. *If T satisfies the hypothesis of Theorem* 2.1, *then $T : D \cap L_Q(\mu) \to L_{P,\infty}(\nu)$ is bounded for every P and Q such that, for some $0 < c \leq 1$,*

$$\sup_{y \leq Mt/(1-c)} \left[A\left(\frac{t}{cy}\right) P(y) \right] \leq Q(t). \tag{3}$$

Proof. Let $f \in D$ and let us write $f = f_1 + f_2$ as in the proof of Theorem 2.1. Then,

$$\begin{aligned}
\sup_y \lambda_{Tf}^\nu(y) P(y) &\leq \sup_y \lambda_{Sf_1}^\nu(cy) P(y) \leq \sup_y \int_{\mathcal{M}} A\left(\frac{|f_1(x)|}{cy}\right) d\mu(x) P(y) \\
&\leq \int_{\mathcal{M}} \sup_y A\left(\frac{|f_1(x)|}{cy}\right) d\mu(x) P(y) \\
&= \int_{\mathcal{M}} \sup_{y \leq |f(x)|M/(1-c)} A\left(\frac{|f(x)|}{cy}\right) P(y) \, d\mu(x),
\end{aligned}$$

and using (3), one immediately obtains that the above expression can be majorized by $\int_{\mathcal{M}} Q(|f(x)|) \, d\mu(x)$. $\square$

Similarly to Corollary 2.3, we obtain the following:

Corollary 2.5. *Let T be an S-subaditive operator such that T is bounded on L^∞ with norm M and let us assume that, for some $\alpha \geq 0$, $A(t) = t^q(1 + (\log^+ t)^\alpha)$ is a D-admissible function with D upper-stable. Then, for every $q < p < \infty$, T is bounded from $L^p \cap D$ into $L^{p,\infty}$, with constant*

$$2^p \left[M^{p-q} + \left(\frac{\alpha}{e(p-q)} \right)^\alpha \right],$$

if $M \geq 1$ *or* $e^{-\alpha/(p-q)} < M \leq 1$, *and*

$$2^p M^{p-q}\left[1 + \left(\log\frac{1}{M}\right)^{\alpha}\right],$$

if $M < \min(1, e^{-\alpha/(p-q)})$.

Let us now consider an example in the setting of the so-called Yano extrapolation theory (see [**Y**]). Let T be such that, for every $1 < p \leq 2$,

$$\lambda_{Tf}(y) \leq \frac{1}{p-1}\frac{\|f\|_p^p}{y^p},$$

and let us assume that $\|f\|_\infty \leq 1$. Then,

$$\lambda_{Tf}(y) \leq \inf_{1<p\leq 2}\frac{1}{p-1}\frac{\|f\|_1}{y^p} \leq \varphi(1/y)\|f\|_1,$$

where $\varphi(t) = t(1 + \log^+ t)$. Therefore, for every f such that $\|f\|_\infty \leq 1$, we have that

$$\lambda_{Tf}(y) \leq \varphi(1/y)\int A(|f(x)|)\, d\mu(x), \tag{4}$$

with $A(t) = t$. This estimate does not imply that the function A is D-admissible for T, with $D = \{\|f\|_\infty \leq 1\}$, but, with the obvious changes the above developed theory can be extended to the case of admissible triples:

DEFINITION 2.6. Let A, B and W be three positive functions so that $A \equiv 1$ or $A(0) = 0$ and the same for B. We say that the triple (A, B, W) is D-admissible for T if, for every $f \in D$,

$$\lambda_{Tf}^{\nu}(y) \leq W(y)\int_{\mathscr{M}} A\left(\frac{|f(x)|}{y}\right)B(|f(x)|)\, d\mu(x).$$

In this context of triples, Theorem 2.1 (for example) reads as follows:

THEOREM 2.7. *Let T be an S-subaditive operator and let $D \subset L_0(\mu)$ be an upper-stable set. Assume that $S : L^\infty(\mu) \to L^\infty(\nu)$ is bounded with constant M and that (A, B, W) is D-admissible for S. Then T satisfies a (P, Q) modular inequality on D, for every P and Q such that, for some $0 < c \leq 1$,*

$$B(t)\int_0^{Mt/(1-c)} W(cy)A\left(\frac{t}{cy}\right) dP(y) \leq Q(t).$$

In particular, if T satisfies (4) for every function in D where $D = \{\|f\|_\infty \leq 1\}$, then T satisfies a (P, Q) modular inequality on D for every P and Q such that

$$t\int_0^{Mt/(1-c)} \frac{1}{cy}\left(1 + \log^+\frac{1}{cy}\right) dP(y) \leq Q(t).$$

From this, it follows that we cannot take $P(y) = y$, but if we take $P(y) = (y - \alpha)_+ := \max(y - \alpha, 0)$, with $\alpha \leq 1/(M + 1)$, we get the following corollary.

Corollary 2.8. *Let T be a subaditive bounded operator on L^∞ with constant M and such that, for every $1 < p \leq 2$,*

$$\lambda^{\nu}_{Tf}(y) \leq \frac{1}{p-1} \frac{\|f\|^p_{L^p(\mu)}}{y^p}.$$

Then there exists a positive constant C_M depending only on M such that for every $\alpha \leq \min(1/(M+1), e)$ and every f such that $\|f\|_\infty \leq 1$,

$$\int_{\mathcal{N}} (|Tf(x)| - \alpha)_+ \, d\nu(x) \leq C_M \frac{1}{\alpha} \left(\log \frac{1}{\alpha}\right)^2 \int_{\mathcal{M}} |f(x)| \left(1 + \log \frac{1}{|f(x)|}\right) d\mu(x).$$

Remark 2.9. Although all the results presented in this paper can be stated in the context of triples, we have preferred, for simplicity in the presentation, to use only the concept of admissible function, since the general extension follows with the obvious changes.

§3. Modular interpolation results for distribution controlled operators.

A second type of interpolation result that was also considered (for a particular cases) in [**CH**] deals with S-subaditive operators such that $T : L_0(\mu) \to L_0(\nu)$ is bounded, in the sense that $\nu(\operatorname{supp} Sf) \leq M\mu(\operatorname{supp} f)$. In this section, we shall consider a closely related class of operators; those that we call distribution controlled operators on D and satisfy that there exists a constant $M > 0$ so that, for every $f \in D$

$$\lambda^{\nu}_{Sf}(0) \leq M \lambda^{\mu}_{f}(0).$$

Examples of distribution controlled operators are the following:

(1) Pointwise multipliers: $Tf(x) = m(x)f(x)$ with $\nu = \mu$.

(2) If $T_1 \leq T_2$ and T_2 is distribution controlled, then T_1 is also a distribution controlled operator. In particular, using (1), we obtained that if $D = \{f \text{ increasing}\}$ and $k(x,t)$ is a positive function such that $\int_0^x k(x,t)\,dt < \infty$ a.e. $x(\mu)$, then

$$Tf(x) = \int_0^x f(t)k(x,t)\,dt,$$

is distribution controlled on D with $\nu = \mu$.

(3) If D is the set of decreasing functions, then any type of the so called generalized Hardy conjugate operator

$$Tf(x) = \int_x^\infty k(x,t)f(t)\,dt$$

is distribution controlled on D, with $\nu = \mu$.

(4) Let φ be a change of variable on $\boldsymbol{R}$. Then, the operators $T : L^1(\boldsymbol{R}) \to L^1(d\varphi^{-1})$ defined by $Tf(x) = f(\varphi(x))$ and $Tf(t) = f^*(\varphi(t))$ are distribution controlled.

(5) $Tf(t) = \phi(f^*_\mu(t))$ where ϕ is subaditive is also distribution controlled with ν equals the Lebesgue measure on $\boldsymbol{R}^+$.

Theorem 3.1. *Let T be an S-subaditive operator.*

(i) *If D is upper-stable, S is a distribution controlled operator on D and A is admissible for S, or*

(ii) *D is lower-stable, S is a distribution controlled operator, and A is D-admissible for S,*

then T satisfies a (P,Q) modular inequality on D for every P and Q such that

$$MP(t) + \int_t^\infty A(t/z)\, dP(z) \le Q(t).$$

PROOF. Let $f \in D$ and let us write $f = f_1 + f_2$ where $f_1(x) = f(x)\chi_{\{x;|f(x)|>y\}}$. Then,

$$\begin{aligned}\lambda^\nu_{Tf}(y) &\le \lambda^\nu_{Sf_1}(0) + \lambda^\nu_{Sf_2}(y)\\ &\le M\lambda^\mu_{f_1}(0) + \int_{\mathcal{M}} A\left(\frac{|f_2(x)|}{y}\right) d\mu(x)\\ &\le M\lambda^\mu_f(y) + \int_{|f|\le y} A\left(\frac{|f(x)|}{y}\right) d\mu(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathcal{N}} P(|Tf(x)|)\, d\nu(x) &= \int_0^\infty \lambda^\nu_{Tf}(y)\, dP(y)\\ &\le \int_0^\infty M\lambda^\mu_f(y)\, dP(y) + \int_{\mathcal{M}} \left[\int_{|f(x)|}^\infty A\left(\frac{|f(x)|}{y}\right) dP(y)\right] d\mu(x)\\ &\le \int_{\mathcal{M}} \left(MP(|f(x)|) + \left[\int_{|f(x)|}^\infty A\left(\frac{|f(x)|}{y}\right) dP(y)\right]\right) d\mu(x)\\ &\le \int_{\mathcal{M}} Q(|f(x)|)\, d\mu(x). \qquad \square\end{aligned}$$

EXAMPLE. Let

$$Tf(x) = \int_x^\infty f(u)u^{-\alpha-1}(u-x)^\alpha\, du,$$

where $-1 < \alpha < 0$ and let us consider D the set of decreasing functions on $\boldsymbol{R}^+$. Then, one can immediately see that T is a distribution controlled on D and that $A(t) = t/(1+\alpha)$ is an admissible function for T. Therefore, T satisfies a (P,Q) modular inequality on D for every P and Q such that

$$P(t) + \frac{t}{1+\alpha}\int_t^\infty \frac{dP(z)}{z} \le Q(t).$$

Similar result holds for the operator

$$Tf(x) = w(x) \int_x^\infty f(u) \frac{du}{W(u)},$$

where $W(u) = \int_0^u w(x)\,dx$.

We also have the weak type version of the previous theorem:

THEOREM 3.2. *Let T satisfy the hypothesis of the previous theorem. Then $T : L_Q \cap D(\mu) \to L_{P,\infty}(v)$ is bounded for every P and Q such that*

$$MP(t) + \sup_{y \geq t} A(t/y)P(y) \leq Q(t).$$

§4. Modular interpolation results.

A third type of interpolation result is the following:

THEOREM 4.1. *Let T be an S-subaditive operator and let A and B be two admissible functions for S. Then, T satisfies a (P, Q) modular inequality for every P and Q such that*

$$\int_0^\infty \min(A, B)(2t/y)\,dP(y) \leq Q(t). \tag{5}$$

PROOF. The proof follows the same ideas than in the classical Marcinkiewicz theorem. By hypothesis,

$$\lambda_{Sf}^v(y) \leq \int_{\mathcal{M}} A\left(\frac{|f(x)|}{y}\right) d\mu(x)$$

and

$$\lambda_{Sf}^v(y) \leq \int_{\mathcal{M}} B\left(\frac{|f(x)|}{y}\right) d\mu(x).$$

Let $E = \{t > 0; A(t) \leq B(t)\}$ and let us write $f = f_1 + f_2$ where $f_1(x) = f(x)\chi_{\{x; 2|f(x)|/y \in E\}}$. Then,

$$\begin{aligned}
\lambda_{Tf}^v(y) &\leq \lambda_{Sf_1}^v(y/2) + \lambda_{Sf_2}^v(y/2) \\
&\leq \int_{\mathcal{M}} A\left(\frac{2|f_1(x)|}{y}\right) d\mu(x) + \int_{\mathcal{M}} B\left(\frac{2|f_2(x)|}{y}\right) d\mu(x) \\
&= \int_{\mathcal{M}} \min(A, B)\left(\frac{2|f(x)|}{y}\right) d\mu(x),
\end{aligned}$$

and the rest of the proof follows easily. □

THEOREM 4.2. *Under the hypothesis of the previous theorem, $T : L_Q(\mu) \to L_{P,\infty}(v)$ is bounded for every P and Q such that*

$$\sup_{y>0}[\min(A, B)(2t/y)P(y)] \leq Q(t).$$

REMARK 4.3.

(a) If $P_\alpha(y) = \chi_{(\alpha,\infty)}$, then the left hand side of (5) is equal to $\min(A,B)(2t/\alpha)$ and therefore the converse of Theorem 4.1 is essentially true whenever $S = T$, in the sense that if T satisfies a (P,Q) modular inequality for every P and Q satisfying (5), then both $A(2t)$ and $B(2t)$ are admissible functions for T.

(b) If $A(t) = C_0 t^{p_0}$ and $B(t) = C_1 t^{p_1}$ as it happens in the classical Marcinkiewicz case, we get that T is bounded on L^p with constant

$$2\left(\frac{p}{p-p_0} + \frac{p}{p-p_1}\right)^{1/p} C_0^{p-p_1/(p(p_1-p_0))} C_1^{p-p_0/(p(p_1-p_0))}.$$

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