

Construction of non-simply connected CMC surfaces via dressing

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Abstract. We investigate surfaces of constant mean curvature with a nontrivial fundamental group in a Weierstraß type representation. As an application we construct a family of complete CMC immersions which are conformally cylinders and have umbilics.

1. Introduction.

In the last 10 years, many results on non-simply connected CMC surfaces have been obtained:

- CMC tori have been classified [35], [1], [31], [14], [20] and explicitly constructed in terms of theta functions [2].
- Kapouleas [22], [21] provided existence proofs for compact CMC surfaces of genus $g \geq 2$.
- Examples of compact and non-compact CMC surfaces with a large symmetry group in \mathbf{R}^3 were proved to exist by Karcher [23] and Große-Brauckmann [16], and constructed numerically in [17].
- Complete embedded CMC surfaces in \mathbf{R}^3 were investigated in [27].

Nevertheless, the status of the investigation of CMC surfaces of nontrivial topological type, with the exception of the case of CMC tori, is still highly unsatisfactory. First of all, Kapouleas' proof gives at most a hint on the visual appearance of compact CMC surfaces of higher genus (for a discussion see [26]). Explicit construction, and with this we mean also visualization, of his surfaces is not yet possible. All existing pictures of (probably) compact CMC surfaces of genus $g \geq 2$ are produced using Große-Brauckmann's approach. Although very carefully implemented, the numerical methods do not provide the viewer with an analytic proof that the surfaces shown are really compact CMC surfaces. Furthermore, both Große-Brauckmann's and Kapouleas' approach describe only very special examples of such CMC surfaces.

Another problem with the results mentioned above is that they do not give much information on the Hopf differential, in particular on the location or number of umbilics of the CMC surfaces constructed. For compact CMC surfaces of genus g the number of umbilics is $4g - 4$. Pinkall and Sterling's classification of CMC tori [31] relies heavily on the fact that CMC tori have no umbilics. Their method and also Bobenko's construction of these surfaces fails as soon as umbilics are introduced. On the other side, umbilics don't play any role in Kapouleas' or Große-Brauckmann's

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work. Since umbilics and the Hopf differential are a fundamental concept in the investigation of CMC surfaces, it is very interesting to note that even the following basic question has not been answered:

Are there complete CMC surfaces with umbilics which are conformally cylinders?
(1.1.1)

The situation for minimal surfaces is quite different. In this case, the Weierstraß representation provides a unified way to represent the surfaces by meromorphic data. Investigation of minimal surfaces of certain topological type can therefore be translated to the problem of constructing appropriate automorphic Weierstraß data. As soon as such data are found they can be used directly for the visualization of the surfaces.

In this paper, we use an analogue of the Weierstraß representation for CMC surfaces, the DPW representation [11], to construct non-simply connected CMC surfaces in \mathbf{R}^3 . To be more precise, it is our goal to construct complete conformal CMC immersions $\Psi : \mathcal{D} \rightarrow \mathbf{R}^3$, \mathcal{D} being the open unit disk or the complex plane, which are invariant under a given Fuchsian group $\Gamma \subset \text{Aut } \mathcal{D}$, i.e. $\Psi \circ \gamma = \Psi$ for all $\gamma \in \Gamma$. Such an immersion yields a CMC surface in \mathbf{R}^3 whose fundamental group contains Γ (see [7]). Since the Hopf differential of the surface is data used in the DPW construction, we can fix an arbitrary quadratic differential form on $\Gamma \backslash \mathcal{D}$ as Hopf differential, i.e., we can effectively prescribe the number and location of umbilics of the CMC surface in the parameter domain.

The main application of the ideas presented in this paper is to construct CMC-cylinders with umbilics. To make sure these cylinders constructed are not only CMC immersions of \mathbf{C} with some point deleted we show that they are complete. To make sure they are not—after straightforward manipulations like a change of coordinates, etc.—just cylinders, which one could just as well obtain by the classical (and particularly beautiful) algebro-geometric method, we have inserted umbilics. Since this paper was originally completed, its ideas have been used and generalized for CMC-cylinders [24], [25] and CMC-trinoids [13], [32]. We are very pleased to note that the referee of this paper sees additional motivation for this paper (“the authors should spell out more clearly that the central issue is the period closing problem, and that the present paper makes some contribution to this”).

Our ultimate goal is—ideally, perhaps never fully reachable—to provide a theory that allows to produce CMC surfaces with prescribed geometric features: as theoretical objects and as concrete pictures. Ideally, again, this should admit concrete applications, like producing surfaces representing answers to specific questions in material sciences or biology. Thus we are looking for a theory which admits “easily” concrete visualizations. On the technical side this means to integrate fundamental groups (and probably also additional symmetries) into a formalism that admits “easy” computer generated visualizations. The path to this goal seems long and arduous at this point. We are convinced that the loop group approach, originally presented in [11], has the potential to help reach this goal. The present paper is a first step towards this goal: we are able to identify a condition that makes sure the surfaces “close up”. Moreover, we show that (at least some of) the constructed surfaces are complete, proving that, e.g., we are not just deleting a point from a previously known surface. Since the submission

of this paper, progress has been made towards our overall goal in several ways: in his dissertation [24], M. Kilian is using and extending the methods of this paper and produces a large family of CMC-cylinders exhibiting various geometric properties. He has also developed algorithms that lead to concrete visualizations of CMC-cylinders with various properties (accessible at this point through www.gang.umass.edu).

In a different development, one of the authors of this paper (JD) and H. Wu have developed, starting from the ideas contained in this paper, an algorithm to construct CMC-trinoids with embedded ends [13]. As a matter of fact, this construction seems to produce all CMC-trinoids with embedded ends—and seems to be able to handle more general “end properties”, like nodoidal behaviour, as well. It seems that more than three ends will also be manageable by this method. An overwhelming amount of questions concerning the relation between geometric properties of CMC-surfaces and their “DPW-potentials” is still open. The authors hope that this paper will make a useful step towards an answer of some of these questions.

The paper starts in Section 2 by recalling the basic facts for the DPW representation of CMC surfaces. In particular, we introduce meromorphic potentials which replace the usual Weierstraß data in the DPW method. We also summarize briefly in Section 2.3, how intrinsic and extrinsic symmetries of a CMC immersion are expressed in terms of meromorphic potentials [7], [8]. In Section 3 we translate the symmetry condition of Section 2 to the so called holomorphic potentials of the DPW method. This leads to conditions on the holomorphic DPW data. While Section 3 deals primarily with dressing from the r -circle, Section 4 shows under which additional conditions the same result can be obtained by dressing from the unit circle. In Section 5 we will use dressing actions to satisfy these conditions for the special case of singly periodic CMC immersions $\Psi : \mathbf{C} \rightarrow \mathbf{R}^3$. We thus construct an infinite dimensional family of conformal CMC cylinders with umbilics, thereby answering the question (1.1.1) in the affirmative. Moreover, we show that a particularly natural class of such cylinders is complete. Part of this work was completed while one of the authors (JD) visited the Sonderforschungsbereich 288. He would like to thank Ulrich Pinkall for his interest in this work and the Sonderforschungsbereich 288 for its hospitality. Finally we would like to thank Ivan Sterling for helpful discussions.

2. Basic definitions.

We begin by collecting some well known results on loop groups and the dressing action. For further reference see [6, Section 2].

2.1. For each real constant r , $0 < r \leq 1$, let $A_r SL(2, \mathbf{C})_\sigma$ denote the group of smooth maps $g(\lambda)$ from C_r , the circle of radius r , to $SL(2, \mathbf{C})$, which satisfy the twisting condition

$$g(-\lambda) = \sigma(g(\lambda)), \quad (2.1.1)$$

where $\sigma : SL(2, \mathbf{C}) \rightarrow SL(2, \mathbf{C})$ is defined by conjugation with the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The Lie algebras of these groups, which we denote by $A_r sl(2, \mathbf{C})_\sigma$,

consist of maps $x : C_r \rightarrow sl(2, \mathbf{C})$, which satisfy a similar twisting condition as the group elements

$$x(-\lambda) = \sigma_3 x(\lambda) \sigma_3. \tag{2.1.2}$$

In order to make these loop groups complex Banach Lie groups, we require that each matrix coefficient, considered as a function on S^1 , is contained in the Wiener Algebra \mathcal{A} , $\mathcal{A} = \{q(\lambda); q = \sum_{n \in \mathbf{Z}} q_n \lambda^n, \sum_{n \in \mathbf{Z}} |q_n| < \infty\}$.

For $r = 1$ we will always omit the subscript “ r ”.

Furthermore, we will use the following subgroups of $A_r SL(2, \mathbf{C})_\sigma$: Let \mathbf{B} be a subgroup of $SL(2, \mathbf{C})$ and $A_{r, \mathbf{B}}^+ SL(2, \mathbf{C})_\sigma$ be the group of maps in $A_r SL(2, \mathbf{C})_\sigma$, which can be extended to holomorphic maps on

$$I^{(r)} = \{\lambda \in \mathbf{C}; |\lambda| < r\}, \tag{2.1.3}$$

the interior of the circle C_r , and take values in \mathbf{B} at $\lambda = 0$. Analogously, let $A_{r, \mathbf{B}}^- SL(2, \mathbf{C})_\sigma$ be the group of maps in $A_r SL(2, \mathbf{C})_\sigma$, which can be extended to the exterior

$$E^{(r)} = \{\lambda \in \mathbf{C}P_1; |\lambda| > r\} \tag{2.1.4}$$

of C_r and take values in \mathbf{B} at $\lambda = \infty$. If $\mathbf{B} = \{I\}$ (based loops) we write the subscript $*$ instead of \mathbf{B} , if $\mathbf{B} = SL(2, \mathbf{C})$ we omit the subscript \mathbf{B} entirely.

Also, by an abuse of notation, we will denote by $A_r SU(2)_\sigma$ the subgroup of maps in $A_r SL(2, \mathbf{C})_\sigma$, which can be extended holomorphically to the open annulus

$$A^{(r)} = \left\{ \lambda \in \mathbf{C}; r < |\lambda| < \frac{1}{r} \right\} \tag{2.1.5}$$

and take values in $SU(2)$ on the unit circle.

Corresponding to these subgroups, we analogously define Lie subalgebras of $A_r sl(2, \mathbf{C})_\sigma$.

We quote the following results from [30] and [11]:

(i) For each solvable subgroup \mathbf{B} of $SL(2, \mathbf{C})$, which satisfies $SU(2) \cdot \mathbf{B} = SL(2, \mathbf{C})$ and $SU(2) \cap \mathbf{B} = \{I\}$, multiplication

$$A_r SU(2)_\sigma \times A_{r, \mathbf{B}}^+ SL(2, \mathbf{C})_\sigma \rightarrow A_r SL(2, \mathbf{C})_\sigma$$

is a diffeomorphism onto. The associated splitting

$$g = Fg_+ \tag{2.1.6}$$

of an element g of $A_r SL(2, \mathbf{C})_\sigma$ such that $F \in A_r SU(2)_\sigma$ and $g_+ \in A_{r, \mathbf{B}}^+ SL(2, \mathbf{C})_\sigma$ will be called Iwasawa decomposition.

(ii) Multiplication

$$A_{r, *}^- SL(2, \mathbf{C})_\sigma \times A_r^+ SL(2, \mathbf{C})_\sigma \rightarrow A_r SL(2, \mathbf{C})_\sigma \tag{2.1.7}$$

is a diffeomorphism onto the open and dense subset $A_{r, *}^- SL(2, \mathbf{C})_\sigma \cdot A_r^+ SL(2, \mathbf{C})_\sigma$ of $A_r SL(2, \mathbf{C})_\sigma$, called the “big cell” [33]. The associated splitting

$$g = g_- g_+ \tag{2.1.8}$$

of an element g of the big cell, where $g_- \in A_{r,*}^- SL(2, \mathbf{C})_\sigma$ and $g_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$, will be called Birkhoff factorization.

2.2. Let $\Psi_\lambda : \mathcal{D} \rightarrow \mathbf{R}^3$, $\lambda \in S^1$ be an associated family of (conformal) immersions of constant mean curvature $\mathbf{H} \neq 0$ (see [7, Section 2.5]). Define the extended frame $F(z, \lambda) : \mathcal{D} \rightarrow ASU(2)_\sigma$, $\lambda \in S^1$, as in [11] (see also the appendix of [9]). Then Ψ_λ can be computed from F using Sym’s formula

$$\Psi_\lambda = -\frac{1}{2\mathbf{H}} \left(\frac{d}{dt} F \cdot F^{-1} + \frac{i}{2} F \sigma_3 F^{-1} \right), \quad \lambda = e^{it}. \tag{2.2.1}$$

Furthermore, define $g_-(z, \lambda) : \mathcal{D} \rightarrow A_{r,*}^- SL(2, \mathbf{C})_\sigma$, $\lambda \in S^1$, by the Birkhoff splitting

$$F(z, \lambda) = g_-(z, \lambda)g_+(z, \lambda). \tag{2.2.2}$$

Then g_- is for every fixed $\lambda \in S^1$ a meromorphic function on \mathcal{D} with poles in the set $\mathcal{S} \subset \mathcal{D}$ of points, where $F(z, \lambda)$ is not in the “big cell”, i.e. where the Birkhoff splitting (2.2.2) of $F(z, \lambda)$ is not defined.

REMARK. By [6, Lemma 2.2], g_- can be analytically continued in λ to $\mathbf{C}P_1 \setminus \{0\}$ for all fixed $z \in \mathcal{D}$, for which it is finite, but is only meromorphic in $z \in \mathcal{D}$ and the maximal analytic continuation of g_- in λ does not depend on the chosen radius $0 < r \leq 1$. I.e., the meromorphic potential of a CMC immersion does not depend on r . We will therefore always consider g_- as a map into $A_*^- SL(2, \mathbf{C})_\sigma = A_{1,*}^- SL(2, \mathbf{C})_\sigma$ which, if necessary, can be continued analytically to all circles C_r with $0 < r \leq 1$.

For given meromorphic g_- we can recover the extended frame F by the ($r = 1$) Iwasawa decomposition

$$g_- = Fg_+^{-1}. \tag{2.2.3}$$

For smoothness questions, see [9].

Next, we define the dressing action of $A_r^+ SL(2, \mathbf{C})_\sigma$, $0 < r \leq 1$, on \mathcal{F} , the set of extended frames of CMC-immersions. For $F(z, \lambda) \in \mathcal{F}$ and $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$ we set

$$h_+(\lambda)F(z, \lambda) = (h_+.F)(z, \lambda)q_+(z, \lambda), \tag{2.2.4}$$

where the right hand side of (2.2.4) is defined by the Iwasawa decomposition in $A_r SL(2, \mathbf{C})_\sigma$ of $h_+.F$, i.e. $q_+ : \mathcal{D} \rightarrow A_r^+ SL(2, \mathbf{C})_\sigma$. In addition, at $\lambda = 0$ the matrix $q_+(z, \lambda)$ takes values in the solvable subgroup B of $SL(2, \mathbf{C})$ such that

$$B \cap SU(2) = \{I\}. \tag{2.2.5}$$

It is easily proved (see e.g. [4]) that $h_+.F$ is again in \mathcal{F} . Therefore, Equation (2.2.4) really defines an action on \mathcal{F} . On the matrices g_- defined by (2.2.2) the dressing is defined by

$$h_+(\lambda)g_-(z, \lambda) = \hat{g}_-(z, \lambda)p_+(z, \lambda). \tag{2.2.6}$$

Here, $\hat{g}_- = h_+.g_-$ and $p_+ : \mathcal{D} \rightarrow A_r^+ SL(2, \mathbf{C})_\sigma$ are defined by the Birkhoff splitting (2.2.2) of $h_+.g_-$. Note that $h_+.F = \hat{g}_-\hat{g}_+$ for some $\hat{g}_+ : \mathcal{D} \rightarrow A_r^+ SL(2, \mathbf{C})_\sigma$. Since g_- and \hat{g}_- are both meromorphic in z , also $p_+ = \hat{g}_+q_+g_+^{-1}$ is meromorphic in z .

The extended frames are normalized by

$$F(0, \lambda) = I, \quad \lambda \in S^1, \quad (2.2.7)$$

which implies

$$g_-(0, \lambda) = I, \quad \lambda \in S^1. \quad (2.2.8)$$

Let now the meromorphic potential be defined by

$$\xi(z, \lambda) = g_-^{-1} dg_-, \quad (2.2.9)$$

then it is of the form

$$\xi(z, \lambda) = \lambda^{-1} \begin{pmatrix} 0 & f \\ E/f & 0 \end{pmatrix} dz, \quad (2.2.10)$$

where f is a nonvanishing meromorphic function. We will always assume $E \neq 0$, i.e. we will exclude the case that the surface is part of a round sphere. To construct a CMC-immersion from a given meromorphic potential of the form (2.2.10), the functions f and E cannot be chosen arbitrarily. They have to satisfy additional conditions, given in [9].

The matrix g_- and therefore also the frame F are uniquely determined by the meromorphic potential and the initial condition (2.2.7).

From Equation (2.2.6) it follows that ξ transforms under dressing with $h_+ \in A_r^+ SL(2, \mathbb{C})_\sigma$ as

$$h_+ \cdot \xi = p_+ \xi p_+^{-1} - dp_+ \cdot p_+^{-1}. \quad (2.2.11)$$

Note that $h_+ \cdot \xi$ is again off-diagonal and that the product of the off-diagonal terms of $h_+ \cdot \xi$ is again E . Thus $E dz^2$, the Hopf differential of the CMC-immersions Ψ_λ , is invariant under dressing, and we can write

$$h_+ \cdot \xi = \lambda^{-1} \begin{pmatrix} 0 & h_+ \cdot f \\ E/(h_+ \cdot f) & 0 \end{pmatrix} dz. \quad (2.2.12)$$

2.3. In [8] we gave a detailed discussion of the transformation properties of the meromorphic potential under a symmetry of the associated CMC immersion. We summarize here those definitions and results of [8] which we will use in this paper.

DEFINITION. Let $\Psi_\lambda : \mathcal{D} \rightarrow \mathbf{R}^3$ be an associated family of CMC immersion. We define the group of symmetries of Ψ_λ as the set $\text{Aut}_{\Psi_\lambda} \mathcal{D} \subset \text{Aut} \mathcal{D}$ of biholomorphic automorphisms γ of \mathcal{D} under which Ψ_λ transforms like

$$\Psi_\lambda \circ \gamma = T(\gamma, \lambda) \circ \Psi_\lambda, \quad (2.3.1)$$

where $T(\gamma, \lambda)$ is in the group of proper Euclidean motions in \mathbf{R}^3 . It was shown in [8, Proposition 2.4] that $\text{Aut}_{\Psi_\lambda} \mathcal{D}$ does not depend on λ , i.e. $\text{Aut}_{\Psi_\lambda} \mathcal{D} = \text{Aut}_{\Psi_1} \mathcal{D} = \text{Aut}_\Psi \mathcal{D}$ for all $\lambda \in S^1$.

THEOREM. Let $\Psi_\lambda : \mathcal{D} \rightarrow \mathbf{R}^3$ be an associated family of CMC immersion and $\gamma \in \text{Aut} \mathcal{D}$. Then the following statements are equivalent:

1. $\gamma \in \text{Aut}_\Psi \mathcal{D}$.
2. The extended frame $F : \mathcal{D} \rightarrow ASU(2)_\sigma$ transforms under γ like

$$F(\gamma(z), \lambda) = \chi(\gamma, \lambda)F(z, \lambda)k(\gamma, z), \tag{2.3.2}$$

where $\chi \in ASU(2)_\sigma$ is unitary and $k : \mathcal{D} \rightarrow A^+U(1) \subset ASL(2, \mathbf{C})_\sigma$ is a λ -independent diagonal matrix.

3. The map $g_- : \mathcal{D} \rightarrow A^-SL(2, \mathbf{C})$ obtained by the Birkhoff splitting $F = g_-g_+$ transforms under γ like

$$g_-(\gamma(z), \lambda) = \chi(\gamma, \lambda)g_-(z, \lambda)r_+(\gamma, z, \lambda), \tag{2.3.3}$$

with χ as above and $r_+ : \mathcal{D} \rightarrow A^+SL(2, \mathbf{C})_\sigma$.

If one of the three conditions above is satisfied, then the meromorphic potential $\xi = g_-^{-1} dg_-$ of Ψ_λ transforms under γ like

$$\gamma^*\xi = \xi \circ \gamma \cdot \gamma^1 = r_+^{-1}\xi r_+ + r_+^{-1} dr_+, \tag{2.3.4}$$

with r_+ as in (2.3.3). Furthermore, the Euclidean motion $T(\gamma, \lambda)$ is given in the spinor representation by conjugation with $\chi(\gamma, \lambda) \in SU(2)$ and subsequent addition of $(d\chi(\gamma, \lambda = e^{it})/dt)\chi(\gamma, \lambda)^{-1} \in su(2)$.

REMARK. 1. The proof of the theorem above can be found in Sections 2.3–2.6 of [8].

2. As was already remarked in [8], comparing Equations (2.3.4) and (2.2.11) shows that as long as ξ is defined at $\gamma(0)$, on the level of the meromorphic potential a symmetry transformation amounts to a dressing transformation with the positive Birkhoff splitting part of the monodromy χ .

3. Invariant holomorphic potentials.

In the last section we recollected the characterization of all meromorphic potentials associated with CMC immersions admitting a given group $\Gamma \subset \text{Aut } \mathcal{D}$ as group of symmetries. Unfortunately, the condition (2.3.3) is difficult to verify. Therefore, in this chapter we pursue another avenue. First we simplify (2.3.3).

3.1. In [11] it was shown that to each extended frame $F : \mathcal{D} \rightarrow ASU(2)_\sigma$ there exists a map $W_+ : \mathcal{D} \rightarrow A^+SL(2, \mathbf{C})_\sigma$ such that

$$H = FW_+ : \mathcal{D} \rightarrow ASL(2, \mathbf{C})_\sigma \tag{3.1.1}$$

is holomorphic. The differential $\eta = H^{-1} dH$ is called holomorphic potential of F . As opposed to the meromorphic potential, it is not uniquely determined by F . Without loss of generality we will always normalize

$$H(z = 0, \lambda) = W_+(z = 0, \lambda) = I. \tag{3.1.2}$$

The connection of η with the meromorphic potential ξ of F is given by

$$\eta = V_+^{-1}\xi V_+ + V_+^{-1} dV_+, \tag{3.1.3}$$

where $V_+ = g_+W_+$, i.e. $H = FW_+ = g_-V_+$. Thus, a holomorphic potential is always of the form

$$\eta = \sum_{n=-1}^{\infty} \lambda^n \eta_n dz, \quad \eta_n : \mathcal{D} \rightarrow sl(2, \mathbf{C}), \tag{3.1.4}$$

where η_{-1} is of the form

$$\eta_{-1} = \begin{pmatrix} 0 & q(z) \\ E(z)/(q(z)) & 0 \end{pmatrix} \tag{3.1.5}$$

with a holomorphic function $q : \mathcal{D} \rightarrow \mathbf{C}$.

Conversely, given a holomorphic differential η of the form (3.1.4) we can define a map $F : \mathcal{D} \rightarrow ASU(2)_\sigma$ by Iwasawa splitting of the solution H of

$$H^{-1} dH = \eta, \quad H(z = 0, \lambda) = I. \tag{3.1.6}$$

As was shown in [11, Lemma 4.2], F is the extended frame of a CMC immersion $\Psi : \mathcal{D} \rightarrow \mathbf{R}^3$ which may have branchpoints over \mathcal{D} . From [9, Theorem A.8] there follows a simple criterion under which condition on η the point $z = z_0 \in \mathcal{D}$ is a branchpoint of the associated immersion Ψ :

THEOREM. *Let $\eta = \sum_{n=-1}^{\infty} \lambda^n \eta_n dz$ be a holomorphic potential and let Ψ be the corresponding CMC immersion constructed above. Moreover, define the holomorphic function $q(z)$ by (3.1.5). Then Ψ has a branchpoint at $z = z_0 \in \mathcal{D}$ if and only if $q(z_0) = 0$.*

PROOF. Let H be the solution of (3.1.6) and define $F = HW_+^{-1}$ with some $W_+ : \mathcal{D} \rightarrow A^+SL(2, \mathbf{C})_\sigma$. Then, as in the proof of [9, Theorem A.8], it is easy to see that the function q is given by

$$q(z) = -\frac{1}{2} \mathbf{H} e^{u(z)/2} w(z)^2, \tag{3.1.7}$$

where $w(z)$ is defined by $W_0(z) = W_+(z, \lambda = 0) = \begin{pmatrix} w(z) & 0 \\ 0 & w(z)^{-1} \end{pmatrix}$ and $-(1/2)\mathbf{H}e^{u(z)/2}$ is the upper right entry of the λ^{-1} -coefficient of $\alpha = F^{-1} dF$. In particular, w is differentiable without zeroes on \mathcal{D} . Since $w(z) \neq 0$ for all $z \in \mathcal{D}$, and since e^u is the real metric factor of Ψ (see the appendix of [9]), it is clear that the metric is singular at $z_0 \in \mathcal{D}$ if and only if $q(z_0) = 0$. □

3.2. We start with the following simple observations:

LEMMA. *Let $\Gamma \subset \text{Aut } \mathcal{D}$ and let η be a holomorphic potential such that $\gamma^*\eta = \eta$ for all $\gamma \in \Gamma$. Then the following holds:*

1. *If H is the solution of (3.1.6) then $H(\gamma(z), \lambda) = \rho(\gamma, \lambda)H(z, \lambda)$, with $\rho(\gamma, \lambda) = H(\gamma(0), \lambda) \in ASL(2, \mathbf{C})_\sigma$ for all $\gamma \in \Gamma$.*
2. *The map $\rho : \Gamma \rightarrow ASL(2, \mathbf{C})_\sigma$ is a group homomorphism, i.e.*

$$\rho(\gamma_2 \circ \gamma_1, \lambda) = \rho(\gamma_2, \lambda)\rho(\gamma_1, \lambda). \tag{3.2.1}$$

3. If $H = FW_+$, $F \in ASU(2)_\sigma$, $W_+ \in A^+SL(2, \mathbf{C})_\sigma$, and $\rho = \chi \in ASU(2)_\sigma$, then for all $\gamma \in \Gamma$

$$F(\gamma(z), \lambda) = \chi(\gamma, \lambda)F(\gamma, \lambda)k(\gamma, z), \tag{3.2.2}$$

where $k(\gamma, z)$ is unitary and diagonal (and independent of λ).

4. If $\rho = \chi \in ASU(2)_\sigma$, then for the immersion $\Psi_\lambda(z)$ associated with $F = F(z, \lambda)$ and all $\gamma \in \Gamma$ we have

$$\Psi_\lambda(\gamma(z)) = \chi(\gamma, \lambda)\Psi_\lambda(z)\chi(\gamma, \lambda)^{-1} - \frac{1}{2\mathbf{H}} \frac{\partial \chi(\gamma, \lambda = e^{it})}{\partial t} \cdot \chi(\gamma, \lambda)^{-1}. \tag{3.2.3}$$

PROOF. The first two parts are clear. The third and fourth part follow from the uniqueness of the Iwasawa splitting and Sym's formula (2.2.1). \square

Thus, we see that for $\gamma \in \text{Aut } \mathcal{D}$ every automorphic meromorphic potential $\eta = \gamma^*\eta$ with unitary monodromy $\rho = H(\gamma(0), \lambda) \in ASU(2)_\sigma$ gives a CMC immersion Ψ such that $\gamma \in \text{Aut}_\Psi \mathcal{D}$. Indeed, the following theorem shows that a large class of symmetric CMC immersions can be obtained in this way.

THEOREM. *Let $M = \Gamma \backslash \mathcal{D}$ be a noncompact Riemann surface with Fuchsian group Γ . Then every CMC immersion $\Psi : \mathcal{D} \rightarrow \mathbf{R}^3$ with $\Gamma \subset \text{Aut}_\Psi \mathcal{D}$ is induced by a holomorphic potential η satisfying*

$$\gamma^*\eta = \eta \text{ for all } \gamma \in \Gamma. \tag{3.2.4}$$

PROOF. The proof follows [11, Appendix]. We want to find a $V_+ : \mathcal{D} \rightarrow A^+SL(2, \mathbf{C})_\sigma$ such that $H = FV_+$ is holomorphic and satisfies

$$H(\gamma(z), \lambda) = \chi(\gamma, \lambda)H(z, \lambda). \tag{3.2.5}$$

First we note that by (2.3.3)

$$g_-(\gamma(z), \lambda) = \chi(\gamma, \lambda)g_-(z, \lambda)w_+(z, \lambda)$$

for some meromorphic $w_+ : \mathcal{D} \rightarrow A^+SL(2, \mathbf{C})_\sigma$. By [11, Lemma 4.5] there exists $u_+ : \mathcal{D} \rightarrow A^+SL(2, \mathbf{C})_\sigma$ such that

$$\tilde{H} = g_-u_+ : \mathcal{D} \rightarrow ASL(2, \mathbf{C})_\sigma$$

is holomorphic. Hence,

$$\begin{aligned} \tilde{H} \circ \gamma &= (g_- \circ \gamma)(u_+ \circ \gamma) = \chi(\gamma)g_-w_+(u_+ \circ \gamma) \\ &= \chi(\gamma)\tilde{H}u_+^{-1}w_+(u_+ \circ \gamma), \end{aligned} \tag{3.2.6}$$

where $p_+ = u_+^{-1}w_+(u_+ \circ \gamma)$ is holomorphic. From [8, Theorem 2.3] we know that χ satisfies

$$\chi(\gamma_2 \circ \gamma_1, \lambda) = \pm \chi(\gamma_2, \lambda)\chi(\gamma_1, \lambda). \tag{3.2.7}$$

From this we see (also compare with [8, 2.5]) that p_+ satisfies the cocycle identity

$$p_+(\gamma_2\gamma_1, z) = \pm p_+(\gamma_1, z)p_+(\gamma_2, \gamma_1(z)). \tag{3.2.8}$$

In other words, for each λ , $p_+(\cdot, \lambda)^{-1}$ is a (group)cocycle in $Z^1(\mathfrak{U}, SL(2, \mathcal{O})/\pm I)$ (notation as in [15]), where $\mathfrak{U} = (U_\gamma)_{\gamma \in \Gamma}$, $U_\gamma = \mathcal{D}$, is a covering of M and \mathcal{O} denotes the sheaf of holomorphic functions on \mathcal{D} .

Thus, by [15, Korollar 30.5], the cocycle $p_+(\cdot, \lambda)^{-1}$ splits, i.e. for each λ there exists a holomorphic map $\alpha(\lambda) : \mathcal{D} \rightarrow SL(2, \mathbf{C})/\pm I$ such that

$$p_+(\gamma, z, \lambda) = \pm \alpha(z, \lambda) \alpha^{-1}(\gamma(z), \lambda). \tag{3.2.9}$$

If we split $\alpha = \alpha_+ \alpha_-$ according to Birkhoff, then (3.2.9) implies $\alpha_- \circ \gamma = \alpha_-$ and $\alpha_+ \circ \gamma = p_+^{-1} \alpha_+$. Therefore,

$$p_+(\gamma, z, \lambda) = \pm \alpha_+(z, \lambda) \alpha_+^{-1}(\gamma(z), \lambda), \tag{3.2.10}$$

i.e., p_+ splits in $A^+(SL(2, \mathbf{C})/\pm I)_\sigma$.

(More directly one can apply [Bungart [3]; Theorem 8.1])

By defining

$$H(z, \lambda) = \tilde{H}(z, \lambda) \alpha_+(z, \lambda) \tag{3.2.11}$$

we get a Γ -automorphic $A(SL(2, \mathbf{C})/\pm I)_\sigma$ -valued holomorphic map H on \mathcal{D} with automorphy factors $\chi(\gamma)$,

$$H(\gamma(z), \lambda) = \chi(\gamma) H(z, \lambda). \tag{3.2.12}$$

For the corresponding holomorphic potential $\eta : \mathcal{D} \rightarrow A\mathfrak{sl}(2, \mathbf{C})_\sigma$ we then get

$$(\eta \circ \gamma) \gamma' = \eta. \tag{3.2.13}$$

□

3.3. While η behaves well under Γ , the meromorphic potential ξ associated with the CMC immersion in question is off-diagonal and does only contain *one* power of λ . The latter property is very restrictive and does in general prohibit ξ to satisfy (3.2.4), but we can obtain the former for η in the sense of the following.

PROPOSITION. *Let η be as in Theorem 4.1. Then there exists a diagonal gauge $k = k(z, \lambda) \in ASU(2)_\sigma$ such that*

$$\tilde{\eta} = k^{-1} \eta k + k^{-1} dk \tag{3.3.1}$$

is off-diagonal. In this case we have

$$\gamma^* \tilde{\eta} = \kappa^{-1} \tilde{\eta} \kappa, \tag{3.3.2}$$

where $\kappa = \kappa(\gamma, \lambda)$ is diagonal and satisfies

$$k(\gamma(z), \lambda) = \kappa(\gamma, \lambda) k(z, \lambda). \tag{3.3.3}$$

PROOF. From (3.3.1) it follows that we need to have

$$\text{diag}(\eta) + k^{-1} dk = 0, \tag{3.3.4}$$

where we also want $k(z = 0, \lambda) = I$. This differential equation with initial condition has a unique solution. Moreover, since $\gamma^*(\text{diag}(\eta)) = \text{diag}(\eta)$, (3.3.3) follows. From this (3.3.2) is straightforward. □

REMARK. In Section 5 we will exhibit a class of interesting examples for which η is off-diagonal (i.e. $\kappa \equiv I$).

3.4. We have seen in Section 2.3 that the crucial condition for $\gamma \in \text{Aut } \mathcal{D}$ to be a symmetry for a CMC immersion is that $\chi(\gamma, \lambda) = F(\gamma(z), \lambda)F(z, \lambda)^{-1}$ is *unitary* and *z-independent*. If the holomorphic potential η for a CMC immersion satisfies $\gamma^*\eta = \eta$ for all $\gamma \in \Gamma$ then

$$H(\gamma(z), \lambda) = \rho(\gamma, \lambda)H(z, \lambda), \tag{3.4.1}$$

where ρ is z -independent but not necessarily unitary. Moreover, if $H(z, \lambda)$ depends analytically on $\lambda \in \mathbf{C}^*$, all coefficients of ρ are analytic for $\lambda \in \mathbf{C}^*$. In view of results for CMC tori [12, 3] we accept η for which ρ turns into a unitary matrix *after dressing* with some h_+ .

3.5. Assume h_+ defines a dressing from the r -circle then $h_+.\rho$ is of the form

$$(h_+.\rho)(\gamma, \lambda) = h_+(\lambda)\rho(\gamma, \lambda)h_+(\lambda)^{-1}, \tag{3.5.1}$$

where $\gamma \in \Gamma$, $|\lambda| = r$. Moreover, if for some $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$, $h_+.\rho = \chi$ is unitary on S^1 , then the eigenvalues $\tau_+(\lambda)$, $\tau_-(\lambda)$ of ρ are unimodular for all $\lambda \in S^1$. In other words, $\text{trace}(\rho)$ is real and satisfies

$$4 - \text{trace}(\rho(\lambda))^2 \geq 0, \quad \text{for all } \lambda \in S^1. \tag{3.5.2}$$

We will show in the Theorem below that under certain assumptions the converse statement also holds true. In preparation of this we consider the eigenvalues of the monodromy matrix ρ in more detail. Let τ_+, τ_- denote the eigenvalues of the monodromy matrix $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\tau_\pm = u \pm iv, \tag{3.5.3}$$

where

$$u = \frac{1}{2} \text{trace}(\rho) = \frac{1}{2}(a + d), \quad v = \pm \frac{1}{2} \sqrt{4 - (\text{trace}(\rho))^2}, \tag{3.5.4}$$

By assumption, we have that v is real on S^1 and

$$\det \rho = \tau_+ \tau_- = u^2 + v^2 = 1. \tag{3.5.5}$$

Since ρ is analytic in $\lambda \in \mathbf{C}^*$

$$\text{trace}(\rho), u, \text{ and } v^2 \text{ are even holomorphic functions of } \lambda \in \mathbf{C}^*. \tag{3.5.6}$$

THEOREM. *Let $\gamma \in \text{Aut } \mathcal{D}$ and η be a holomorphic potential, which is holomorphic for $\lambda \in \mathbf{C}^*$ and which satisfies $\gamma^*\eta = \eta$. Let furthermore H be defined as in (3.1.6) and set $\rho(\lambda) = H(\gamma(z), \lambda)H(z, \lambda)^{-1} = H(\gamma(0), \lambda)$. Assume further that $\rho(\lambda)$ is semisimple for all $\lambda \in S^1$. Then the following are equivalent:*

- 1) *There exists some $0 < r \leq 1$ and $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$, such that $\chi = h_+ \rho h_+^{-1} \in A_r SU(2)_\sigma$.*
- 2) *There exists some $0 < r \leq 1$ and $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$, such that the symmetry group $\text{Aut}_\Psi(\mathcal{D})$ of the surface Ψ obtained from η by dressing with h_+ contains γ .*

- 3) The eigenvalues of $\rho(\lambda)$ are unimodular for all $\lambda \in S^1$.
- 4) The trace of ρ is real and satisfies $4 - (\text{trace}(\rho(\lambda)))^2 \geq 0$ for all $\lambda \in S^1$.

If neither of b, c , nor v vanish identically, then the assumption that ρ is semisimple on S^1 can be omitted.

PROOF. We have already seen that $1) \Rightarrow 3) \Rightarrow 4)$ and $1) \Leftrightarrow 2)$ follows from Theorem 2.3. It therefore suffices to show $4) \Rightarrow 1)$.

Note that the assumptions imply that $\rho(\lambda)$ is holomorphic for all $\lambda \in \mathbf{C}^*$ and that (3.5.6) holds. It is also easy to see that a function v can be defined via the binomial series as a function on S^1 which is in the algebra of functions considered in this paper and satisfies equation (3.5.5). However, we want to show that such a function v is actually holomorphic in \mathbf{S}^* , an open, connected, and dense subset of \mathbf{C}^* : Since by assumption the holomorphic function v^2 is real and nonnegative on S^1 , it cannot have zeroes of odd order on S^1 . Thus

$$v^2 = \left(\prod_{j=1}^m (\lambda - \lambda_j)^{2k_j} \right) s \tag{3.5.7}$$

where $\lambda_1, \dots, \lambda_m$ are all the roots of v^2 on S^1 . Clearly, s is analytic on \mathbf{C}^* , whence $s \in \mathcal{A}$. By construction, s does not vanish on S^1 . Therefore, in the algebra \mathcal{A} we have the decomposition

$$s = c_0 s_- \lambda^r s_+ \tag{3.5.8}$$

where s_+ is holomorphic inside the unit disc and does not vanish there, while s_- is holomorphic outside the unit disc, including $\lambda = \infty$, and does not vanish there. Moreover, $s_+ = 1 + \lambda s_1 + \dots$ and $s_- = 1 + \lambda s_{-1} \dots$ and $c_0 \in \mathbf{C}$. Thus v^2 can be written

$$v^2 = s_1 s_2, \quad s_1 = c_0 \lambda^r \prod_{j=1}^m (\lambda - \lambda_j)^{2k_j}, \quad s_2 = s_- s_+. \tag{3.5.9}$$

We evaluate the condition that v^2 is real on S^1 :

$$\begin{aligned} \overline{v^2} &= \bar{c}_0 \lambda^{-r} \prod_{j=1}^m (\lambda^{-1} - \lambda_j^{-1})^{2k_j} \cdot \bar{s}_- \bar{s}_+ = \bar{c}_0 \lambda^{-r} \lambda^{-\sum 2k_j} \prod_{j=1}^m \lambda_j^{-2k_j} \prod_{j=1}^m (\lambda - \lambda_j)^{2k_j} \bar{s}_- \bar{s}_+ \\ &= c_0 \lambda^r \prod_{j=1}^m (\lambda - \lambda_j)^{2k_j} s_- s_+ = v^2. \end{aligned}$$

Comparing the last two products we obtain equivalently

$$r = - \sum_{j=1}^m k_j, \quad \bar{s}_- = s_+, \quad \bar{c}_0 = c_0 \prod_{j=1}^m \lambda_j^{2k_j}. \tag{3.5.10}$$

Altogether, (3.5.9) and (3.5.10) describe equivalently that v^2 is real on S^1 and does not change the sign there. Finally, using in addition that v^2 is even in λ we see

$$r \text{ is even.} \tag{3.5.11}$$

It is well-known (and easy to see) that s_+ and s_- can be written in the form $s_+ = \exp h_+$, $s_- = \exp h_-$ with $h_+, h_- \in \mathcal{A}$. Therefore, $\tilde{s}_+ = \exp(1/2)h_+$ and $\tilde{s}_- = \exp(1/2)h_-$ are roots of s_+ and s_- respectively. Due to (3.5.10) a root of s_1 is given by

$$\tilde{s}_1 = \sqrt{c_0} \lambda^{r/2} \prod_{j=1}^m (\lambda - \lambda_j)^{k_j}. \tag{3.5.12}$$

This shows altogether

$$v = \tilde{s}_1 \tilde{s}_2 \in \mathcal{A} \text{ is holomorphic on } \mathbf{S}^* \tag{3.5.13}$$

where \mathbf{S}^* is a connected, open and dense subset of \mathbf{C}^* containing an open neighbourhood of S^1 . Moreover, of every circle parallel to the unit circle, \mathbf{S}^* contains all but finitely many points. One can obtain such an \mathbf{S}^* as follows: for every root of s_+ (which lies necessarily outside of the unit disk) one removes the geodesic segment between the root and the point at infinity from the Riemann sphere, and for every root of s_- (which sits necessarily inside of the unit disk) one removes the geodesic segment from the root to 0 from the Riemann sphere. Note that actually \mathbf{S}^* is invariant under reflection in S^1 .

And in view of (3.5.6) we have in addition

$$v \text{ is even or odd in } \lambda. \tag{3.5.14}$$

Let us consider first the case, where the function $b \cdot c \cdot v$ vanishes identically on \mathbf{S}^* . If $b = 0$, then $d = \bar{a}$ on S^1 and this function has absolute value 1 there. If $a = \bar{a}$, then $c = 0$, since ρ is assumed to be semisimple and the claim is obvious. If $a \neq \bar{a}$, then we choose $0 < r \leq 1$, so that $a - \bar{a} \neq 0$ on C_r . Now the matrix

$$Y = \begin{pmatrix} 1 & 0 \\ c/(a - \bar{a}) & 1 \end{pmatrix} \tag{3.5.15}$$

is contained in $A_r SL(2, \mathbf{C})_\sigma$ and satisfies $\rho = YDY^{-1}$ with $D = \text{diag}(a, a^{-1})$.

Iwasawa splitting on the circle C_r gives $Y^{-1} = Uh_+$, with $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$ and $U \in A_r SU(2)_\sigma$, whence

$$h_+^{-1} \chi h_+ = \rho, \tag{3.5.16}$$

with

$$\chi = U^{-1}DU \in A_r SU(2)_\sigma, \tag{3.5.17}$$

proving the statement in the first case.

Assume next $c = 0$. In this case one can argue as above and the claim follows.

Finally, assume $v = 0$. In this case $\tau_+ = \tau_- = \pm 1$ and $\rho = \tau_+ I$, since ρ is semisimple. Again, the claim is trivial in this case.

Assume now $b \cdot c \cdot v$ does not vanish identically.

We choose $0 < r \leq 1$ such that the holomorphic function $b \cdot c \cdot v$ has no zeroes on the circle C_r . Note that this implies $u - iv - a \neq 0$ on the circle C_r . Otherwise $\tau_-(\lambda_0) = a(\lambda_0)$ or $\tau_+(\lambda_0) = a(\lambda_0)$ for some $|\lambda_0| = r$. The characteristic equation $(a - \tau)(d - \tau) - bc = 0$ then yields a contradiction, since we have chosen r so that $bc \neq 0$ on the circle of radius r .

1. Assume first that v is even. If we set

$$Y = \begin{pmatrix} (u + iv - d)/(2iv) & b/(u - iv - a) \\ c/(2iv) & 1 \end{pmatrix} \quad (3.5.18)$$

then $Y \in A_r SL(2, \mathbf{C})_\sigma$ and a direct computation shows that Y satisfies

$$\rho = YDY^{-1}, \quad (3.5.19)$$

where, by (3.5.5)–(3.5.13),

$$D = \begin{pmatrix} u + iv & 0 \\ 0 & u - iv \end{pmatrix} \in A_r SU(2)_\sigma. \quad (3.5.20)$$

Iwasawa splitting at the circle C_r gives $Y^{-1} = Uh_+$, with $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$ and $U \in A_r SU(2)_\sigma$, whence

$$h_+^{-1} \chi h_+ = \rho, \quad (3.5.21)$$

with

$$\chi = U^{-1}DU \in A_r SU(2)_\sigma, \quad (3.5.22)$$

proving the statement in the first case.

2. Assume now that v is an odd holomorphic function of λ . The function $f^2(\lambda) = ib(\lambda)v(\lambda)$ is even and holomorphic on \mathbf{S}^* without zeroes on C_r . It follows that $f = \sqrt{ibv}$ is a holomorphic function of λ on \mathbf{S}^{**} , which is either even or odd. The open dense subset $\mathbf{S}^{**} \subset \mathbf{S}^*$ of \mathbf{C}^* can be defined similar to the set \mathbf{S}^* . However, \mathbf{S}^{**} does contain an open neighbourhood of C_r , but may not contain S^1 , since some of the cuts may start on S^1 . If f is even we set

$$Y = \frac{i}{f} \begin{pmatrix} 0 & b \\ iv & u - a \end{pmatrix}. \quad (3.5.23)$$

If f is odd we set

$$Y = \frac{1}{f} \begin{pmatrix} b & 0 \\ u - a & iv \end{pmatrix}. \quad (3.5.24)$$

In both cases $Y \in A_r SL(2, \mathbf{C})_\sigma$ and

$$\rho = YVY^{-1}, \quad (3.5.25)$$

where, by (3.5.5)–(3.5.13),

$$V = \begin{pmatrix} u & iv \\ iv & u \end{pmatrix} \in A_r SU(2)_\sigma. \quad (3.5.26)$$

Iwasawa splitting at the circle C_r gives $Y^{-1} = Uh_+$, with $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$ and $U \in A_r SU(2)_\sigma$. Then,

$$h_+^{-1} \chi h_+ = \rho, \quad (3.5.27)$$

with

$$\chi = U^{-1}VU \in A_rSU(2)_\sigma, \tag{3.5.28}$$

proving the statement in the case of odd v . □

COROLLARY 1. *Under the assumptions of the Theorem and with the notation used in its proof, the matrices D and V are holomorphic in \mathcal{S}^* and Y is meromorphic on \mathcal{S}^{**} . Moreover, h_+ and U are meromorphic on some open dense subset of \mathcal{C}^* .*

PROOF. First note that the coefficients of ρ and the function u are holomorphic in \mathcal{C}^* . Moreover, v is holomorphic in \mathcal{S}^* by (3.5.13). Since f is holomorphic in \mathcal{S}^{**} —for a definition see the proof above—, Y is meromorphic on \mathcal{S}^{**} in all cases and does not have any singularities on the circle $|\lambda| = r$. Since $\det Y = 1$, the same holds for Y^{-1} . Consider next the Iwasawa decomposition of Y^{-1} , $Y^{-1} = Uh_+$. Since h_+ is holomorphic for $0 < |\lambda| < r$, we see that U has a meromorphic extension to $\mathcal{S}^{**} \cap \{0 < |\lambda| < r\}$. Since U is holomorphic for $r < |\lambda| < r^{-1}$, we derive that U has a meromorphic extension to $\mathcal{S}^{**} \cap \{0 < |\lambda| < r^{-1}\}$, and it is nonsingular on S^1 . The unitarity condition for U finally implies that U is meromorphic on an open dense subset of \mathcal{C}^* . As a consequence, also h_+ is meromorphic on some open dense subset of \mathcal{C}^* . □

COROLLARY 2. *We retain the assumptions of Corollary 1 and consider χ as defined in the proof of Theorem 3.5. Then χ, U, V and D have meromorphic extensions to some open dense subset of \mathcal{C}^* and are holomorphic in a neighbourhood of S^1 .*

PROOF. From Corollary 1 above we know that χ and U are meromorphic in some open dense subset of \mathcal{C}^* and that V and D are holomorphic in \mathcal{S}^* . In particular, V and D are holomorphic on a neighbourhood of S^1 . Since $U, \chi \in A_rSU(2)_\sigma$, the last claim follows. □

3.6. In the last section we gave a necessary and sufficient condition for the trace of the monodromy matrix $\rho \in A_rSL(2, \mathcal{C})_\sigma$ of an invariant holomorphic potential η to produce an associated family of CMC immersions Ψ_λ with invariant metric after dressing. For the immersion Ψ_λ to factor through $M = \Gamma \backslash \mathcal{D}$ we need additional conditions.

THEOREM. *We retain the assumptions of Theorem 3.5. Then for any $h_+ \in A_rSL(2, \mathcal{C})_\sigma$, $0 < r \leq 1$, such that $\chi = h_+\rho h_+^{-1} \in A_rSU(2)_\sigma$ and any $\lambda_0 \in S^1$ the following are equivalent:*

1. *The surface Ψ_{λ_0} factors through γ , i.e. $\Psi_{\lambda_0} \circ \gamma = \Psi_{\lambda_0}$.*
2. *$\chi(\lambda_0) = \pm I$ and $\left. \frac{d\chi}{d\lambda} \right|_{\lambda=\lambda_0} = 0$.*
3. *$\text{trace}(\rho)^2 - 4$ has a zero of at least fourth order at $\lambda = \lambda_0$.*

If the matrix Y as used in the proof of Theorem 3.5 is nonsingular at $\lambda = 1$, then 2. is equivalent with

4. *$\rho(\lambda_0) = \pm I$ and $\left. \frac{d\rho}{d\lambda} \right|_{\lambda=\lambda_0} = 0$.*

PROOF. 1. \Rightarrow 2.: The assumption implies for $\lambda = \lambda_0 : \chi \cdot \psi(z) \cdot \chi^{-1} + (d/dt)\chi \cdot \chi^{-1} = \psi(z)$ for all z . Then both, the z -derivative of ψ and the \bar{z} -derivative of ψ commute with $\chi(\lambda_0)$ and $\chi(\lambda_0) = \pm I$ follows. But then we also obtain $(d/dt)\chi = 0$ at $\lambda = \lambda_0$. 2. \Rightarrow 1.: Clear from the formula above. 2. \Leftrightarrow 3.: Using (3.5.16), (3.5.21) and (3.5.27) we have

$$\chi = U^{-1} \cdot B \cdot U \tag{3.6.1}$$

with U and $B \in \{D, V\}$ holomorphic in an open neighbourhood of S^1 . Hence

$$\chi(\lambda_0) = \pm I \Leftrightarrow B(\lambda_0) = \pm I. \tag{3.6.2}$$

We recall that χ and ρ have the same eigenvalue functions. Therefore, the last statement is equivalent with

$$4v(\lambda_0)^2 = 4(1 - u(\lambda_0)^2) = 4 - (\text{trace}(\rho(\lambda_0)))^2 = 0. \tag{3.6.3}$$

Let \dots' denote differentiation with respect to λ . Then under the assumption (3.6.2) we obtain by differentiating (3.6.1):

$$\chi'(\lambda_0) = U(\lambda_0)B'(\lambda_0)U(\lambda_0)^{-1}. \tag{3.6.4}$$

From the form of B we conclude that $B(\lambda_0) = \pm I$ and $B'(\lambda_0) = 0$ imply that v has a zero of at least second order at λ_0 . This in turn shows that $v^2 = 4 - (\text{trace}(\rho))^2$ has a zero of at least fourth order at λ_0 . Conversely, if v^2 has a zero of at least fourth order at λ_0 , then v and v' vanish at λ_0 and $u(\lambda_0)^2 = 1 - v(\lambda_0)^2 = 1$, proving $B(\lambda_0) = \pm I$ and $B'(\lambda_0) = 0$. This shows 2. \Leftrightarrow 3. □

4. Dressing from S^1 and symmetries.

4.1. We would like to construct complete CMC-immersions of cylinders. In Theorem 5.3 we will show that dressing (=1-dressing) preserves completeness. Therefore we would like to obtain a result similar to Theorem 3.5 for the case $r = 1$. It is easy to verify that in Theorem 3.5 the implications 1) \Leftrightarrow 2) \Rightarrow 3) \Rightarrow 4) are true for any fixed $0 < r \leq 1$. The conclusion 4) \Rightarrow 1) seems to require additional assumptions if one insists on $r = 1$.

THEOREM. *Let $\gamma \in \text{Aut } \mathbf{D}$ and η be a holomorphic potential, which is holomorphic for $\lambda \in \mathbf{C}^*$ and which satisfies $\gamma * \eta = \eta$. Furthermore, let H be defined as in (3.1.6) and set $\rho(\lambda) = H(\gamma(z), \lambda)H(z, \lambda)^{-1} = H(\gamma(0), \lambda)$. Assume that $\rho(\lambda)$ satisfies the following conditions for every $\lambda \in S^1$:*

$$\rho(\lambda) \text{ is semisimple} \tag{4.1.1}$$

$$\text{trace}(\rho(\lambda)) \text{ is real} \tag{4.1.2}$$

$$4 - (\text{trace}(\rho(\lambda)))^2 \geq 0 \tag{4.1.3}$$

For every root $\lambda_0 \in S^1$ of $v^2 = 1 - \left(\frac{1}{2} \text{trace}(\rho(\lambda))\right)^2$, the matrix

$$\left(1 - \left(\frac{1}{2} \text{trace}(\rho(\lambda))\right)^2\right)^{-1/2} (\rho(\lambda) - \rho(\lambda_0)) \text{ has all coefficients in } \mathcal{A}. \tag{4.1.4}$$

Then there exists some $h_+ \in A^+SL(2, \mathbf{C})_\sigma$ such that $\chi = h_+ \rho h_+^{-1} \in ASU(2)_\sigma$.

REMARK. Note that in this Theorem, stronger than in Theorem 3.5, h_+ is defined on the circle with radius 1.

PROOF. The main objective of the proof is to show that the matrices Y used in the proof of Theorem 3.5 are still in $ASL(2, \mathbf{C})_\sigma$ under the conditions listed above.

We say that $\rho(\lambda)$ and $\tilde{\rho}(\lambda)$ are equivalent if there exist finitely many matrices h_1, \dots, h_N of the form $h_*(\lambda) = \begin{pmatrix} 1 & t_*\lambda \\ 0 & 1 \end{pmatrix}$ or $h_*(\lambda) = \begin{pmatrix} 1 & 0 \\ t_*\lambda & 1 \end{pmatrix}$ such that $\tilde{\rho}(\lambda) = h_1(\lambda) \cdots h_N(\lambda)\rho(\lambda)h_N(\lambda)^{-1} \cdots h_1(\lambda)^{-1}$. Clearly, (4.1.1), (4.1.2) and (4.1.3) are invariant under conjugation with any matrix in $ASL(2, \mathbf{C})_\sigma$. Moreover, since v^2 , and therefore v (3.5.4), is independent of conjugation, also the condition (4.1.4) is independent of conjugation. In view of 1) and $h_* \in A^+SL(2, \mathbf{C})_\sigma$ for all choices of h_* considered, it suffices to prove the claim for any $\tilde{\rho}$ equivalent with ρ .

First we consider the case where in the notation of the proof of Theorem 3.5 we have $v = 0$, i.e., $u = \varepsilon = \pm 1$. Since ρ is semisimple this implies $\rho(\lambda) = \varepsilon I$. But then ρ is already unitary and our claim holds trivially. Therefore, from now on we will assume $v \neq 0$.

For our argument it will be convenient to have the following two transformation formulas available

$$\begin{pmatrix} 1 & t\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -t\lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + t\lambda c & b - (t\lambda)^2 c + (t\lambda)(d - a) \\ c & d - t\lambda c \end{pmatrix}, \tag{4.1.5}$$

$$\begin{pmatrix} 1 & 0 \\ s\lambda & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -s\lambda & 1 \end{pmatrix} = \begin{pmatrix} a - s\lambda b & b \\ c - (s\lambda)^2 b + (s\lambda)(a - d) & d + s\lambda b \end{pmatrix}. \tag{4.1.6}$$

Using the notation of the proof of Theorem 3.5 we assume that λ_0 is a root of v . Then $u(\lambda_0) = \varepsilon = \pm 1$, whence $\tau_\pm = \varepsilon$. Since $\rho(\lambda_0)$ is semisimple, $\rho(\lambda_0) = \varepsilon I$. This implies $a(\lambda_0) = d(\lambda_0) = \varepsilon$ and $b(\lambda_0) = c(\lambda_0) = 0$. Since $\rho(\lambda)$ is analytic in λ we can write

$$\begin{aligned} a(\lambda) &= \varepsilon + (\lambda - \lambda_0)^\alpha \tilde{a}, & d(\lambda) &= \varepsilon + (\lambda - \lambda_0)^\delta \tilde{d} \\ b(\lambda) &= (\lambda - \lambda_0)^\beta \tilde{b}, & c(\lambda) &= (\lambda - \lambda_0)^\gamma \tilde{c} \end{aligned} \tag{4.1.7}$$

where $\alpha, \beta, \gamma, \delta$ are positive integers and $\tilde{a}, \tilde{d}, \tilde{b}, \tilde{c}$ are either non-zero at λ_0 or identically zero as functions of λ .

The condition $\det \rho(\lambda) = 1$ is equivalent with

$$\varepsilon(\lambda - \lambda_0)^\delta \tilde{d}(\lambda) + \varepsilon(\lambda - \lambda_0)^\alpha \tilde{a}(\lambda) + (\lambda - \lambda_0)^{\alpha+\delta} \tilde{a}(\lambda) \tilde{d}(\lambda) = (\lambda - \lambda_0)^{\beta+\gamma} \tilde{b}(\lambda) \tilde{c}(\lambda). \tag{4.1.8}$$

We will distinguish the following cases:

$$1. b = 0; \quad 2. c = 0; \quad 3. a = \varepsilon; \quad 4. d = \varepsilon; \quad 5. \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \neq 0. \tag{4.1.9}$$

Let us consider the case (1). Here (4.1.8) reduces to

$$\varepsilon(\lambda - \lambda_0)^\delta \tilde{d}(\lambda) + \varepsilon(\lambda - \lambda_0)^\alpha \tilde{a}(\lambda) + (\lambda - \lambda_0)^{\alpha+\delta} \tilde{a}(\lambda) \tilde{d}(\lambda) = 0. \tag{4.1.10}$$

If in addition $\tilde{a} = 0$, then also $\tilde{d} = 0$. Therefore, ρ is of the form $\rho(\lambda) = \begin{pmatrix} \varepsilon & 0 \\ c(\lambda) & \varepsilon \end{pmatrix}$.

This implies $u = \varepsilon$ and $v = 0$, a contradiction. Similarly $\tilde{d} = 0$ is impossible. But now $\tilde{a} \neq 0, \tilde{d} \neq 0$ implies $\alpha, \delta < \alpha + \delta$, whence $\tilde{a} = 0$ or $\tilde{d} = 0$, showing that the case (1) does not occur. Similarly one sees that the case (2) does not occur. Let us assume from now on $b \neq 0$ and $c \neq 0$. If $\tilde{a} = 0$, then (4.1.8) reduces to

$$\varepsilon(\lambda - \lambda_0)^\delta \tilde{d}(\lambda) = (\lambda - \lambda_0)^{\beta+\gamma} \tilde{b}(\lambda) \tilde{c}(\lambda). \tag{4.1.11}$$

Similarly, if $\tilde{d} = 0$, then (4.1.8) reduces to

$$\varepsilon(\lambda - \lambda_0)^\alpha \tilde{a}(\lambda) = (\lambda - \lambda_0)^{\beta+\gamma} \tilde{b}(\lambda) \tilde{c}(\lambda). \tag{4.1.12}$$

In particular, in both cases, $\tilde{a} = \tilde{d} = 0$ is impossible.

Our goal is to show that $h_+ \rho h_+^{-1} = \chi$ is unitary for all $\lambda \in S^1$ if we choose $h_+ \in A^+ SL(2, \mathbf{C})_\sigma$ properly. For this purpose we can replace ρ by any $A^+ SL(2, \mathbf{C})_\sigma$ -conjugate of ρ . We know $b \neq 0$ and $c \neq 0$. We want to obtain w.r.g. $\beta = \gamma \neq 0$. If $\gamma < \beta$, then we use (4.1.5) and see that the right upper coefficient is of the form

$$\begin{aligned} & -t^2 \lambda_0^2 (\lambda - \lambda_0)^\gamma \tilde{c} + t \lambda_0 \{ (\lambda - \lambda_0)^\delta \tilde{d} - (\lambda - \lambda_0)^\alpha \tilde{a} \} \\ & + (\lambda - \lambda_0)^\beta \tilde{b} + \text{higher order in } (\lambda - \lambda_0). \end{aligned} \tag{4.1.13}$$

Therefore, the new coefficient “ b ” in the matrix $\tilde{\rho} = h + \rho h_+^{-1}$ has order $\tilde{\beta} \leq \gamma$, where t can be chosen arbitrary in an open and dense set in \mathbf{C} (more precisely in $\mathbf{C} \setminus \{pt.\}$).

If $\beta < \gamma$ then the analogous argument using (4.1.6) shows $\tilde{\gamma} \leq \beta$.

Iterating this procedure if necessary we thus can assume

$$0 < \beta = \gamma \text{ and the value of } \beta = \gamma \text{ cannot be decreased by transformations of the form (4.1.5) or (4.1.6).} \tag{4.1.14}$$

From (4.1.7) we obtain that $d - a$ can be written in the form

$$d - a = (\lambda - \lambda_0)^s \tilde{q}(\lambda), \text{ where } s = \min(\alpha, \delta) \text{ if } \alpha \neq \delta \text{ and } s \geq \alpha = \delta \text{ otherwise.} \tag{4.1.15}$$

Since $\beta = \gamma$ is minimal, an additional application of (4.1.5) shows

$$s \geq \beta = \gamma \text{ and } \alpha, \delta \leq \beta = \gamma. \tag{4.1.16}$$

Since $\beta = \gamma > 0$, $\tilde{a} = 0$ now yields a contradiction in view of (4.1.10). Similarly, $\tilde{d} = 0$ is impossible. Therefore only case (5) above needs to be considered. Assume $\alpha = \min\{\alpha, \delta\}$ and $0 < \alpha < \beta = \gamma$. Then we write $a = \varepsilon + (\lambda - \lambda_0)^\alpha (a_0 + \check{a}(\lambda))$ and $d = \varepsilon + (\lambda - \lambda_0)^\delta (d_0 + \check{d}(\lambda))$. Then (4.1.8) yields $a_0 = 0$, if $\alpha \neq \delta$, a contradiction, and $a_0 + d_0 = 0$ if $\alpha = \delta$. Since $s \geq \beta = \gamma$ we also have $a_0 - d_0 = 0$. Therefore, $a_0 = d_0 = 0$, a contradiction. The analogous argument in the case $\delta = \min\{\alpha, \delta\}$ yields the same result. As a consequence we obtain

$$\alpha = \delta = \beta = \gamma. \tag{4.1.17}$$

Now (4.1.8) implies $\tilde{a}(\lambda) + \tilde{d}(\lambda) = (\lambda - \lambda_0)^r \tilde{\rho}(\lambda)$, where $r \geq 2\alpha$. Therefore $u = \varepsilon + (1/2)(\lambda - \lambda_0)^r \tilde{\rho}$ and $1 - u^2 = \varepsilon(\lambda - \lambda_0)^r \tilde{\rho}(\lambda) + \text{higher order}$. Since $1 - u^2$ has an analytic square root (3.5.12), r is even and $v = (\lambda - \lambda_0)^{r/2} \hat{\rho}(\lambda)$, where $r/2 \geq \alpha$. Assumption (4.1.4) now implies $r/2 = \alpha$.

This enables us now to prove our claim.

1. Assume that v is an even function of λ . Consider the matrix Y given in (3.5.15). We need to show that $(u + iv - d)/(2iv)$, $b/(u - iv - a)$ and $c/(2iv)$ are in \mathcal{A} and that the first of these functions is even in λ while the last two are odd in λ . This last statement is obvious. To prove the first one we note that $u - d = (1/2)(a - d) = (1/2)(\lambda - \lambda_0)^\alpha(\tilde{a}(\lambda) - \tilde{d}(\lambda))$ has degree $\geq \alpha$ at λ_0 , while v has degree α . Similarly one sees that $c/(2iv)$ has degree 0 at λ_0 . We claim that we can assume that $u - iv - a = (1/2)(d - a) - iv$ has degree α at λ_0 . From (4.1.5) we see that we can replace the $(\lambda - \lambda_0)^\alpha$ -coefficient in a by $a_0 + t\lambda_0 c_0$ and in d by $d_0 - t\lambda_0 c_0$. Therefore, $(1/2)(d - a)$ has $(\lambda - \lambda_0)^\alpha$ -coefficient $(1/2)(d_0 - a_0 - 2t\lambda_0 c_0)$ and can be assumed to be different from iv_0 . As a consequence, all coefficients of Y are defined at λ_0 and analytic in λ .
2. If v is odd, then we consider the matrices given in (3.5.20) and (3.5.21). We need to show that b/f , v/f and $(u - a)/f$, $f^2 = ibv$, are in \mathcal{A} and are even or odd as required. The latter statement is easily verified. For the first we note that f^2 has degree 2α at λ_0 , whence f has degree α at λ_0 , making b/f and v/f defined at λ_0 and analytic in λ . Since also $u - a = (1/2)(d - a)$ has degree α at λ_0 , Y is defined at λ_0 and analytic in λ .

It is easy to verify that the arguments above can be carried out simultaneously at all the finitely many roots of v on S^1 , whence

$$Y \in ASL(2, \mathbf{C})_\sigma, \quad D \in ASU(2)_\sigma.$$

From this point on we continue as in the proof of Theorem 3.5. □

4.2. From [9, Theorem A8] we see that the coefficients of η cannot all be odd functions of z . But it is possible that these coefficients are all even in z .

THEOREM. *Let η be a holomorphic potential of the form $\eta = \begin{pmatrix} 0 & A(z, \lambda) \\ B(z, \lambda) & 0 \end{pmatrix}$ satisfying $j_0 * \eta = -\eta$ where $j_0(z) = -z$. Then $j_0 * \eta = R\eta R^{-1}$, where $R = i\sigma_3$, and we have $H(-z, \lambda) = RH(z, \lambda)R^{-1}$. If $\gamma \in \text{Aut } \mathbf{D}$ is a translation, $\gamma.z = z + q$ and satisfies $\gamma^*\eta = \eta$, then $H(\gamma.z, \lambda) = \rho(\lambda)H(z, \lambda)$. Moreover, $\rho(\lambda)^{-1} = R\rho(\lambda)R^{-1}$, $F(-z, \lambda) = RF(z, \lambda)R^{-1}$ and $\psi(-z, \lambda) = R\psi(z, \lambda)R^{-1}$, where the last expression describes a rotation around the e_3 -axis in \mathbf{R}^3 by the angle π .*

PROOF. The relation $R\eta R^{-1} = -\eta$ is obvious for $R = i\sigma_3$. But then $R^{-1}j_0 * \eta R = \eta$, and $R^{-1}H(-z, \lambda)R = LH(z)$ follows. Evaluating this relation at $z = 0$ yields $L = I$. If $\gamma \in \text{Aut } \mathbf{D}$ satisfies $\gamma^*\eta = \eta$, then $H(\gamma.z, \lambda) = \rho(\lambda)H(q, \lambda)$. Using that γ is a translation we see $\rho(\lambda) = H(q, \lambda)$, whence $\rho(\lambda)^{-1} = H(-q, \lambda) = R\rho(\lambda)R^{-1}$. Moreover, since R is unitary and independent of λ , the remaining statements follow by a straightforward argument. □

PROPOSITION. *Let η be a holomorphic potential of the form $\eta = \begin{pmatrix} 0 & A(z, \lambda) \\ B(z, \lambda) & 0 \end{pmatrix}$ satisfying $j_0 * \eta = -\eta$. Let $\tilde{\eta} = h_+\eta h_+^{-1}$, $h_+ \in ASL(2, \mathbf{C})_\sigma$, be a potential obtained from η by dressing. Then $j_0 * \tilde{\eta} = -\tilde{\eta}$ and $\tilde{H}(-z, \lambda) = \tilde{R}\tilde{H}(z, \lambda)\tilde{R}^{-1}$, where $d\tilde{H} = \tilde{H}\tilde{\eta}$, $\tilde{H}(0, \lambda) =$*

I , and $\tilde{R} = h_+ R h_+^{-1}$. If γ is as in the Theorem above, then $\tilde{H}(\gamma.z, \lambda) = \tilde{\rho}(\lambda) \tilde{H}(z, \lambda)$ and $\tilde{\rho}(\lambda)^{-1} = \tilde{R} \tilde{\rho}(\lambda) \tilde{R}^{-1}$ where $\tilde{\rho}(\lambda) = h_+(\lambda) \rho(\lambda) h_+(\lambda)^{-1}$.

PROOF. Clearly, $j_0 * \tilde{\eta} = j_0 * (h_+ \eta h_+^{-1}) = h_+(j_0 * \eta) h_+^{-1} = -h_+ \eta h_+^{-1} = -\tilde{\eta}$. Moreover, $\tilde{H}(z, \lambda) = h_+ H(z, \lambda) h_+^{-1}$ satisfies $d\tilde{H} = h_+(dH) h_+^{-1} = h_+ H \eta h_+^{-1} = \tilde{H} \tilde{\eta}$ and $\tilde{H}(z=0, \lambda) = I$. Moreover, $\tilde{H}(-z, \lambda) = h_+ H(-z, \lambda) h_+^{-1} = h_+ R H(z, \lambda) R^{-1} h_+^{-1} = \tilde{R} \tilde{H}(z, \lambda) \tilde{R}^{-1}$. Finally, $\tilde{H}(\gamma.z, \lambda) = h_+ H(\gamma.z, \lambda) h_+^{-1} = h_+ \rho H h_+^{-1} = \tilde{\rho} \tilde{H}$ and $\tilde{\rho}(\lambda)^{-1} = h_+ \rho(\lambda)^{-1} h_+^{-1} = h_+ R \rho(\lambda) R^{-1} h_+^{-1} = \tilde{R} \tilde{\rho}(\lambda) \tilde{R}^{-1}$. \square

REMARK. The Proposition is an immediate consequence of the Theorem. However, since \tilde{R} is no longer unitary, the last statements of the Theorem above are in general no longer valid for \tilde{F} and $\tilde{\psi}$.

5. Examples.

Next, we want to exhibit concrete examples of invariant potentials, for which the assumptions of Theorem 3.5 or Theorem 4.1 and Theorem 3.6 are satisfied.

5.1. To this end we consider holomorphic potentials of the form

$$\eta_0 = \begin{pmatrix} 0 & f(z) \\ g(z) & 0 \end{pmatrix} w \, dz, \quad w = \cos(t) = \frac{1}{2}(\lambda^{-1} + \lambda), \tag{5.1.1}$$

with entire functions f and g which are periodic and real on the real axis, i.e.

$$f(\bar{z}) = \overline{f(z)}, \quad g(\bar{z}) = \overline{g(z)} \tag{5.1.2}$$

and

$$f(z + L) = f(z), \quad g(z + L) = g(z), \quad L \in \mathbf{R}^+. \tag{5.1.3}$$

In other words, we consider the group $\Gamma = L\mathbf{Z} \subset \text{Aut } \mathbf{C}$ acting as a group of translations on the complex plane.

THEOREM. Let η_0 be defined as above.

1) If

$$\int_0^L f(z) \, dz \cdot \int_0^L g(z) \, dz < 0 \tag{5.1.4}$$

then there exists $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$, $0 < r \leq 1$, such that for Ψ_λ defined by dressing η_0 with h_+ we have $\Gamma \subset \text{Aut}_{\Psi_\lambda} \mathbf{C}$. Furthermore, for $\lambda = \pm i$ we have

$$\Psi_{\pm i}(z + Ln) = \Psi_{\pm i}(z) + nV, \tag{5.1.5}$$

for some $V \in \mathbf{R}^3$ and all $n \in \mathbf{Z}$.

2) If

$$\int_0^L f(z) \, dz \cdot \int_0^L g(z) \, dz = 0 \tag{5.1.6}$$

and

$$\int_0^L f(z) \int_0^z g(z') \int_0^{z'} f(z'') \int_0^{z''} g(z''') dz''' \cdots dz + \int_0^L g(z) \int_0^z f(z') \int_0^{z'} g(z'') \int_0^{z''} f(z''') dz''' \cdots dz < 0 \tag{5.1.7}$$

then there exists $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$, $0 < r \leq 1$, such that for Ψ_λ defined by dressing η_0 with h_+ we have $\Gamma \subset \text{Aut}_{\Psi_\lambda} \mathbf{C}$. Furthermore, for $\lambda = \pm i$ we have

$$\Psi_{\pm i}(z + Ln) = \Psi_{\pm i}(z), \tag{5.1.8}$$

for all $n \in \mathbf{Z}$.

PROOF. Since Γ is generated by $\gamma : z \rightarrow z + L$, we restrict our attention to γ . By (5.1.2) we know that H defined by (3.1.6) has real entries for $z \in \mathbf{R}$ and $\lambda \in S^1$. Thus, the monodromy $\rho(\lambda) = H(L, \lambda)$ and its trace are real valued for all $\lambda \in S^1$. Obviously, η_0 and ρ are entire functions of $w = (1/2)(\lambda + \lambda^{-1})$. Since the trace of ρ is even in λ it depends only on w^2 . Therefore, we may expand $\text{trace}(\rho)$ into a power series with respect to w^2 . Since $\eta_0(w = 0) \equiv 0$, we have $H(z, w = 0) \equiv I$, whence $\text{trace}(\rho)(w = 0) = 2$. The expansion therefore looks like

$$\text{trace}(\rho) = 2 + a_1 w^2 + a_2 w^4 + \cdots \tag{5.1.9}$$

A simple calculation gives that a_1 is the left hand side of (5.1.4) and (5.1.6), and that a_2 is the left hand side of (5.1.7). Thus, if we restrict $\text{trace}(\rho)(w)$ to the real axis in the w -plane, then in an interval I around $w = 0$ we have $4 - \text{trace}(\rho)(w)^2 \geq 0$, i.e. the eigenvalues are unimodular for $w \in I$. If we define the Riemann surface \mathcal{C}' by

$$(\mu')^2 = \text{trace}(\rho(w))^2 - 4, \quad w \in \mathbf{C}, \tag{5.1.10}$$

then the eigenvalues of $\rho(w)$ combine to a holomorphic function τ' on \mathcal{C}' . The function $f(w) = \tau'(w)\overline{\tau'(\bar{w})} - 1$ is holomorphic on \mathcal{C}' and vanishes over I . Thus, $f(w) \equiv 0$ on \mathcal{C}' . In particular, this implies that the eigenvalues of $\rho(w)$ are unimodular for all $w \in [-1, 1]$, i.e. for all $\lambda \in S^1$.

Therefore, in both parts of the theorem, the conditions of part 3) of Theorem 3.5 are satisfied for our η_0 . Thus, there exists $0 < r \leq 1$ and $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$, such that the surfaces Ψ_λ obtained by dressing η_0 with h_+ contain Γ in their symmetry group $\text{Aut}_{\Psi_\lambda} \mathbf{C}$, if we can show that neither b, c nor v vanish identically.

But by assumption $a_1 \neq 0$ or $a_2 \neq 0$, whence $v \neq 0$ in both cases. To see that neither b nor c vanish identically we consider the expansion

$$H(L, w) = I + w \int_0^L \eta_0(z) dz + w^2 \int_0^L \eta_0(z) \int_0^z \eta_0(z') dz' dz + \cdots \tag{5.1.11}$$

In the first case the coefficient at w yields what we want. In the second case we assume w.l.g. that $\int_0^L \eta_0(z) dz = 0$ holds. Then rewriting the first summand by iterated integrations by part we obtain for the leftside of (5.1.7) the expression

$$\left(\int_0^L g(z) \int_0^z f(z') dz' dz \right)^2 + \int_0^L g(z) dz \cdot \int_0^L f(z') \int_0^{z'} g(z'') \int_0^{z''} f(z''') dz''' dz'' dz'. \tag{5.1.12}$$

Since this needs to be negative by assumption, the second summand does not vanish. But this shows that both, b and c do not vanish identically. Therefore Theorem 3.5 is applicable.

Since in addition $\rho(w = 0) = \rho(\lambda = \pm i) = I$ we have that $\chi(\pm i) = h_+(\pm i)\rho(\pm i)h_+^{-1}(\pm i) = I$. Sym's formula (2.2.1) then gives (5.1.5) with $V = -(1/(2\mathbf{H}))(\mathrm{d}\chi(\lambda = e^{it})/\mathrm{d}t)|_{\lambda=\pm i}\chi(\pm i)^{-1}$. This proves the first part.

For the second part, by Theorem 3.6, it only remains to prove that $T = \text{trace}(\rho)^2 - 4$ has a zero of at least fourth order at $\lambda = \pm i$. An elementary calculation shows that the first nonvanishing derivative of T with respect to λ at $\lambda = \pm i$ coincides with the first nonvanishing derivative of T with respect to w at $w = 0$. However, by the power series expansion (5.1.9) it is clear that T has a zero of fourth order at $w = 0$, which finishes the proof. \square

5.2. To construct examples we first note that there are infinite dimensional families of periodic real functions f and g such that either (5.1.4) or (5.1.6) is satisfied. To avoid branched immersions we restrict our attention to $f = A = \text{const.}$ with $A \in \mathbf{R} \setminus \{0\}$. Then condition (5.1.6) reduces to

$$a_1 = \int_0^L g(z) \, dz = 0 \tag{5.2.1}$$

and (5.1.7) reduces to

$$a_2 = \int_0^L \int_0^z g(z') \int_0^{z'} \int_0^{z''} g(z''') \, dz''' \cdots dz + \int_0^L g(z) \int_0^z \int_0^{z'} g(z'') \int_0^{z''} dz''' \cdots dz < 0. \tag{5.2.2}$$

In fact, if we set $f = A, g = B \cos(2\pi z/L), A, B \in \mathbf{R} \setminus \{0\}$, then a simple calculation gives $a_1 = 0, a_2 = -(1/2)(ABL)^2 < 0$. The conditions of Theorem 5.1, part 2), are therefore satisfied for all constants A, B . Since f has no zeroes in \mathbf{C} each holomorphic potential in the dressing orbit of η_0 generates an unbranched immersion under the construction of [11]. By Theorem 5.1 there exists a holomorphic potential η in the r -dressing orbit of η_0 such that the associated CMC immersion Ψ_i factors through the group $\Gamma = LZ$. In other words, η describes a CMC surface which is a topological cylinder. In each fundamental domain of Γ there are precisely two zeroes of g . Thus, the surface Ψ_i has two umbilics, located at $\Psi_i(\pm L/4)$. Altogether this proves

THEOREM. *Let $A, B \in \mathbf{R} \setminus \{0\}$ and $\eta_0 = W \begin{pmatrix} 0 & A \\ B \cos(2\pi z/L) & 0 \end{pmatrix} dz, w = (1/2)(\lambda + \lambda^{-1})$. Then (5.1.6) and (5.1.7) are satisfied. Hence there exists some $0 < r \leq 1$ and some $h_+ \in A_r^+ SL(2, \mathbf{C})_\sigma$ such that for ψ_λ defined by dressing η_0 with h_+ we have $\Gamma = LZ \subset \text{Aut}_{\psi_\lambda} \mathbf{C}$. Moreover, $\psi_{\pm i}$ is a CMC-immersion of the cylinder \mathbf{C}/Γ with umbilics at $z = L/4$ and $z = 3L/4$.*

REMARK. 1. We would like to point out that the associated family of the CMC-immersion $\psi_0(z, \lambda)$, associated with η_0 has according to Theorem 4.2 the additional symmetry

$$\psi_0(-z, \lambda) = R_3^\pi(\psi_0(z, \lambda)), \tag{5.2.3}$$

where R_3^π is the rotation around the e_3 -axis in \mathbf{R}^3 by the angle π . In the spinor representation it corresponds to conjugation with $R = \text{diag}(i, -i)$.

2. Since the condition (5.2.1) is satisfied for all linear combinations of $\sin(2\pi n z/L)$ and $\cos(2\pi n z/L)$, $n \in \mathbf{N}$, we can perturb $\cos(2\pi z/L)$ inside an infinite dimensional space of real periodic functions such that both conditions (5.2.1) and (5.2.2) are satisfied. This construction therefore proves the existence of an infinite dimensional family of CMC cylinders with umbilics.

3. Since the original submission of this paper, Martin Kilian [24] has found many more CMC-cylinders with umbilics along the lines of the example above (see also [25]).

5.3. In section 5.5 we will sharpen 5.2 and prove the completeness of the constructed surfaces. To this end we start with the following result which is interesting in its own right:

THEOREM. *Let $\Psi : \mathcal{D} \rightarrow \mathbf{R}^3$, \mathcal{D} being the complex plane or the open unit disk, be a complete CMC immersion. Then all surfaces in the same $r = 1$ -dressing orbit are complete.*

PROOF. Let $F : \mathcal{D} \rightarrow \Lambda SU(2)_\sigma$ be the extended frame for the immersion Ψ . By (2.2.4), for each $h_+ \in \Lambda^+ SL(2, \mathbf{C})_\sigma$ the extended frame \tilde{F} for the dressed surface is given by

$$\tilde{F} = h_+ F p_+, \tag{5.3.1}$$

where p_+ is a map from \mathcal{D} into $\Lambda^+ SL(2, \mathbf{C})_\sigma$ such that $w(z)$, the upper left entry of the λ^0 -coefficient of $p_+(z, \lambda)$, is a positive real number. It follows that

$$\tilde{F}^{-1} d\tilde{F} = p_+^{-1} dp_+ + p_+^{-1} F^{-1} dF p_+. \tag{5.3.2}$$

Comparing the λ^{-1} -coefficient of both sides of (5.3.2) and using [8, Equation (A.6.4)] we get

$$w = e^{(u-\tilde{u})/4}, \tag{5.3.3}$$

where e^u and $e^{\tilde{u}}$ are the conformal metric factors for the original and the dressed surface, respectively. To prove the theorem it now suffices to show that $|u - \tilde{u}|$ is uniformly bounded or, by (5.3.3), that w and w^{-1} are uniformly bounded.

Let $C(\mathcal{D}, \Lambda^+ gl(2, \mathbf{C}))$ denote the space of continuous maps from \mathcal{D} into $\Lambda^+ gl(2, \mathbf{C})$. For each $M \in \Lambda^+ gl(2, \mathbf{C})$ let $\|M\|$ be the supremum over $\lambda \in S^1$ of the operator norm of $M(\lambda)$. For $v \in C(\mathcal{D}, \Lambda^+ gl(2, \mathbf{C}))$ define $\|v\|$ as the supremum over \mathcal{D} of all $|v(z)|$. This obviously defines a norm on $C(\mathcal{D}, \Lambda^+ gl(2, \mathbf{C}))$. Furthermore, $\|F\| = 1$ for each extended frame F , since F takes unitary values for all $z \in \mathcal{D}$ and all $\lambda \in S^1$. Thus, by (5.3.1), we obtain $\|p_+\| = \|h_+^{-1}\| = q < \infty$. In particular, each matrix entry of p_+ is bounded by q for all $\lambda \in S^1$ and all $z \in \mathcal{D}$. Furthermore, since p_+ can be extended holomorphically to the interior of the unit circle, the maximum principle implies that the entries w and w^{-1} of $p_+(z, \lambda = 0)$ are also bounded by q for all $z \in \mathcal{D}$, which finishes the proof. \square

REMARK. Note that this result is restricted to the dressing action on the unit circle. The proof above does not hold for $r < 1$ since the extended frames then do not take unitary values for $\lambda \in C_r$.

To prove the existence of complete surfaces among our examples above it will therefore be our strategy to show that there exists a periodic potential η_0 such that

1. the construction of Section 5.1 can be done for η_0 by dressing on the unit circle,
2. η_0 produces a complete CMC surface.

Theorem 5.3 then proves the completeness of the CMC cylinder which is obtained by the construction in Section 5.1.

5.4. In section 4.1 we have seen that under additional assumptions on η_0 the dressing matrix h_+ can be chosen on the unit circle. We will follow here a similar procedure to show

THEOREM. *There exists some $\varepsilon_0 > 0$ such that for all $A, B \in \mathbf{R}$, $0 < |AB| < \varepsilon_0$, the holomorphic potential $\eta_0 = w \begin{pmatrix} 0 & A \\ B \cos(2\pi z/L) & 0 \end{pmatrix} dz$, $w = (1/2)(\lambda + \lambda^{-1})$, satisfies the conditions (4.1.1), (4.1.2) and (4.1.3). In addition for every A and every such B there exists some $h_+ \in \Lambda^+ SL(2, \mathbf{C})_\sigma$ such that the CMC-immersions ψ_λ , $\lambda \in S^1$, obtained from η_0 by dressing with h_+ satisfy*

$$\Gamma = LZ \subset \text{Aut}_{\psi_\lambda} \mathbf{C}.$$

Moreover, $\psi_{\pm i}$ is a CMC-immersion of the cylinder \mathbf{C}/Γ and has umbilics at $z = L/4$ and $z = 3L/4$.

PROOF. In 5.2 it was shown that the assumptions (5.1.6) and (5.1.7) of Theorem 5.1 are satisfied. As in the proof of Theorem 5.1 we see that (4.1.2) and (4.1.3) hold. Next we want to show that $\rho(\lambda)$ is semisimple for every $\lambda \in S^1$. First we note that the identity $\rho(\lambda)^{-1} = R\rho(\lambda)R^{-1}$, proven in Theorem 4.2, implies that ρ is of the form $\rho = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$, where $a^2 - bc = 1$. As in 3.5 we denote the eigenvalues of ρ by $\tau_\pm = u \pm iv$, $u = a$ and $v = \sqrt{1 - a^2}$. If $v(\lambda) \neq 0$, then $\rho(\lambda)$ is semisimple. If $v = 0$ and $\lambda = \lambda_0$, then $a(\lambda_0) = \varepsilon$, $\varepsilon = \pm 1$. This implies $bc = 0$, whence $b(\lambda_0) = 0$ or $c(\lambda_0) = 0$. It is easy to see now that $\rho(\lambda)$ is semisimple if and only if b and c both vanish at $\lambda = \lambda_0$ or both are different from 0. Let $H = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}$ satisfy $H' = H\eta$, $H(z = 0, \lambda) = I$. Then $\hat{b}'' = w^2 AB \cos(2\pi z/L)\hat{b}$, $\hat{b}' = wA\hat{a}$, $\hat{d}' = wA\hat{c}$ and $\hat{d}'' = w^2 AB \cos(2\pi z/L)\hat{d}$, where $\hat{b}(0, w) = 0$, $\hat{b}'(0, w) = wA$, $\hat{d}(0, w) = 1$ and $\hat{d}'(0, w) = 0$. Moreover, Theorem 4.2 shows that \hat{a} and \hat{d} are even in z and λ and \hat{b} and \hat{c} are odd in z and λ . Consider the differential equation $y'' = t^2 \delta \cos(2\pi z/L)y$, $y(0, t) = 0$, $y'(0, t) = A$, where $\delta = \text{sign}(AB)$. Then $\hat{b}(z, w) = wy(z, \sqrt{AB}w)$. The function $y(L, t)$ has only finitely many zeroes in the interval $|t| < |AB|$. Therefore also $b(\lambda) = \hat{b}(L, w)$ has only finitely many roots in the interval $|w| < 1$. If $|AB|$ is so small that $y(L, t)$ has no zeroes in $|t| < |AB|$, except possibly at $t = 0$, then b only vanishes for $w = 0$, i.e., for $\lambda = \pm i$. Similarly, $d'(\lambda) = \hat{d}'(L, \sqrt{|AB|}w)$ will only vanish for $w = 0$, if $\hat{d}'' = t^2 \delta \cos(2\pi z/L)\hat{d}$, $\hat{d}(0, t) = 1$, $\hat{d}'(0, t) = 0$ and $|AB|$ is sufficiently small. This shows that for such a choice of AB the functions b and c vanish on S^1 only at $\pm i$. This shows that $\rho(\lambda)$ is semisimple for all $\lambda \in S^1$.

As in the proof of Theorem 4.1 we need to show that ρ can be diagonalized with a matrix Y that has coefficients in \mathcal{A} . To this end we develop \hat{a} , \hat{b} and \hat{c} into

power series of $w = (1/2)(\lambda + \lambda^{-1})$ near $w = 0$. The discussion in section 5.2 shows $\hat{a} = (1/2) \text{traces} = 1 - (1/2)(ABL)^2 w^2 + \dots$. From the differential equation $H' = H\eta$ we obtain directly $\hat{b} = wzA + \dots$ and $\hat{c} = w^3 c_3 + \dots$. Using these expansions it is easy to verify that the coefficients of all the matrices Y occurring in 3.5 are in \mathcal{A} and functions of w . Now one argues as before to see that there exists some $h_+ \in A^+SL(2, \mathbf{C})_\sigma$ such that $h_+ \rho h_+^{-1}$ is unitary. That $\psi_{\pm i}$ descends to \mathbf{C}/Γ follows as in the proof of Theorem 5.1. □

5.5. The goal of this section is to prove

THEOREM. *Let η_0 and ε_0 be as in Theorem 5.4 and assume $0 < |AB| < \varepsilon_0$. Then the surface associated with η_0 is complete.*

PROOF. We can assume $L = \pi$. From the proof of Theorem 5.4 we know that there exists some $Y(w) \in ASL(2, \mathbf{C})_\sigma$ such that

$$Y(w)\rho(w)Y(w)^{-1} = \text{diag}(e^{iv}, e^{-iv}). \tag{5.5.1}$$

Set $\hat{H}(z, w) = Y(w)H(z, w) = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}$. Then $\hat{H}' = \hat{H}\eta_0$, whence \hat{d} and \hat{b} satisfy the Mathieu differential equation

$$y'' = w^2 AB(\cos z)y. \tag{5.5.2}$$

For \hat{a} and \hat{c} we have

$$\hat{a} = \frac{1}{w}\hat{b}', \quad \hat{c} = \frac{1}{w}\hat{d}'. \tag{5.5.3}$$

Moreover, (5.5.1) implies

$$\hat{b}(z + \pi) = e^{iv}\hat{b}(z), \quad \hat{d}(z + \pi) = e^{-iv}\hat{d}(z) \tag{5.5.4}$$

i.e., \hat{b} and \hat{d} are Floquet solutions to (5.5.2). From [29, 2.21 Satz 1] we obtain another set of Floquet solutions to (5.5.2), $me_\nu(z)$ and $me_{-\nu}(z)$. It is clear that all functions depend on w . We will indicate the dependence on w only where this is essential. We also note that in [29] always

$$h = \sqrt{\frac{AB}{2}} w \tag{5.5.5}$$

is used.

Since Floquet solutions are uniquely determined up to z -independent factors, we obtain

$$\hat{d} = s(w)me_{-\nu}, \quad \hat{b} = r(w)me_\nu. \tag{5.5.6}$$

We will need to know how \hat{d} and \hat{b} behave for $\text{Im } z \rightarrow -\infty$. For convenience we agree to use only $w > 0$ when discussing asymptotic behavior.

In terms of the Bessel functions J_ν of level ν [34, Chapter III] we obtain

$$\hat{d}(z) = \frac{s(w)}{v_2(w)} J_{-v}(2w\alpha_0 \cos z) \left(1 + O\left(\frac{1}{\cos z}\right)\right) \quad (5.5.7)$$

$$\hat{b}(z) = \frac{r(w)}{v_1(w)} J_v(2w\alpha_0 \cos z) \left(1 + O\left(\frac{1}{\cos z}\right)\right) \quad (5.5.8)$$

where $\alpha_0 = \sqrt{AB/2}$ and $v_j(w)$ are the coefficients in [28, 2.65.21].

For a proof we note that [28, 2.65.21] states $me_v(z) = (1/(v_1(w)))M_v(iz)$ and $me_{-v}(z) = (1/(v_2(w)))M_{-v}(iz)$, where the functions $M_{\pm v} = M_{\pm v}^{(1)}$ have been defined in [28, 2.41] and satisfy

$$M_{\pm v}(\tau) = J_{\pm v}(2w\alpha_0 \cosh \tau) \left(1 + O\left(\frac{1}{\cos \tau}\right)\right) \quad (5.5.9)$$

for $\operatorname{Re} \tau > 0$. Moreover, (see also [28, 2.42.18]).

$$M'_{\pm v}(\tau) = 2w\alpha_0 \sinh \tau \cdot J'_{\pm v}(2w\alpha_0 \cosh \tau) \left(1 + O\left(\frac{1}{\cosh \tau}\right)\right). \quad (5.5.10)$$

From these two equations it is clear that we need to know the behavior of $J_{\pm v}(s)$ and $J'_{\pm v}(s)$ for $s = 2w\alpha_0 \cosh(iz)$ when $\operatorname{Im} z \rightarrow -\infty$. It will turn out that it suffices to restrict to $0 < \operatorname{Re} z < \pi$, whence we need to investigate $\operatorname{Im} s \rightarrow +\infty$, $|\arg s| < \pi$.

For this range of s we can apply [34, 7.21] and obtain

$$J_v(s) = \left(\frac{2}{\pi s}\right)^{1/2} \left[\cos\left(s - \frac{v\pi}{2} - \frac{\pi}{4}\right)(1 + S) - \sin\left(s - \frac{v\pi}{2} - \frac{\pi}{4}\right)T \right] \quad (5.5.11)$$

where $S = O(1/s)$, $T = O(1/s)$.

The formula for $J_{-v}(s)$ follows from (5.5.11) by the substitution $v \rightarrow -v$. Differentiating (5.5.11) yields, via product rule, three terms. Differentiating $s^{-1/2}$ yields $-(1/2)s^{-1/2} \cdot s^{-1}$. Differentiating $1 + S$ and T produces expressions of the same type. So altogether we obtain

$$J'_v(s) = \left(\frac{2}{\pi s}\right)^{1/2} \cdot \left\{ \cos\left(s - \frac{v\pi}{2} - \frac{\pi}{4}\right) \cdot \tilde{S} - \sin\left(s - \frac{v\pi}{2} - \frac{\pi}{4}\right)(1 + \tilde{T}) \right\}, \quad (5.5.12)$$

where \tilde{S}, \tilde{T} are of type $O(1/s)$. Again, the expression for J'_{-v} follows by substitution $v \rightarrow -v$.

We consider $J_v(s)$ more closely. Note that we have $\operatorname{Im} s \rightarrow +\infty$ and actually $s = 2\alpha_0 w \cosh iz = 2\alpha_0 w \cos z$. The leading term in (5.5.11) therefore is $(2/(\pi s))^{1/2} e^{-i(s-v\pi/2-\pi/4)}$ and we obtain

$$J_v(s) = \left(\frac{2}{\pi s}\right)^{1/2} \frac{1}{2} e^{-i(s-v\pi/2-\pi/4)} C \quad (5.5.13)$$

where

$$C = (1 + e^{2i(s-v\pi/2-\pi/4)})(1 + S) + (1 - e^{2i(s-v\pi/2-\pi/4)})T.$$

Substituting $s = 2\alpha_0 w \cos z$ we thus obtain for J_{-v} , when writing $s = \alpha_0 w e^{+iz}(1 + e^{-2iz})$

$$J_{-v}(2\alpha_0 w \cos z) = \frac{1}{\sqrt{2\pi\alpha_0 w}} e^{-i(z/2+2\alpha_0 w \cos z)} e^{i(v\pi/2+\pi/4)} \cdot (1 + e^{-2iz})^{-1/2} \cdot \tilde{C}, \quad (5.5.14)$$

where \tilde{C} is obtained from C by substitution for s . Altogether these expressions show that for $\text{Im } z \rightarrow -\infty$ we have $(1 + e^{-2iz})^{-1/2} \tilde{C}(z) \rightarrow 1$.

We point out that the two exponential factors occurring in (5.5.14) both exist for every $z \in C$. The choices of the coefficients A and B of η_0 ensure that v is real. The second factor therefore is in S^1 . The first factor will be split according to λ -behavior:

$$-i\left(\frac{z}{2} + 2\alpha_0 w \cos z\right) = (-2i\alpha_0 w \text{Re } \cos z + \alpha_0(\lambda^{-1} - \lambda) \text{Im } \cos z) + \left(-\frac{iz}{2} + 2\alpha_0 \lambda \text{Im } \cos z\right). \tag{5.5.15}$$

Let A_0 denote the first bracket and B_0 the second. Then $e^{A_0} \in S^1$ and $e^{B_0} \in \mathcal{A}_+$.

We set

$$U_0 = \text{diag}(e^{A_0} \cdot e^{-i(v\pi/2-\pi/4)}, e^{-A_0} e^{i(v\pi/2-\pi/4)}) \tag{5.5.16}$$

$$D_0 = \text{diag}(e^{B_0}, e^{-B_0}). \tag{5.5.17}$$

Then the (22)-entry of $\tilde{H} = U_0 \hat{H} D_0$ is of the form \tilde{d} , where $\tilde{d}(z, \bar{z})$ tends to 1 as $\text{Im } z \rightarrow -\infty$. Moreover, since $J_\nu(2\alpha_0 w \cos z)$ and $J_\nu(2\alpha_0 w \cos z)$ only differ for large $\text{Im } z < 0$ in the unitary factor $e^{-(\pm v\pi/2-\pi/4)}$, which is taken care of by U_0 , also the (12)-entry \tilde{b} of \tilde{H} has a simple behavior: it stays bounded as $\text{Im } z \rightarrow -\infty$. As a consequence, $U_0 Y H D_0 = \tilde{H}$ has a second column that stays bounded as $\text{Im } z \rightarrow -\infty$. But then also $H D_0$ has a second column that stays bounded as $\text{Im } z \rightarrow -\infty$. This implies that with $H = FV_+$ we see that the second column of $V_+ D_0 F^{-1}(H D_0)$ stays bounded as $\text{Im } z \rightarrow -\infty$. In particular, the λ^0 -term in $V_+ D_0$ stays bounded as $\text{Im } z \rightarrow -\infty$. Setting $V_+ = V_0 + \lambda V_1 + \dots$, $V_0 = (\beta, \beta^{-1})$ we obtain $|\beta^{-1} e^{iz/2}| \leq \varepsilon_1$, where $\varepsilon_1 > 0$ is some constant. Since $|e^{iz/2}| = e^{-y/2}$ this yields

$$\varepsilon_1^{-1} \cdot e^{-y/2} \leq |\beta|. \tag{5.5.18}$$

Comparing the two sides of the Maurer-Cartan differential

$$H^{-1} H' = V_+^{-1} F^{-1} \partial_z F V_+ + V_+^{-1} \partial_z V_+$$

we obtain

$$e^{u/2} = \varepsilon_2 |\beta|^2, \tag{5.5.19}$$

where e^u is the conformal factor for the CMC-surface M associated with η_0 and ε_2 some positive constant. This implies with some constant $0 < \varepsilon_3$

$$\varepsilon_3 e^{-2y} \leq e^u \quad \text{for } \text{Im } z \rightarrow -\infty. \tag{5.5.20}$$

This proves that every curve tending on M towards $-i\infty$ has infinite length in the induced metric. To verify that the same holds when approaching $+i\infty$ one can carry out a similar analysis or one simply notices that M has a reflection property: $\overline{H(\bar{z}, w)} = H(z, w)$, whence $\overline{F(\bar{z}, z, \lambda^{-1})} = F(z, \bar{z}, \lambda)$ and for $\lambda = \pm i$ also $\overline{\psi_{-i}(\bar{z})} = -\psi_i(z)$ since $\overline{\psi_{\lambda^{-1}}(\bar{z})} = \partial_t F(\bar{z}, z, \lambda^{-1}) \cdot F(\bar{z}, z, \lambda^{-1})^{-1} + F(\bar{z}, z, \lambda^{-1}) \cdot (i/2)\sigma_3 \cdot F(\bar{z}, z, \lambda^{-1})^{-1} = -\partial_t F(z, \bar{z}, \lambda) \cdot F(z, \bar{z}, \lambda)^{-1} - F(z, \bar{z}, \lambda)(i/2)\sigma_3 F(z, \bar{z}, \lambda)^{-1} = -\psi_\lambda(z)$. \square

COROLLARY. *The CMC-cylinders constructed from the potentials $\eta_0 = w \begin{pmatrix} 0 & 0 \\ B \cos(2\pi z/L) & 0 \end{pmatrix} dz$, $0 < |AB| < \varepsilon_0$, are complete and have umbilics.*

5.6. For a number of reasons it is very desirable to visualize surfaces of constant mean curvature. The most basic goal certainly is to be able to draw a large variety of surfaces of constant mean curvature. For this purpose the method of [11] is very convenient, since, at least theoretically, it permits to produce all surfaces of constant mean curvature. Unfortunately, while being theoretically straightforward, the implementation on computers is not immediate. The first visualization program, written by Lerner and Sterling [28], was fairly slow. Nevertheless, they proved that the procedure could be implemented on a computer, contradicting conventional wisdom at that time. A short time later, in response to the work of Lerner and Sterling, Pinkall and Gunn improved the numerical algorithm considerably. This made it possible to draw reasonably large pieces of surfaces of constant mean curvature with a reasonable number of umbilics at a reasonable distance from each other in very short time.

For a few specific surfaces the effect of dressing and the change of the location of umbilics was visualized via videos by the Sonderforschungsbereich 288 at the TU-Berlin.

In the long run, however, it is clearly important to be able to produce surfaces with a given fundamental group. While some steps in this direction have been done [6], [7], [8], the incorporation of fundamental groups into the procedure of [11] is still in its beginnings.

A preliminary draft of this article, originally finished in the fall of 1997, produced the first cylinders of constant mean curvature with umbilics. Based on this draft and the subsequently expanded, originally submitted version of this paper, Martin Kilian wrote a dissertation [24], where he produced cylinders of constant mean curvature from potentials which are skewhermitian along the unit circle, whence the unitarity of the monodromy matrices is automatic (also see [25]). Unfortunately, Kilian's beautiful trick only worked for very special cases. However, the techniques developed in the previous sections of this paper turned out to be applicable to a larger class of surfaces.

Restricting to non-compact surfaces one can start, due to Theorem 3.2, from some invariant potential η satisfying $\gamma^* \eta = \eta$ for all $\gamma \in \Gamma = \pi_1(M)$. This implies for the "holomorphic extended frame" H , defined by $dH = H\eta$, $H(z_0, \lambda) = I$, the equation $H(\gamma.z, \lambda) = \rho(\gamma, \lambda) \cdot H(z, \lambda)$ for $\gamma \in \Gamma$, $z \in \mathbf{D}$, $\lambda \in S^1$. If ρ is unitary for all $\gamma \in \Gamma$ and all $\lambda \in S^1$, then one obtains for the extended frame F associated to H , $H = F \cdot V_+$, the equation $F(\gamma.z, \lambda) = \rho(\gamma, \lambda) \cdot F(z, \lambda) \cdot k(z)$ and the Sym-Bobenko formula implies formula (3.2.3) for the immersion ψ derived from F . So it only remains to satisfy the closing conditions stated in Theorem 3.6, which in many concrete cases is not an overly difficult task. The crucial problem is that it is fairly difficult to choose η so that ρ is unitary for all $\gamma \in \Gamma$. An instance where this problem has been solved successfully is the work of Kilian [24]. A more general situation occurs, if ρ is not unitary, but can be represented in the form $\rho = h_+ \cdot \rho_u \cdot h_+^{-1}$ with $h_+ \in A_r^+ SI(2, C)_\sigma$ and $\rho_u \in A_r SU(2)_\sigma$. In this case, after dressing with h_+^{-1} , one obtains unitary monodromy, and the comments above apply. This way one can produce surfaces with fundamental group Γ from Γ -invariant potentials after some dressing. The dressing trick, developed in this article, has been

carried out successfully in chapter 5 of this paper for the construction of cylinders of constant mean curvature, initiating the study [24]. In [13] the same approach is used for the construction of trinoids and more generally N-noids. Based on the dressing idea outlined above and a special representation of the holomorphic extended frame [13], Schmitt has developed a numerical algorithm which produces pictures of cylinders, trinoids and N-noids [25], [32], www.gang.umass.edu. The fact that these pictures actually do represent surfaces of the indicated topological type has not been shown in [25], or [32]. In some cases, however, this follows from [24] or from this paper. We would expect that the other cases will follow from [13].

It should be noted also that by completely different methods (almost) embedded trinoids and planar N-noids have been investigated (see e.g. [18], [19], and the references listed there). Also some pictures have been produced following this approach.

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