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Harmonic functions on finitely sheeted unlimited covering surfaces

Dedicated to Professor Masayuki Itô on his sixtieth birthday

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Abstract. We denote by $HP(R)$ and ($HB(R)$, resp.) the class of positive (bounded, resp.) harmonic functions on a Riemann surface R. Consider an open Riemann surface W possessing a Green's function and a p-sheeted $(1 < p < \infty)$ unlimited covering surface $\tilde{W}$ of W with projection map φ. We give a necessary and sufficient condition, in terms of Martin boundary, for $HX(W) \circ \varphi = HX(\tilde{W})$ $(X = P, B)$. We also give some examples illustrating the above result when W is the unit disc.

1. Introduction.

Let W be an open Riemann surface possessing a Green's function. Consider a p-*sheeted unlimited* covering surface $\tilde{W}$ of W with projection map φ. It is easily seen that $\tilde{W}$ also possesses a Green's function (cf. e.g. [**AS**]). We denote by $HP(R)$ ($HB(R)$, resp.) the class of positive (bounded, resp.) harmonic functions on an open Riemann surface R. It is obvious that the inclusion relation

$$HX(W) \circ \varphi := \{h \circ \varphi : h \in HX(W)\} \subset HX(\tilde{W})$$

holds for $X = P, B$. The main purpose of this paper is to give a necessary and sufficient condition, in terms of Martin boundary, in order that the relation $HX(W) \circ \varphi = HX(\tilde{W})$ holds for $X = P, B$.

For an open Riemann surface R, we denote by R^*, Δ^R and Δ_1^R the Martin compactification, the Martin boundary and the minimal Martin boundary of R, respectively. It is known that the projection map φ of $\tilde{W}$ to W has the unique continuous extension to $\tilde{W}^*$, which is also denoted by φ, and $\varphi(\Delta^{\tilde{W}}) = \Delta^W$ (cf. [**MS2**]). For each $\zeta \in \Delta^W$, put

$$\Delta_1^{\tilde{W}}(\zeta) = \Delta_1^{\tilde{W}} \cap \varphi^{-1}(\zeta) = \{\tilde{\zeta} \in \Delta_1^{\tilde{W}} : \varphi(\tilde{\zeta}) = \zeta\},$$

which is the set of minimal boundary points of $\tilde{W}$ lying over $\zeta \in \Delta^W$. Our main results are the followings.

Theorem 1. *In order that the relation $HP(W) \circ \varphi = HP(\tilde{W})$ holds, it is necessary and sufficient that $\Delta_1^{\tilde{W}}(\zeta)$ consists of a single point for every $\zeta \in \Delta_1^W$.*

Theorem 2. *In order that the relation $HB(W) \circ \varphi = HB(\tilde{W})$ holds, it is necessary and sufficient that $\Delta_1^{\tilde{W}}(\zeta)$ consists of a single point for ω_z^W—almost all $\zeta \in \Delta_1^W$, where ω_z^W is a harmonic measure on Δ^W with respect to W and $z \in W$.*

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Proofs of Theorems 1 and 2 will be given in §3 and §4, respectively.

Let D be the unit disc $\{|z| < 1\}$. In §5, we will be concerned with p-sheeted unlimited covering surfaces of D which illustrate Theorems 1 and 2. We will prove the following.

Proposition. *Set* $A = \{(1 - 2^{-n-1})e^{i2\pi k/2^{n+2}} : n = 1, 2, \dots, k = 1, \dots, 2^{n+2}\}$. *If* $\tilde{D}$ *is a p-sheeted unlimited covering surface of D with projection map* φ *such that there is a branch point of* $\tilde{D}$ *of order* $p - 1$ (*or multiplicity* p) *over every* $z \in A$ *and there are no branch points of* $\tilde{D}$ *over* $D \backslash A$, *then* $HP(D) \circ \varphi = HP(\tilde{D})$.

We will show a bit more (cf. Theorem 5.1). Modifying the above $\tilde{D}$, we will also give a p-sheeted unlimited covering surface $\tilde{D}_1$ of D with projection map φ such that $HB(D) \circ \varphi = HB(\tilde{D}_1)$ and $HP(D) \circ \varphi \neq HP(\tilde{D}_1)$.

2. Martin boundary of p-sheeted unlimited covering surfaces.

Let W be an open Riemann surface possessing a Green's function and $\tilde{W}$ a p-sheeted unlimited covering surface of W with projection map φ. Since the pullback of a Green's function on W by φ is a nonconstant positive superharmonic function on $\tilde{W}$, we see that $\tilde{W}$ possesses a Green's function (cf. e.g. [**AS**], [**SN**]). For the Martin compactifications, Martin boundaries and minimal Martin boundaries, we follow the notation in Introduction. We first note the following (cf. [**MS2**]).

Proposition 2.1. *The projection map* φ *of* $\tilde{W}$ *onto* W *has the unique continuous extension to the Martin compactification* $\tilde{W}^*$ *of* $\tilde{W}$, *which is also denoted by* φ, *and* $\varphi(\Delta^{\tilde{W}}) = \Delta^W$.

We recall the definition of $\Delta_1^{\tilde{W}}(\zeta)$ $(\zeta \in \Delta^W)$ in Introduction:

$$\Delta_1^{\tilde{W}}(\zeta) = \Delta_1^{\tilde{W}} \cap \varphi^{-1}(\zeta) = \{\tilde{\zeta} \in \Delta_1^{\tilde{W}} : \varphi(\tilde{\zeta}) = \zeta\}.$$

We denote by $v_{\tilde{W}}(\zeta)$ the (cardinal) number of $\Delta_1^{\tilde{W}}(\zeta)$. We next fix a point $a \in W$ and a point $\tilde{a} \in \tilde{W}$ with

$$\varphi(\tilde{a}) = a. \tag{2.1}$$

We consider the Martin kernel $k_\zeta^W(\cdot)$ ($k_{\tilde{\zeta}}^{\tilde{W}}(\cdot)$, resp.) on W ($\tilde{W}$, resp.) with pole at ζ ($\tilde{\zeta}$, resp.) and with reference point a ($\tilde{a}$, resp.), that is,

$$k_\zeta^W(z) = \frac{g^W(z,\zeta)}{g^W(a,\zeta)} \quad \left(k_{\tilde{\zeta}}^{\tilde{W}}(\tilde{z}) = \frac{g^{\tilde{W}}(\tilde{z},\tilde{\zeta})}{g^{\tilde{W}}(\tilde{a},\tilde{\zeta})}, \text{ resp.}\right)$$

for $\zeta \in W$ ($\tilde{\zeta} \in \tilde{W}$, resp.), where $g^W(\cdot,\zeta)$ ($g^{\tilde{W}}(\cdot,\tilde{\zeta})$, resp.) is a Green's function on W ($\tilde{W}$, resp.) with pole at ζ ($\tilde{\zeta}$, resp.). Note that

$$k_\zeta^W(a) = k_{\tilde{\zeta}}^{\tilde{W}}(\tilde{a}) = 1. \tag{2.2}$$

We also note that the proof of Proposition 2.1 yields the following.

Proposition 2.2. *Let $\tilde{\zeta}$ be a point of $\Delta^{\tilde{W}}$ and $\varphi(\tilde{\zeta}) = \zeta$. Then there exists a constant c depending only on $\tilde{\zeta}$ and ζ such that*

$$\sum_{\tilde{z} \in \varphi^{-1}(z)} m(\tilde{z}) k_{\tilde{\zeta}}^{\tilde{W}}(\tilde{z}) = c k_{\zeta}^{W}(z)$$

on W, where $m(\tilde{\zeta})$ is multiplicity of φ at $\tilde{\zeta}$.

In our previous paper [**MS2**], we proved the following.

Proposition 2.3. *Suppose $\zeta \in \Delta^W$. Then*
(i) *If $\zeta \in \Delta^W \backslash \Delta_1^W$, then $v_{\tilde{W}}(\zeta) = 0$;*
(ii) *If $\zeta \in \Delta_1^W$, then $1 \le v_{\tilde{W}}(\zeta) \le p$;*
(iii) *If $\zeta \in \Delta_1^W$ and $\Delta_1^{\tilde{W}}(\zeta) = \{\tilde{\zeta}_1, \dots, \tilde{\zeta}_n\}$, then there exist positive numbers $c_1, \dots, c_n$ such that*

$$k_{\zeta}^{W} \circ \varphi = c_1 k_{\tilde{\zeta}_1}^{\tilde{W}} + \cdots + c_n k_{\tilde{\zeta}_n}^{\tilde{W}}. \tag{2.3}$$

In the relation (2.3) above, by (2.1) and (2.2), we have

$$\sum_{i=1}^{n} c_n = 1. \tag{2.4}$$

Let s be a positive superharmonic function on W and E a subset of W. We denote by ${}^W\hat{R}_s^E$ the *balayage* of s with respect to E on W. We here give the definitions of *minimal thinness* and *minimal fine neighborhood* (cf. [**B**]).

Definition 2.1. Let ζ be a point of Δ_1^W and E a subset of W. We say that E is *minimally thin* at ζ if ${}^W\hat{R}_{k_\zeta^W}^E \neq k_\zeta^W$.

Definition 2.2. Let ζ be a point of Δ_1^W and U a subset of W. We say that $U \cup \{\zeta\}$ is a *minimal fine neighborhood* of ζ if $W \backslash U$ is minimally thin at ζ.

The following is easily verified from Proposition 3.1 of our previous paper [**MS2**] (see also [**M**]).

Proposition 2.4. *Let $\tilde{\zeta}$ be $\in \Delta_1^{\tilde{W}}$ and $\tilde{U}$ a subset of $\tilde{W}$. Then $\tilde{U} \cup \{\tilde{\zeta}\}$ is a minimal fine neighborhood of $\tilde{\zeta}$ if and only if $\varphi(\tilde{U}) \cup \{\varphi(\tilde{\zeta})\}$ is a minimal fine neighborhood of $\varphi(\tilde{\zeta})$.*

For $\zeta \in \Delta_1^W$, we denote by $\mathscr{M}_W(\zeta)$ the class of connected open sets M such that $W \backslash M$ is minimally thin at ζ. Moreover, for $M \in \mathscr{M}_W(\zeta)$ and a p-sheeted unlimited covering surface $\tilde{W}$ of W with projection map φ, we denote by $n_{\tilde{W}}(M)$ the number of connected components of $\varphi^{-1}(M)$. Then $v_{\tilde{W}}(\zeta)$ is characterized by $n_{\tilde{W}}(M)$ as follows, which is a main result of our previous paper [**MS2**].

Proposition 2.5. *Suppose $\zeta \in \Delta_1^W$. Then $v_{\tilde{W}}(\zeta) = \max_{M \in \mathscr{M}_W(\zeta)} n_{\tilde{W}}(M)$.*

3. Proof of Theorem 1.

In this section, we give the proof of Theorem 1. For the sake of simplicity, we introduce the following notation:

$$\Delta = \Delta^W, \quad \Delta_1 = \Delta_1^W, \quad \tilde{\Delta} = \Delta^{\tilde{W}}, \quad \tilde{\Delta}_1 = \Delta_1^{\tilde{W}}, \quad \tilde{\Delta}_1(\zeta) = \Delta_1^{\tilde{W}}(\zeta)$$

and

$$k_\zeta = k_\zeta^W, \quad \tilde{k}_{\tilde{\zeta}} = k_{\tilde{\zeta}}^{\tilde{W}}.$$

PROOF OF THEOREM 1. Assume that $HP(W) \circ \varphi = HP(\tilde{W})$. Let ζ be an arbitrary point in Δ_1. We need to show that $\tilde{\Delta}_1(\zeta)$ consists of a single point. Take a point $\tilde{\zeta} \in \tilde{\Delta}_1(\zeta)$. By Proposition 2.3 (iii), there exists a positive constant c such that

$$c\tilde{k}_{\tilde{\zeta}} \le k_\zeta \circ \varphi \tag{3.1}$$

on $\tilde{W}$. By assumption, there exists an $h \in HP(W)$ such that

$$\tilde{k}_{\tilde{\zeta}} = h \circ \varphi \tag{3.2}$$

on $\tilde{W}$. Hence, by (3.1), we see that $ch \le k_\zeta$ on W. This with minimality of k_ζ implies that there exists a positive constant c_1 such that

$$h = c_1 k_\zeta \tag{3.3}$$

on W. Hence, by (3.2), we see that $\tilde{k}_{\tilde{\zeta}} = c_1 k_\zeta \circ \varphi$ on $\tilde{W}$. From this with (2.1) and (2.2), it follows that $c_1 = 1$. Therefore we obtain

$$\tilde{k}_{\tilde{\zeta}} = k_\zeta \circ \varphi \tag{3.4}$$

on $\tilde{W}$. This yields that $\tilde{\Delta}_1(\zeta) = \{\tilde{\zeta}\}$.

Conversely, assume that $v_{\tilde{W}}(\zeta) = 1$ for every $\zeta \in \Delta_1$. We only need to show $HP(\tilde{W}) \subset HP(W) \circ \varphi$, since the reversed inclusion is trivial. By assumption, we set $\tilde{\Delta}_1(\zeta) = \{\tilde{\zeta}\}$ for each $\zeta \in \Delta_1$. By Proposition 2.3 (iii) and (2.4), we have

$$\tilde{k}_{\tilde{\zeta}} = k_\zeta \circ \varphi \tag{3.5}$$

for every $\zeta \in \Delta_1$. Take an arbitrary $\tilde{h}$ in $HP(\tilde{W})$. By the Martin representation theorem (cf. e.g. [**CC**], [**HL**] and [**B**]), there exists a Radon measure $\tilde{\mu}$ on $\tilde{\Delta}$ with $\tilde{\mu}(\tilde{\Delta} \backslash \tilde{\Delta}_1) = 0$ such that

$$\tilde{h} = \int \tilde{k}_{\tilde{\xi}}\, d\tilde{\mu}(\tilde{\xi}). \tag{3.6}$$

Choose arbitrary two points $\tilde{z}_1$ and $\tilde{z}_2$ in $\tilde{W}$ with $\varphi(\tilde{z}_1) = \varphi(\tilde{z}_2)$. In view of (3.5) and (3.6), we obtain

$$\tilde{h}(\tilde{z}_1) = \int \tilde{k}_{\tilde{\xi}}(\tilde{z}_1)\, d\tilde{\mu}(\tilde{\xi}) = \int \tilde{k}_{\tilde{\xi}}(\tilde{z}_2)\, d\tilde{\mu}(\tilde{\xi}) = \tilde{h}(\tilde{z}_2).$$

Therefore we deduce that $\tilde{h} \in HP(W) \circ \varphi$ for every $\tilde{h} \in HP(\tilde{W})$, and hence $HP(\tilde{W}) \subset HP(W) \circ \varphi$.

The proof is herewith complete. □

In view of Theorem 1, we obtain the following.

COROLLARY 3.1. *In order that the relation $HP(W) \circ \varphi = HP(\tilde{W})$ holds, it is necessary and sufficient that $\varphi^{-1}(\zeta)$ consists of a single point for every $\zeta \in \Delta$ $(=\Delta^W)$.*

Proof. Assume that $\varphi^{-1}(\zeta)$ consists of a single point for every $\zeta \in \Delta$. Then Proposition 2.3 (ii) yields that $\tilde{\Delta}_1(\zeta)$ consists of a single point for every $\zeta \in \Delta_1$, since $\tilde{\Delta}_1(\zeta) \subset \varphi^{-1}(\zeta)$. Hence, by Theorem 1, we have $HP(W) \circ \varphi = HP(\tilde{W})$.

Conversely, assume $HP(W) \circ \varphi = HP(\tilde{W})$. Let $\zeta \in \Delta$ and take an arbitrary point $\tilde{\zeta} \in \varphi^{-1}(\zeta)$. Then, by assumption, there exists an $h \in HP(W)$ such that $\tilde{k}_{\tilde{\zeta}} = h \circ \varphi$ on $\tilde{W}$. Hence, in view of Proposition 2.2 and (2.2), we see that $\tilde{k}_{\tilde{\zeta}} = k_\zeta \circ \varphi$ on $\tilde{W}$. This means that $\varphi^{-1}(\zeta)$ consists of a single point $\tilde{\zeta}$. □

4. Proof of Theorem 2.

In this section, we give the proof of Theorem 2. Let $\omega_z(\cdot)$ ($\tilde{\omega}_{\tilde{z}}(\cdot)$, resp.) be the harmonic measure on Δ ($\tilde{\Delta}$, resp.) with respect to W ($\tilde{W}$, resp.) and $z \in W$ ($\tilde{z} \in \tilde{W}$, resp.). It is well-known that harmonic measure is a Radon measure (cf. e.g. [**CC**]). It is also well-known that $\omega_z(\cdot)$ ($\tilde{\omega}_{\tilde{z}}(\cdot)$, resp.) can be extended to the outer measure on Δ ($\tilde{\Delta}$, resp.) by

$$\omega_z(E) = \inf\{\omega_z(B) \colon B \text{ is an open set with } E \subset B\}$$

$$(\tilde{\omega}_{\tilde{z}}(\tilde{E}) = \inf\{\tilde{\omega}_{\tilde{z}}(\tilde{B}) \colon \tilde{B} \text{ is an open set with } E \subset B\}, \text{ resp.})$$

for a subset E ($\tilde{E}$, resp.) of Δ ($\tilde{\Delta}$, resp.). By definition, $h(z) = \omega_z(E)$ is a nonnegative harmonic function on W for every $E \subset \Delta$. By minimum principle, it is obvious that, for an arbitrary E ($\subset \Delta$) ($\tilde{E} \subset \tilde{\Delta}$, resp.), $\omega_z(E) = 0$ ($\tilde{\omega}_{\tilde{z}}(\tilde{E}) = 0$, resp.) for a $z \in W$ ($\tilde{z} \in \tilde{W}$, resp.) if and only if $\omega_z(E) = 0$ ($\tilde{\omega}_{\tilde{z}}(\tilde{E}) = 0$, resp.) for all $z \in W$ ($\tilde{z} \in \tilde{W}$, resp.). Let f be a real-valued function on the Martin boundary Δ^R of an open Riemann surface R. We denote by H_f^R ($\overline{H}_f^R$, resp.) the solution (upper solution, resp.) of Dirichlet problem on R ($=W$ or $\tilde{W}$) with boundary values f in the sense of Perron-Wiener-Brelot. We first prove the following.

Lemma 4.1. *Let $\tilde{E}$ be a subset of $\tilde{\Delta}$. Then $\tilde{\omega}_{\tilde{z}}(\tilde{E}) = 0$ if and only if $\omega_z(\varphi(\tilde{E})) = 0$.*

Proof. Suppose that $\tilde{\omega}_{\tilde{z}}(\tilde{E}) = 0$. By definition, there exists a Borel set $\tilde{B} \subset \tilde{\Delta}$ with $\tilde{E} \subset \tilde{B}$ such that

$$\tilde{\omega}_{\tilde{z}}(\tilde{B}) = H_{1_{\tilde{B}}}^{\tilde{W}}(\tilde{z}) = 0, \tag{4.1}$$

where $1_{\tilde{B}}$ is the characteristic function of $\tilde{B}$ on $\tilde{\Delta}$. Let $\tilde{s}$ be an arbitrary positive superharmonic function on $\tilde{W}$ such that $\liminf_{\tilde{z} \to \tilde{\zeta}} \tilde{s}(\tilde{z}) \geq 1$ for every $\tilde{\zeta} \in \tilde{B}$. Set

$$s(z) := \sum_{\tilde{z} \in \varphi^{-1}(z)} m(\tilde{z}) \tilde{s}(\tilde{z}),$$

where $m(\tilde{z})$ is multiplicity of φ at $\tilde{z}$. Then $s(z)$ is a positive superharmonic function on W and $\liminf_{z \to \zeta} s(z) \geq 1$ for every $\zeta \in \varphi(\tilde{B})$. Hence $s(z) \geq \overline{H}_{1_{\varphi(\tilde{B})}}^W(z)$. From this and the fact $\overline{H}_{1_{\varphi(\tilde{B})}}^W(z) \geq \omega_z(\varphi(\tilde{B}))$ (cf. e.g. [**CC**]), it follows that

$$s(z) \geq \omega_z(\varphi(\tilde{B})) \geq \omega_z(\varphi(\tilde{E})).$$

Therefore, by letting $s(z)$ arbitrarily small in view of (4.1), we obtain $\omega_z(\varphi(\tilde{E})) = 0$.

Suppose $\omega_z(\varphi(\tilde{E})) = 0$. By definition, there exists a Borel set $B \subset \Delta$ with $B \supset \varphi(\tilde{E})$ such that

$$\omega_z(B) = H^W_{1_B}(z) = 0. \tag{4.2}$$

Let s be an arbitrary positive superharmonic function on W such that $\liminf_{z\to\zeta} s(z) \geq 1$ for every $\zeta \in B$. Then $s \circ \varphi(\tilde{z})$ is a positive superharmonic function on $\tilde{W}$ and

$$\liminf_{\tilde{z}\to\tilde{\zeta}} s \circ \varphi(\tilde{z}) \geq 1$$

for every $\tilde{\zeta} \in \varphi^{-1}(B)$. Hence $s \circ \varphi(\tilde{z}) \geq \overline{H}^{\tilde{W}}_{1_{\varphi^{-1}(B)}}(\tilde{z})$. From this and the fact $\overline{H}^{\tilde{W}}_{1_{\varphi^{-1}(B)}}(\tilde{z}) \geq \tilde{\omega}_{\tilde{z}}(\varphi^{-1}(B))$, it follows that

$$s \circ \varphi(\tilde{z}) \geq \tilde{\omega}_{\tilde{z}}(\varphi^{-1}(B)) \geq \tilde{\omega}_{\tilde{z}}(\varphi^{-1}(\varphi(\tilde{E}))) \geq \tilde{\omega}_{\tilde{z}}(\tilde{E}).$$

Therefore, letting $s \circ \varphi(\tilde{z})$ arbitrarily small in view of (4.2), we obtain $\tilde{\omega}_{\tilde{z}}(\tilde{E}) = 0$.

The proof is herewith complete. □

We next consider the sets

$$N_1 := \{\zeta \in \Delta_1 : \nu_{\tilde{W}}(\zeta) = 1\}$$

and

$$N_2 := \Delta_1 \backslash N_1 = \{\zeta \in \Delta_1 : \nu_{\tilde{W}}(\zeta) \geq 2\}.$$

Put $\tilde{N}_1 = \varphi^{-1}(N_1) \cap \tilde{\Delta}_1$ and $\tilde{N}_2 = \varphi^{-1}(N_2) \cap \tilde{\Delta}_1$. By means of Proposition 2.3, it is easily seen that $\tilde{N}_1 \cup \tilde{N}_2 = \tilde{\Delta}_1$ and $\varphi(\tilde{N}_i) = N_i$ $(i = 1, 2)$. We denote by $\tilde{d}(\cdot\,,\cdot)$ the metric on $\tilde{W}^*$ defined by

$$\tilde{d}(\tilde{z}, \tilde{\zeta}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \frac{\tilde{k}_{\tilde{z}}(\tilde{z}_n)}{1 + \tilde{k}_{\tilde{z}}(\tilde{z}_n)} - \frac{\tilde{k}_{\tilde{\zeta}}(\tilde{z}_n)}{1 + \tilde{k}_{\tilde{\zeta}}(\tilde{z}_n)} \right|,$$

where $\{\tilde{z}_n : n = 1, 2, \ldots\}$ is a dense subset of $\tilde{W}$. Set $\tilde{U}_r(\tilde{z}_0) = \{\tilde{z} \in \tilde{W}^* : \tilde{d}(\tilde{z}, \tilde{z}_0) < r\}$ for $\tilde{z}_0 \in \tilde{W}^*$ and $r > 0$.

Lemma 4.2. *Suppose $\omega_z(N_2) > 0$. Then there exists a $\tilde{\zeta}_0 \in \tilde{N}_2$ such that*

$$\tilde{\omega}_{\tilde{z}}(\tilde{N}_2 \cap \tilde{U}_r(\tilde{\zeta}_0)) > 0$$

for every $r > 0$.

Proof. By virtue of Lemma 4.1, we have $\tilde{\omega}_{\tilde{z}}(\tilde{N}_2) > 0$, since $\varphi(\tilde{N}_2) = N_2$. Contrary to the assertion, assume that, for every $\tilde{\zeta} \in \tilde{N}_2$, there exists an $r_{\tilde{\zeta}} > 0$ such that $\tilde{\omega}_{\tilde{z}}(\tilde{N}_2 \cap \tilde{U}_{r_{\tilde{\zeta}}}(\tilde{\zeta})) = 0$. Then, by the Lindelöf covering theorem, there exists a sequence $\{\tilde{\zeta}_j\}_{j=1}^{\infty}$ in $\tilde{N}_2$ such that $\tilde{N}_2 \subset \bigcup_{j=1}^{\infty} \tilde{U}_{r_{\tilde{\zeta}_j}}(\tilde{\zeta}_j)$. Hence we have

$$\tilde{\omega}_{\tilde{z}}(\tilde{N}_2) \leq \sum_{j=1}^{\infty} \tilde{\omega}_{\tilde{z}}(\tilde{N}_2 \cap \tilde{U}_{r_{\tilde{\zeta}_j}}(\tilde{\zeta}_j)) = 0,$$

which is a contradiction. □

Here, we again recall the definition of $\tilde{\Delta}_1(\zeta)$:

$$\tilde{\Delta}_1(\zeta) = \tilde{\Delta}_1 \cap \varphi^{-1}(\zeta) = \{\tilde{\zeta} \in \tilde{\Delta}_1 : \varphi(\tilde{\zeta}) = \zeta\}.$$

Lemma 4.3. *Let $\tilde{\xi}$ be a point in $\tilde{N}_2$. Then there exists a $\rho > 0$ such that $\tilde{\Delta}_1(\zeta) \backslash \tilde{U}_\rho(\tilde{\xi})$ is not empty for every $\zeta \in N_2 \cap \varphi(\tilde{U}_\rho(\tilde{\xi}))$.*

Proof. Set $\varphi(\tilde{\xi}) = \xi$. Then, by definition, $\xi \in N_2$. Assume that the assertion is false. Then there exists a sequence $\{\zeta_j\}_{j=1}^{\infty}$ in $N_2 \backslash \{\varphi(\tilde{\xi})\}$ such that

$$\max_{\tilde{\eta} \in \tilde{\Delta}_1(\zeta_j)} \tilde{d}(\tilde{\eta}, \tilde{\xi}) < 1/j. \tag{4.3}$$

From this and Proposition 2.2 it follows that

$$\lim_{j \to \infty} k_{\zeta_j} = k_{\xi}. \tag{4.4}$$

For each j, put $\tilde{\Delta}_1(\zeta_j) = \{\tilde{\zeta}_{j1}, \dots, \tilde{\zeta}_{jn_j}\}$. By Proposition 2.3 and (2.4), there exist positive constants $c_{j1}, \dots, c_{jn_j}$ with $\sum_{i=1}^{n_j} c_{ji} = 1$ such that

$$k_{\zeta_j} \circ \varphi = \sum_{i=1}^{n_j} c_{ji} \tilde{k}_{\tilde{\zeta}_{ji}}. \tag{4.5}$$

On the other hand, in view of (4.3), we see that

$$\lim_{j \to \infty} \tilde{k}_{\tilde{\zeta}_{ji_j}} = \tilde{k}_{\tilde{\xi}}$$

independently of choice of i_j in $\{1, \dots, n_j\}$. This with (4.4) and (4.5) implies that

$$k_{\xi} \circ \varphi = \tilde{k}_{\tilde{\xi}}.$$

Therefore, by means of Proposition 2.3, we obtain $\tilde{\Delta}_1(\xi) = \{\tilde{\xi}\}$, which contradicts $\xi \in N_2$. This completes the proof. □

We can restate Theorem 2, in terms of the set N_2, as follows: *The relation $HB(W) \circ \varphi = HB(\tilde{W})$ holds if and only if $\omega_z(N_2) = 0$.*

Proof of Theorem 2. We first prove 'if' part. Suppose $\omega_z(N_2) = 0$. Then, by Lemma 4.1,

$$\tilde{\omega}_{\tilde{z}}(\tilde{N}_2) = 0. \tag{4.6}$$

Take an arbitrary $\tilde{h} \in HB(\tilde{W})$. We only need to show $\tilde{h} \in HB(W) \circ \varphi$. Adding a constant to $\tilde{h}$, we may assume that $\tilde{h} > 0$ on $\tilde{W}$. Let c (>0) be the supremum of $\tilde{h}$ on $\tilde{W}$. By the Martin representation theorem, there exist Radon measures $\tilde{\mu}$ and $\tilde{\chi}$ on $\tilde{\Delta}$ with $\tilde{\mu}(\tilde{\Delta} \backslash \tilde{\Delta}_1) = 0$ and $\tilde{\chi}(\tilde{\Delta} \backslash \tilde{\Delta}_1) = 0$ such that

$$\tilde{h}(\tilde{z}) = \int \tilde{k}_{\tilde{\zeta}}(\tilde{z})\, d\tilde{\mu}(\tilde{\zeta}) \tag{4.7}$$

and

$$1 = \int \tilde{k}_{\tilde{\zeta}}(\tilde{z})\, d\tilde{\chi}(\tilde{\zeta}). \tag{4.8}$$

Then

$$c\int \tilde{k}_{\tilde{\zeta}}(\tilde{z})\, d\tilde{\chi}(\tilde{\zeta}) = c \geq \tilde{h}(\tilde{z}) = \int \tilde{k}_{\tilde{\zeta}}(\tilde{z})\, d\tilde{\mu}(\tilde{\zeta}).$$

Hence, by uniqueness of representing measure, we have

$$c\tilde{\chi} \geq \tilde{\mu}. \tag{4.9}$$

Note that $\tilde{k}_{\tilde{\zeta}}(\tilde{z})\, d\tilde{\chi}(\tilde{\zeta}) = d\tilde{\omega}_{\tilde{z}}(\tilde{\zeta})$ (cf. [**CC**, p. 140]). From this and (4.9) it follows that

$$\int_{\tilde{N}_2} \tilde{k}_{\tilde{\zeta}}(\tilde{z})\, d\tilde{\mu}(\tilde{\zeta}) \leq c\int_{\tilde{N}_2} \tilde{k}_{\tilde{\zeta}}(\tilde{z})\, d\tilde{\chi}(\tilde{\zeta}) = c\int_{\tilde{N}_2} d\tilde{\omega}_{\tilde{z}}(\tilde{\zeta}) = c\tilde{\omega}_{\tilde{z}}(\tilde{N}_2).$$

This with (4.6) yields that

$$\int_{\tilde{N}_2} \tilde{k}_{\tilde{\zeta}}(\tilde{z})\, d\tilde{\mu}(\tilde{\zeta}) = 0.$$

Therefore, by (4.7) and the fact $\tilde{N}_1 \cup \tilde{N}_2 = \tilde{\Delta}_1$, we have

$$\tilde{h}(\tilde{z}) = \int_{\tilde{N}_1} \tilde{k}_{\tilde{\zeta}}(\tilde{z})\, d\tilde{\mu}(\tilde{\zeta}). \tag{4.10}$$

By Proposition 2.3 (iii) and (2.4), we see that $\tilde{k}_{\tilde{\zeta}} \in HP(W) \circ \varphi$ for every $\tilde{\zeta} \in \tilde{N}_1$. Hence, by (4.10) and the same argument as in the proof of Theorem 1, we obtain

$$\tilde{h} \in HP(W) \circ \varphi \cap HB(\tilde{W}) \subset HB(W) \circ \varphi.$$

We next prove 'only if' part. Suppose $\omega_z(N_2) > 0$. Then, by Lemma 4.2, there exists a $\tilde{\xi} \in \tilde{N}_2$ such that

$$\tilde{\omega}_{\tilde{z}}(\tilde{N}_2 \cap \tilde{U}_r(\tilde{\xi})) > 0 \tag{4.11}$$

for every $r > 0$. Moreover, by Lemma 4.3, there exists $\rho > 0$ such that

$$\tilde{\Delta}_1(\zeta) \backslash \tilde{U}_\rho(\tilde{\xi}) \neq \varnothing \tag{4.12}$$

for every $\zeta \in N_2 \cap \varphi(\tilde{U}_\rho(\tilde{\xi}))$. Set

$$\tilde{E}_1 = \tilde{N}_2 \cap \tilde{U}_{\rho/2}(\tilde{\xi}).$$

Then, by (4.11) and Lemma 4.1, we have

$$\omega_z(\varphi(\tilde{E}_1)) > 0. \tag{4.13}$$

Set

$$\tilde{E}_2 = \tilde{N}_2 \cap \varphi^{-1}(\varphi(\tilde{U}_{\rho/2}(\tilde{\xi}))) \backslash \tilde{U}_\rho(\tilde{\xi}).$$

In view of (4.12), we find that

$$\varphi(\tilde{E}_1) = \varphi(\tilde{E}_2). \tag{4.14}$$

Put $\tilde{h}(\tilde{z}) = \tilde{\omega}_{\tilde{z}}(\tilde{E}_1)$. Then $\tilde{h}(\tilde{z})$ is a bounded harmonic function on $\tilde{W}$. We only need to show $\tilde{h} \notin HB(W) \circ \varphi$. By the Fatou-Naïm-Doob theorem (cf. [**CC**, p. 152]), $\tilde{h}(\tilde{z})$ has the minimal fine limit 1 (0, resp.) at almost all $\tilde{\zeta}$ in $\tilde{E}_1$ ($\tilde{E}_2$, resp.) with respect to $\tilde{\omega}_{\tilde{z}}$, since $\overline{\tilde{E}_1} \cap \overline{\tilde{E}_2} = \emptyset$. Accordingly there exists a subset $\tilde{F}_1$ ($\tilde{F}_2$, resp.) of $\tilde{E}_1$ ($\tilde{E}_2$, resp.) with $\tilde{\omega}_{\tilde{z}}(\tilde{F}_1) = 0$ ($\tilde{\omega}_{\tilde{z}}(\tilde{F}_2) = 0$, resp.) such that, for every $\tilde{\zeta}$ in $\tilde{E}_1 \backslash \tilde{F}_1$ ($\tilde{E}_2 \backslash \tilde{F}_2$, resp.),

$$\mathscr{F} - \lim_{\tilde{z} \to \tilde{\zeta}} \tilde{h}(\tilde{z}) = 1 \quad (\mathscr{F} - \lim_{\tilde{z} \to \tilde{\zeta}} \tilde{h}(\tilde{z}) = 0, \text{ resp.}) \tag{4.15}$$

where we denote by $\mathscr{F} - \lim$ minimal fine limit. Then, by Lemma 4.1, $\omega_z(\varphi(\tilde{F}_1) \cup \varphi(\tilde{F}_2)) = 0$. Hence, by (4.13) and (4.14), there exist points $\tilde{\zeta}_1 \in \tilde{E}_1 \backslash \tilde{F}_1$ and $\tilde{\zeta}_2 \in \tilde{E}_2 \backslash \tilde{F}_2$ with $\varphi(\tilde{\zeta}_1) = \varphi(\tilde{\zeta}_2)$. This with (4.15) implies that there exists an open subset $\tilde{O}_1$ ($\tilde{O}_2$, resp.) of $\tilde{W}$ such that $\tilde{O}_1 \cup \{\tilde{\zeta}_1\}$ ($\tilde{O}_2 \cup \{\tilde{\zeta}_2\}$, resp.) is a minimal fine neighborhood of $\tilde{\zeta}_1$ ($\tilde{\zeta}_2$, resp.) and that

$$\inf_{\tilde{z} \in \tilde{O}_1} \tilde{h}(\tilde{z}) \geq \frac{2}{3} \quad (\sup_{\tilde{z} \in \tilde{O}_2} \tilde{h}(\tilde{z}) \leq \frac{1}{3}, \text{ resp.}). \tag{4.16}$$

Then, by virtue of Proposition 2.4, we see that $(\varphi(\tilde{O}_1) \cap \varphi(\tilde{O}_2)) \cup \{\varphi(\tilde{\zeta}_1)\}$ is a minimal fine neighborhood of $\varphi(\tilde{\zeta}_1) = \varphi(\tilde{\zeta}_2)$, and hence $\varphi(\tilde{O}_1) \cap \varphi(\tilde{O}_2) \neq \emptyset$. Therefore, by (4.16), there exists a subset $\tilde{U}_j$ of $\tilde{O}_j$ $(j = 1, 2)$ with $\varphi(\tilde{U}_1) = \varphi(\tilde{U}_2)$ such that

$$\inf_{\tilde{z} \in \tilde{U}_1} \tilde{h}(\tilde{z}) \geq \frac{2}{3} \quad (\sup_{\tilde{z} \in \tilde{U}_2} \tilde{h}(\tilde{z}) \leq \frac{1}{3}, \text{ resp.}).$$

This means that $\tilde{h} \notin HB(W) \circ \varphi$.

The proof is herewith complete. □

Corollary 4.1. *In order that the relation* $HB(W) \circ \varphi = HB(\tilde{W})$ *holds, it is necessary and sufficient that* $\varphi^{-1}(\zeta)$ *consists of a single point for* ω_z^W*—almost all* $\zeta \in \Delta$ $(=\Delta^W)$.

Proof. Note that $\omega_z^W(\Delta \backslash \Delta_1) = 0$ (cf. [**CC**]). Hence, by virtue of Theorem 2, it suffices to show that, for each $\zeta \in \Delta_1$, $\tilde{\Delta}_1(\zeta)$ consists of a single point if and only if $\varphi^{-1}(\zeta)$ consists of a single point.

If $\varphi^{-1}(\zeta)$ consists of a single point, then it instantly follows from Proposition 2.3 (ii) that $\tilde{\Delta}_1(\zeta)$ consists of a single point, since $\tilde{\Delta}_1(\zeta) \subset \varphi^{-1}(\zeta)$. Assume that $\tilde{\Delta}_1(\zeta)$ consists of a single point $\tilde{\zeta}$. Take an arbitrary point $\tilde{\xi} \in \varphi^{-1}(\zeta)$. Then, in view of Proposition 2.2 and Proposition 2.3 (iii), there exists a positive constant c such that $\tilde{k}_{\tilde{\xi}} \leq c\tilde{k}_{\tilde{\zeta}}$ on $\tilde{W}$. Hence, by minimality of $\tilde{k}_{\tilde{\zeta}}$ and (2.2), we have $\tilde{k}_{\tilde{\xi}} = \tilde{k}_{\tilde{\zeta}}$. This means that $\varphi^{-1}(\zeta)$ consists of a single point $\tilde{\zeta}$. □

5. Harmonic functions on covering surfaces of the unit disc.

Let D be the unit disc $\{|z| < 1\}$. In this section, we are concerned with application of Theorems 1 and 2 in case base surface is D. As is well-known, the Martin compactification D^* of D is identified with the closure $\bar{D}$ of D with respect to Euclidian topology and the Martin boundary Δ^D of D consists of only minimal points. In this view, we regard $\partial D = \{|z| = 1\}$ as the (minimal) Martin boundary of D.

To state our main result of this section, we introduce some notations. For a discrete subset A of D, we denote by $\mathscr{B}_p(A)$ the class of p-sheeted unlimited covering surface $\tilde{D}$ of D such that there exists a branch point in $\tilde{D}$ of order $p - 1$ (or multiplicity

p) over every $z \in A$ and there exist no branch points in $\tilde{D}$ over $D \backslash A$. In addition to the Euclidean metric, we consider the pseudohyperbolic metric on D given by

$$\rho(z, w) = \left|\frac{z - w}{1 - \overline{w}z}\right|.$$

For $\zeta \in \partial D$ and a positive number C (<1), we also consider the Stolz type domain with vertex ζ given by

$$S_C(\zeta) = \{z \in D : C|z - \zeta| < 1 - |z|\}.$$

Theorem 5.1. *Let $A = \{a_n : n \in \boldsymbol{N}\}$ be a discrete subset of D and $\tilde{D}$ belong to $\mathscr{B}_p(A)$. Suppose that there exists a positive constant C (<1) satisfying the following two conditions*

(i) *for every pair (a_m, a_n) in A with $a_m \neq a_n$, $\rho(a_m, a_n) \geq C$;*

(ii) *for every $\zeta \in \partial D$, there exists a subset $B_\zeta = \{b_n : n \geq n_0\}$ $(n_0 = n_0(\zeta))$ of A such that $b_n \in \{z : \sigma^{n+1} \leq |z - \zeta| \leq \sigma^n\} \cap S_C(\zeta)$ for every $n \geq n_0$, where σ is a positive number with $\sigma < 1$.*

Then $HP(\tilde{D}) = HP(D) \circ \varphi$, where φ is the projection map.

For a bounded Borel subset K of $\boldsymbol{C}$, we denote by $\lambda(K)$ the logarithmic capacity. As a necessary condition for minimal thinness, the following is available (cf. [**L**], [**J**]).

Lemma 5.1. *Let ζ be in $\partial D = \Delta_1^D$ and E a relatively closed subset of $S_C(\zeta)$. If E is minimally thin at ζ, then*

$$\sum_{n=1}^{\infty} \frac{1}{\log(1/(\lambda(E_n)))} < \infty,$$

where $E_n = E \cap \{z : \tau^{n+1} \leq |z - \zeta| \leq \tau^n\}$ and τ is a positive number with $\tau < 1$.

Proof of Theorem 5.1. Let ζ be an arbitrary point in ∂D. By virtue of Theorem 1, we only have to show that $\Delta_1^{\tilde{D}}(\zeta)$ consists of a single point. Take an arbitrary $M \in \mathscr{M}_D(\zeta)$. Our goal is to show that $\varphi^{-1}(M)$ *is connected.* In fact, in view of Proposition 2.5, connectivity of $\varphi^{-1}(M)$ for all $M \in \mathscr{M}_D(\zeta)$ implies $\Delta_1^{\tilde{D}}(\zeta)$ consists of a single point.

We first assume that there exists an $a_n \in M \cap A$. Then, it is easily seen that $\varphi^{-1}(M)$ is connected, since $\tilde{D}$ has a branch point of order $p - 1$ over $a_n \in M$ and M is connected.

We next assume $M \cap A = \varnothing$. Put $F = D \backslash M$. Note that F is minimally thin at ζ and relatively closed in D. For each n $(\geq n_0)$, let F_n be the connected component of F which contains $b_n \in B_\zeta$. We first consider the case that there exists an F_n $(n \geq n_0)$ such that

$$d(F_n) < C^2 \sigma^{n+1}, \tag{5.1}$$

where $d(F_n)$ indicates the diameter of F_n. Then there exists a closed Jordan curve γ_n in M such that γ_n surrounds F_n and

$$d(F_n) < d(\gamma_n) < C^2 \sigma^{n+1}. \tag{5.2}$$

By (i) and (ii), we have

$$|a_m - b_n| \geq C|1 - \overline{b_n} a_m| \geq C(1 - |b_n|) \geq C^2 |b_n - \zeta| \geq C^2 \sigma^{n+1},$$

for every $a_m \in A \backslash \{b_n\}$. Hence, by means of (5.2), we see that γ_n surrounds only one point b_n in A. Therefore, $\varphi^{-1}(\gamma_n)$ is connected, since $\tilde{D}$ has a branch point of order $p-1$ over b_n. This with $\gamma_n \in M$ and connectivity of M yields that $\varphi^{-1}(M)$ is connected. Accordingly, we complete the proof if we show that there exists an F_n $(n \geq n_0)$ satisfying (5.1).

Now we may assume that

$$d(F_n) \geq C^2 \sigma^{n+1} \tag{5.3}$$

for every n $(\geq n_0)$. Set $E = F \cap S_{C/2}(\zeta)$. Note that E is minimally thin at ζ. We denote by F_n^* the connected component of E which contains b_n. Then, in view of (ii) and (5.3), we find that there exists a positive constant C_1 $(\leq C^2\sigma)$ such that

$$d(F_n^*) \geq C_1 \sigma^n \tag{5.4}$$

for every n $(\geq n_0)$. Set $E_m = E \cap \{z : \sigma^{3(m+1)} \leq |z - \zeta| \leq \sigma^{3m}\}$. Note that $b_{3m+1} \in E_m$. Then, by (5.4), taking an appropriate constant C_2 $(< C_1)$, we see that, for every m with $3m+1 \geq n_0$, E_m contains a continuum whose diameter is equal to or greater than $C_2\sigma^{3m+1}$. From this it follows that

$$\lambda(E_m) \geq 4^{-1} C_2 \sigma^{3m+1}$$

for every m with $3m+1 \geq n_0$ (cf. [**T**]). Hence we see that

$$\frac{1}{\log(1/(\lambda(E_m)))} \geq \frac{1}{(3m+1)\log(1/\sigma) + \log(4/C_2)}$$

for every m with $3m+1 \geq n_0$. Therefore we deduce

$$\sum_{3m+1 \geq n_0} \frac{1}{\log(1/(\lambda(E_m)))} \geq \sum_{3m+1 \geq n_0} \frac{1}{(3m+1)\log(1/\sigma) + \log(4/C_2)} = \infty.$$

By Lemma 5.1, this is absurd, since E is minimally thin at ζ.

The proof is herewith complete. □

Using the notation above, we restate Proposition in Introduction as follows:

Corollary 5.1. *Let* $A = \{(1 - 2^{-n-1})e^{i2\pi k/2^{n+2}} : n = 1, 2, \ldots, k = 1, \ldots, 2^{n+2}\}$ *and* $\tilde{D}$ *belong to* $\mathscr{B}_p(A)$. *Then* $HP(D) \circ \varphi = HP(\tilde{D})$, *where* φ *is the projection map.*

Proof. For a pair $(z, w) = (1 - 2^{-n}, 1 - 2^{-n-1})$ or $(1 - 2^{-n}, (1 - 2^{-n})e^{i2\pi/2^{n+1}})$, by a calculation, it is easily checked that there exists a positive constant C independent of $n = 1, 2, \ldots$ such that $\rho(z, w) \geq C$. This implies that A and the above constant C satisfy the condition (i) of Theorem 5.1. Let ζ be an arbitrary point in ∂D. For every positive integer n, we can choose a positive integer k_n with $1 \leq k_n \leq 2^{n+2}$ such that

$$\left| \arg \zeta - \frac{2\pi k_n}{2^{n+2}} \right| \leq \frac{\pi}{2^{n+2}}. \tag{5.5}$$

Set

$$b_n = (1 - 2^{-n-1})e^{i2\pi k_n/2^{n+2}} \quad (n = 1, 2, \ldots).$$

Then, by (5.5), we have

$$(2^{-n-1})^2 \le |b_n - \zeta|^2 \le (2^{-n-1})^2 + 4\sin^2\frac{\pi}{2^{n+3}}.$$

In view of this with (5.5), it is easily seen that $B_\zeta := \{b_n : n \ge 1\}$ and a positive constant C satisfy the condition (ii) of Theorem 5.1 for $\sigma = 2^{-1}$. □

At the last, we give a p-sheeted unlimited covering surface $\tilde{D}_1$ of D with projection map φ such that $HB(D) \circ \varphi = HB(\tilde{D}_1)$ and $HP(D) \circ \varphi \ne HP(\tilde{D}_1)$. Let A be the same as in Corollary 5.1. Set $M = \{|z - 1/2| < 1/2\}$ and $A_1 = A \backslash M$. Consider a covering surface $D_1 \in \mathscr{B}_p(A_1)$ with projection map φ. We now show that $HB(D) \circ \varphi = HB(\tilde{D}_1)$ and $HP(D) \circ \varphi \ne HP(\tilde{D}_1)$. As is proved in the proof of Corollary 5.1, A_1 and a positive constant C satisfy the following two conditions:

(i) for every pair (a_m, a_n) in A_1 with $a_m \ne a_n$, $\rho(a_m, a_n) \ge C$;

(ii) for every $\zeta \in \partial D \backslash \{1\}$, there exists a subset $B_\zeta = \{b_n : n \ge n_0\}$ $(n_0 = n_0(\zeta))$ of A_1 such that $b_n \in \{z : 2^{-n-1} \le |z - \zeta| \le 2^{-n}\} \cap S_C(\zeta)$ for every $n \ge n_0$.

Therefore the proof of Theorem 5.1 yields that $v_{\tilde{D}_1}(\zeta) = 1$ for every $\zeta \in \partial D \backslash \{1\}$. Hence, by virtue of Theorem 2, we have $HB(D) \circ \varphi = HB(\tilde{D}_1)$. On the other hand, it is easily seen that M belongs to $\mathscr{M}_D(1)$ and $\varphi^{-1}(M)$ consists of p components. Hence, by Proposition 2.5 and Proposition 2.3 (ii), $v_{\tilde{D}_1}(1) = p$. Therefore, by Theorem 1, we see that $HP(D) \circ \varphi \ne HP(\tilde{D}_1)$.

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