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A central limit theorem on a covering graph with a transformation group of polynomial growth

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Abstract. We prove a central limit theorem for the transition operator of the symmetric random walk on a covering graph with a covering transformation group of polynomial growth. As the limit, the continuous semigroup of the sub-Laplacian on a nilpotent Lie group is obtained.

1. Introduction.

Let $X = (V, E)$ be a locally finite connected graph, V being the set of vertices and E being the set of oriented edges. For $e \in E$, the origin and the terminus of e are denoted by $o(e)$ and $t(e)$, respectively, and the inverse edge is denoted by $\bar{e}$. We shall assume that X is a covering graph of a finite graph whose covering transformation group Γ is a finitely generated group of polynomial growth. A symmetric random walk on X with a weight $m : V \to \boldsymbol{R}_+$ is given by a transition probability $p : E \to \boldsymbol{R}_+$ satisfying $\sum_{e \in E_x} p(e) = 1$ and $p(e)m(o(e)) = p(\bar{e})m(t(e))$, where $E_x = \{e \in E \mid o(e) = x\}$. We assume m and p are Γ-invariant. The transition operator L associated with the random walk is the operator acting on functions on V defined by

$$Lf(x) = \sum_{e \in E_x} f(t(e))p(e).$$

Suppose that X is realized in a continuous model M. Let $C_\infty(X)$ be the set of functions on V vanishing at infinity and $C_\infty(M)$ the set of continuous functions on M vanishing at infinity. The purpose of this article is to show that, the n-th iteration of L on $C_\infty(X)$ approaches a continuous semigroup on $C_\infty(M)$ as n goes to infinity with a suitable scale change on M. M. Kotani and T. Sunada considered the case of a crystal lattice, which is an abelian covering of a finite graph ([**6**], [**8**]). A central limit theorem for magnetic transition operators on a crystal lattice is obtained in [**6**]. As a special case of [**6**], when a vector potential is zero, the following central limit theorem is deduced.

Theorem (M. Kotani [**6**]). *Let X be a crystal lattice with an abelian covering transformation group Γ and $\Phi : X \to \Gamma \otimes \boldsymbol{R}$ a piecewise linear Γ-equivariant map. Put $X_0 = \Gamma \backslash X$ and $m(X_0) = \sum_{x \in X_0} m(x)$. Then for any $f \in C_\infty(\Gamma \otimes \boldsymbol{R})$, as $n \uparrow \infty$, $\delta \downarrow 0$ and $n\delta^2 \to m(X_0)t$, we have*

$$\|L^n(f \circ (\delta\Phi)) - (e^{-t\Delta} f) \circ (\delta\Phi)\|_\infty \to 0,$$

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where Δ is the Laplacian for the Albanese metric on $\Gamma \otimes \boldsymbol{R}$. In particular, for a sequence $\{x_\delta\}_{\delta>0}$ in X with $\lim_{\delta\downarrow 0}\delta\Phi(x_\delta) = x$,

$$\lim L^n(f \circ (\delta\Phi))(x_\delta) = e^{-t\Delta} f(x).$$

Let Γ be a finitely generated group of polynomial growth. G. Alexopoulos obtained a local central limit theorem for the convolution powers on Γ ([**1**]). Limit theorems for compositions of distribution on certain nilpotent Lie groups are obtained by P. Crépel and A. Raugi [**2**], H. Hennion [**4**], G. Pap [**11**], [**12**], A. Raugi [**14**], V. N. Tutubalin [**17**], A. D. Virtser [**19**]. We remark that a covering graph with a covering transformation group of polynomial growth can be considered as a generalization of a crystal lattice or the Cayley graph of a finitely generated group of polynomial growth. Let X be a covering graph whose covering transformation group is Γ. By a theorem of M. Gromov [**3**], Γ has a finitely generated torsion free nilpotent subgroup N of finite index so that X is a covering of the finite quotient graph $N\backslash X$ with the covering transformation group N. Therefore we may always assume that X is a covering graph of a finite graph $X_0 = (V_0, E_0)$ whose covering transformation group Γ is a finitely generated torsion free nilpotent group.

As the continuous model, we take the *limit group* $(G_\Gamma, *)$ of a connected, simply connected nilpotent Lie group $(G_\Gamma, \cdot)$ such that Γ is isomorphic to a lattice of $(G_\Gamma, \cdot)$. We have the following diagram.

$$\begin{array}{ccc}
G_\Gamma/[G_\Gamma, G_\Gamma] & \longleftarrow & \mathrm{H}_1(X_0, \boldsymbol{R}) \\
\Big\updownarrow \text{dual} & & \Big\updownarrow \text{dual} \\
\mathrm{Hom}(G_\Gamma/[G_\Gamma, G_\Gamma], \boldsymbol{R}) & \hookrightarrow & \mathrm{H}^1(X_0, \boldsymbol{R})
\end{array}$$

where $\mathrm{H}^1(X_0, \boldsymbol{R})$ is the first cohomology of X_0. By identifying $\mathrm{H}^1(X_0, \boldsymbol{R})$ with the space of harmonic 1-forms on X_0, we introduce an inner product on $\mathrm{H}^1(X_0, \boldsymbol{R})$. Let $\mathfrak{g}$ be the Lie algebra of G_Γ and $\mathfrak{g}^{(1)}$ a subspace of $\mathfrak{g}$ satisfying $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus [\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{g}^{(1)} \simeq G_\Gamma/[G_\Gamma, G_\Gamma]$, we can induce the metric from $\mathrm{H}^1(X_0, \boldsymbol{R})$ to $\mathfrak{g}^{(1)}$ by this diagram. We call the induced metric the *Albanese metric* on $\mathfrak{g}^{(1)}$. We define a sub-Laplacian Ω_* on G_Γ by setting

$$\Omega_* = -\sum_{i=1}^{d_1} X_{i*}^{(1)} X_{i*}^{(1)},$$

where $\{X_1^{(1)}, \dots, X_{d_1}^{(1)}\}$ is an orthonormal basis for the Albanese metric on $\mathfrak{g}^{(1)}$ and $X_{i*}^{(1)}$ is the extension of $X_i^{(1)} \in \mathfrak{g}$ to a left invariant vector field on the limit group $(G_\Gamma, *)$ of $(G, \cdot)$.

A piecewise smooth Γ-equivariant map $\Phi : X \to G_\Gamma$ is said to be a realization. By using Trotter's approximation theory [**16**] and Theorem 3, we have

Theorem 1 (The central limit theorem). *Let X be a covering graph of a finite graph X_0 whose covering transformation group Γ is a finitely generated torsion free nilpotent group and $\Phi : X \to G_\Gamma$ a realization. Then for any $f \in C_\infty(G_\Gamma)$, as $n \uparrow \infty$, $\delta \downarrow 0$ and $n\delta^2 \to m(X_0)t$, we have*

$$\|L^n(f \circ (\tau_\delta\Phi)) - (e^{-t\Omega_*} f) \circ (\tau_\delta\Phi)\|_\infty \to 0,$$

where τ_δ *is the dilation on* G_Γ. *In particular, for a sequence* $\{x_\delta\}_{\delta>0}$ *in* X *with* $\lim_{\delta\downarrow 0}\tau_\delta\Phi(x_\delta)=x$,

$$\lim L^n(f\circ(\tau_\delta\Phi))(x_\delta)=e^{-t\Omega_*}f(x).$$

The proof of Theorem 1 is reduced to the case when the composite $\pi\circ\Phi: X\to G_\Gamma/[G_\Gamma,G_\Gamma]$ is harmonic, where π is the canonical surjective homomorphism from G_Γ to the abelian group $G_\Gamma/[G_\Gamma,G_\Gamma]$ (see the proof).

Definition (M. Kotani and T. Sunada [**7**]). A piecewise linear map $F: X\to G_\Gamma/[G_\Gamma,G_\Gamma]$ is said to be *harmonic* if for each $x\in X$,

$$\Delta F(x)=m(x)^{-1}\sum_{e\in E_x}m(e)\{F(t(e))-F(o(e))\}=0, \tag{1}$$

where $m(e)=m(o(e))p(e)$.

Since $\mathfrak{g}^{(1)}\simeq G_\Gamma/[G_\Gamma,G_\Gamma]$, the composite $\pi\circ\Phi^h$ is harmonic if and only if

$$\sum_{e\in E_x}m(e)\{\exp^{-1}\Phi^h(t(e))|_{\mathfrak{g}^{(1)}}-\exp^{-1}\Phi^h(o(e))|_{\mathfrak{g}^{(1)}}\}=0$$

for each $x\in X$.

According to the argument of harmonic maps from a graph to a Riemannian manifold [**7**], we have the existence and uniqueness of Φ^h.

Theorem 2 (M. Kotani and T. Sunada [**7**]). *There exists a realization* $\Phi^h: X\to G_\Gamma$ *such that the composite* $\pi\circ\Phi^h$ *is harmonic. If* $\pi\circ\Phi_1^h$ *and* $\pi\circ\Phi_2^h$ *are harmonic*,

$$\pi\circ\Phi_1^h-\pi\circ\Phi_2^h=\text{constant}.$$

We prove that the sub-Laplacian Ω_* can be written in terms of Φ^h.

Theorem 3. *Let* $\Phi^h: X\to G_\Gamma$ *be a realization such that the composite* $\pi\circ\Phi^h$ *is harmonic. Then we have*

$$\Omega_*=-\frac{1}{2}\sum_{e\in E_0}m(e)(\exp^{-1}\Phi^h(o(e))^{-1}\Phi^h(t(e))|_{\mathfrak{g}^{(1)}})_*^2.$$

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2. Limit group.

We will introduce the notion of limit groups, which is given by a deformation of the product on a nilpotent Lie group. We can find the definition of the limit group in G. Alexopoulos [**1**] (see also A. D. Virtser [**19**], P. Crépel and A. Raugi [**2**], A. Raugi [**14**]). We remark that the limit group is isomorphic to G_∞ defined by P. Pansu [**10**]. The invariance under the deformation of product (Lemma 2.3) and stratification (Lemma 2.1) play an important role in the proof of the central limit theorem.

Let $(G,\cdot)$ be a connected, simply connected nilpotent Lie group and $\mathfrak{g}$ its Lie algebra. We set $n_1 = \mathfrak{g}$ and $n_{i+1} = [\mathfrak{g}, n_i]$ for $i \geq 1$. Since $\mathfrak{g}$ is nilpotent, we have the filtration: $\mathfrak{g} = n_1 \supset n_2 \supset \cdots \supset n_r \neq \{0\} \supset n_{r+1} = \{0\}$. We consider subspaces $\mathfrak{g}^{(1)}, \dots, \mathfrak{g}^{(r)} \subset \mathfrak{g}$ such that

$$n_k = \mathfrak{g}^{(k)} \oplus n_{k+1}.$$

By this decomposition, each elements $X \in \mathfrak{g}$ can be represented uniquely as $X = X^{(1)} + X^{(2)} + \cdots + X^{(k)} + \cdots + X^{(r)}$ for some $X^{(k)} \in \mathfrak{g}^{(k)}$. For $\varepsilon > 0$, we define a linear operator $T_\varepsilon : \mathfrak{g} \to \mathfrak{g}$ by

$$T_\varepsilon(X^{(1)} + X^{(2)} + \cdots + X^{(k)} + \cdots + X^{(r)}) = \varepsilon X^{(1)} + \varepsilon^2 X^{(2)} + \cdots + \varepsilon^k X^{(k)} + \cdots + \varepsilon^r X^{(r)}.$$

We also define a Lie product $[\,,]^*$ on $\mathfrak{g}$, by setting

$$[X, Y]^* = \lim_{\varepsilon \to 0} T_\varepsilon [T_{\varepsilon^{-1}} X, T_{\varepsilon^{-1}} Y].$$

For any $X^{(k)} \in \mathfrak{g}^{(k)}$, $X^{(\ell)} \in \mathfrak{g}^{(\ell)}$, we have

$$[X^{(k)}, X^{(\ell)}]^* = [X^{(k)}, X^{(\ell)}]|_{\mathfrak{g}^{(k+\ell)}}. \tag{2}$$

We denote the dilation $\tau_\varepsilon : G \to G$ by

$$\tau_\varepsilon(x) = \exp(T_\varepsilon(\exp^{-1} x))$$

for the exponential map $\exp : \mathfrak{g} \to G$. On G, we define a product $*$, by setting

$$x * y = \lim_{\varepsilon \to 0} \tau_\varepsilon(\tau_{\varepsilon^{-1}} x \cdot \tau_{\varepsilon^{-1}} y).$$

Then $(G, *)$ is a nilpotent Lie group and its Lie algebra is isomorphic to $(\mathfrak{g}, [\,,]^*)$. We call $(G, *)$ the limit group of $(G, \cdot)$. The limit group $(G, *)$ has the following properties.

Lemma 2.1.

(a) *For* $X, Y \in \mathfrak{g}$, $\exp X * \exp Y = \exp(X + Y + 1/2[X, Y]^* + \cdots [\,,]^* \cdots)$.

(b) *The exponential map from* $(\mathfrak{g}, [\,,]^*)$ *to* $(G, *)$ *is equal to the original exponential map.*

(c) $(G, *)$ *is a stratified Lie group. Namely, the Lie algebra* $(\mathfrak{g}, [\,,]^*)$ *of* $(G, *)$ *has a direct sum decomposition* $\bigoplus_{k=1}^r \mathfrak{g}^{(k)}$ *which satisfies*
   - (i) *If* $k + \ell \leq r$, $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(\ell)}]^* \subset \mathfrak{g}^{(k+\ell)}$.
     If $k + \ell > r$, $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(\ell)}]^* = \{0\}$.
   - (ii) $\mathfrak{g}^{(1)}$ *generates* $\mathfrak{g}$.

(d) $\tau_\delta(x * y) = \tau_\delta x * \tau_\delta y$.

Proof. (a) Let $x = \exp X$ and $y = \exp Y$. From the Campbell-Hausdorff formula,

$$x * y = \lim_{\varepsilon \to 0} \exp\bigg(X + Y + \frac{1}{2} T_\varepsilon [T_{\varepsilon^{-1}} X, T_{\varepsilon^{-1}} Y] + \frac{1}{12} T_\varepsilon [[T_{\varepsilon^{-1}} X, T_{\varepsilon^{-1}} Y], T_{\varepsilon^{-1}} Y] - \frac{1}{12} T_\varepsilon [[T_{\varepsilon^{-1}} X, T_{\varepsilon^{-1}} Y], T_{\varepsilon^{-1}} X] \cdots \bigg).$$

By the definition of T_ε, we have

$$\lim_{\varepsilon\to 0} T_\varepsilon[[T_{\varepsilon^{-1}}X, T_{\varepsilon^{-1}}Y], T_{\varepsilon^{-1}}Y] = \lim_{\varepsilon,\delta\to 0} T_\varepsilon[T_{\varepsilon^{-1}}(T_\delta[T_{\delta^{-1}}X, T_{\delta^{-1}}Y]), T_{\varepsilon^{-1}}Y].$$

So we conclude

$$x * y = \exp\left(X + Y + \frac{1}{2}[X, Y]^* + \frac{1}{12}[[X, Y]^*, Y]^* + \cdots\right).$$

(b) Let $\phi(t) = \exp tX$ for $t \in \boldsymbol{R}$ and $X \in \mathfrak{g}$. Since

$$\begin{aligned}\phi(t_1) * \phi(t_2) &= \exp t_1 X * \exp t_2 X \\ &= \exp\left(t_1 X + t_2 X + \frac{1}{2}[t_1 X, t_2 X]^* + \cdots\right) \\ &= \exp(t_1 + t_2)X = \phi(t_1 + t_2),\end{aligned}$$

ϕ is a one-parameter subgroup of $(G, *)$. Hence the exponential map of $(G, *)$ is equal to the original exponential map.

(c) We will show that $(\mathfrak{g}, [\,,]^*)$ satisfies the properties of the stratified Lie group. By (2), for $k + \ell \le r$, we have

$$[\mathfrak{g}^{(k)}, \mathfrak{g}^{(\ell)}]^* \subset \mathfrak{g}^{(k+\ell)}.$$

For $m \ge 2$, we assume that $\mathfrak{g}^{(1)}$ generates $\mathfrak{g}^{(m-1)}$. From the definition of $\mathfrak{g}^{(m)}$ and $[\,,]^*$, we have

$$\mathfrak{g}^{(m)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(m-1)}]^*.$$

By the induction, $(G, *)$ is a stratified Lie group.

(d) For a fixed $\delta > 0$, we have

$$\begin{aligned}\tau_\delta(x * y) &= \tau_\delta \lim_{\varepsilon\to 0} \tau_\varepsilon(\tau_{\varepsilon^{-1}}x \cdot \tau_{\varepsilon^{-1}}y) \\ &= \lim_{\varepsilon\to 0} \tau_{\delta\varepsilon}(\tau_{(\delta\varepsilon)^{-1}}\tau_\delta x \cdot \tau_{(\delta\varepsilon)^{-1}}\tau_\delta y) \\ &= \tau_\delta x * \tau_\delta y.\end{aligned}$$

□

By the definition of $*$ and Lemma 2.1, we easily obtain

$$\begin{aligned}\exp^{-1}(x * y)|_{\mathfrak{g}^{(1)}} &= \exp^{-1}(x \cdot y)|_{\mathfrak{g}^{(1)}}, \\ \exp^{-1}(x * y)|_{\mathfrak{g}^{(2)}} &= \exp^{-1}(x \cdot y)|_{\mathfrak{g}^{(2)}}\end{aligned}$$

for any $x, y \in G$. For $k \ge 3$, $\exp^{-1}(x * y)|_{\mathfrak{g}^{(k)}}$ is not equal to $\exp^{-1}(x \cdot y)|_{\mathfrak{g}^{(k)}}$ in general. These invariances for $k = 1, 2$ are important for the central limit theorem.

We consider a basis $\{X_1^{(k)}, X_2^{(k)}, \ldots, X_{d_k}^{(k)}\}$ of $\mathfrak{g}^{(k)}$ for each $k \le r$. We have two identifications of G with $\boldsymbol{R}^n$ as differential manifold given by

$$(x_{d_r}^{(r)}, x_{d_r-1}^{(r)}, \ldots, x_1^{(1)}) \mapsto \exp x_{d_r}^{(r)} X_{d_r}^{(r)} \cdot \exp x_{d_r-1}^{(r)} X_{d_r-1}^{(r)} \cdot \cdots \cdot \exp x_1^{(1)} X_1^{(1)}$$

and

$$(x_{d_r*}^{(r)}, x_{d_r-1*}^{(r)}, \dots, x_{1*}^{(1)}) \mapsto \exp x_{d_r*}^{(r)} X_{d_r}^{(r)} * \exp x_{d_r-1*}^{(r)} X_{d_r-1}^{(r)} * \cdots * \exp x_{1*}^{(1)} X_1^{(1)}.$$

We call them $(\cdot)$-coordinates and $(*)$-coordinates of second kind respectively. For $x \in G$, we denote $P_i^{(k)}(x) = x_i^{(k)}$ and $P_{i*}^{(k)}(x) = x_{i*}^{(k)}$. The following lemma gives a comparison of the two coordinates.

Lemma 2.2. *For $x \in G$, we have*

$$P_{i*}^{(1)}(x) = P_i^{(1)}(x), \tag{3}$$

$$P_{i*}^{(2)}(x) = P_i^{(2)}(x), \tag{4}$$

$$P_{i*}^{(k)}(x) = P_i^{(k)}(x) + \sum_{0<|K|\le k-1} C_K P^K(x) \tag{5}$$

for some constants C_K, where K denotes a multi-index $((i_1, k_1), \dots, (i_n, k_n))$ and $P^K(x) = P_{i_1}^{(k_1)}(x) P_{i_2}^{(k_2)}(x) \cdots P_{i_n}^{(k_n)}(x)$. We call $|K| = \sum_{i=1}^n k_n$ the order of $P^K(x)$.

Proof. (3) and (4) are obtained immediately by comparing $(\cdot)$-coordinates and $(*)$-coordinates of $x \in G$. We will show (5) by induction for k of $P_{i*}^{(k)}(x)$. Indeed the cases $k = 1$ and $k = 2$ are obvious. We assume that it is true in the case $P_{i*}^{(\ell)}(x)$ for $\ell \le k-1$. Then the (i,k)-component of x is

$$\begin{aligned} \exp^{-1} x|_{X_i^{(k)}} &= P_{i*}^{(k)}(x) + \sum_{|K|=k} C_K Pr_i^{(k)}[X^K]^* P_*^K(x) \\ &= P_i^{(k)}(x) + \sum_{0<|K|\le k} C_K Pr_i^{(k)}[X^K] P^K(x) \end{aligned}$$

for some constants C_K, where $[X^K] = [X_{i_1}^{(k_1)}, [X_{i_2}^{(k_2)}, [X_{i_3}^{(k_3)}, \dots, X_{i_n}^{(k_n)}]] \cdots]$, $[X^K]^* = [X_{i_1}^{(k_1)}, [X_{i_2}^{(k_2)}, [X_{i_3}^{(k_3)}, \dots, X_{i_n}^{(k_n)}]^*]^* \cdots]^*$ and $Pr_i^{(k)} X = X|_{X_i^{(k)}}$. By the hypothesis of induction, the lower order terms do not affect for this claim. Since $C_K Pr_i^{(k)}[X^K]^* = C_K Pr_i^{(k)}[X^K]$ for $|K| = k$ by (2), the terms of order k are cancelled. Consequently,

$$P_{i*}^{(k)}(x) = P_i^{(k)}(x) + \sum_{0<|K|\le k-1} C_K P^K(x). \qquad \square$$

As an invariance under the deformation of the product on G, we conclude

Lemma 2.3.

$$P_{i*}^{(1)}(x * y) = P_i^{(1)}(x \cdot y), \tag{6}$$

$$P_{i*}^{(2)}(x * y) = P_i^{(2)}(x \cdot y), \tag{7}$$

$$P_{i*}^{(k)}(x * y) = P_i^{(k)}(x \cdot y) + \sum_{\substack{|K_1|+|K_2|\le k-1, \\ |K_2|>0}} C_{K_1 K_2} P_*^{K_1}(x) P^{K_2}(x \cdot y). \tag{8}$$

PROOF. From (2), Lemma 2.2 and the Campbell-Hausdorff formula, (6) and (7) are obtained easily. We will show (8) inductively. By the definition of $*$, Lemma 2.2 and the hypothesis of induction, the difference of $P_{i*}^{(k)}(x*y)$ and $P_i^{(k)}(x\cdot y)$ is the terms whose order is less than k. Namely,

$$P_{i*}^{(k)}(x*y) = P_i^{(k)}(x\cdot y) + \sum_{0<|K_1|+|K_2|\le k-1} C_{K_1K_2}P^{K_1}(x)P^{K_2}(y). \tag{9}$$

We can replace $P^{K_2}(y)$ with

$$P^{K_2}(x\cdot y) - \sum_{0<|K_3|+|K_4|\le |K_2|} C_{K_3K_4}P^{K_3}(x)P^{K_4}(x\cdot y) + \sum_{0<|K|\le|K_2|} C_K P^K(x)$$

by using

$$P_i^{(k)}(y) = P_i^{(k)}(x\cdot y) - P_i^{(k)}(x) - \sum_{0<|K_1|+|K_2|\le k} C_{K_1K_2}P^{K_1}(x)P^{K_2}(y).$$

Hence we refine (9) to

$$P_{i*}^{(k)}(x*y) = P_i^{(k)}(x\cdot y) + \sum_{\substack{|K_1|+|K_2|\le k-1,\\ |K_2|>0}} C_{K_1K_2}P^{K_1}(x)P^{K_2}(x\cdot y) + \sum_{0<|K|\le k-1} C_K P^K(x).$$

But $\sum_{0<|K|\le k-1} C_K P^K(x)$ vanish because if $y = x^{-1}$, then $x*y = x\cdot y = e$. Moreover $P^{K_1}(x)$ can be replaced with $P_*^{K_1}(x)$ because of Lemma 2.2. So we conclude

$$P_{i*}^{(k)}(x*y) = P_i^{(k)}(x\cdot y) + \sum_{\substack{|K_1|+|K_2|\le k-1,\\ |K_2|>0}} C_{K_1K_2}P_*^{K_1}(x)P^{K_2}(x\cdot y). \qquad \square$$

EXAMPLE 2.4. *For* $k = 3$, *we have*

$$P_{i*}^{(3)}(x) = P_i^{(3)}(x) - \frac{1}{2}\sum_{i_1>i_2} Pr_i^{(3)}[X_{i_1}^{(1)}, X_{i_2}^{(1)}]P_{i_1}^{(1)}(x)P_{i_2}^{(1)}(x),$$

$$\begin{aligned} P_{i*}^{(3)}(x*y) = P_i^{(3)}(x\cdot y) - \frac{1}{2}\sum_{i_1>i_2} Pr_i^{(3)}[X_{i_1}^{(1)}, X_{i_2}^{(1)}]\{&P_{i_1*}^{(1)}(x)P_{i_2}^{(1)}(x\cdot y) \\ &- P_{i_1}^{(1)}(x\cdot y)P_{i_2*}^{(1)}(x) + P_{i_1}^{(1)}(x\cdot y)P_{i_2}^{(1)}(x\cdot y)\}. \end{aligned}$$

3. The central limit theorem.

We shall prove a convergence of the transition operator by using the approximation theory of H. F. Trotter [**16**]. Let G_Γ be the nilpotent Lie group such that Γ is isomorphic to a lattice of G_Γ. There exists uniquely such a connected and simply connected nilpotent Lie group up to isomorphism by A. I. Malćev [**9**] and Γ is a cocompact lattice (cf. M. S. Raghunathan [**13**]).

Let $\mathfrak{g}$ be the Lie algebra of G_Γ and denote $\mathfrak{g}^{(1)}, \dots, \mathfrak{g}^{(r)}$, subspaces of $\mathfrak{g}$ as in Section 1. We define a map $P_\delta : C_\infty(G_\Gamma) \to C_\infty(X)$ by $P_\delta f(x) = f(\tau_\delta \Phi(x))$, where $\tau_\delta : G_\Gamma \to G_\Gamma$ is a dilation. We remark that $(C_\infty(G_\Gamma), \|\cdot\|_\infty)$ and $(C_\infty(X), \|\cdot\|_\infty)$ are Banach spaces, where $\|\cdot\|_\infty$ is the sup. norm. Take a basis $\{X_1^{(k)}, \dots, X_{d_k}^{(k)}\}$ of $\mathfrak{g}^{(k)}$ for each $k \le r$ and we identify $X_i^{(k)}$ with the left invariant vector field on G_Γ. We denote by d the Carnot-Carathéodory distance. More precisely, let C be the set of all absolutely continuous paths $c : [0,1] \to G_\Gamma$, satisfying $\dot{c}(t) = \sum_{i \le d_1} a_i(t) X_i^{(1)}(c(t))$, for almost every $t \in [0,1]$. Put

$$|c| = \int_0^1 \left(\sum_{i \le d_1} a_i^2(t) \right)^{1/2} dt,$$

and for $x, y \in G_\Gamma$,

$$d(x,y) = \inf\{|c| \mid c \in C, c(0) = x, c(1) = y\}.$$

Then d is a left invariant distance, which induces the topology of G_Γ (see [**18**]).

Lemma 3.1. *$\{(C_\infty(X), P_\delta)\}_{\delta>0}$ is a sequence of Banach spaces approximating to $C_\infty(G_\Gamma)$. Namely, for any $f \in C_\infty(G_\Gamma)$, we have*

$$\|P_\delta f\|_\infty \le \|f\|_\infty, \tag{10}$$

$$\|P_\delta f\|_\infty \to \|f\|_\infty \quad \text{as } \delta \to 0. \tag{11}$$

Proof. (10) is trivial. We consider (11). Fix $a \in G_\Gamma$ which satisfies $|f(a)| = \|f\|_\infty$. Then

$$\begin{aligned} \|P_\delta f\| &= \sup_{x \in X} |f(\tau_\delta \Phi(x)) - f(a) + f(a)| \\ &\ge |f(a)| - \inf_{x \in X} |f(a) - f(\tau_\delta \Phi(x))|. \end{aligned}$$

On the other hand, since $\Gamma \subset G_\Gamma$ is a cocompact lattice and Φ is Γ-equivariant, we have

$$\inf_{x \in X} d(a, \tau_\delta \Phi(x)) = \delta \inf_{x \in X} d(\tau_{\delta^{-1}} a, \Phi(x)) < \delta M$$

for $M = \sup_{g \in F, x \in F_X} d(g, \Phi(x)) < \infty$, where $F \subset G_\Gamma$ and $F_X \subset X$ are these fundamental domains. Since f is continuous at a, for any $\varepsilon > 0$, there exists $\delta' > 0$ such that if $d(a,y) < \delta'$, then $|f(a) - f(y)| < \varepsilon$. For $\delta = \delta'/M$, there exists $x' \in X$ such that $d(a, \tau_\delta(x')) < \delta'$. Hence for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\inf_{x \in X} |f(a) - f(\tau_\delta \Phi(x))| \le |f(a) - f(\tau_\delta \Phi(x'))| < \varepsilon.$$

Consequently we have $\|P_\delta f\|_\infty \to \|f\|_\infty$ as $\delta \to 0$. □

According to the theorem of H. F. Trotter ([**16**], Theorem 5.3), to deduce the assertion of Theorem 1, it suffices to show the following lemma which gives the convergence of the sequence of the infinitesimal generators.

Lemma 3.2. *Let $\Phi^h : X \to G_\Gamma$ be a realization such that the composite $\pi \circ \Phi^h$ is harmonic. Then for any $f \in C_0^\infty(G_\Gamma)$ and $N \uparrow \infty$, $\delta \downarrow 0$ with $N^2\delta \to 0$, we have*

$$\left\| \frac{m(X_0)}{N\delta^2}(I - L^N)P_\delta^h f - P_\delta^h \Omega_* f \right\|_\infty \to 0,$$

where $P_\delta^h f(x) = f(\tau_\delta \Phi^h(x))$.

Proof. By the definition of the transition operator, we have

$$\frac{m(X_0)}{N\delta^2}(I - L^N)P_\delta^h f(x) = \frac{m(X_0)}{N\delta^2} \sum_{c \in C_{x,N}} p(c)\{f(\Phi_\delta^h(x)) - f(\Phi_\delta^h(t(c)))\},$$

where $C_{x,N}$ is a set of paths $(e_1, \dots, e_N)$ with $o(e_1) = x$, $p(c) = p(e_1)p(e_2)\cdots p(e_N)$ and $\Phi_\delta^h = \tau_\delta \Phi^h$. By the same arguments as G. Alexopoulos [**1**] and M. Kotani [**6**], we apply the Taylor formula for the $(*)$-coordinates of second kind to $f'(g) = f(\Phi_\delta^h(x) * g)$ with $g = \Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c))$. Then we have

$$\begin{aligned}
&\frac{m(X_0)}{N\delta^2}(I - L^N)P_\delta^h f(x) \\
&= \frac{m(X_0)}{N\delta^2} \sum_{c \in C_{x,N}} p(c) \Bigg\{ -\sum_{(i,k)} X_{i*}^{(k)} f(\Phi_\delta^h(x)) P_{i*}^{(k)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c))) \\
&\quad - \frac{1}{2}\left(\sum_{(i_1,k_1) \ge (i_2,k_2)} X_{i_1*}^{(k_1)} X_{i_2*}^{(k_2)} + \sum_{(i_2,k_2) > (i_1,k_1)} X_{i_2*}^{(k_2)} X_{i_1*}^{(k_1)} \right) f(\Phi_\delta^h(x)) \\
&\qquad \cdot P_{i_1*}^{(k_1)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c))) P_{i_2*}^{(k_2)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c))) \\
&\quad - \frac{1}{6} \sum_{(i_1,k_1),(i_2,k_2),(i_3,k_3)} \frac{\partial^3 f'}{\partial x_{i_1*}^{(k_1)} \partial x_{i_2*}^{(k_2)} \partial x_{i_3*}^{(k_3)}}(\theta) P_{i_1*}^{(k_1)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c))) \\
&\qquad \cdot P_{i_2*}^{(k_2)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c))) P_{i_3*}^{(k_3)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c))) \Bigg\}
\end{aligned} \tag{12}$$

for some $\theta \in G_\Gamma$ satisfying $|P_{i*}^{(k)}(\theta)| \le |P_{i*}^{(k)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c)))|$, where $(i_1,k_1) > (i_2,k_2)$ means $k_1 > k_2$ or $k_1 = k_2$, $i_1 > i_2$. Since $(G_\Gamma, *)$ is a stratified Lie group,

$$P_{i*}^{(k)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c))) = \delta^k P_{i*}^{(k)}(\Phi^h(x)^{-1} * \Phi^h(t(c))).$$

We denote by $\mathrm{Ord}_\delta(k)$ the terms of (12) whose order of δ is k. Then (12) is rewritten as

$$\frac{m(X_0)}{N\delta^2}(I - L^N)P_\delta^h f(x) = \mathrm{Ord}_\delta(-1) + \mathrm{Ord}_\delta(0) + \sum_{k \ge 1} \mathrm{Ord}_\delta(k). \tag{13}$$

We will consider three terms in (13) separately.

Estimate of $\mathrm{Ord}_\delta(-1)$. From Lemma 2.2, 2.3 and the harmonicity of $\pi \circ \Phi^h$, we have inductively

$$\sum_{c\in C_{x,N}} p(c)P_{i*}^{(1)}(\Phi^h(x)^{-1} * \Phi^h(t(c)))$$

$$= \sum_{c'\in C_{x,N-1}} p(c') \sum_{e\in E_{t(c')}} p(e)\{\exp^{-1}\Phi^h(x)^{-1}\cdot\Phi^h(t(c'))|_{X_i^{(1)}}$$

$$+ \exp^{-1}\Phi^h(o(e))^{-1}\cdot\Phi^h(t(e))|_{X_i^{(1)}}\}$$

$$= \sum_{c'\in C_{x,N-1}} p(c')P_i^{(1)}(\Phi^h(x)^{-1} * \Phi^h(t(c')))$$

$$= 0.$$

This shows that $\mathrm{Ord}_\delta(-1)$ vanishes.

Estimate of $\mathrm{Ord}_\delta(0)$. Let us first observe the coefficient of $X_{i*}^{(2)} f(\Phi_\delta^h(x))$. Then we have

$$\frac{m(X_0)}{N}\sum_{c\in C_{x,N}} p(c)\Bigg\{P_{i*}^{(2)}(\Phi^h(x)^{-1} * \Phi^h(t(c))) - \frac{1}{2}\sum_{i_2>i_1} Pr_i^{(2)}[X_{i_1}^{(1)}, X_{i_2}^{(1)}]^*$$

$$\cdot P_{i_1*}^{(1)}(\Phi^h(x)^{-1} * \Phi^h(t(c)))P_{i_2*}^{(1)}(\Phi^h(x)^{-1} * \Phi^h(t(c)))\Bigg\}$$

$$= \frac{m(X_0)}{N}\sum_{c\in C_{x,N}} p(c)\exp^{-1}\Phi^h(x)^{-1} * \Phi^h(t(c))|_{X_i^{(2)}}$$

$$= \frac{m(X_0)}{N}\sum_{k=0}^{N-1}\sum_{c\in C_{x,k}} p(c)\sum_{e\in E_{t(c)}} p(e)\exp^{-1}\Phi^h(o(e))^{-1}\cdot\Phi^h(t(e))|_{X_i^{(2)}}$$

$$= \frac{m(X_0)}{N}\sum_{k=0}^{N-1}\sum_{c\in C_{x,k}} p(c)F(t(c)), \tag{14}$$

where $F(x) = \sum_{e\in E_x} p(e)\exp^{-1}\Phi^h(o(e))^{-1}\cdot\Phi^h(t(e))|_{X_i^{(2)}}$. Since $F(\gamma x) = F(x)$, there exists a function $f_0 : X_0 \to \boldsymbol{R}$ such that $f_0(\kappa(x)) = F(x)$, where $\kappa : X \to X_0$ is the covering map. Let L_0 be the transition operator on $C(X_0)$. By the ergodicity (cf. [**6**]), we have

$$\frac{m(X_0)}{N}\sum_{k=0}^{N-1}\sum_{c\in C_{x,k}} p(c)F(t(c)) = \frac{m(X_0)}{N}\sum_{k=0}^{N-1} L_0^k f_0(\kappa(x))$$

$$= \sum_{x_0\in X_0} f_0(x_0)m(x_0) + O\left(\frac{1}{N}\right)$$

$$= \sum_{e\in E_0} m(e)\exp^{-1}\Phi^h(o(e))^{-1}\cdot\Phi^h(t(e))|_{X_i^{(2)}} + O\left(\frac{1}{N}\right).$$

However, $\sum_{e\in E_0} m(e)\exp^{-1}\Phi^h(o(e))^{-1}\cdot\Phi^h(t(e))|_{X_i^{(2)}} = 0$. Hence (14) goes to 0.

By the harmonicity and ergodicity, the coefficient of $X^{(1)}_{i_1 *} X^{(1)}_{i_2 *} f(\Phi^h_\delta(x))$ is given by

$$-\frac{m(X_0)}{N} \sum_{i_1, i_2 \le d_1} \frac{1}{2} X^{(1)}_{i_1 *} X^{(1)}_{i_2 *} f(\Phi^h_\delta(x))$$

$$\cdot \sum_{c \in C_{x,N}} p(c) P^{(1)}_{i_1 *}(\Phi^h(x)^{-1} * \Phi^h(t(c))) P^{(1)}_{i_2 *}(\Phi^h(x)^{-1} * \Phi^h(t(c)))$$

$$= -\sum_{i_1, i_2 \le d_1} \frac{1}{2} \sum_{e \in E_0} m(e) P^{(1)}_{i_1}(\Phi^h(o(e))^{-1} \cdot \Phi^h(t(e))) P^{(1)}_{i_2}(\Phi^h(o(e))^{-1} \cdot \Phi^h(t(e)))$$

$$\cdot X^{(1)}_{i_1 *} X^{(1)}_{i_2 *} f(\Phi^h_\delta(x)) + O\left(\frac{1}{N}\right).$$

From Theorem 3, $\mathrm{Ord}_\delta(0)$ converges to $P^h_\delta \Omega_* f(x)$.

Estimate of $\sum_{k \ge 1} \mathrm{Ord}_\delta(k)$. We observe the coefficient of $X^{(k)}_{i*} f(\Phi^h_\delta(x))$. By Lemma 2.3 and

$$|P^{(k)}_i(\Phi^h(x)^{-1} \cdot \Phi^h(t(c)))| \le CN^k,$$

for a continuous function $M^{(k)}_i$ on G_Γ, we have

$$\frac{m(X_0)\delta^{k-2}}{N} \sum_{c \in C_{x,N}} p(c) P^{(k)}_{i*}(\Phi^h(x)^{-1} * \Phi^h(t(c)))$$

$$= \frac{m(X_0)\delta^{k-2}}{N} \sum_{c \in C_{x,N}} p(c) \left\{ P^{(k)}_i(\Phi^h(x)^{-1} \cdot \Phi^h(t(c))) \right.$$

$$\left. + \sum_{\substack{|K_1|+|K_2| \le k-1, \\ |K_2|>0}} C_{K_1 K_2} P^{K_1}_*(\Phi^h(x)^{-1}) P^{K_2}(\Phi^h(x)^{-1} \cdot \Phi^h(t(c))) \right\}$$

$$\le M^{(k)}_i(\Phi^h_\delta(x)) \left(\delta^{k-2} N^{k-1} + \sum_{\substack{|K_1|+|K_2| \le k-1, \\ |K_2| \ge 2}} \delta^{k-2-|K_1|} N^{|K_2|-1} \right) \tag{15}$$

because

$$\sum_{c \in C_{x,N}} p(c) P^{K_2}(\Phi^h(x)^{-1} \cdot \Phi^h(t(c))) = 0$$

when $|K_2| = 1$. By the assumptions of N and δ, (15) converges to 0.

By the same argument as above, the coefficient of $X^{(k_1)}_{i_1 *} X^{(k_2)}_{i_2 *} f(\Phi^h_\delta(x))$ for $k_1 + k_2 \ge 3$ converges to 0.

Finally we consider the coefficient of $(\partial^3 f' / (\partial x^{(k_1)}_{i_1 *} \partial x^{(k_2)}_{i_2 *} \partial x^{(k_3)}_{i_3 *}))(\theta)$. Since $f \in C^\infty_0(G_\Gamma)$ and

$$\operatorname{supp} \frac{\partial^3 f'}{\partial x_{i_1 *}^{(k_1)} \partial x_{i_2 *}^{(k_2)} \partial x_{i_3 *}^{(k_3)}} \subset \operatorname{supp} f' = \Phi_\delta^h(x)^{-1} * \operatorname{supp} f,$$

it suffices to show that, for a continuous function $M_i^{(k)}$ on G_Γ,

$$|P_{i*}^{(k)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c)))| \le M_i^{(k)}(\Phi_\delta^h(x) * \theta)\delta N$$

if $\delta N < 1$. For $k = 1$ and 2, this is true. Assume it holds for less than k. Then

$$\begin{aligned} P_{i*}^{(k)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c))) &= \delta^k P_{i*}^{(k)}(\Phi^h(x)^{-1} * \Phi^h(t(c))) \\ &= \delta^k \left(P_i^{(k)}(\Phi^h(x)^{-1} \cdot \Phi^h(t(c))) \right. \\ &\quad \left. + \sum_{\substack{|K_1|+|K_2| \le k-1, \\ |K_2|>0}} C_{K_1 K_2} P_*^{K_1}(\Phi^h(x)^{-1}) P^{K_2}(\Phi^h(x)^{-1} \cdot \Phi^h(t(c))) \right). \end{aligned}$$

Since

$$\begin{aligned} P_{i_1 *}^{(k_1)}(\Phi_\delta^h(x)^{-1}) &= P_{i_1 *}^{(k_1)}(\theta * (\Phi_\delta^h(x) * \theta)^{-1}) \\ &= P_{i_1 *}^{(k_1)}(\theta) + P_{i_1 *}^{(k_1)}((\Phi_\delta^h(x) * \theta)^{-1}) \\ &\quad + \sum_{\substack{|L_1|+|L_2|=k_1, \\ |L_1|,|L_2|>0}} C_{L_1 L_2} P_*^{L_1}(\theta) P_*^{L_2}((\Phi_\delta^h(x) * \theta)^{-1}), \end{aligned}$$

we have inductively $|P_{i_1 *}^{(k_1)}(\Phi_\delta^h(x)^{-1})| \le M(\Phi_\delta^h(x) * \theta)$ for $k_1 \le k - 1$. So we conclude

$$\begin{aligned} &|P_{i*}^{(k)}(\Phi_\delta^h(x)^{-1} * \Phi_\delta^h(t(c)))| \\ &\quad \le C \left(\delta^k N^k + \sum_{\substack{|K_1|+|K_2| \le k-1, \\ |K_2|>0}} M(\Phi_\delta^h(x) * \theta) \delta^{k-|K_1|} N^{|K_2|} \right) \\ &\quad \le M_i^{(k)}(\Phi_\delta^h(x) * \theta)\delta N. \end{aligned}$$

From these estimates, it follows that $\sum_{k \ge 1} \mathrm{Ord}_\delta(k)$ converges to 0. Hence the proof of the lemma is completed. □

We remark that Ω_* has the following property.

Lemma 3.3 (D. W. Robinson [**15**], p. 304). *For $\lambda > 0$, the range of $\Omega_* + \lambda$ in $C_\infty(G_\Gamma)$ is dense.*

By the same argument as M. Kotani [**6**], we conclude

Theorem 1 (The central limit theorem). *Let $\Phi : X \to G_\Gamma$ be a realization. For any $f \in C_\infty(G_\Gamma)$, as $n \uparrow \infty$, $\delta \downarrow 0$ and $n\delta^2 \to m(X_0)t$, we have*

$$\|L^n P_\delta f - P_\delta e^{-t\Omega_*} f\|_\infty \to 0. \tag{16}$$

For any $x \in G_\Gamma$, choose $\{x_\delta\} \subset X$ such that $\Phi_\delta(x_\delta) \to x$ as $\delta \downarrow 0$. Then

$$L^n P_\delta f(x_\delta) \to e^{-t\Omega_*} f(x). \tag{17}$$

Proof. Let Φ^h be a realization such that the composite $\pi \circ \Phi^h$ is harmonic. Then

$$\|L^n P_\delta f - P_\delta e^{-t\Omega_*} f\|_\infty \le \|L^n (P_\delta f - P_\delta^h f)\|_\infty \tag{18}$$

$$+ \|L^n P_\delta^h f - P_\delta^h e^{-t\Omega_*} f\|_\infty \tag{19}$$

$$+ \|P_\delta^h e^{-t\Omega_*} f - P_\delta e^{-t\Omega_*} f\|_\infty. \tag{20}$$

Since f and $e^{-t\Omega_*} f$ are uniformly continuous and

$$d(\tau_\delta \Phi(x), \tau_\delta \Phi^h(x)) = \delta d(\Phi(x), \Phi^h(x)) \le \delta M$$

for $M = \sup_{x \in X} d(\Phi(x), \Phi^h(x)) < \infty$, (18) and (20) converges to 0 as $\delta \to 0$.

Take $N \uparrow \infty$ and $\delta \downarrow 0$ such that $N^2\delta \to 0$. Then Lemma 3.2, 3.3 and Trotter ([**16**], Theorem 5.3) imply for any $f \in C_\infty(G_\Gamma)$,

$$\|(L^N)^{k_N} P_\delta^h f - P_\delta^h e^{-t\Omega_*} f\|_\infty \to 0 \tag{21}$$

as $k_N N\delta^2 \to m(X_0)t$. Now we will prove that (19) converges to 0. Let $N(n)$ be the integer with $n^{1/5} \le N(n) \le n^{1/5} + 1$ and k_N and r_N are the quotient and remainder of n/N respectively. $n \uparrow \infty$ and $\delta \downarrow 0$ imply $N \to \infty$, $N^2\delta \le (n^{1/5}+1)^2\delta \to 0$ and $k_N N\delta^2 = n\delta^2 - r_N\delta^2$. We also see $k_N N\delta^2 \to m(X_0)t$, since $r_N < N$ and $r_N\delta^2 \le N\delta^2 \le (n^{1/5}+1)\delta^2 \to 0$. Then we have

$$\begin{aligned}\|L^n P_\delta^h f - P_\delta^h e^{-t\Omega_*} f\|_\infty &= \|L^{k_N N + r_N} P_\delta^h f - P_\delta^h e^{-t\Omega_*} f\|_\infty \\ &\le \|L^{k_N N}(L^{r_N} - \mathrm{I}) P_\delta^h f\|_\infty + \|L^{N k_N} P_\delta^h f - P_\delta^h e^{-t\Omega_*} f\|_\infty.\end{aligned}$$

From the property of N, δ and k_N, (21) holds. Since $r_N^2\delta \le (n^{1/5}+1)^2\delta \to 0$ and by Lemma 3.2,

$$\left\| \frac{m(X_0)}{r_N\delta^2} (\mathrm{I} - L^{r_N}) P_\delta^h \varphi - P_\delta^h \Omega_* \varphi \right\|_\infty \to 0$$

for any $\varphi \in C_0^\infty(G_\Gamma)$. This implies $\|L^{k_N N}(L^{r_N} - \mathrm{I}) P_\delta^h f\|_\infty \to 0$. Hence we conclude (16).

Finally (17) is given by

$$\begin{aligned}&|L^n P_\delta f(x_\delta) - e^{-t\Omega_*} f(x)| \\ &\quad \le \|L^n P_\delta f - P_\delta e^{-t\Omega_*} f\|_\infty + |e^{-t\Omega_*} f(\Phi_\delta(x_\delta)) - e^{-t\Omega_*} f(x)| \to 0.\end{aligned}$$

□

4. Existence and uniqueness of a realization such that the composite with π is harmonic.

Let $\pi : G_\Gamma \to G_\Gamma/[G_\Gamma, G_\Gamma]$ be the canonical surjective homomorphism. It is known that $\pi(\Gamma) \subset G_\Gamma/[G_\Gamma, G_\Gamma]$ is also lattice (A. I. Mal'cev [**9**], M. S. Raghunathan [**13**]). We apply the arguments of harmonic map from X_0 to the torus $T = \pi(\Gamma)\backslash(G_\Gamma/[G_\Gamma, G_\Gamma])$. For a flat metric on the torus T, we consider an energy functional E of the piecewise smooth map $F : X_0 \to T$ defined by

$$E(F) = \frac{1}{2}\sum_{e \in E_0} m(e) \int_0^1 \left\| \frac{dF_e}{dt}(t) \right\|^2 dt,$$

where $F_e : [0,1] \to T$ is the restriction of F to $e \in E_0$ such that $F_e(0) = o(e)$, $F_e(1) = t(e)$. Then we have the following result (cf. [**7**]):

Theorem (M. Kotani and T. Sunada). *A piecewise smooth map $F : X_0 \to T$ is a critical map if and only if F_e is a geodesic for every $e \in E_0$ and at each $x \in V_0$,*

$$\sum_{e \in E_x} m(e) \frac{dF_e}{dt}(0) = 0.$$

Then the critical map does not depend on the choice of a flat metric on T. We remark that the composite $\pi \circ \Phi : X \to G_\Gamma/[G_\Gamma, G_\Gamma]$ is harmonic if and only if the map $(\pi \circ \Phi)_0 : X_0 \to T$, whose lift is equal to $\pi \circ \Phi$ is a critical map. From these results, we have

Theorem 2 (M. Kotani and T. Sunada [**7**]).

(a) *Each homotopy class of piecewise smooth maps of X_0 into T contains at least one harmonic map.*

(b) *If two harmonic maps $F_i : X_0 \to T$, $(i = 1, 2)$ are homotopic, then there exists $a \in T$ such that $F_1 - F_2 = a$.*

(c) *There exists a realization $\Phi^h : X \to G_\Gamma$ such that the composite $\pi \circ \Phi^h$ is harmonic. If $\pi \circ \Phi_1^h$ and $\pi \circ \Phi_2^h$ are harmonic, then*

$$\pi \circ \Phi_1^h - \pi \circ \Phi_2^h = \text{constant}.$$

Proof. We will show (c) by using (a), (b). Let C be a homotopy class of X_0 into T such that for any $F \in C$, $F_* : \pi_1(X_0) \to \pi_1(T) = \pi(\Gamma)$ satisfies

$$F_*([c]) = \pi(\sigma_c).$$

Here $\sigma_c \in \Gamma$ satisfies $\sigma_c o(\tilde{c}) = t(\tilde{c})$, where $\tilde{c}$ is a lift of c to X. From **(i)**, there exists a harmonic map F^h in C. By the definition of C, $\widetilde{F^h} : X \to G_\Gamma/[G_\Gamma, G_\Gamma]$, the lift of F^h is π-equivariant. Namely, $\widetilde{F^h}(\gamma x) = \widetilde{F^h}(x) + \pi(\gamma)$ for any $x \in X$ and $\gamma \in \Gamma$.

We define $\Phi^h(x)$ such that $\pi \circ \Phi^h(x) = \widetilde{F^h}(x)$ for a vertex x in a fundamental domain $\mathscr{D} \subset X$. Next we define $\Phi^h(\gamma x) = \gamma\Phi^h(x)$ for all $\gamma \in \Gamma$. Iterating these processes for all vertices in $\mathscr{D}$, we can realize all vertices of X to G_Γ. Finally for any $e \in E$, we define a smooth map $\Phi_e^h : [0,1] \to G_\Gamma$ which satisfies $\pi \circ \Phi_e^h(t) = \widetilde{F_e^h}(t)$ $(t \in [0,1])$ with $\Phi_e^h(0) = \Phi^h(o(e))$, $\Phi_e^h(1) = \Phi^h(t(e))$ and $\Phi_{\gamma e}^h = \gamma \Phi_e^h$. Consequently, Φ^h is a realization such that the composite $\pi \circ \Phi^h$ is harmonic.

From the result of (b), if $\pi \circ \Phi_1^h$, $\pi \circ \Phi_2^h$ are both harmonic, then

$$\pi \circ \Phi_1^h - \pi \circ \Phi_2^h = \text{constant}. \qquad \square$$

5. Sub-Laplacian for the Albanese metric.

First we consider the following diagram.

$$\begin{array}{ccccc} G_\Gamma/[G_\Gamma, G_\Gamma] & \simeq & \pi(\Gamma) \otimes \boldsymbol{R} & \longleftarrow & \mathrm{H}_1(X_0, \boldsymbol{R}) \\ \updownarrow \text{dual} & & \updownarrow \text{dual} & & \updownarrow \text{dual} \\ \mathrm{Hom}(G_\Gamma/[G_\Gamma, G_\Gamma], \boldsymbol{R}) & \simeq & \mathrm{Hom}(\pi(\Gamma), \boldsymbol{R}) & \hookrightarrow & \mathrm{H}^1(X_0, \boldsymbol{R}) \end{array}$$

where $G_\Gamma/[G_\Gamma, G_\Gamma]$ is identified with $\mathfrak{g}^{(1)}$ by a homomorphism $\exp^{-1}|_{\mathfrak{g}^{(1)}} : G_\Gamma \to \mathfrak{g}^{(1)}$. We identify $\mathrm{H}^1(X_0, \boldsymbol{R})$ with the set of harmonic 1-forms on X_0 by the discrete analogue of Hodge-Kodaira's theorem. Namely,

$$\mathrm{H}^1(X_0, \boldsymbol{R}) \simeq \left\{ \omega : E_0 \to \boldsymbol{R} \,|\, \omega(\bar{e}) = -\omega(e), \sum_{e \in E_x} \omega(e) = 0 \right\}.$$

We have an inner product on the set of harmonic 1-forms given by

$$\langle\!\langle \omega, \eta \rangle\!\rangle = \frac{1}{2} \sum_{e \in E_0} m(e)\omega(e)\eta(e)$$

for any harmonic 1-forms ω, η. By the identification, we define an inner product on $\mathrm{H}^1(X_0, \boldsymbol{R})$.

The surjective homomorphism $\rho : \mathrm{H}_1(X_0, \boldsymbol{Z}) \to \pi(\Gamma)$ is given by $\rho([c]) = \pi(\sigma_c)$, where $\sigma_c \in \Gamma$ satisfies $\sigma_c o(\tilde{c}) = t(\tilde{c})$. Since $\pi(\Gamma)$ is a lattice in the abelian group $G_\Gamma/[G_\Gamma, G_\Gamma]$, we have $G_\Gamma/[G_\Gamma, G_\Gamma] \simeq \pi(\Gamma) \otimes \boldsymbol{R}$. Hence the surjective homomorphism $\rho : \mathrm{H}_1(X_0, \boldsymbol{R}) \to G_\Gamma/[G_\Gamma, G_\Gamma]$ is defined. We induce the metric from $\mathrm{H}^1(X_0, \boldsymbol{R})$ to $\mathrm{Hom}(G_\Gamma/[G_\Gamma, G_\Gamma], \boldsymbol{R})$ by ${}^t\rho : \mathrm{Hom}(G_\Gamma/[G_\Gamma, G_\Gamma], \boldsymbol{R}) \hookrightarrow \mathrm{H}^1(X_0, \boldsymbol{R})$, the transpose of ρ. The dual metric on $G_\Gamma/[G_\Gamma, G_\Gamma]$ is said to be the *Albanese metric*.

We define the Albanese map $\mathrm{Alb} : X \to G_\Gamma/[G_\Gamma, G_\Gamma]$ by

$$\mathrm{Alb}(x)\omega = \int_{x_0}^{x} \tilde{\omega} \quad (\omega \in \mathrm{Hom}(G_\Gamma/[G_\Gamma, G_\Gamma], \boldsymbol{R}))$$

for a base point $x_0 \in V$, where $\tilde{\omega}$ is the lift of ω to X. For an orthonormal basis $\{\omega_1, \dots, \omega_{d_1}\}$ on $\mathrm{Hom}(G_\Gamma/[G_\Gamma, G_\Gamma], \boldsymbol{R})$ and the dual basis $\{X_1^{(1)}, \dots, X_{d_1}^{(1)}\}$ on $G_\Gamma/[G_\Gamma, G_\Gamma]$, we have

$$\mathrm{Alb}(x) = \left(\int_{x_0}^{x} \tilde{\omega}_1, \dots, \int_{x_0}^{x} \tilde{\omega}_{d_1} \right) = \sum_{i \le d_1} \int_{x_0}^{x} \tilde{\omega}_i X_i^{(1)}.$$

Because $\int_c \tilde{\omega} = 0$ for any closed path c in X and $\omega \in \mathrm{Hom}(G_\Gamma/[G_\Gamma, G_\Gamma], \boldsymbol{R})$, Alb is well-defined. For any $x \in X$, $\gamma \in \Gamma$, and $\omega \in \mathrm{Hom}(G_\Gamma/[G_\Gamma, G_\Gamma], \boldsymbol{R})$, Alb satisfies

$$\mathrm{Alb}(\gamma x)\omega = \int_{x_0}^{x} \tilde{\omega} + \int_{x}^{\gamma x} \tilde{\omega} = \mathrm{Alb}(x)\omega + \int_{[c_\gamma]} \omega,$$

where c_γ is a loop in X_0 satisfying $t(\tilde{c}_\gamma) = \gamma o(\tilde{c}_\gamma)$. Since $\omega \in \mathrm{Hom}(G_\Gamma/[G_\Gamma, G_\Gamma], \boldsymbol{R})$, we have $\int_{[c_\gamma]} \omega = \pi(\gamma)\omega$. Thus Alb is a π-equivariant map. Moreover, Alb is harmonic. Hence we conclude

Theorem 3. *Let $\Phi^h : X \to G_\Gamma$ be a realization such that the composite $\pi \circ \Phi^h$ is harmonic. Then*

$$\Omega_* = -\frac{1}{2} \sum_{e \in E_0} m(e) (\exp^{-1} \Phi^h(o(e))^{-1} \Phi^h(t(e))|_{\mathfrak{g}^{(1)}})_*^2.$$

Proof. From Theorem 2 and the identification of $G_\Gamma/[G_\Gamma, G_\Gamma]$ with $\mathfrak{g}^{(1)}$, there exists $X^{(1)} \in \mathfrak{g}^{(1)}$ such that $\mathrm{Alb} = \exp^{-1} \Phi^h|_{\mathfrak{g}^{(1)}} + X^{(1)}$. Hence we have

$$\begin{aligned}
\Omega_* &= - \sum_{i,j \le d_1} \frac{1}{2} \sum_{e \in E_0} m(e) \omega_i(e) \omega_j(e) X_{i*}^{(1)} X_{j*}^{(1)} \\
&= -\frac{1}{2} \sum_{e \in E_0} m(e) (\mathrm{Alb}(t(e)) - \mathrm{Alb}(o(e)))_*^2 \\
&= -\frac{1}{2} \sum_{e \in E_0} m(e) (\exp^{-1} \Phi^h(o(e))^{-1} \Phi^h(t(e))|_{\mathfrak{g}^{(1)}})_*^2.
\end{aligned}$$

□

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