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# Stationary preserving ideals over $\mathscr{P}_\kappa\lambda$

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**Abstract.** We investigate the properties of ideals such that their corresponding partial orders preserve stationarity. We show that these ideals exhibit many large cardinal-like consequences. We also prove the existence of a certain non-reflecting stationary subset of $\mathscr{P}_\kappa\lambda$ under some hypotheses.

## 1. Introduction.

In this paper we say that $I$ is an ideal over $S$ when $I$ is a countably complete fine (i.e. for every $i \in \bigcup S$ $\{x \in S : i \notin x\} \in I$) ideal over $S$. For an ideal $I$, we denote the poset of $I$-positive sets ordered by inclusion by $\boldsymbol{P}_I$. We refer the reader to Jech [**12**] for background about ideals, and Kunen [**14**] and Kanamori [**13**] for terminology from the theory of forcing (e.g. poset, antichain, predense, generic filter, etc.).

For an uncountable regular cardinal $\kappa$ and a set $X$ with $|X| \geq \kappa$, we let $\mathscr{P}_\kappa X$ denote the set $\{s \subseteq X : |s| < \kappa\}$. A set $W \subseteq \mathscr{P}_\kappa X$ is said to be *stationary in* $\mathscr{P}_\kappa X$ if for every function $f : \lambda^{<\omega} \to \mathscr{P}_\kappa X$, there exists a set $s \in W$ such that $s$ is closed under $f$ i.e. $\forall a \in s^{<\omega} f(a) \subseteq s$. The collection of non-stationary subsets of $\mathscr{P}_\kappa X$ forms a $\kappa$-complete normal ideal over $\mathscr{P}_\kappa X$. This ideal is known as the *non-stationary ideal* over $\mathscr{P}_\kappa X$. We denote the non-stationary ideal over $\mathscr{P}_\kappa X$ by $NS_{\kappa X}$. We refer the reader to Kanamori [**13**, Section 25] for basic facts about the combinatorics of $\mathscr{P}_\kappa X$.

In this paper we investigate ideals for which their corresponding posets preserve stationarity. We remind the reader that a poset $\boldsymbol{P}$ is said to be *proper* if $\boldsymbol{P}$ preserves stationarity of subsets of $\mathscr{P}_{\aleph_1}\lambda$ for every uncountable cardinal $\lambda$, i.e. for every $\lambda \geq \aleph_1$, if $X \subseteq \mathscr{P}_{\aleph_1}\lambda$ is stationary, then $\Vdash_{\boldsymbol{P}_I}$ "$\check{X}$ is stationary in $\mathscr{P}_{\aleph_1}\check{\lambda}$". We will show that an ideal whose corresponding poset is proper possesses many interesting consequences. As an example we state one of our theorems.

Theorem. *Let $\lambda$ be an uncountable cardinal and $\delta$ be a cardinal $\geq 2^{2^{2^\lambda}}$. If there is a $\lambda^+$-complete normal ideal $I$ over $\mathscr{P}_{\lambda^+}\delta$ such that $\boldsymbol{P}_I$ is proper, then $NS_{\aleph_1\lambda}$ is precipitous. Furthermore $NS_{\aleph_1\lambda}$ is presaturated if $2^{\lambda^{\aleph_0}} = \lambda^+$.*

We will restate and prove this theorem as Theorem 7 later. Let us discuss some terminology. We say that an ideal $I$ is *precipitous* if for every generic filter $G$ for $\boldsymbol{P}_I$ over the ground model $V$, $Ult_G(V)$, the generic ultrapower of $V$ with respect to $G$, is well-founded. We refer the reader to Jech [**12**] for the definition of $Ult_G(V)$. The

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existence of a precipitous ideal is known as a large cardinal hypothesis. For example, see Theorem 86 of [**12**]. An ideal $I$ over $\mathscr{P}_\kappa\lambda$ is said to be *presaturated* if it is precipitous and forcing with $\boldsymbol{P}_I$ preserves $\lambda^+$.

We also present a result concerning "non-reflecting" stationary sets.

Theorem. *Let $\kappa$ and $\lambda$ be regular cardinals such that $\aleph_2 \leq \kappa < \lambda$ and $2^{\lambda^{<\kappa}} = \lambda^+$. If $\theta$ is a regular cardinal $\geq \lambda^+$ with a measurable cardinal above, then there exists a stationary $S \subseteq \mathscr{P}_\kappa H_\theta$ such that for any $X \in [H_\theta]^\lambda$, $\lambda \subseteq X$ implies that $S \cap \mathscr{P}_\kappa X$ is not stationary in $\mathscr{P}_\kappa X$.*

Here $H_\theta$ stands for the collection of sets which are hereditarily of cardinality $<\theta$. We refer the reader to Kunen [**14**] for background about $H_\theta$. The last theorem will be presented as Theorem 9.

The next theorem due to Gitik and Shelah [**8**] shows that the existence of an ideal $I$ such that $\boldsymbol{P}_I$ is proper is always a large cardinal property. We present a different proof which shows this result can be strengthened slightly.

Theorem 1. *If $I$ is an ideal such that $\boldsymbol{P}_I$ is proper, then $I$ is precipitous.*

Proof. For a poset $\boldsymbol{P}$ we consider the following game $\Gamma(\boldsymbol{P})$ due to Gray [**11**]. $\Gamma(\boldsymbol{P})$ is a two player game of length $\omega$. Player I as her first move chooses $p_0 \in \boldsymbol{P}$ and a maximal antichain $A_0$ in $\boldsymbol{P}$. Then Player II chooses a countable subset $B_0^0$ of $A_0$. Next Player I chooses another maximal antichain $A_1$ and in response Player II chooses countable $B_0^1 \subseteq A_0$ and $B_1^1 \subseteq A_1$. In general as their $n$th moves, with the exception of the first move, Player I chooses a maximal antichain $A_n$ and Player II chooses countable $B_i^n \subseteq A_i$ for each $i \leq n$. After $\omega$ moves Player II wins $\Gamma(\boldsymbol{P})$ if and only if there exists $q \leq p_0$ such that for each $i$, $B_i = \bigcup_{n \in \omega - i} B_i^n$ is predense below $q$. Gray proved that $\boldsymbol{P}$ is proper if and only if Player II has a winning strategy in $\Gamma(\boldsymbol{P})$. For the proof of this see Baumgartner [**2**].

We will show that if an ideal $I$ over $Z$ is not precipitous then Player I has a winning strategy in $\Gamma(\boldsymbol{P}_I)$. Suppose an ideal $I$ is not precipitous. Assume for some $I$-positive set $X$ that $X \Vdash_{\boldsymbol{P}_I}$ "$\langle [\dot{\tau}_n] \mid n \in \omega \rangle$ is a descending sequence in $\langle Ult_G(\check{V}), \mathrm{E} \rangle$." In particular for each $n \in \omega$, $X \Vdash_{\boldsymbol{P}_I} \dot{\tau}_n \in \check{V}$. Let $A_0'$ be a maximal antichain in $\{Y \in \boldsymbol{P}_I : Y \subseteq X,\ Y \Vdash [\check{f_Y}] = [\dot{\tau}_0]$ for some $f_Y \in V\}$. *Let* $A_0 = A_0' \cup \{Z - X\}$ if $Z - X \notin I$. If $Z - X \in I$, then let $A_0 = A_0'$. It is easy to see that $A_0$ is a maximal antichain in $\boldsymbol{P}_I$. As her first move let Player I choose $X$ and $A_0$. If $Z - X \notin I$, as her $n$th move ($n \geq 1$), let Player I choose $A_n = A_n' \cup \{Z - X\}$ where $A_n'$ is a maximal antichain in $\{Y \in \boldsymbol{P}_I : Y \subseteq X,\ \exists f_Y \in V$ such that $Y \Vdash [\check{f_Y}] = [\dot{\tau}_n]\}$. If $Z - X \in I$ let Player I choose $A_n = A_n'$.

We want to show this is a winning strategy for Player I. Suppose Player II chooses a countable $B_i^n \subseteq A_i$ for all $i \leq n$ as his $n$th move.

Claim. *There is no $Y \leq X$ such that for each $i$, $\bigcup_{n \in \omega - i} B_i^n$ is predense below $Y$.*

Proof of Claim. Suppose otherwise. Assume that $Y \leq X$ and $B_i = \bigcup_{n \in \omega - i} B_i^n$ is predense below $Y$ for all $i$. For each $i$ let $\langle X_m : m < \omega \rangle$ enumerate elements of $B_i$. Let $\overline{X}_0 = X_0$ and for $m \geq 1$, $\overline{X}_m = X_m - \bigcup_{k<m} X_k$. By the definition of $B_i$ for each $m$ there exists $f_{X_m} \in V$ such that $X_m \Vdash_{\boldsymbol{P}_I} [\check{f}_{X_m}] = [\dot{\tau}_i]$. Define $g_i$ by $g_i = \bigcup_{m \in \omega} f_{X_m} \upharpoonright \overline{X}_m$. Now we claim that $Y \Vdash_{\boldsymbol{P}_I} [\check{g}_i] = [\dot{\tau}_i]$. Suppose otherwise. Then $\exists Z \leq Y : Z \Vdash_{\boldsymbol{P}_I} [\check{g}_i] \neq [\dot{\tau}_i]$. Since $B_i$ is predense below $Y$ there is some $X_m \in B_i$ such that $X_m \cap Z$ is $I$-positive.

Then $X_m \cap Z \Vdash_{\boldsymbol{P}_I} [\check{f}_{X_m}] = [\dot{\tau}_i]$. Since $X_m \cap Z \subseteq \bigcup_{k\le m} \overline{X}_k$ there is some $k \le m$ such that $X_m \cap Z \cap \overline{X}_k$ is $I$-positive. Clearly $X_m \cap Z \cap \overline{X}_k \Vdash_{\boldsymbol{P}_I} [\check{f}_{X_k}] = [\dot{\tau}_i] \wedge [\check{g}_i] = [\check{f}_{X_k}]$. So $X_m \cap Z \cap \overline{X}_k \Vdash_{\boldsymbol{P}_I} [\check{g}_i] = [\dot{\tau}_i]$ contradicting $Z \Vdash_{\boldsymbol{P}_I} [\check{g}_i] \neq [\dot{\tau}_i]$.

Therefore for each $i \in \omega$ there is some $g_i \in V$ such that $Y \Vdash_{\boldsymbol{P}_I} [\check{g}_i] = [\dot{\tau}_i]$. For each $i \in \omega$ let $W_i = \{a \in Y : g_{i+1}(a) \in g_i(a)\}$. We know that for each $i \in \omega$, $Y \Vdash [\check{g}_{i+1}] E [\check{g}_i]$. Thus $Y \cap \bigcap_{i\in\omega} W_i$ is $I$-positive. But if $a \in \bigcap_{i\in\omega} W_i$ then $g_0(a) \ni g_1(a) \ni g_2(a) \cdots$. This contradiction completes the proof of Claim. $\square$

Therefore Player I has a winning strategy in $\Gamma(\boldsymbol{P}_I)$. So by Gray's thorem $\boldsymbol{P}_I$ is not proper. $\square$

Remark. The above proof of Theorem 1 shows that if $I$ is an ideal such that Player I does not have a winning strategy in $\Gamma(\boldsymbol{P}_I)$, then $I$ is precipitous. Since this hypothesis is weaker than "$\boldsymbol{P}_I$ is proper", we proved a slightly stronger result.

## 2. Stationary preserving ideals.

In Matsubara [**15**] we adopted the proof of Theorem 4 in Galvin-Jech-Magidor [**7**] to show the following result:

Theorem 2. *If $\kappa$ is a supercompact cardinal $>$ an uncountable regular cardinal $\delta$, then $\Vdash_{\mathrm{Coll}(\delta,<\kappa)}$ "For every $\lambda \ge \delta^+$ there is a $\delta^+$-complete normal fine ideal $I$ over $\mathscr{P}_{\delta^+}\lambda$ such that $\boldsymbol{P}_I$ is $\prec\delta$-strategically closed" where* $\mathrm{Coll}(\delta,<\kappa)$ *is the Levy collapse causing $\kappa$ to be $\delta^+$.*

For the definition of "$\prec\delta$-strategically closed" we refer the reader to Apter-Shelah [**1**]. It is well-known that every $\omega$-strategically closed poset is proper. Therefore after Levy collapsing a supercompact cardinal to $\delta^+$, $\mathscr{P}_{\delta^+}\lambda$ carries a $\delta^+$-complete normal fine ideal $I$ such that $\boldsymbol{P}_I$ is proper. From now on by a $\kappa$-ideal over $\mathscr{P}_\kappa A$ we mean a $\kappa$-complete normal fine ideal over $\mathscr{P}_\kappa A$. Furthermore we say that an ideal $I$ is "proper" if $\boldsymbol{P}_I$ is proper. The existence of a "proper" $\delta^+$-ideal on $\mathscr{P}_{\delta^+}\lambda$ implies a certain reflection principle.

Definition. If $S$ is a stationary subset of $\mathscr{P}_{\aleph_1}\lambda$ and $X \subseteq \lambda$, then we say that $S$ reflects to $X$ if $S \cap \mathscr{P}_{\aleph_1}X$ is stationary in $\mathscr{P}_{\aleph_1}X$.

The following is a slight generalization of the Reflection Principle due to Foreman, Magidor, and Shelah [**6**].

Definition. The Reflection Principle at $\lambda$ to size $\delta$ holds if the following proposition is true:

Every stationary set $S \subseteq \mathscr{P}_{\aleph_1}\lambda$ reflects to some $X \in [\lambda]^\delta$ such that $\delta \subseteq X$.

Theorem 3. *If there exists a "proper" $\delta^+$-ideal over $\mathscr{P}_{\delta^+}\theta$, then for every $\lambda$ such that $\delta^+ \le \lambda \le \theta$ the Reflection Principle at $\lambda$ to size $\delta$ holds.*

Proof. If $T$ is a stationary subset of $\mathscr{P}_{\aleph_1}\lambda$ then $\{x \in \mathscr{P}_{\aleph_1}\theta : x \cap \lambda \in T\}$ is a stationary subset of $\mathscr{P}_{\aleph_1}\theta$. It is also known that the projection of a stationary subset of $\mathscr{P}_{\aleph_1}\theta$ onto $\mathscr{P}_{\aleph_1}X$ with $\aleph_1 \subseteq X \subseteq \theta$ is stationary in $\mathscr{P}_{\aleph_1}X$. For the proofs of these facts, see Foreman-Magidor-Shelah [**5**]. From these facts it is easy to see that the Reflection Principle at $\theta$ to size $\delta$ implies the Reflection Principle at $\lambda$ to size $\delta$ for every $\lambda$ such that $\delta^+ \le \lambda \le \theta$.

Let $S$ be a stationary subset of $\mathscr{P}_{\aleph_1}\theta$ in $V$. Let $I$ be a $\delta^+$-ideal over $\mathscr{P}_{\delta^+}\theta$ such that $\boldsymbol{P}_I$ is proper. Let $G$ be a $\boldsymbol{P}_I$-generic filter over $V$. Let $j : V \to M \cong Ult_G(V)$ be the corresponding generic elementary embedding.

Now work in $V[G]$. Since $\boldsymbol{P}_I$ is proper, $S$ is stationary in $\mathscr{P}_{\aleph_1}\theta$. Define $S^* = \{j''x : x \in S\}$. Since $j \restriction \theta$ is a bijection between $\theta$ and $j''\theta$, we know that $S^*$ is stationary in $\mathscr{P}_{\aleph_1} j''\theta$.

Define a map $f$ on $(\mathscr{P}_{\delta^+}\theta)^V$ by $f(a) = S \cap \mathscr{P}_{\aleph_1} a$. It is easy to see that $S^* \subseteq [f] \subseteq (\mathscr{P}_{\aleph_1} j''\theta)^M$. Thus $M \models$ "$[f]$ is stationary in $\mathscr{P}_{\aleph_1} j''\theta$." Therefore $\{a \in (\mathscr{P}_{\delta^+}\theta)^V : S \cap \mathscr{P}_{\aleph_1} a$ is stationary in $\mathscr{P}_{\aleph_1} a\} \in G$. So it is clear that $\{a \in \mathscr{P}_{\delta^+}\theta : S \cap \mathscr{P}_{\aleph_1} a$ is stationary in $\mathscr{P}_{\aleph_1} a\} \notin I$. Hence there exists some $X \in [\theta]^\delta$ such that $S$ reflects to $X$ and $\delta \subseteq X$. $\square$

The Reflection Principle at $\aleph_2$ to size $\aleph_1$ implies that the continuum is at most $\aleph_2$ (Shelah [**17**] and Todorčević [**18**]). This gives the following corollary.

Corollary 4. *If there exists a "proper" $\aleph_2$-ideal over $\mathscr{P}_{\aleph_2}\lambda$, then $2^{\aleph_0} \le \aleph_2$.*

Magidor proved that if for every regular $\lambda \ge \aleph_2$ the Reflection Principle at $\lambda$ to size $\aleph_1$ holds then $NS_{\aleph_1}$, the nonstationary ideal on $\aleph_1$, is presaturated i.e. $NS_{\aleph_1}$ is precipitous and $\aleph_2$ remains a cardinal through forcing with $\boldsymbol{P}_{NS_{\aleph_1}}$. The following corollary is also immediate.

Corollary 5. *If for cofinally many $\lambda$'s $\mathscr{P}_{\aleph_2}\lambda$ carries a "proper" $\aleph_2$-ideal, then $NS_{\aleph_1}$ is presaturated.*

It turns out that under some conditions for $\delta$ sufficiently large the existence of a "proper" $\lambda^+$-ideal over $\mathscr{P}_{\lambda^+}\delta$ implies the presaturation of $NS_{\aleph_1\lambda}$. We will present a result that is more general, using the following concept:

Definition. Let $\delta$ and $\kappa$ be uncountable regular cardinals such that $\kappa < \delta$. We say that a $\delta$-ideal over $\mathscr{P}_\delta A$ is $\mathscr{P}_\kappa A$-stationary preserving if the following hold:

(1) $I$ is precipitous,

(2) If $S$ is a stationary subset of $\mathscr{P}_\kappa A$, then there exists some $X \in \boldsymbol{P}_I$ such that

$$X \Vdash_{\boldsymbol{P}_I} \text{“}S \text{ remains stationary in } \mathscr{P}_\kappa A\text{”}.$$

Remark. By an argument similar to the one given at the beginning of the proof of Theorem 3, if $\boldsymbol{P}_I$ preserves every stationary subset of $\mathscr{P}_\kappa A$ as in the sense of (2) then it preserves every stationary subset of $\mathscr{P}_\kappa B$ for every $B \subseteq A$.

The proof of the next lemma follows Foreman-Magidor-Shelah [**6**], Goldring [**9**] and [**10**] closely. Although all of the ingredients of the proof of the following lemma can be found in [**9**] and [**10**], since the theorems below depend on its proof we have decided to present the lemma below.

Lemma 6. *Let $\kappa$ and $\theta$ be uncountable regular cardinals and $\lambda$ a cardinal such that $2^{\lambda^{<\kappa}} < \theta$. If there exists a $\mathscr{P}_\kappa H_\theta$-stationary preserving $\lambda^+$-ideal over $\mathscr{P}_{\lambda^+} H_\theta$, then $NS_{\kappa\lambda}$ is precipitous. Furthermore $NS_{\kappa\lambda}$ is presaturated if $2^{\lambda^{<\kappa}} = \lambda^+$.*

Proof. Let $I$ be a $\mathscr{P}_\kappa H_\theta$-stationary preserving $\lambda^+$-ideal over $\mathscr{P}_{\lambda^+} H_\theta$.

Let $\mathfrak{H} = \langle H_\theta, \in, \varDelta_\theta, \dots \rangle$ where $\varDelta_\theta$ stands for a well-ordering of $H_\theta$.

Claim 1. *$C = \{N \in \mathscr{P}_\kappa H_\theta : N \prec \mathfrak{H} \wedge \forall \vec{A} \in N$ [$\vec{A}$ enumerates a maximal antichain in $NS_{\kappa\lambda} \Rightarrow \exists\beta \in dom(\vec{A})(N \cap \lambda \in \vec{A}(\beta) \wedge N \cap \lambda = Sk^{\mathfrak{H}}(N \cup \{\beta\}) \cap \lambda)]\}$ contains a club subset of $\mathscr{P}_\kappa H_\theta$. Here $Sk^{\mathfrak{H}}(N \cup \{\beta\})$ stands for the Skolem hull of $N \cup \{\beta\}$ in $\mathfrak{H}$.*

Proof of Claim 1. Suppose otherwise i.e. $\mathscr{P}_\kappa H_\theta - C$ is stationary in $\mathscr{P}_\kappa H_\theta$. By the usual normality argument there is some $\vec{A}^*$ which enumerates a maximal antichain in $NS_{\kappa\lambda}$ such that $S = \{N \in \mathscr{P}_\kappa H_\theta : \vec{A}^* \in N \wedge \forall\beta \in dom(\vec{A}^*)(N \cap \lambda \in \vec{A}^*(\beta) \Rightarrow N \cap \lambda \neq Sk^{\mathfrak{H}}(N \cup \{\beta\}) \cap \lambda)\}$ is stationary in $\mathscr{P}_\kappa H_\theta$. Let $G$ be a $\boldsymbol{P}_I$-generic filter over $V$ such that $S$ remains stationary in $V[G]$. Let $j$ be the corresponding generic elementary embedding from $V$ into $M$. It is clear that $M \models$ "$S$ is stationary in $\mathscr{P}_\kappa H_\theta^V$ and $|H_\theta^V| = \lambda$." Let $f$ be a function in $M$ such that $f$ is a bijection from $\lambda$ to $H_\theta^V$.

Now work in $M$. Let $T = \{N \in \mathscr{P}_\kappa\lambda : f''N \in S \wedge (f''N) \cap \lambda = N\}$. Then $T$ is a stationary subset of $\mathscr{P}_\kappa\lambda$. Since $crit(j) = (\lambda^+)^V$, $j(\vec{A}^*)$ is an enumeration of a maximal antichain in $NS_{\kappa\lambda}$. Therefore there exists some $\alpha^* \in dom(j(\vec{A}^*))$ such that $j(\vec{A}^*)(\alpha^*) \cap T \notin NS_{\kappa\lambda}$.

Let $D = \{N \in j(\mathscr{P}_\kappa H_\theta) : N \prec j(\mathfrak{H}), \alpha^* \in N$, and $N$ is closed under $f, f^{-1}$ and $j \restriction H_\theta^V\}$. It is clear that $D$ is club in $\mathscr{P}_\kappa j(H_\theta^V)$. Thus there exists some $N \in D$ such that $N \cap \lambda \in j(\vec{A}^*)(\alpha^*) \cap T$. Let $N^* = f''(N \cap \lambda)$. From $N \cap \lambda \in T$ we get $N^* \in S$. Since $N$ is closed under $f$ and $j \restriction H_\theta^V$ we know that $j''N^*(= j(N^*)) \subseteq N$. Note that since $N$ is also closed under $f^{-1}$, we have $j(N^*) \cap \lambda = N \cap \lambda$. So $j(N^*) \cap j(\lambda)$ $(= j(N^*) \cap \lambda)$ belongs to $j(\vec{A}^*)(\alpha^*)$. From $N \prec j(\mathfrak{H})$, $j(N^*) \subseteq N$, and $\alpha^* \in N$ we see that $Sk^{j(\mathfrak{H})}(j(N^*) \cup \{\alpha^*\}) \cap j(\lambda) \subseteq N \cap j(\lambda) = N \cap \lambda = j(N^*) \cap j(\lambda)$. So clearly $Sk^{j(\mathfrak{H})}(j(N^*) \cup \{\alpha^*\}) \cap j(\lambda) = j(N^*) \cap j(\lambda)$. We can now conclude $j(N^*) \notin j(S)$. By the elementarity of $j$ we have $N^* \notin S$. This contradiction completes the proof of Claim 1. $\square$

Claim 2. *$NS_{\kappa\lambda}$ is precipitous.*

Proof. Let $S$ be a stationary subset of $\mathscr{P}_\kappa\lambda$. Let $\langle W_n \mid n < \omega\rangle$ be a sequence of maximal antichains in $\boldsymbol{P}_{NS_{\kappa\lambda}} \restriction S = \{X : X \subseteq S$ and $X$ is stationary in $\mathscr{P}_\kappa\lambda\}$. Assume $W_0 \geq W_1 \geq \cdots \geq W_n \geq \cdots$ where $W_i \geq W_{i+1}$ denotes that every $X \in W_{i+1}$ is a subset of some $Y \in W_i$. We would like to show that there exists a sequence of sets $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$ such that $X_n \in W_n$ for each $n$, and $\bigcap_{n \in \omega} X_n \neq \emptyset$.

Let $\eta$ be a regular cardinal $> \theta$ such that $C \in H_\eta$ where $C$ is the club set from Claim 1. Let $\Delta_\eta$ be a well-ordering extending $\Delta_\theta$. Let $\mathfrak{H}^* = \langle H_\eta, \in, \Delta_\eta, \ldots\rangle$. Let $C^* = \{N \in \mathscr{P}_\kappa H_\eta : N \prec \mathfrak{H}^*, C \in N, N \cap \kappa \in \kappa\}$. Clearly $C^*$ is a club subset of $\mathscr{P}_\kappa H_\eta$.

Subclaim. *If $N \in C^*$ and $\vec{A} \in N$ is an enumeration of a maximal antichain in $NS_{\kappa\lambda}$, then there is some $\beta \in dom(\vec{A})$ such that $N \cap \lambda \in \vec{A}(\beta)$ and $N \cap \lambda = Sk^{\mathfrak{H}^*}(N \cup \{\beta\}) \cap \lambda$.*

For a proof of the Subclaim we refer the reader to p. 292 of Goldring [**9**].

Let $S$ and $\langle W_n \mid n < \omega\rangle$ be as above. Suppose $N_0 \in C^*$ such that $S, \langle W_n \mid n < \omega\rangle \in N_0$, and $N_0 \cap \lambda \in S$. Let $\vec{A}_0 \in N_0$ be an enumeration of the maximal antichain $W_0 \cup \{\mathscr{P}_\kappa\lambda - S\}$ (or simply $W_0$ if $\mathscr{P}_\kappa\lambda - S$ is nonstationary.) By the Subclaim there is some $\beta_0 \in dom(\vec{A}_0)$ such that $N_0 \cap \lambda \in \vec{A}_0(\beta_0)$ and $N_0 \cap \lambda = Sk^{\mathfrak{H}^*}(N_0 \cup \{\beta_0\}) \cap \lambda$. Since $N_0 \cap \lambda \in S$ we must have $\vec{A}_0(\beta_0) \in W_0$. Let $X_0 = \vec{A}_0(\beta_0)$ and $N_1 = Sk^{\mathfrak{H}^*}(N_0 \cup \{\beta_0\})$. So $X_0 \in N_1$. Since $N_1 \in C^*$ we can repeat the procedure as follows:

By $W_1 \upharpoonright X_0$ we denote the set $\{X \in W_1 : X \subseteq X_0\}$. Then $W_1 \upharpoonright X_0 \cup \{\mathscr{P}_\kappa\lambda - X_0\}$ is a maximal antichain. Let $\vec{A}_1 \in N_1$ enumerate this antichain. Then there exists $\beta_1 \in dom(\vec{A}_1)$ such that $N_1 \cap \lambda \in \vec{A}_1(\beta_1)$ and $N_1 \cap \lambda = Sk^{\mathfrak{H}^*}(N_1 \cup \{\beta_1\}) \cap \lambda$. Since $N_1 \cap \lambda \in X_0$ we know $\vec{A}_1(\beta_1) \in W_1 \upharpoonright X_0$. Let $X_1 = \vec{A}_1(\beta_1)$ and $N_2 = Sk^{\mathfrak{H}^*}(N_1 \cup \{\beta\})$. The above procedure will produce a sequence of sets $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$ such that $X_n \in W_n$ for each $n$ and $N_0 \cap \lambda \in \bigcap_{n\in\omega} X_n$. (Claim 2) □

CLAIM 3. *Suppose $\langle \vec{A}_\alpha \mid \alpha < \lambda \rangle$ is a sequence of enumerations of maximal antichains in $NS_{\kappa\lambda}$. Assume $B$ is a stationary subset of $\mathscr{P}_\kappa\lambda$. Then $S = \{N \in \mathscr{P}_\kappa H_\theta : N \prec \mathfrak{H} \wedge \forall\alpha \in N \cap \lambda\ \exists\beta \in N\ N \cap \lambda \in \vec{A}_\alpha(\beta) \wedge N \cap \lambda \in B\}$ is stationary.*

PROOF OF CLAIM 3. Let $D$ be an arbitrary club subset of $\mathscr{P}_\kappa H_\theta$. We will show that $S \cap D \neq \varnothing$. Let $\mathfrak{H}^*$ be as in the proof of Claim 2 and $C$ be as in Claim 1. Let $E = \{N \in \mathscr{P}_\kappa H_\eta : N \prec \mathfrak{H}^*, \langle \vec{A}_\alpha \mid \alpha < \lambda\rangle \in N, \theta, C, D \in N, N \cap H_\theta \in D$ and $N \cap \kappa \in \kappa\}$. Clearly $E$ is a club subset of $\mathscr{P}_\kappa H_\eta$ and $E \subseteq C^*$. Since $B$ is stationary, there is some $N_0 \in E$ such that $N_0 \cap \lambda \in B$. By the Subclaim if $N \in E$ and $\vec{A} \in N$ is an enumeration of a maximal antichain in $NS_{\kappa\lambda}$, then there is some $\beta \in dom(\vec{A})$ such that $N \cap \lambda \in \vec{A}(\beta)$ and $N \cap \lambda = Sk^{\mathfrak{H}^*}(N \cup \{\beta\}) \cap \lambda$. By induction we can define a sequence $\langle N_\gamma \mid \gamma < o.t.(N_0 \cap \lambda)\rangle$ of elements of $E$ such that $N_0 \cap \lambda = N_\gamma \cap \lambda$ for each $\gamma < o.t.(N_0 \cap \lambda)$ and $\forall\alpha \in N_0 \cap \lambda\ \exists\beta \in \bigcup_{\gamma<o.t.(N_0\cap\lambda)} N_\gamma\ N_0 \cap \lambda \in \vec{A}_\alpha(\beta)$. Let $N^* = \bigcup_{\gamma<o.t.(N_0\cap\lambda)} N_\gamma \cap H_\theta$. From $\bigcup_{\gamma<o.t.(N_0\cap\lambda)} N_\gamma \in E$ we see $N^* \in D$. Since $\forall\alpha \in N^* \cap \lambda (= N_0 \cap \lambda)\ \exists\beta \in N^*\ N_0 \cap \lambda \in \vec{A}_\alpha(\beta)$, and $N^* \cap \lambda = N_0 \cap \lambda \in B$, we conclude $N^* \in S \cap D$. (Claim 3) □

From now on we will assume that $2^{\lambda^{<\kappa}} = \lambda^+$. Therefore we can assume that if $\vec{A}$ is an enumeration of a maximal antichain in $NS_{\kappa\lambda}$ then $dom(\vec{A}) \leq \lambda^+$. Presaturation of $NS_{\kappa\lambda}$ follows immediately from the next claim.

CLAIM 4. *Let $\langle \vec{A}_\alpha \mid \alpha < \lambda\rangle$ be a sequence of enumerations of maximal antichains in $NS_{\kappa\lambda}$ and $B$ be a stationary subset of $\mathscr{P}_\kappa\lambda$. There exists some stationary subset $D$ of $B$ such that for each $\alpha < \lambda$*

$$|\{\beta \in dom(\vec{A}_\alpha) : D \cap \vec{A}_\alpha(\beta) \text{ is stationary}\}| \leq \lambda.$$

PROOF. Let $S$ be as in Claim 3. Let $G$ be a $\boldsymbol{P}_I$-generic filter over $V$ and $j : V \to M$ be the corresponding generic elementary embedding such that

$$M \models \text{“}S \text{ is stationary in } \mathscr{P}_\kappa H_\theta^V\text{”}.$$

Since $M \models |H_\theta^V| = \lambda$ there is some $g \in M$ such that $g$ is a bijection from $\lambda$ to $H_\theta^V$. We will work in $M$ for a while. Let $S^* = \{N \in S : N$ is closed under $g$ and $g^{-1}\}$. Note that $S^*$ is a stationary set such that if $N \in S^*$ then $N \cap \lambda = g^{-1}{}''N$.

Let $E = \{N \cap \lambda : N \in S^*\} = \{g^{-1}{}''N : N \in S^*\}$. So $E$ is a stationary subset of $B (\subseteq j(B)) \subseteq \mathscr{P}_\kappa\lambda$.

We will show that for every $\alpha < \lambda = j(\lambda)$

$$|\{\beta < dom(j(\vec{A}_\alpha)) : j(\vec{A}_\alpha)(\beta) \cap E \text{ is stationary}\}| \leq \lambda. \tag{†}$$

We will show that if $j(\vec{A}_\alpha)(\beta) \cap E$ is stationary for some $\beta < dom(j(\vec{A}_\alpha)) \leq j((\lambda^+)^V)$ then $\beta < (\lambda^+)^V$ i.e. $\{\beta < dom(j(\vec{A}_\alpha)) : j(\vec{A}_\alpha)(\beta) \cap E$ is stationary$\} \subseteq (\lambda^+)^V$. Then since $|(\lambda^+)^V| = \lambda$ (in $M$), this gives (†). Let $T = \{g''x : x \in j(\vec{A}_\alpha)(\beta) \cap E\} \cap \{N \in \mathscr{P}_\kappa H_\theta :$

$\alpha \in N\}$. Since $E = \{g^{-1\prime\prime}N : N \in S^*\}$, $T$ is a stationary subset of $S^*$. Therefore if $N \in T$ then there is some $\gamma \in N$ such that $N \cap \lambda \in \vec{A}_\alpha(\gamma)$. By the usual normality argument there are stationary $T^* \subseteq T$ and $\beta^* < (\lambda^+)^V$ such that if $N \in T^*$ then $\beta^* \in N$ and $N \cap \lambda \in \vec{A}_\alpha(\beta^*)$.

Note that $\vec{A}_\alpha(\beta^*) \subseteq j(\vec{A}_\alpha(\beta^*)) = j(\vec{A}_\alpha)(\beta^*)$. So $\{N \cap \lambda : N \in T^*\} \subseteq j(\vec{A}_\alpha)(\beta^*)$. On the other hand $\{N \cap \lambda : N \in T^*\} = \{g^{-1\prime\prime}N : N \in T^*\} \subseteq \{g^{-1\prime\prime}g''x : x \in j(\vec{A}_\alpha)(\beta) \cap E\} = j(\vec{A}_\alpha)(\beta) \cap E$. Thus $\beta = \beta^* < (\lambda^+)^V$.

We now have $M \models \exists E \subseteq j(B)$ [$E$ is stationary $\wedge \forall \alpha < j(\lambda)(|\{\beta < dom(j(\vec{A}_\alpha)) : j(\vec{A}_\alpha)(\beta) \cap E$ is stationary$\}| \leq j(\lambda))$]. Thus by the elementarity of $j$, we are done. (Claim 4) $\square$

(Lemma 6) $\square$

To carry out the arguments in the proof of Lemma 6, it is enough to have a precipitous ideal over $\mathscr{P}_{\lambda^+}\delta$ for $\delta \geq |H_\theta| = 2^{<\theta}$ that preserves stationary subsets of $\mathscr{P}_\kappa\delta$.

Theorem 7. *Let $\lambda$ be an uncountable cardinal and $\delta$ be a cardinal $\geq 2^{2^{2^\lambda}}$. If there is a "proper" $\lambda^+$-ideal over $\mathscr{P}_{\lambda^+}\delta$, then $NS_{\aleph_1\lambda}$ is precipitous. Furthermore $NS_{\aleph_1\lambda}$ is presaturated if $2^{\lambda^{\aleph_0}} = \lambda^+$.*

Unfortunately, for $\kappa \geq \aleph_2$ the hypotheses of Lemma 6 cannot be realized. For $\kappa = \lambda = \aleph_2$ the following theorem of Feng and Magidor shows the impossibility of our hypotheses.

Theorem of Feng-Magidor [**5**]. *If $\theta \geq (2^{\aleph_2})^{++}$ is regular, then there exists a stationary $S \subseteq \mathscr{P}_{\aleph_2}H_\theta$ such that for any $X \in [H_\theta]^{\aleph_2}$, $\omega_2 \subseteq X$ implies that $S \cap \mathscr{P}_{\aleph_2}X$ is not stationary in $\mathscr{P}_{\aleph_2}X$.*

Let $S$ be a "non-reflecting" stationary subset of $\mathscr{P}_{\aleph_2}H_\kappa$ as in Feng-Magidor. Suppose $I$ is a precipitous $\aleph_3$-ideal over $\mathscr{P}_{\aleph_3}H_\theta$. Let $F : \mathscr{P}_{\aleph_3}H_\theta \to V$ be defined by $F(X) = S \cap \mathscr{P}_{\aleph_2}X$. So $1 \Vdash_{\boldsymbol{P}_I}$ "$Ult_V(G) \models$ '$[F]$ is not stationary in $\mathscr{P}_{\aleph_2}j''H_\theta^V$'". Let $G$ be an arbitrary $\boldsymbol{P}_I$-generic filter over $V$. Then in $V[G]$, $[F]$ is a nonstationary subset of $\mathscr{P}_{\aleph_2}j''H_\theta^V$. It is easy to see that $\{j''y : y \in S\} \subseteq [F]$. Thus $S$ cannot stay stationary in $V[G]$. Feng and Magidor obtained their theorem by proving that the negation of their conclusion implies the presaturation of $NS_{\aleph_2}$ and invoking Shelah's theorem [**16**] which refutes the presaturation of $NS_\kappa$ for every successor cardinal $\geq \aleph_2$. For general $\kappa \geq \aleph_2$ and $\lambda$ we will imitate their proof using the following result.

Theorem 8 (Burke-Matsubara [**3**]). *If $\kappa$ is a regular cardinal $\geq \aleph_2$ and $\lambda$ a regular cardinal $>\kappa$, then $NS_{\kappa\lambda}$ cannot be presaturated.*

Remark. Even when $\kappa = \lambda$ the same conclusion as in Theorem 8 holds provided $\kappa$ is a successor cardinal $\geq \aleph_2$ using the above mentioned theorem of Shelah.

The following is our result on "non-reflecting" stationary sets.

Theorem 9. *Let $\kappa$ and $\lambda$ be regular cardinals such that $\aleph_2 \leq \kappa < \lambda$ and $2^{\lambda^{<\kappa}} = \lambda^+$. If $\theta$ is a regular cardinal $\geq \lambda^+$ with a measurable cardinal above, then there exists a stationary $S \subseteq \mathscr{P}_\kappa H_\theta$ such that for any $X \in [H_\theta]^\lambda$, $\lambda \subseteq X$ implies that $S \cap \mathscr{P}_\kappa X$ is not stationary in $\mathscr{P}_\kappa X$. Furthermore the same conclusion holds even if we replace "$\kappa < \lambda$" by "$\kappa = \lambda = \delta^+$ for some $\delta$".*

Proof. Suppose otherwise. Let $\kappa$, $\lambda$, and $\theta$ be as in the hypothesis of our theorem. Let $\delta > \theta$ be a measurable cardinal. We are assuming the following:

$(*)$ *For every stationary* $S \subseteq \mathscr{P}_\kappa H_\theta$, *there exists* $X \in [H_\theta]^\lambda$ *such that* $S \cap \mathscr{P}_\kappa X$ *is stationary in* $\mathscr{P}_\kappa X$ *and* $\lambda \subseteq X$.

We will show that $(*)$ implies the presaturation of $NS_{\kappa\lambda}$ thus deriving the desired contradiction.

We will use the following notion:

Definition. $C \subseteq \mathscr{P}_\mu H_\theta$ is *$\lambda$-club* if

(i) if $\langle a_\alpha : \alpha < \lambda \rangle$ is a $\subseteq$-chain from $C$ then $\bigcup_{\alpha<\lambda} a_\alpha \in C$,

(ii) $C$ is unbounded in $\mathscr{P}_\mu H_\theta$.

We say $T \subseteq \mathscr{P}_\mu H_\theta$ is *$\lambda$-stationary* if $T \cap C \neq \emptyset$ for every $\lambda$-club $C \subseteq \mathscr{P}_\mu H_\theta$.

By adopting the proof of Theorem 3.1 and Corollary 3.2 in Feng-Jech [**4**], we can show the following claim.

Claim 1. *Assume* $(*)$ *holds. If* $S \subseteq \mathscr{P}_\kappa H_\theta$ *is stationary then* $R(S) = \{X \in [H_\theta]^\lambda :$ $S \cap \mathscr{P}_\kappa X$ *is stationary in* $\mathscr{P}_\kappa X$ *and* $X \supseteq \lambda\}$ *is a $\lambda$-stationary subset of* $\mathscr{P}_{\lambda^+} H_\theta$.

Let $h$ be a bijection from $H_\theta$ to $\mathscr{P}_\kappa H_\theta$. Given a stationary $S \subseteq \mathscr{P}_\kappa H_\theta$, let $R(S)^* = R(S) \cap \{X \in [H_\theta]^\lambda : \forall a \in X\ h(a) \subseteq X \wedge \forall t \in \mathscr{P}_\kappa X\ h^{-1}(t) \in X\}$. So $R(S)^*$ is stationary in $\mathscr{P}_{\lambda^+} H_\theta$ for every stationary $S \subseteq \mathscr{P}_\kappa H_\theta$.

Now we carry out the arguments given in the proof of Lemma 6. Note that if for every stationary $S \subseteq \mathscr{P}_\kappa H_\theta$ there exists some generic elementary embedding $j_s : V \to M_s \cong Ult(V, G_s)$ such that $M_s \models$ "$S$ is stationary in $\mathscr{P}_\kappa H_\theta^V$", then we can carry out the same proof. Furthermore this $M_s$ need not be entirely well-founded. If $j_s(\theta) \in wfp(\langle M_s, \mathrm{E}\rangle)$ where $wfp(\langle M_s, \mathrm{E}\rangle)$ is the well-founded part of $\langle M_s, \mathrm{E}\rangle$ and $\langle M_s, \mathrm{E}\rangle \models$ "$S$ is stationary", then these arguments will survive.

In order to obtain such a generic "elementary embedding" we use Woodin's stationary tower forcing [**19**].

Let $\delta$ be a measurable cardinal $>\theta$. Suppose $S \subseteq \mathscr{P}_\kappa H_\theta$ is stationary. There is some Ramsey cardinal $\xi$ such that $\theta < \xi < \delta$. So $p = \{X \subseteq V_\xi : X \cap H_\theta \in R(S)^* \wedge |X \cap \xi| = \xi\} \in \boldsymbol{P}_{<\delta}$. We refer the reader to Woodin [**20**] for the definition of $\boldsymbol{P}_{<\delta}$. Let $G$ be a $\boldsymbol{P}_{<\delta}$-generic object containing $p$. Let $j : V \to M$ be the corresponding generic "elementary embedding." Note that $j(\xi) = \xi$. Thus all of the pertinent parts are in $wfp(M)$.

The following claim completes the proof:

Claim 2. *The function* $F_S : p \to V$ *defined by* $F_S(X) = S \cap \mathscr{P}_\kappa(X \cap H_\theta)$ *represents* $\{j''y : y \in S\}$ *in* $M$.

Proof. Clearly $\{j''y : y \in S\} \subseteq [F_S]$. Suppose $[g] \in [F_S]$. We may assume that $g : \mathscr{P}(V_\beta) \to V$ where $\xi < \beta < \delta$.

Then $q = \{X \subseteq V_\beta : X \cap V_\xi \in p \wedge g(X) \in S \cap \mathscr{P}_\kappa(X \cap H_\theta)\} \in G$. If $X \in q$ then $h^{-1}(g(X)) \in X \cap H_\theta$. Thus by the normality, there exists some $a \in H_\theta$ such that $q^* = \{X \subseteq V_\beta : X \in q \wedge h^{-1}(g(X)) = a\} \in G$. Therefore $[g] = j''h(a)$ where $h(a) \in S$.

(Claim 2) $\square$

(Theorem 9) $\square$

We conclude this paper by raising the following questions:

Question 1. *Is it consistent to have a "proper" $\lambda^+$-ideal over $\mathscr{P}_{\lambda^+}H_\theta$ for some singular $\lambda$?**

Of course by Theorem 7 a positive answer to this question implies the consistency of the precipitousness of $NS_{\aleph_1\lambda}$ for some singular $\lambda$.

Question 2. *Can we drop some conditions from the hypothesis of Theorem* 9?

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*This question was answered by S. Shelah negatively.