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On the steady flow of compressible viscous fluid and its stability with respect to initial disturbance

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Abstract. We consider a compressible viscous fluid effected by general form external force in $\boldsymbol{R}^3$. In part 1, an analysis of the linearized problem based on the weighted-L_2 method implies a condition on the external force for the existence and some regularities of the steady flow. In part 2, we study the stability of the steady flow with respect to the initial disturbance. What we proved is: if H^3-norm of the initial disturbance is small enough, then the solution to the non-stationary problem exists uniquely and globally in time.

1. Introduction.

The motion of a compressible viscous isotropic Newtonian fluid is formulated by the following initial value problem of the Navier-Stokes equation for viscous compressible fluid:

$$\begin{cases} \rho_t + \nabla \cdot (\rho v) = G(x), \\ v_t + (v \cdot \nabla) v = \dfrac{\mu}{\rho} \Delta v + \dfrac{\mu + \mu'}{\rho} \nabla (\nabla \cdot v) - \dfrac{\nabla (P(\rho))}{\rho} + F(x), \\ (\rho, v)(0, x) = (\rho_0, v_0)(x), \end{cases} \tag{1.1}$$

where $t \geq 0$, $x = (x_1, x_2, x_3) \in \boldsymbol{R}^3$; $\rho = \rho(t, x)$ (>0) and $v = (v_1(t, x), v_2(t, x), v_3(t, x))$ denote the density and velocity respectively, which are unknown; $P(\cdot)$ $(P' > 0)$ denotes the pressure; μ and μ' are the viscosity coefficients which satisfy the condition: $\mu > 0$ and $\mu' + 2\mu/3 \geq 0$; $F(x) = (F_1(x), F_2(x), F_3(x))$ is a given external force and $G(x)$ is a given mass source. The stationary problem corresponding to the initial value problem (1.1) is

$$\begin{cases} \nabla \cdot (\rho v) = G(x), \\ (v \cdot \nabla) v = \dfrac{\mu}{\rho} \Delta v + \dfrac{\mu + \mu'}{\rho} \nabla (\nabla \cdot v) - \dfrac{\nabla (P(\rho))}{\rho} + F(x), \end{cases} \tag{1.2}$$

where $x = (x_1, x_2, x_3) \in \boldsymbol{R}^3$; $\rho = \rho(x)$ (>0) and $v = (v_1(x), v_2(x), v_3(x))$ are unknown functions; $F(x), G(x)$ and the other symbols are the same as in (1.1). Here and hereafter, we use the standard notation in the vector analysis. For example, we put for scalar u, vectors $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$ and matrix $f = (f_{ij})_{1 \leq i, j \leq 3}$

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$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}, \quad \Delta v = (\Delta v_1, \Delta v_2, \Delta v_3), \quad (v \cdot \nabla) u = \sum_{i=1}^{3} v_i \frac{\partial u}{\partial x_i},$$

$$(v \cdot \nabla) w = ((v \cdot \nabla) w_1, (v \cdot \nabla) w_2, (v \cdot \nabla) w_3),$$

$$\nabla^k u = (\partial_x^\alpha u \mid |\alpha| = k), \quad \nabla^k v = (\partial_x^\alpha v_i \mid |\alpha| = k, i = 1, 2, 3),$$

$$\nabla \cdot v = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i}, \quad \nabla \cdot f = \left(\sum_{j=1}^{3} \frac{\partial f_{1j}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial f_{2j}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial f_{3j}}{\partial x_j} \right),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\partial_x^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$.

Before stating our results, we introduce some function spaces. We put for scalars u_1, u_2 and vectors $v = (v_1, \dots, v_n)$, $w = (w_1, \dots, w_n)$

$$\|u_1\|_{L_p} = \left(\int_{\boldsymbol{R}^3} |u_1(x)|^p \, dx \right)^{1/p}, \quad \|v\|_{L_p} = \left(\sum_{i=1}^{n} \|v_i\|_{L_p}^p \right)^{1/p} \quad (1 \le p < \infty),$$

$$\|u_1\|_{L_\infty} = \sup_{\boldsymbol{R}^3} |u_1(x)|, \quad \|v\|_{L_\infty} = \max_{1 \le i \le n} \|v_i(x)\|_{L\infty}, \quad (u_1, u_2) = \int_{\boldsymbol{R}^3} u_1 u_2 \, dx,$$

$$(v, w) = \sum_{i=1}^{n} (v_i, w_i), \quad \|v\|_k = \left(\sum_{0 \le \nu \le k} \|\nabla^\nu v\|^2 \right)^{1/2}, \quad \| \cdot \| = \| \cdot \|_{L_2}.$$

Let L_p denote the usual L_p space, $\mathscr{S}'$ the set of all tempered distributions both on $\boldsymbol{R}^3$ and

$$H^k = \{u \in L_{1,loc} \mid \|u\|_k < \infty\} = \{u \in \mathscr{S}' \mid \|\mathscr{F}^{-1}[(1 + |\xi|^2)^{k/2} \hat{u}]\| < \infty\},$$

$$\hat{H}^k = \{u \in L_{1,loc} \mid \nabla u \in H^{k-1}\}, \quad H^\infty = \bigcap_{k \ge 0} H^k,$$

where u is either vector or scalar. Further we put

$$\mathscr{H}^{k,\ell} = \{(\sigma, v) \mid \sigma \in H^k, v \in H^\ell\}, \quad \hat{\mathscr{H}}^{k,\ell} = \{(\sigma, v) \mid \sigma \in \hat{H}^k, v \in \hat{H}^\ell\},$$

$$\mathscr{H}^{j,k,\ell} = \{(\sigma, v, w) \mid \sigma \in H^j, v \in H^k, w \in H^\ell\},$$

and

$$\|(\sigma, v)\|_{k,\ell} = \|\sigma\|_k + \|v\|_\ell, \quad \|(\sigma, v, w)\|_{j,k,\ell} = \|\sigma\|_j + \|v\|_k + \|w\|_\ell.$$

DEFINITION 1.1.

$$I_\varepsilon^k = \{\sigma \in H^k \mid \|\sigma\|_{I^k} < \varepsilon\}, \quad J_\varepsilon^k = \{u \in \hat{H}^k \mid \|v\|_{J^k} < \varepsilon\},$$

where

$$\|\sigma\|_{I^k} = \|\sigma\|_{L_6} + \left\| \frac{\sigma}{|x|} \right\| + \sum_{\nu=1}^{k} \|(1 + |x|)^\nu \nabla^\nu \sigma\| + \|(1 + |x|)^2 \sigma\|_{L_\infty},$$

$$\|v\|_{J^k} = \|v\|_{L_6} + \left\| \frac{v}{|x|} \right\| + \sum_{\nu=1}^{k} \|(1 + |x|)^{\nu-1} \nabla^\nu v\| + \sum_{\nu=0}^{1} \|(1 + |x|)^{\nu+1} \nabla^\nu v\|_{L_\infty}.$$

Moreover we put

$$\mathscr{I}_\varepsilon^{k,\ell} = \{(\sigma, v) \,|\, \sigma \in I_\varepsilon^k, v \in J_\varepsilon^\ell\},$$

$$\dot{\mathscr{I}}_\varepsilon^{k,\ell} = \{(\sigma, v) \in \mathscr{I}_\varepsilon^{k,\ell} \,|\, \nabla \cdot v = \nabla \cdot V_1 + V_2 \text{ for some } V_1, V_2$$
$$\text{such that } \|(1+|x|)^3 V_1\|_{L_\infty} + \|(1+|x|)^{-1} V_2\|_{L_1} \le \varepsilon\},$$

$$\|(\sigma, v)\|_{\mathscr{I}^{k,\ell}} = \|\sigma\|_{I^k} + \|v\|_{J^\ell}.$$

In this paper, we consider the case where the external force F is given by following form

$$F = \nabla \cdot F_1 + F_2, \tag{1.3}$$

where $F_1 = (F_{1,ij}(x))_{1 \le i,j \le 3}$ and $F_2 = (F_{2,i}(x))_{1 \le i \le 3}$. The first theorem is concerning the existence of stationary solution of (1.2) and its weighted-L_2, L_∞ estimates.

Theorem 1.1. *Let $\bar{\rho}$ be any positive constant and $P(\cdot)$ is smooth (at least C^2) in a neighborhood of $\bar{\rho}$. Then, there exist small constants $c_0 > 0$ and $\varepsilon_0 > 0$ depending on $\bar{\rho}$ such that if (F, G) satisfies the estimate*:

$$\sum_{\nu=0}^{3} \|(1+|x|)^{\nu+1} \nabla^\nu F\| + \|(1+|x|)^3 F\|_{L_\infty} + \|(1+|x|)^2 F_1\|_{L_\infty} + \|F_2\|_{L_1}$$
$$+ \|(1+|x|) G\| + \sum_{\nu=1}^{4} \|(1+|x|)^\nu \nabla^\nu G\|$$
$$+ \sum_{\nu=0}^{1} \|(1+|x|)^{\nu+2} \nabla^\nu G\|_{L_\infty} + \|(1+|x|)^{-1} G\|_{L_1} \le c_0 \varepsilon$$

for some $\varepsilon \le \varepsilon_0$, then (1.2) admits a solution of the form: $(\rho, v) = (\bar{\rho} + \sigma, v)$ where $(\sigma, v) \in \mathscr{I}_\varepsilon^{4,5}$. Furthermore the solution is unique in the following sense:

There exists an ε_1 with $0 < \varepsilon_1 \le \varepsilon$ such that if $(\bar{\rho} + \sigma_1, v_1)$ and $(\bar{\rho} + \sigma_2, v_2)$ satisfy (1.2) with the same (F, G), and $\|(\sigma_1, v_1)\|_{\mathscr{I}^{3,4}}, \|(\sigma_1, v_1)\|_{\mathscr{I}^{3,4}} \le \varepsilon_1$, then $(\sigma_1, v_1) = (\sigma_2, v_2)$.

Next we consider the stability of the stationary solution of (1.2) with respect to initial disturbance. Let (ρ^*, v^*) be the solution of (1.2) obtained in Theorem 1.1. The stability of (ρ^*, v^*) means the solvability of the non-stationary problem (1.1). Let us introduce the class of functions which solutions of (1.1) belong to.

Definition 1.2.

$$\mathscr{C}(0, T; \mathscr{H}^{k,\ell}) = \{(\sigma, v) \,|\, \sigma(t, x) \in C^0(0, T; H^k) \cap C^1(0, T; H^{k-1}),$$
$$w(t, x) \in C^0(0, T; H^\ell) \cap C^1(0, T; H^{\ell-2})\}.$$

Then, we have the following theorem.

Theorem 1.2. *There exist $C > 0$ and $\delta > 0$ such that if $\|(\rho_0 - \rho^*, v_0 - v^*)\|_{3,3} \le \delta$ then (1.1) admits a unique solution: $(\rho, v) = (\rho^* + \sigma, v^* + w)$ globally in time, where*

$(\sigma, w) \in \mathscr{C}(0, \infty; \mathscr{H}^{3,3})$, $\nabla\sigma, w_t \in L_2(0, \infty; H^2)$, $\nabla w \in L_2(0, \infty; H^3)$. *Moreover the* (σ, w) *satisfies the estimate*:

$$\|(\sigma, w)(t)\|_{3,3}^2 + \int_0^t \|(\nabla\sigma, \nabla w, w_t)(s)\|_{2,3,2}^2 \, ds \le C\|(\rho_0 - \rho^*, v_0 - v^*)\|_{3,3}^2$$

for any $t \ge 0$.

Remark 1.1. When Theorem 1.2 holds, we shall say that the stationary solution (ρ^*, v^*) of (1.2) is stable in the H^3-framework with respect to small initial disturbance.

Matsumura and Nishida [**13**], [**14**] first proved the stability of a constant state $(\bar{\rho}, 0)$ in H^3-framework with respect to initial disturbance, namely they proved Theorem 1.2 in the case where $(\rho^*, v^*) = (\bar{\rho}, 0)$. When the external force is given by the potential: $F = -\nabla\Phi$, $F_2 = G = 0$ in (1.2) and (1.3), where Φ is a scalar function, the stationary solution $(\rho^*, v^*)(x)$ of (1.2) in a neighborhood of $(\bar{\rho}, 0)$ in $\mathscr{H}^{2,2}$ has the form:

$$\int_{\bar{\rho}}^{\rho^*(x)} \frac{P'(\eta)}{\eta} \, d\eta + \Phi(x) = 0, \quad v^*(x) = 0.$$

In this case, Matsumura and Nishida [**15**] proved the stability of $(\rho^*(x), 0)$ in the H^3-framework with respect to initial disturbance in an exterior domain. Recently, even in the large potential force case, the stability theorem was studied by Feireisl and Petzeltová [**7**]; Matsumura and Yamagata [**17**] within a class of weak solutions.

The purpose of this paper is to consider the case where the external force is given by the general formula (1.3) and also mass source G appears. In this case, the stationary solution $(\rho^*, v^*)(x)$ is non-trivial in general, especially $v^* \not\equiv 0$. We are interested only in strong solutions. Then, when F is small enough in a certain norm and $G = 0$, Matsumura and Nishida [**16**]; Novotny and Padula [**19**] proved a unique existence theorem of solutions to (1.2) in an exterior domain. In the proof of Novotny and Padula [**19**], they decomposed the equations into the Stokes equation, transport equation and Laplace equation. Since we consider the problem in $\boldsymbol{R}^3$, that is, the boundary condition is not imposed, we can solve (1.2) without any such decomposition technique. In fact, in §2 we establish the corresponding linear theory to (1.2) in the L_2-framework by the usual Banach closed range theorem, after obtaining some weighted-L_2 estimates for solutions. On the other hand, Matsumura and Nishida [**16**] used a regularization method. More precisely, for an approximation of the linearized equation of the continuum equation: $\nabla \cdot v + (a \cdot \nabla)\sigma = g$, they used not only $\varepsilon(1 - \Delta)\sigma$ but also $\varepsilon'\sigma^3$ to control $(a \cdot \nabla)\sigma$ and they consider the limit $\varepsilon \downarrow 0$ and $\varepsilon' \downarrow 0$ after obtaining the suitable estimates. But in this paper we only use $\varepsilon(1 - \Delta)\sigma$.

Another purpose of this paper is to prove the stability of the stationary solution $(\rho^*, v^*)(x)$ of (1.2) in H^3-framework. Matsumura and Nishida [**16**] mentioned a possibility of proof of the stability, but they did not give any proof. We shall give a proof of the stability of $(\rho^*, v^*)(x)$ in §3. The main step of our proof of Theorem 1.2 is to obtain a priori estimates for solutions of (1.1) as usual. We shall obtain a priori estimates by choosing several multipliers and using the integration by parts. Compared with the case where $v^* = 0$, we have to give more consideration to choice of multipliers.

Recently, Benabidallah [**1**], [**2**], Kawashita [**9**] and Danchin [**3**], [**4**] considered the optimal class of initial data regarding the regularity. We think that our result will be improved in this direction.

It is interesting to investigate the convergence rate of non-stationary solutions when time goes to infinity. So far, this problem has been studied when $F = G = 0$ or $F = -\nabla\Phi$ and $F_2 = G = 0$. In fact, Ponce [**20**]; Deckelnick [**5**], [**6**]; Hoff and Zumbrun [**8**]; Wang [**21**]; Kobayashi and Shibata [**12**]; Kobayashi [**10**], [**11**] obtained a rate of convergence. These works have been done in the small strong solution case. But to obtain the rate of convergence in our case, we need more delicate argument to treat the term: $(v^* \cdot \nabla)w(t)$ in (3.4). We consider this problem in a fore-coming paper.

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2. Stationary problem.

We study the stationary problem (1.2). Take any constant $\bar\rho > 0$. Substituting $\rho = \bar\rho + \sigma$ into (1.2) and putting $\gamma = P'(\bar\rho)$, (1.2) is reduced to the equation:

$$\begin{cases} \nabla\cdot v + \left(\dfrac{v}{\bar\rho+\sigma}\cdot\nabla\right)\sigma = \dfrac{G}{\bar\rho+\sigma}, \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma = -(\bar\rho+\sigma)(v\cdot\nabla)v \\ \qquad - [P'(\bar\rho+\sigma) - P'(\bar\rho)]\nabla\sigma + (\bar\rho+\sigma)F. \end{cases} \tag{2.1}$$

Our goal in this part is to prove Theorem 1.1 by application of weighted-L_2 method to the linearized problem for (2.1).

2.1. Weighted-L_2 theory for linearized problem.

In this section, let k be an integer fixed to $k = 3$ or $k = 4$. We shall consider the linearized equation of (2.1):

$$\begin{cases} \nabla\cdot v + (a\cdot\nabla)\sigma = g, & (2.2) \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma = -(b\cdot\nabla)c + f, & (2.3) \end{cases}$$

where $a = (a_1(x), a_2(x), a_3(x))$, $b = (b_1(x), b_2(x), b_3(x))$, $c = (c_1(x), c_2(x), c_3(x))$ and $(f, g) \in \mathscr{H}^{k-1,k}$ are given. Throughout this section, we assume that

$$a \in \hat H^4, \quad \|(1+|x|)a\|_{L_\infty} + \sum_{\nu=1}^{4} \|(1+|x|)^{\nu-1}\nabla^\nu a\| \le \delta, \quad b, c \in J_\delta^{k+1}, \tag{2.4}$$

$$\sum_{\nu=0}^{k-1} \|(1+|x|)^{\nu+1}\nabla^\nu f\| + \|(1+|x|)g\| + \sum_{\nu=1}^{k} \|(1+|x|)^\nu \nabla^\nu g\| < \infty. \tag{2.5}$$

2.1.1. Solution to approximate problem.

First, we solve the approximate problem:

$$\begin{cases} \nabla\cdot v + (a\cdot\nabla)\sigma - \varepsilon\Delta\sigma + \varepsilon\sigma = g, & (2.6) \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma + \varepsilon v = h & (2.7) \end{cases}$$

in $\mathscr{H}^{2,2}$, where h is defined by

$$h = -(b \cdot \nabla)c + f.$$

It immediately follows from (2.4) and the Sobolev inequality that

$$\|(1+|x|)h\| + \|\nabla h\|_{k-2} \leq C\left[\delta^2 + \sum_{\nu=0}^{k-1} \|(1+|x|)^{\nu+1}\nabla^\nu f\|\right] < \infty. \tag{2.8}$$

In the next lemma, we shall prove some fundamental a priori estimate needed later.

Lemma 2.1. *There exists* $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ *such that if* δ *in* (2.4) *satisfies* $\delta \leq \delta_0$ *then we have the following estimates*:

(i) *If* $0 < \varepsilon \leq 1$ *and* $(\sigma, v) \in \mathscr{H}^{2,2}$ *is a solution to* (2.6)–(2.7), *then*

$$\|\nabla v\|_1^2 + \|\nabla\sigma\|^2 + \varepsilon\{\|v\|^2 + \|\sigma\|^2 + \|\nabla^2\sigma\|^2\} \leq C\varepsilon^{-1}\|(h,g)\|^2. \tag{2.9}$$

(ii) *If* $0 \leq \varepsilon \leq 1$ *and* $(\sigma, v) \in \mathscr{H}^{2,2}$ *is a solution to* (2.6)–(2.7), *then*

$$\|(\nabla\sigma, \nabla^2 v)\| \leq C\{\|v\| + \|(h, \nabla g)\|\}. \tag{2.10}$$

Here, $C > 0$ *is a constant depending only on* μ, μ' *and* γ.

Proof. (i) Multiplying (2.6) and (2.7) by σ and v respectively; using integration by parts, we have

$$(h, v) = \mu\|\nabla v\|^2 + (\mu + \mu')\|\nabla \cdot v\|^2 + \gamma(\nabla\sigma, v) + \varepsilon\|v\|^2,$$

$$(g, \sigma) = -(v, \nabla\sigma) + (a \cdot \nabla\sigma, \sigma) + \varepsilon\|\nabla\sigma\|^2 + \varepsilon\|\sigma\|^2.$$

Canceling the term of $(\nabla\sigma, v)$ in the above two relations, we obtain

$$\mu\|\nabla v\|^2 + \varepsilon\gamma\|\sigma\|^2 + \varepsilon\|v\|^2 \leq \gamma|(a \cdot \nabla\sigma, \sigma)| + |(h, v)| + \gamma|(g, \sigma)|. \tag{2.11}$$

Differentiating (2.6) and (2.7), and employing the same argument, we have

$$\mu\|\nabla^2 v\|^2 + \varepsilon\gamma\|\nabla^2\sigma\|^2 \leq \gamma|(\nabla(a \cdot \nabla\sigma), \nabla\sigma)| + |(\nabla h, \nabla v)| + \gamma|(\nabla g, \nabla\sigma)|. \tag{2.12}$$

Adding (2.11) and (2.12), we have

$$\mu\|\nabla v\|_1^2 + \varepsilon\{\|v\|^2 + \gamma\|\sigma\|^2 + \gamma\|\nabla^2\sigma\|^2\}$$
$$\leq \sum_{\nu=0}^{1}[\gamma|(\nabla^\nu(a \cdot \nabla\sigma), \nabla^\nu\sigma)| + |(\nabla^\nu h, \nabla^\nu v)| + \gamma|(\nabla^\nu g, \nabla^\nu\sigma)|]. \tag{2.13}$$

Since

$$\|\nabla\sigma\|^2 \leq C_{\gamma,\mu,\mu'}\{\|\nabla^2 v\|^2 + \varepsilon\|v\|^2 + \|h\|^2\} \tag{2.14}$$

as follows from (2.7), it follows from (2.13) that

$$\|\nabla v\|_1^2 + \|\nabla\sigma\|^2 + \varepsilon\{\|v\|^2 + \|\sigma\|^2 + \|\nabla^2\sigma\|^2\}$$
$$\leq C_1 \sum_{\nu=0}^{1} |(\nabla^\nu(a \cdot \nabla\sigma), \nabla^\nu\sigma)|$$
$$+ C_2\left[\|h\|^2 + \sum_{\nu=0}^{1}\{|(\nabla^\nu h, \nabla^\nu v)| + |(\nabla^\nu g, \nabla^\nu\sigma)|\}\right] \equiv I_1 + I_2, \tag{2.15}$$

where the constants $C_j > 0$ $(j = 1, 2)$ depend only on μ, μ' and γ. Now, integration by parts and the Hardy inequality imply that

$$I_1 \le C_1\left[\left|\left(|x|a\cdot\nabla\sigma, \frac{\sigma}{|x|}\right)\right| + \sum_{i=1}^{3}\left\{\left|\left(\frac{\partial a}{\partial x_i}\cdot\nabla\sigma, \frac{\partial\sigma}{\partial x_i}\right)\right| + \frac{1}{2}\left|\left((\nabla\cdot a)\frac{\partial\sigma}{\partial x_i}, \frac{\partial\sigma}{\partial x_i}\right)\right|\right\}\right]$$
$$\le C_3\{\|(1+|x|)a\|_{L_\infty} + \|\nabla a\|_{L_\infty}\}\|\nabla\sigma\|^2$$
$$\le C_3\delta\|\nabla\sigma\|^2 \quad (C_3 \text{ depends only on } \mu, \mu' \text{ and } \gamma), \tag{2.16}$$

whereas

$$I_2 \le C_2\|h\|^2 + \frac{1}{2}\{\varepsilon\|v\|^2 + \|\nabla^2 v\|^2 + \varepsilon\|\sigma\|^2 + \varepsilon\|\nabla^2\sigma\|^2\}$$
$$+ \frac{C_2^2}{2}\{\varepsilon^{-1}\|h\|^2 + \|h\|^2 + \varepsilon^{-1}\|g\|^2 + \varepsilon^{-1}\|g\|^2\}. \tag{2.17}$$

Combining (2.15)–(2.17), we have (2.9) if $\delta \le 1/4C_3$.

(ii) Using the Friedrichs mollifier (cf. Mizohata [**18**], p. 313), we may assume that $(\sigma, v) \in \mathscr{H}^{\infty,\infty}$. Employing the same argument as in the beginning of proof for (i), we have (2.12) and (2.14). Adding (2.12) and (2.14), we have

$$\|(\nabla\sigma, \nabla^2 v)\|^2 \le C_1[\|v\|^2 + \|h\|^2 + |(\nabla(a\cdot\nabla\sigma), \nabla\sigma)|$$
$$+ \{|(\nabla h, \nabla v)| + |(\nabla g, \nabla\sigma)|\}] \equiv C_1\{\|v\|^2 + \|h\|^2 + I_1 + I_2\}, \tag{2.18}$$

where the constant $C_1 > 0$ depends only on μ, μ' and γ. By the same calculation as in (2.16)

$$I_1 \le C_2\delta\|\nabla\sigma\|^2 \quad (C_2 \text{ depends only on } \mu, \mu' \text{ and } \gamma), \tag{2.19}$$

whereas integration by parts implies that

$$I_2 \le \frac{1}{2C_1}\{\|\nabla^2 v\|^2 + \|\nabla\sigma\|^2\} + \frac{C_1}{2}\{\|h\|^2 + \|\nabla g\|^2\}. \tag{2.20}$$

Combining (2.18)–(2.20), we have (2.10) if $\delta \le 1/4C_2$. □

Now, we employ the closed range theorem to prove the existence of solution. We introduce the operator A defined on $D(A) \subset L_2$ into L_2 by

$$A(\sigma, v) = (A_1(\sigma, v), A_2(\sigma, v)),$$

where $D(A) = \mathscr{H}^{2,2}$ and

$$A_1(\sigma, v) = \nabla\cdot v + (a\cdot\nabla)\sigma - \varepsilon\Delta\sigma + \varepsilon\sigma,$$
$$A_2(\sigma, v) = -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma + \varepsilon v.$$

Clearly A is closed operator. Furthermore, Lemma 2.1 (i) implies that for each $0 < \varepsilon \le 1$ the range of A is closed.

Proposition 2.1. *There exists* $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ *such that if* δ *in* (2.4) *satisfies* $\delta \le \delta_0$ *then for* $0 < \varepsilon \le 1$, (2.6)–(2.7) *has a solution* $(\sigma, v) \in \mathscr{H}^{2,2}$, *which satisfies*

$$\|(\sigma, v)\|_{2,2} \le C(\varepsilon)\|(h, g)\|, \tag{2.21}$$

where the constant $C(\varepsilon)$ *depends on* $\mu, \mu', \gamma, \varepsilon$ *and* $C(\varepsilon) \to \infty$ *as* $\varepsilon \downarrow 0$.

Proof. Let $(\theta, w) \in \mathscr{H}^{2,2}$. Then for any $(\sigma, v) \in \mathscr{H}^{2,2}$, integration by parts implies that

$$\begin{aligned}(A(\sigma, v), (\theta, w)) &= (\nabla \cdot v, \theta) + (a \cdot \nabla\sigma, \theta) - \varepsilon(\Delta\sigma, \theta) + \varepsilon(\sigma, \theta) \\ &\qquad - \mu(\Delta v, w) - (\mu + \mu')(\nabla(\nabla \cdot v), w) + \gamma(\nabla\sigma, w) + \varepsilon(v, w) \\ &= (\sigma, -\gamma\nabla \cdot w - \nabla \cdot (a\theta) - \varepsilon\Delta\theta + \varepsilon\theta) \\ &\qquad + (v, -\mu\Delta w - (\mu + \mu')\nabla(\nabla \cdot w) - \nabla\theta + \varepsilon w).\end{aligned}$$

Therefore $\mathscr{H}^{2,2} \subset D(A^*)$ and for $(\theta, w) \in \mathscr{H}^{2,2}$, $A^*(\theta, w) = (A_1^*(\theta, w), A_2^*(\theta, w))$ where

$$\begin{aligned}A_1^*(\theta, w) &= -\gamma\nabla \cdot w - \nabla \cdot (a\theta) - \varepsilon\Delta\theta + \varepsilon\theta, \\ A_2^*(\theta, w) &= -\mu\Delta w - (\mu + \mu')\nabla(\nabla \cdot w) - \nabla\theta + \varepsilon w.\end{aligned}$$

Employing the same argument as in the proof of Lemma 2.1 (i), we have

$$\|\nabla w\|_1^2 + \|\nabla\theta\|^2 + \varepsilon\{\|w\|^2 + \|\theta\|^2 + \|\nabla^2\theta\|^2\} \le C\varepsilon^{-1}\|A^*(\theta, w)\|^2$$

for $(\theta, w) \in \mathscr{H}^{2,2}$. Hence the closed range theorem implies the existence of solution. The estimate (2.21) is given by (2.9) directly. □

By the regularity theorem of the properly elliptic operator, we have

Corollary 2.1. *Let* $(\sigma, v) \in \mathscr{H}^{2,2}$ *be solution to* (2.6)–(2.7) *obtained in Proposition* 2.1. *Then* $(\sigma, v) \in \mathscr{H}^{k+1,k+1}$ *and*

$$\|(\sigma, v)\|_{k+1,k+1} \le C(\varepsilon)\|(h, g)\|_{k-1,k-1}, \tag{2.22}$$

where the constant $C(\varepsilon) > 0$ *depends on* $\mu, \mu', \gamma, \varepsilon$ *and* $C(\varepsilon) \to \infty$ *as* $\varepsilon \downarrow 0$.

2.1.2. Solution to linearized problem (2.2)–(2.3) and its L_2 estimate.

Next, we shall discuss the estimate for solution to (2.6)–(2.7) independent of $0 < \varepsilon \le 1$.

Lemma 2.2. *Let* $0 < \varepsilon \le 1$ *and* $(\sigma, v) \in \mathscr{H}^{k+1,k+1}$ *be solution to* (2.6)–(2.7) *which satisfies* (2.22). *Then, there exists* $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ *such that if* δ *in* (2.4) *satisfies* $\delta \le \delta_0$, *then we have the estimate*:

$$\|(\nabla\sigma, \nabla v)\|_{k-1,k} \le C\{\|(1 + |x|)(h, g)\| + \|(\nabla h, \nabla g)\|_{k-2,k-1}\}, \tag{2.23}$$

where the constant C *depends only on* μ, μ' *and* γ.

Proof. By aid of the Friedrichs mollifier, we may assume that $(\sigma, v) \in \mathscr{H}^{\infty,\infty}$. The same argument as in the proof of Lemma 2.1 (i) implies that

$$\|\nabla v\|_1^2 + \|\nabla\sigma\|^2 \le C\left[\|h\|^2 + \sum_{\nu=0}^{1}\{|(\nabla^\nu h, \nabla^\nu v)| + |(\nabla^\nu g, \nabla^\nu\sigma)|\}\right].$$

For the right hand side, using the Hardy inequality, we have

$$\sum_{\nu=0}^{1}\{|(\nabla^\nu h,\nabla^\nu v)|+|(\nabla^\nu g,\nabla^\nu \sigma)|\}\le \frac{1}{2C}\{\|\nabla v\|_1^2+\|\nabla\sigma\|^2\}$$
$$+C'\{\|(1+|x|)h\|^2+\|(1+|x|)g\|^2+\|\nabla g\|^2\}.$$

So we obtain

$$\|(\nabla\sigma,\nabla v)\|_{0,1}\le C\{\|(1+|x|)(h,g)\|+\|\nabla g\|\}, \tag{2.24}$$

where the constant C depends only on μ,μ' and γ.

Moreover, for any multi-index α with $1\le|\alpha|\le k-1$, applying ∂_x^α to (2.6)–(2.7), we have

$$\begin{cases}\nabla\cdot\partial_x^\alpha v+(a\cdot\nabla)\partial_x^\alpha\sigma-\varepsilon\Delta\partial_x^\alpha\sigma+\varepsilon\partial_x^\alpha\sigma=\partial_x^\alpha g-I_\alpha,\\ -\mu\Delta\partial_x^\alpha v-(\mu+\mu')\nabla(\nabla\cdot\partial_x^\alpha v)+\gamma\nabla\partial_x^\alpha\sigma+\varepsilon\partial_x^\alpha v=\partial_x^\alpha h,\end{cases} \tag{2.25}$$

where I_α is defined by the formula:

$$I_\alpha=\sum_{\beta<\alpha}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}a\cdot\nabla)\partial_x^\beta\sigma.$$

Employing Lemma 2.1 (ii) for (2.25), we have

$$\|(\nabla\partial_x^\alpha\sigma,\nabla^2\partial_x^\alpha v)\|\le C\{\|\partial_x^\alpha v\|+\|(\partial_x^\alpha h,\nabla(\partial_x^\alpha g-I_\alpha))\|\},$$

where the constant $C>0$ depends only on μ,μ' and γ. Since

$$\|\nabla I_\alpha\|\le C\delta\|\nabla\sigma\|_{|\alpha|}$$

as follows from (2.4) and by the Sobolev inequality, we obtain

$$\|(\nabla^{|\alpha|+1}\sigma,\nabla^{|\alpha|+2}v)\|\le C\{\|\nabla^{|\alpha|}v\|+\|\nabla\sigma\|_{|\alpha|-1}+\|(\nabla^{|\alpha|}h,\nabla^{|\alpha|+1}g)\|\}, \tag{2.26}$$

if $\delta>0$ is small enough. Combining (2.24) and (2.26), we obtain (2.23). □

Now using (2.23), we shall show the existence of solution to linearized problem (2.2)–(2.3).

Proposition 2.2. *There exists $\delta_0=\delta_0(\gamma,\mu,\mu')>0$ such that if δ in* (2.4) *satisfies $\delta\le\delta_0$ then for $0<\lambda\le1$,* (2.2)–(2.3) *admits a solution $(\sigma,v)\in\hat{\mathscr{H}}^{k,k+1}$ which satisfies the estimate*:

$$\|(\sigma,v)\|_{L_6}+\left\|\frac{(\sigma,v)}{|x|}\right\|+\|(\nabla\sigma,\nabla v)\|_{k-1,k}\le C\{\|(1+|x|)(h,g)\|+\|(\nabla h,\nabla g)\|_{k-2,k-1}\}, \tag{2.27}$$

where the constant $C>0$ depends only on μ,μ' and γ.

Proof. We put

$$K=\|(1+|x|)(h,g)\|+\|(\nabla h,\nabla g)\|_{k-2,k-1}.$$

From Proposition 2.1, Corollary 2.1 and Lemma 2.2, it follows that for each $0<\varepsilon\le1$, (2.6)–(2.7) admits a solution $(\sigma^\varepsilon,v^\varepsilon)\in\mathscr{H}^{k+1,k+1}$ such that

$$\|(\nabla\sigma^\varepsilon,\nabla v^\varepsilon)\|_{k-1,k}\le CK.$$

The Gagliard-Nirenberg inequality and the Hardy inequality imply that

$$\|(\sigma^\varepsilon, v^\varepsilon)\|_{L_6} + \left\|\frac{(\sigma^\varepsilon, v^\varepsilon)}{|x|}\right\| \le C\|(\nabla\sigma^\varepsilon, \nabla v^\varepsilon)\| \le CK.$$

Choosing an appropriate subsequence, there exist $(\sigma, v) \in L_6$, $(\theta, w) \in L_2$, $(\theta^i, w^i) \in \mathscr{H}^{k-1,k}$ such that

$$(\sigma^\varepsilon, v^\varepsilon) \rightharpoonup (\sigma, v) \text{ weakly in } L_6, \quad \frac{(\sigma^\varepsilon, v^\varepsilon)}{|x|} \rightharpoonup (\theta, w) \text{ weakly in } L_2,$$

$$\left(\frac{\partial\sigma^\varepsilon}{\partial x_i}, \frac{\partial v^\varepsilon}{\partial x_i}\right) \rightharpoonup (\theta^i, w^i) \text{ weakly in } \mathscr{H}^{k-1,k}$$

as $\varepsilon \downarrow 0$. Then we can easily check that

$$(\sigma, v) = (|x|\theta, |x|w), \quad \left(\frac{\partial\sigma}{\partial x_i}, \frac{\partial v}{\partial x_i}\right) = (\theta^i, w^i),$$

$$\|(\sigma, v)\|_{L_6} + \left\|\frac{(\sigma, v)}{|x|}\right\| + \|(\nabla\sigma, \nabla v)\|_{k-1,k} \le CK.$$

On the other hand, we have

$$\nabla\cdot v^\varepsilon + (a\cdot\nabla)\sigma^\varepsilon - \varepsilon\Delta\sigma^\varepsilon + \varepsilon\sigma^\varepsilon \to \nabla\cdot v + (a\cdot\nabla)\sigma,$$

$$-\mu\Delta v^\varepsilon - (\mu+\mu')\nabla(\nabla\cdot v^\varepsilon) + \gamma\nabla\sigma^\varepsilon + \varepsilon v^\varepsilon \to -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma$$

in distribution sense. This completes the proof of Proposition 2.2. □

2.1.3. Weighted-L_2 estimate for solution to the linearized equation (2.2)–(2.3).

At last, we shall give weighted-L_2 estimate for the solution to (2.2)–(2.3).

Lemma 2.3. *Let $(\sigma, v) \in \hat{\mathscr{H}}^{k,k+1}$ be solution to* (2.2)–(2.3) *which satisfies* (2.27). *Then, there exists $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ such that if δ in* (2.4) *satisfies $\delta \le \delta_0$ then for any integer with $1 \le \ell \le k$, we have the estimate*:

$$\sum_{\nu=1}^{\ell} \|(1+|x|)^\nu(\nabla^\nu\sigma, \nabla^{\nu+1}v)\| \le C\left[\|b\|_{J^{k+1}}\|c\|_{J^{k+1}} + \|\nabla v\| + \sum_{\nu=1}^{\ell}\|(1+|x|)^\nu(\nabla^{\nu-1}f, \nabla^\nu g)\|\right], \tag{2.28}$$

where C is a constant depending only on μ, μ' and γ.

Proof. Let $(\sigma, v) \in \hat{\mathscr{H}}^{k,k+1}$ be a solution to (2.2)–(2.3) satisfying (2.27). We shall prove the lemma by induction on ℓ. Let ℓ be any integer with $1 \le \ell \le k$ and if $\ell \ge 2$, we assume that

$$\sum_{\nu=1}^{\ell-1} \|(1+|x|)^\nu(\nabla^\nu\sigma, \nabla^{\nu+1}v)\| \le C\left[\|b\|_{J^{k+1}}\|c\|_{J^{k+1}} + \|\nabla v\| + \sum_{\nu=1}^{\ell-1}\|(1+|x|)^\nu(\nabla^{\nu-1}f, \nabla^\nu g)\|\right]. \tag{2.29}$$

Using the Fiedrichs mollifier and a cut-off function, we may assume that $(\sigma, v) \in C_0^\infty(\boldsymbol{R}^3)$. For any multi-index α with $|\alpha| = \ell$, applying ∂_x^α to (2.3) and multiplying $(1+|x|)^{2\ell}\partial_x^\alpha v$, we have (after using integration by parts)

$$\mu\|(1+|x|)^\ell \nabla \partial_x^\alpha v\|^2 + \mu\left(\nabla \partial_x^\alpha v, 2\ell(1+|x|)^{2\ell-1}\frac{x}{|x|}\partial_x^\alpha v\right)$$
$$+ (\mu+\mu')\|(1+|x|)^\ell \nabla\cdot \partial_x^\alpha v\|^2 + (\mu+\mu')\left(\nabla\cdot\partial_x^\alpha v, 2\ell(1+|x|)^{2\ell-1}\frac{x}{|x|}\cdot\partial_x^\alpha v\right)$$
$$+ \gamma(\nabla\partial_x^\alpha \sigma, (1+|x|)^{2\ell}\partial_x^\alpha v) = (\partial_x^\alpha\{-(b\cdot\nabla)c + f\}, (1+|x|)^{2\ell}\partial_x^\alpha v).$$

Also applying ∂_x^α to (2.2) and multiplying $(1+|x|)^{2\ell}\partial_x^\alpha \sigma$, we have (after using integration by parts)

$$-\left(\partial_x^\alpha v, (1+|x|)^{2\ell}\nabla\partial_x^\alpha\sigma + 2\ell(1+|x|)^{2\ell-1}\frac{x}{|x|}\partial_x^\alpha \sigma\right)$$
$$+ (\partial_x^\alpha\{(a\cdot\nabla)\sigma\}, (1+|x|)^{2\ell}\partial_x^\alpha\sigma) = (\partial_x^\alpha g, (1+|x|)^{2\ell}\partial_x^\alpha\sigma).$$

Summing up the above two relations, canceling the term of $(\nabla\partial_x^\alpha\sigma, (1+|x|)^{2\ell}\partial_x^\alpha v)$ and taking summation with respect to α, we obtain

$$\begin{aligned}\|(1+|x|)^\ell\nabla^{\ell+1}v\|^2 \le C[&(|\nabla^{\ell+1}v|, (1+|x|)^{2\ell-1}|\nabla^\ell v|)\\ &+ (|\nabla^\ell v|, (1+|x|)^{2\ell-1}|\nabla^\ell\sigma|) + |(\nabla^\ell(a\cdot\nabla\sigma), (1+|x|)^{2\ell}\nabla^\ell\sigma)|\\ &+ |(\nabla^\ell f, (1+|x|)^{2\ell}\nabla^\ell v)| + |(\nabla^\ell g, (1+|x|)^{2\ell}\nabla^\ell\sigma)|\\ &+ |(\nabla^\ell\{(b\cdot\nabla)c\}, (1+|x|)^{2\ell}\nabla^\ell v)|],\end{aligned} \tag{2.30}$$

where the constant C depends only on μ, μ' and γ. Since

$$\begin{aligned}\|(1+|x|)^\ell\nabla^\ell\sigma\|^2 \le C_{\gamma,\mu,\mu'}[&\|(1+|x|)^\ell\nabla^{\ell+1}v\|^2\\ &+ \|(1+|x|)^\ell\nabla^{\ell-1}f\|^2 + |(\nabla^{\ell-1}\{(b\cdot\nabla)c\}, (1+|x|)^{2\ell}\nabla^\ell\sigma)|].\end{aligned}$$

as follows from (2.3), combining this with (2.30), we have

$$\begin{aligned}\|(1+|x|)^\ell(\nabla^{\ell+1}v, \nabla^\ell\sigma)\|^2 &\le C_1|(\nabla^\ell(a\cdot\nabla\sigma), (1+|x|)^{2\ell}\nabla^\ell\sigma)|\\ &\quad + C_2[\|(1+|x|)^{\ell-1}\nabla^\ell v\|^2 + \|(1+|x|)^\ell\nabla^{\ell-1}f\|^2\\ &\qquad + |(\nabla^\ell f, (1+|x|)^{2\ell}\nabla^\ell v)| + |(\nabla^\ell g, (1+|x|)^{2\ell}\nabla^\ell\sigma)|]\\ &\quad + C_3|(\nabla^\ell\{(b\cdot\nabla)c\}, (1+|x|)^{2\ell}\nabla^\ell v)|\\ &\quad + C_4|(\nabla^{\ell-1}\{(b\cdot\nabla)c\}, (1+|x|)^{2\ell}\nabla^\ell\sigma)|\\ &\equiv I_1 + I_2 + I_3 + I_4,\end{aligned} \tag{2.31}$$

where the constants C_j $(j = 1, 2, 3)$ depend only on μ, μ' and γ.

Now, we estimate the right hand side of (2.31) respectively. Integration by parts and the Sobolev inequality imply that

$$I_1 \le C\varepsilon \sum_{\nu=1}^{\ell} \|(1+|x|)^{\nu}\nabla^{\nu}\sigma\|^2 \text{ in the same way as in (2.16),}$$

$$\begin{aligned} I_2 \le\ & \frac{1}{5}\|(1+|x|)^{\ell}(\nabla^{\ell}\sigma, \nabla^{\ell+1}v)\|^2 \\ & + C\{\|(1+|x|)^{\ell-1}\nabla^{\ell}v\|^2 + \|(1+|x|)^{\ell}(\nabla^{\ell-1}f, \nabla^{\ell}g)\|^2\}. \end{aligned} \tag{2.32}$$

Moreover, noting that

$$\|(1+|x|)^{|\alpha|+|\beta|+1}|\partial^{\alpha}b|\,|\partial^{\beta}c|\,\| \le C\|b\|_{J^{k+1}}\|c\|_{J^{k+1}}, \tag{2.33}$$

for multi-index α, β with $|\alpha| \le 1$ or $|\beta| \le 1$ (and $|\alpha|, |\beta| \le k+1$), we can show that

$$\begin{aligned} I_3 + I_4 \le\ & \frac{1}{5}\|(1+|x|)^{\ell}(\nabla^{\ell}\sigma, \nabla^{\ell+1}v)\|^2 \\ & + C\begin{cases} \|(1+|x|)^{\ell-1}\nabla^{\ell}v\|^2 + \|b\|_{J^{k+1}}^2\|c\|_{J^{k+1}}^2 & \ell = 1,2,3, \\ \|(1+|x|)^3(\nabla^3\sigma, \nabla^4 v)\|^2 + \|b\|_{J^{k+1}}^2\|c\|_{J^{k+1}}^2 & \ell = 4. \end{cases} \end{aligned} \tag{2.34}$$

Indeed, I_3 is estimated as follows: If $\ell = 1$ or 2, since $(1+|x|)^{\ell+1}\nabla^{\ell}\{(b\cdot\nabla)c\} \in L_2$ as follows from (2.33), we have

$$I_3 \le C\{\|(1+|x|)^{\ell-1}\nabla^{\ell}v\|^2 + \|b\|_{J^{k+1}}^2\|c\|_{J^{k+1}}^2\}.$$

If $\ell = 3$ or $\ell = 4$, reforming I_3 into the following two parts:

$$\begin{aligned} I_3 =\ & C_3 \sum_{|\alpha|=\ell}\left(\sum_{\substack{\beta\le\alpha \\ |\beta|=\ell-2}}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}b\cdot\nabla)\partial_x^{\beta}c + \sum_{\substack{\beta\le\alpha \\ |\beta|=1}}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}b\cdot\nabla)\partial_x^{\beta}c, (1+|x|)^{2\ell}\partial_x^{\alpha}v\right) \\ & + C_3\sum_{|\alpha|=\ell}\left(\left\{\sum_{\substack{\beta\le\alpha \\ |\beta|=0}} + \sum_{\substack{\beta\le\alpha \\ |\beta|=\ell-1}} + \sum_{\substack{\beta\le\alpha \\ |\beta|=\ell}}\right\}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}b\cdot\nabla)\partial_x^{\beta}c, (1+|x|)^{2\ell}\partial_x^{\alpha}v\right) \\ \equiv\ & I_{31} + I_{32}. \end{aligned} \tag{2.35}$$

Using integration by parts for I_{31}, we have

$$\begin{aligned} I_{31} \le\ & C\|(1+|x|)^2\nabla b\|_{L_\infty}\Bigg[\|(1+|x|)^{\ell-2}\nabla^{\ell-1}c\|\,\|(1+|x|)^{\ell}\nabla^{\ell+1}v\| \\ & + \sum_{\nu=\ell-2}^{\ell-1}\|(1+|x|)^{\nu}\nabla^{\nu+1}c\|\,\|(1+|x|)^{\ell-1}\nabla^{\ell}v\|\Bigg] \\ & + (\text{the same term except for the exchange of } b \text{ and } c) \\ \le\ & \frac{1}{5}\|(1+|x|)^{\ell}\nabla^{\ell+1}v\|^2 + C\{\|(1+|x|)^{\ell-1}\nabla^{\ell}v\|^2 + \|b\|_{J^{k+1}}^2\|c\|_{J^{k+1}}^2\}, \end{aligned}$$

and for I_{32} we can use (2.33) directly as in the case $\ell = 1$ or 2,

$$I_{32} \le C\{\|(1+|x|)^{\ell-1}\nabla^{\ell} v\|^2 + \|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}}\},$$

where the constant C depends only on μ, μ' and γ. Further, as for I_4: If $\ell = 1, 2$, or 3, since $(1+|x|)^{\ell}\nabla^{\ell-1}\{(b\cdot\nabla)c\} \in L_2$ as follows from (2.33), we have

$$I_4 \le \frac{1}{5}\|(1+|x|)^{\ell}\nabla^{\ell}\sigma\|^2 + C\|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}}.$$

If $\ell = 4$, integration by parts implies that

$$I_4 \le C_4 \sum_{|\alpha|=3}\left[|(\nabla\cdot\partial_x^{\alpha}\{(b\cdot\nabla)c\}, (1+|x|)^8\partial_x^{\alpha}\sigma)| + \left|\left(\partial_x^{\alpha}\{(b\cdot\nabla)c\}, 8(1+|x|)^7\frac{x}{|x|}\partial_x^{\alpha}\sigma\right)\right|\right].$$

Then, decomposing each term as in (2.35) (the first term same as I_3 with $\ell = 4$ and the second term same as I_3 with $\ell = 3$) and using integration by parts, we have

$$I_4 \le \frac{1}{5}\|(1+|x|)^4\nabla^4\sigma\|^2 + C\{\|(1+|x|)^3\nabla^3\sigma\|^2 + \|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}}\},$$

where the constant C depends only on μ, μ' and γ.

Combining (2.31), (2.32), (2.34) and (2.29) if $\ell \ge 2$, we obtain

$$\|(1+|x|)^{\ell}(\nabla^{\ell}\sigma, \nabla^{\ell+1}v)\|^2 \le C\left[\|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}} + \|\nabla v\|^2 + \sum_{\nu=1}^{\ell}\|(1+|x|)^{\nu}(\nabla^{\nu-1}f, \nabla^{\nu}g)\|^2\right].$$

This completes the proof of Lemma 2.3. □

Now combining Proposition 2.2 and Lemma 2.3, we have the following theorem.

Theorem 2.1. *There exists* $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ *such that if* δ *in* (2.4) *satisfies* $\delta \le \delta_0$, *then* (2.2)–(2.3) *admits a solution* $(\sigma, v) \in \hat{\mathscr{H}}^{k,k+1}$ *which satisfies the estimate*:

$$\|(\sigma,v)\|_{L_6} + \left\|\frac{(\sigma,v)}{|x|}\right\| + \sum_{\nu=1}^{k}\|(1+|x|)^{\nu}\nabla^{\nu}\sigma\| + \sum_{\nu=1}^{k+1}\|(1+|x|)^{\nu-1}\nabla^{\nu}v\|$$
$$\le C\left[\|b\|^2_{J^{k+1}} + \sum_{\nu=0}^{k-1}\|(1+|x|)^{\nu+1}\nabla^{\nu}f\| + \|(1+|x|)g\| + \sum_{\nu=1}^{k}\|(1+|x|)^{\nu}\nabla^{\nu}g\|\right],$$

where the constant $C > 0$ *is depending only on* μ, μ' *and* γ. *Furthermore the uniqueness is held in* $\hat{\mathscr{H}}^{1,2} \cap L_6$.

Proof. The existence and the estimate follows from Proposition 2.2 and Lemma 2.3 directly. The uniqueness follows from an argument similar to Lemma 2.1 (ii). □

2.2. A Proof of Theorem 1.1.

In this section, we shall construct a solution to (2.1), by use of the contraction mapping principle in $\mathscr{I}_{\varepsilon}^{4,5}$. We employ the following system of equations:

$$\begin{cases} \nabla \cdot v + \left(\dfrac{\tilde{v}}{\bar{\rho}+\tilde{\sigma}} \cdot \nabla\right)\sigma = \dfrac{G}{\bar{\rho}+\tilde{\sigma}}, & (2.36) \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla \cdot v) + \gamma\nabla\sigma = -(\bar{\rho}+\tilde{\sigma})(\tilde{v}\cdot\nabla)\tilde{v} \\ \qquad - [P'(\bar{\rho}+\tilde{\sigma}) - P'(\bar{\rho})]\nabla\tilde{\sigma} + (\bar{\rho}+\tilde{\sigma})F, & (2.37) \end{cases}$$

where $(\tilde{\sigma}, \tilde{v})(x) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$ is given.

2.2.1. Introduction of the solution map T for (2.36)–(2.37).

First and foremost, we put

$$a = \frac{\tilde{v}}{\bar{\rho}+\tilde{\sigma}}, \quad b = c = \bar{\rho}^{1/2}\tilde{v}, \quad g = \frac{G}{\bar{\rho}+\tilde{\sigma}},$$

$$f = -\tilde{\sigma}(\tilde{v}\cdot\nabla)\tilde{v} - [P'(\bar{\rho}+\tilde{\sigma}) - P'(\bar{\rho})]\nabla\tilde{\sigma} + (\bar{\rho}+\tilde{\sigma})F. \tag{2.38}$$

If we assume that $\varepsilon \le \bar{\rho}/2$ and

$$K_0 \equiv \|(1+|x|)G\| + \sum_{\nu=0}^{3} \|(1+|x|)^{\nu+1}\nabla^\nu F\| + \sum_{\nu=1}^{4} \|(1+|x|)^\nu \nabla^\nu G\| < \infty, \tag{2.39}$$

then we can check (2.4)–(2.5) easily and additionally we have

$$\|(1+|x|)g\| + \sum_{\nu=0}^{3} \|(1+|x|)^{\nu+1}\nabla^\nu f\| + \sum_{\nu=1}^{4} \|(1+|x|)^\nu \nabla^\nu g\| \le C\{\varepsilon^2 + K_0\} \tag{2.40}$$

for some constant $C = C(\bar{\rho}, \mu, \mu')$. Applying Theorem 2.1 with $k = 4$ for (2.36)–(2.37), we have the following lemma.

Lemma 2.4. *Let $(F, G) \in \mathscr{H}^{3,4}$ satisfies* (2.39). *Then, there exists ε_0 such that if $\varepsilon \le \varepsilon_0$ then* (2.36)–(2.37) *with $(\tilde{\sigma}, \tilde{v}) \in \mathscr{I}_\varepsilon^{4,5}$ has a solution $(\sigma, v) \in \hat{\mathscr{H}}^{4,5}$ which satisfies the estimate*:

$$\|(\sigma, v)\|_{L_6} + \left\|\frac{(\sigma, v)}{|x|}\right\| + \sum_{\nu=1}^{5} \|(1+|x|)^{\nu-1}\nabla^\nu v\| + \sum_{\nu=1}^{4} \|(1+|x|)^\nu \nabla^\nu \sigma\| \le C\{\varepsilon^2 + K_0\}, \tag{2.41}$$

where the constant $C > 0$ depends only on μ, μ' and $\bar{\rho}$.

Hence, we can consider the solution map $T : (\tilde{\sigma}, \tilde{v}) \mapsto (\sigma, v); \mathscr{I}_\varepsilon^{4,5} \to \hat{\mathscr{H}}^{4,5}$ for (2.36)–(2.37).

Next, we have to show that $(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$ leads to $(\sigma, v) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$. The following lemma plays an important role when we estimate the solution by L_∞-norm.

Lemma 2.5. *Let $E(x)$ be a scalar function satisfying*

$$|\partial_x^\alpha E(x)| \le \frac{C_\alpha}{|x|^{|\alpha|+1}} \quad (|\alpha| = 0, 1, 2).$$

(i) *If $\phi(x)$ is a smooth scalar function of the form: $\phi = \nabla \cdot \phi_1 + \phi_2$ satisfying*

$$L_1(\phi) \equiv \|(1+|x|)^3\phi\|_{L_\infty} + \|(1+|x|)^2\phi_1\|_{L_\infty} + \|\phi_2\|_{L_1} < \infty,$$

then we have for any multi-index α with $|\alpha| = 0, 1$

$$|\partial_x^\alpha (E * \phi)(x)| \le \frac{C_\alpha}{|x|^{|\alpha|+1}} L_1(\phi).$$

(ii) *If $\phi(x)$ is a smooth scalar function of the form: $\phi = \phi_1\phi_2$ satisfying*

$$L_2(\phi) \equiv \|(1+|x|)^2\phi\|_{L_\infty} + \|(1+|x|)^3(\nabla\phi_1)\phi_2\|_{L_\infty} + \|(1+|x|)^3\phi_1(\nabla\phi_2)\|_1 < \infty,$$

then we have for any multi-index α with $|\alpha| = 1, 2$

$$|\partial_x^\alpha (E * \phi)(x)| \le \frac{C_\alpha}{|x|^{|\alpha|}} L_2(\phi).$$

Here, C_α denotes a constant depending only on α.

The proof is by direct calculation and here we may omit it.

Now, with aid of the Helmholtz decomposition and the Fourier transform, we shall estimate L_∞-norm of the solution to (2.36)–(2.37).

Lemma 2.6. *Let (F, G) satisfy following estimate (for K_0 defined by (2.39)):*

$$K \equiv K_0 + \|(1+|x|)^3 F\|_{L_\infty} + \|(1+|x|)^2 F_1\|_{L_\infty} + \|F_2\|_{L_1} + \sum_{\nu=0}^{1} \|(1+|x|)^{\nu+2}\nabla^\nu G\|_{L_\infty} < \infty.$$

Then, if $(\sigma, v) \in \mathscr{H}^{4,5}$ is a solution to (2.36)–(2.37) with $(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$ and satisfies (2.41) then (σ, v) satisfies the estimate:

$$\|(1+|x|)^2\sigma\|_{L_\infty} + \sum_{\nu=0}^{1} \|(1+|x|)^{\nu+1}\nabla^\nu v\|_{L_\infty} \le C\{\varepsilon^2 + K\}, \tag{2.42}$$

where the constant $C > 0$ depends only on μ, μ' and $\bar{\rho}$.

Proof. In view of the Helmholtz decomposition, v is written of the form:

$$v = w + \nabla p \quad (w \in \dot{L}_6, \nabla p \in M_6). \tag{2.43}$$

Here and hereafter

$$M_6 = \{\nabla p \mid p \in L_{6,loc}, \nabla p \in L_6\}, \quad \dot{L}_6 = \overline{\{w \in C_0^\infty \mid \nabla \cdot w = 0\}}^{\|\cdot\|_{L_6}},$$

where $\overline{\cdot}^{\|\cdot\|_{L_6}}$ means the completion of $\cdot$ with respect to the L_6-norm. Substituting (2.43) into (2.36)–(2.37), we have

$$\begin{cases} \Delta p + \left(\dfrac{\tilde{v}}{\bar{\rho} + \tilde{\sigma}} \cdot \nabla\right)\sigma = \dfrac{G}{\bar{\rho} + \tilde{\sigma}}, & (2.44) \\ -\mu\Delta w + \nabla\Phi = h, & (2.45) \\ \Phi = \gamma\sigma - (2\mu + \mu')\Delta p, & (2.46) \end{cases}$$

where h is defined by

$$h = -\bar{\rho}(\tilde{v} \cdot \nabla)\tilde{v} + f \quad (f \text{ is what we put at (2.38)})$$

and Φ is introduced by (2.46) for the first time. Calculating the divergence of (2.45), we get

$$\Delta\Phi = \nabla \cdot h.$$

Thus we have the representation for Φ:

$$\Phi = \sum_{i=1}^{3} \frac{\partial E_0}{\partial x_i} * h_i \quad \left(E_0(x) = -\frac{1}{4\pi}|x|^{-1}\right). \tag{2.47}$$

Again from (2.45), we have

$$\begin{aligned} -\mu\Delta w_j &= h_j - \frac{\partial\Phi}{\partial x_j} = h_j - \frac{\partial}{\partial x_j}(\Delta^{-1}\nabla \cdot h) \\ &= h_j - \mathscr{F}^{-1}\left[\sum_{i=1}^{3} \frac{\xi_i\xi_j}{|\xi|^2}\hat{h}_i(\xi)\right]. \end{aligned}$$

So we get the representation for w:

$$w_j(x) = \frac{1}{\mu}\sum_{i=1}^{3} \mathscr{F}^{-1}\left[\left(\frac{\delta_{ij}}{|\xi|^2} - \frac{\xi_i\xi_j}{|\xi|^4}\right)\hat{h}_i\right](x) = \sum_{i=1}^{3} E_{ij} * h_i(x), \tag{2.48}$$

where

$$E_{ij}(x) = \frac{1}{\mu}\left(\frac{1}{4\pi}\frac{\delta_{ij}}{|x|} - \frac{1}{8\pi}\frac{\partial^2}{\partial x_i \partial x_j}|x|\right) = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}\right).$$

We shall apply Lemma 2.5 (i) to estimate Φ and w. Therefore, in order to estimate (2.47) and (2.48), we need to take a look at h. By $(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$, there exist $\tilde{V}_1 = (\tilde{V}_{1,i})_{1\le i\le 3}$ and $\tilde{V}_2$ such that

$$\nabla \cdot \tilde{v} = \nabla \cdot \tilde{V}_1 + \tilde{V}_2, \quad \|(1+|x|)^3\tilde{V}_1\|_{L_\infty} + \|(1+|x|)^{-1}\tilde{V}_2\|_{L_1} \le \varepsilon \tag{2.49}$$

and so we can calculate

$$\begin{aligned} h_i &= -(\bar{\rho}+\tilde{\sigma})(\tilde{v}\cdot\nabla)\tilde{v}_i - [P'(\bar{\rho}+\tilde{\sigma}) - P'(\bar{\rho})]\frac{\partial\tilde{\sigma}}{\partial x_i} + (\bar{\rho}+\tilde{\sigma})F_i \\ &= \left[\bar{\rho}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\{-\tilde{v}_i\tilde{v}_j + \tilde{v}_i\tilde{V}_{1,j}\} + \nabla\cdot\{(\bar{\rho}+\tilde{\sigma})F_{1,i\cdot}\}\right] \\ &\quad + \left\{-\bar{\rho}(\tilde{V}_1\cdot\nabla)\tilde{v}_i + \bar{\rho}\tilde{V}_2\tilde{v}_i - \tilde{\sigma}(\tilde{v}\cdot\nabla)\tilde{v}_i - Q(\sigma)\sigma\frac{\partial\tilde{\sigma}}{\partial x_i} - \nabla\sigma\cdot F_{1,i\cdot} + (\bar{\rho}+\tilde{\sigma})F_{2,i}\right\} \\ &\equiv \nabla\cdot h_1^i + h_2^i, \end{aligned}$$

where

$$Q(\sigma) = \int_0^1 P''(\bar{\rho}+\theta\sigma)\,d\theta.$$

By $(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_{\varepsilon}^{4,5}$ and (2.49) using the Sobolev inequality, we obtain

$$\|(1+|x|)^3 h_i\|_{L_\infty} + \|(1+|x|)^2 h_1^i\|_{L_\infty} + \|h_2^i\|_{L_1} \le C\{\varepsilon^2 + K_1\},$$

where K_1 is defined by

$$K_1 \equiv \|(1+|x|)^3 F\|_{L_\infty} + \|(1+|x|)^2 F_1\|_{L_\infty} + \|F_2\|_{L_1}$$

and $C > 0$ is a constant depending only on $\bar{\rho}$. Thus, applying Lemma 2.5 (i) to (2.47) and (2.48), we have

$$|x|^2|\Phi(x)| + \sum_{\nu=0}^{1} |x|^{\nu+1}|\nabla^\nu w(x)| \le CK_1. \tag{2.50}$$

As for p, we have from (2.44)

$$p = E_0 * \left(-\sum_{i=1}^{3} \frac{\tilde{v}_i}{\bar{\rho}+\tilde{\sigma}} \frac{\partial \sigma}{\partial x_i} + \frac{G}{\bar{\rho}+\tilde{\sigma}} \right) \equiv -E_0 * \sum_{i=1}^{3} q_1^i q_2^i + E_0 * r. \tag{2.51}$$

Since $(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_{\varepsilon}^{4,5}$, it follows from (2.41) and the Sobolev inequality that

$$\|(1+|x|)^2 q_1^i q_2^i\|_{L_\infty} + \|(1+|x|)^3 (\nabla q_1^i) q_2^i\|_{L_\infty} + \|(1+|x|)^3 q_1^i (\nabla q_2^i)\|_1 \le C\{\varepsilon^2 + K_0\},$$

$$\sum_{\nu=0}^{1} \|(1+|x|)^{\nu+2} \nabla^\nu r\|_{L_\infty} \le C \sum_{\nu=0}^{1} \|(1+|x|)^{\nu+2} \nabla^\nu G\|_{L_\infty} \equiv K_2,$$

where the constant $C > 0$ depends only on $\bar{\rho}$. Applying Lemma 2.5 (ii) to each term of (2.51) respectively, we also have

$$\sum_{\nu=1}^{2} |x|^\nu |\nabla^\nu p(x)| \le C\{\varepsilon^2 + K_0 + K_2\}. \tag{2.52}$$

Now, we are ready to estimate v and σ. First, we consider the case where $|x| \ge 1$. Returning to (2.43) and combining (2.50) and (2.52), we obtain

$$\sum_{\nu=0}^{1} (1+|x|)^{\nu+1} |\nabla^\nu v(x)| \le C\{\varepsilon^2 + K_0 + K_1 + K_2\}. \tag{2.53}$$

Besides by (2.46) we have

$$\sigma = \gamma^{-1}\{(2\mu + \mu')\Delta p + \Phi\}.$$

Combining (2.50) and (2.52), we get

$$(1+|x|)^2 |\sigma(x)| \le C\{\varepsilon^2 + K_0 + K_1 + K_2\}. \tag{2.54}$$

Next, we consider the case where $|x| < 1$. The Sobolev inequality and the Hardy inequality imply that

$$(1+|x|)^2|\sigma(x)| + \sum_{\nu=0}^{1}(1+|x|)^{\nu+1}|\nabla^\nu v(x)|$$

$$\leq 8\left\|\frac{\sigma}{1+|x|}\right\|_2 + 4\|\nabla v\|_2 + 4\left\|\frac{v}{1+|x|}\right\|_2 \leq C\|(\nabla\sigma, \nabla v)\|_{1,2} \leq C\{\varepsilon^2 + K_0\}. \quad (2.55)$$

Consequently by (2.53), (2.54) and (2.55), we have

$$\|(1+|x|)^2\nabla^2\sigma\|_{L_\infty} + \sum_{\nu=0}^{1}\|(1+|x|)^{\nu+1}\nabla^\nu v\|_{L_\infty} \leq C\left[\varepsilon^2 + \sum_{j=0}^{2}K_j\right] \leq C\{\varepsilon^2 + K\}.$$

This completes the proof of Lemma 2.6. □

We combine Lemmas 2.4 and 2.6 to prove that the solution $(\sigma, v) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$.

Proposition 2.3. *There exist* $c_0 > 0$ *and* $\varepsilon > 0$ *such that if* $(F, G) \in \mathscr{H}^{3,4}$ *satisfies*

$$K + \|(1+|x|)^{-1}G\|_{L_1} \leq c_0\varepsilon \quad (K \text{ is defined in Lemma 2.6}), \quad (2.56)$$

then (2.36)–(2.37) *with* $(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$ *admits a solution* $(\sigma, v) = T(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$.

Proof. By Lemmas 2.4, 2.6 and (2.56), it follows that (2.36)–(2.37) has a solution $(\sigma, v) \in \hat{\mathscr{H}}^{4,5}$, which satisfies

$$\|(\sigma, v)\|_{\mathscr{I}^{4,5}} \leq C\{\varepsilon^2 + K\} \leq C\{\varepsilon^2 + c_0\varepsilon\},$$

where the constant $C > 0$ depends only on μ, μ' and $\bar{\rho}$. Thus if we take $c_0 \leq 1/2C$ and $\varepsilon > 0$ sufficiently small, it follows that $(\sigma, v) \in \mathscr{I}_\varepsilon^{4,5}$. At last, we define V_1 and V_2 by

$$V_1 = -\frac{\tilde{v}}{\bar{\rho} + \tilde{\sigma}}\sigma, \quad V_2 = \left(\nabla \cdot \frac{\tilde{v}}{\bar{\rho} + \tilde{\sigma}}\right)\sigma + \frac{G}{\bar{\rho} + \tilde{\sigma}}.$$

Then immediately from (2.36)

$$\nabla \cdot v = \nabla \cdot V_1 + V_2.$$

Moreover, by $(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$ and (2.42), using Sovolev inequality, we have

$$\|(1+|x|)^3 V_1\|_{L_\infty} + \|(1+|x|)^{-1}V_2\|_{L_1} \leq C\{\varepsilon^2 + K + \|(1+|x|)^{-1}G\|_{L_1}\},$$

further by (2.56)

$$\leq C\{\varepsilon^2 + c_0\varepsilon\} \leq C\varepsilon^2 + \frac{\varepsilon}{2} \leq \varepsilon,$$

if $c_0 \leq 1/2C$ and $\varepsilon > 0$ is sufficiently small. This completes the proof of Proposition 2.3. □

2.2.2. Contraction of the solution map T.

Finally, we shall show that the solution map T for (2.36)–(2.37) is contract. We suppose that $(\tilde{\sigma}^j, \tilde{v}^j) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$ and $(\sigma^j, v^j) = T(\tilde{\sigma}^j, \tilde{v}^j)$ for $j = 1, 2$. Then it immediately follows from (2.36)–(2.37) that

$$
\begin{cases}
\nabla\cdot(v^1-v^2)+\left(\dfrac{\tilde{v}^1}{\bar{\rho}+\tilde{\sigma}^1}\cdot\nabla\right)(\sigma^1-\sigma^2)=g,\\
-\mu\Delta(v^1-v^2)-(\mu+\mu')\nabla\{\nabla\cdot(v^1-v^2)\}+\gamma\nabla(\sigma^1-\sigma^2)\\
\qquad=-\bar{\rho}(\tilde{v}^1\cdot\nabla)\tilde{v}^1+\bar{\rho}(\tilde{v}^2\cdot\nabla)\tilde{v}^2+f,
\end{cases}
\tag{2.57}
$$

where $(f,g)\in\mathscr{H}^{3,3}$ is defined by

$$
g=-\left(\frac{\tilde{v}^1}{\bar{\rho}+\tilde{\sigma}^1}-\frac{\tilde{v}^2}{\bar{\rho}+\tilde{\sigma}^2}\right)\cdot\nabla\sigma^2+\left(\frac{G}{\bar{\rho}+\tilde{\sigma}^1}-\frac{G}{\bar{\rho}+\tilde{\sigma}^2}\right),
$$

$$
f=-\tilde{\sigma}^1(\tilde{v}^1\cdot\nabla)\tilde{v}^1+\tilde{\sigma}^2(\tilde{v}^2\cdot\nabla)\tilde{v}^2-[P'(\bar{\rho}+\tilde{\sigma}^1)-P'(\bar{\rho})]\nabla\tilde{\sigma}^1
$$
$$
+[P'(\bar{\rho}+\tilde{\sigma}^2)-P'(\bar{\rho})]\nabla\tilde{\sigma}^2+(\tilde{\sigma}^1-\tilde{\sigma}^2)F.
$$

Since

$$
\sum_{\nu=0}^{2}\|(1+|x|)^{\nu+1}\nabla^\nu f\|+\|(1+|x|)g\|+\sum_{\nu=1}^{3}\|(1+|x|)^\nu\nabla^\nu g\|
$$
$$
\le C\{\varepsilon+K_0\}\|(\tilde{\sigma}^1-\tilde{\sigma}^2,\tilde{v}^1-\tilde{v}^2)\|_{\mathscr{I}^{3,4}}.
$$

as follows from the Sobolev inequality for K_0 defined in (2.39), by application of Theorem 2.1 with $k=3$ to (2.57), we obtain

$$
\|(\sigma^1-\sigma^2,v^1-v^2)\|_{L_6}+\left\|\frac{(\sigma^1-\sigma^2,v^1-v^2)}{|x|}\right\|
$$
$$
+\sum_{\nu=1}^{3}\|(1+|x|)^\nu\nabla^\nu(\sigma^1-\sigma^2)\|+\sum_{\nu=1}^{4}\|(1+|x|)^{\nu-1}\nabla^\nu(v^1-v^2)\|
$$
$$
\le C\{\varepsilon+K_0\}\|(\tilde{\sigma}^1-\tilde{\sigma}^2,\tilde{v}^1-\tilde{v}^2)\|_{\mathscr{I}^{3,4}}.
\tag{2.58}
$$

Next, we decompose (2.57) as in the proof of Lemma 2.6: Putting $v^1-v^2=w+\nabla p$ $(w\in\dot{L}_6,\ \nabla p\in M_6)$, we have

$$
\begin{cases}
\Delta p+\left(\dfrac{\tilde{v}^1}{\bar{\rho}+\tilde{\sigma}^1}\cdot\nabla\right)(\sigma^1-\sigma^2)=g,\\
-\mu\Delta w+\nabla\Phi=-\bar{\rho}(\tilde{v}^1\cdot\nabla)\tilde{v}^1+\bar{\rho}(\tilde{v}^2\cdot\nabla)\tilde{v}^2+f\equiv h,\\
\Phi=\gamma(\sigma^1-\sigma^2)-(2\mu+\mu')\Delta p.
\end{cases}
$$

The same argument as in the proof of Lemma 2.6 implies that

$$
\|(1+|x|)^2(\sigma^1-\sigma^2)\|_{L_\infty}+\sum_{\nu=0}^{1}\|(1+|x|)^{\nu+1}\nabla^\nu(v^1-v^2)\|_{L_\infty}
$$
$$
\le C\{\varepsilon+K\}\|(\tilde{\sigma}^1-\tilde{\sigma}^2,\tilde{v}^1-\tilde{v}^2)\|_{\mathscr{I}^{3,4}}
$$
$$
+C\varepsilon[\|(1+|x|)^3(\tilde{V}_1^1-\tilde{V}_1^2)\|_{L_\infty}+\|(1+|x|)^{-1}(\tilde{V}_2^1-\tilde{V}_2^2)\|_{L_1}],
\tag{2.59}
$$

where $\tilde{V}_1^j, \tilde{V}_2^j$ $(j = 1, 2)$ are functions satisfying

$$\nabla \cdot \tilde{v}^j = \nabla \cdot \tilde{V}_1^j + \tilde{V}_2^j, \quad \|(1+|x|)^3 \tilde{V}_1^j\|_{L_\infty} + \|(1+|x|)^{-1} \tilde{V}_2^j\|_{L_1} \le \varepsilon. \tag{2.60}$$

Moreover, if we define V_1^j, V_2^j $(j = 1, 2)$ as

$$V_1^j = -\frac{\tilde{v}^j}{\bar{\rho} + \tilde{\sigma}^j}\sigma^j, \quad V_2^j = \left(\nabla \cdot \frac{\tilde{v}^j}{\bar{\rho} + \tilde{\sigma}^j}\right)\sigma^j + \frac{G}{\bar{\rho} + \tilde{\sigma}^j}, \tag{2.61}$$

then

$$\begin{aligned} &\|(1+|x|)^3(V_1^1 - V_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(V_2^1 - V_2^2)\|_{L_1} \\ &\quad \le C\{\varepsilon + \|(1+|x|)^{-3}G\|_{L_1}\}\|(\tilde{\sigma}^1 - \tilde{\sigma}^2, \tilde{v}^1 - \tilde{v}^2)\|_{\mathscr{I}^{3,4}}. \end{aligned} \tag{2.62}$$

Combining (2.58), (2.59) and (2.62), we obtain

$$\begin{aligned} &\|(\sigma^1 - \sigma^2, v^1 - v^2)\|_{\mathscr{I}^{3,4}} + \|(1+|x|)^3(V_1^1 - V_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(V_2^1 - V_2^2)\|_{L_1} \\ &\quad \le C\{\varepsilon + K\}\|(\tilde{\sigma}^1 - \tilde{\sigma}^2, \tilde{v}^1 - \tilde{v}^2)\|_{\mathscr{I}^{3,4}} \\ &\qquad + C\varepsilon[\|(1+|x|)^3(\tilde{V}_1^1 - \tilde{V}_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(\tilde{V}_2^1 - \tilde{V}_2^2)\|_{L_1}]. \end{aligned}$$

Therefore, we have the following proposition.

Proposition 2.4. *There exist $c_0 > 0$ and $\varepsilon > 0$ such that if $(F, G) \in \mathscr{H}^{3,4}$ satisfies*

$$K \le c_0\varepsilon \quad (K \text{ is defined in Lemma } 2.6),$$

then for $(\tilde{\sigma}^j, \tilde{v}^j) \in \dot{\mathscr{I}}_\varepsilon^{4,5}$ and $(\sigma^j, v^j) = T(\tilde{\sigma}^j, \tilde{v}^j)$ $(j = 1, 2)$ we have the following estimate:

$$\begin{aligned} &\|(\sigma^1 - \sigma^2, v^1 - v^2)\|_{\mathscr{I}^{3,4}} + \|(1+|x|)^3(V_1^1 - V_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(V_2^1 - V_2^2)\|_{L_1} \\ &\quad \le \frac{1}{2}[\|(\tilde{\sigma}^1 - \tilde{\sigma}^2, \tilde{v}^1 - \tilde{v}^2)\|_{\mathscr{I}^{3,4}} + \|(1+|x|)^3(\tilde{V}_1^1 - \tilde{V}_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(\tilde{V}_2^1 - \tilde{V}_2^2)\|_{L_1}], \end{aligned}$$

where $(\tilde{V}_1^j, \tilde{V}_2^j)$ $(j = 1, 2)$ is a function satisfying (2.60) *and (V_1^j, V_2^j) $(j = 1, 2)$ is defined by* (2.61).

Hence, by Propositions 2.3 and 2.4, the contraction mapping principle implies the existence and uniqueness of solution to (1.2) which we have stated in Theorem 1.1.

3. Non-stationary problem.

In this part, we consider stability of the stationary solution with respect to the initial disturbance (ρ_0, v_0). Let $\bar{\rho}$ be a positive constant and let (F, G) be small in the sense of Theorem 1.1. We denote the corresponding stationary solution obtained in Theorem 1.1 by (ρ^*, v^*). Putting

$$\rho(t, x) = \rho^*(x) + \sigma(t, x), \quad v(t, x) = v^*(x) + w(t, x)$$

into (1.1), we have the system of equations for (σ, w):

$$
\begin{cases}
\sigma_t(t) + \nabla \cdot \{(\rho^* + \sigma(t))w(t)\} = -\nabla \cdot (v^*\sigma(t)), & (3.1) \\
w_t(t) - \dfrac{1}{\rho^*}[\mu\Delta w(t) + (\mu + \mu')\nabla(\nabla \cdot w(t))] + A(t)\nabla\sigma(t) = f(t), & (3.2) \\
(\sigma, w)(0, x) = (\rho_0 - \rho^*, v_0 - v^*)(x), & (3.3)
\end{cases}
$$

where

$$
\begin{aligned}
f(t) &= -(v^* \cdot \nabla)w(t) - (w(t) \cdot \nabla)(v^* + w(t)) \\
&\quad - \frac{1}{\rho^*}\{P'(\rho^* + \sigma(t)) - P'(\rho^*)\}\nabla\rho^* - \frac{\sigma(t)}{\rho^*(\rho^* + \sigma(t))} \\
&\quad \times [\mu\Delta(v^* + w(t)) + (\mu + \mu')\nabla\{\nabla \cdot (v^* + w(t))\} - P'(\rho^* + \sigma(t))\nabla\rho^*], \qquad (3.4) \\
A(t) &= \frac{P'(\rho^* + \sigma(t))}{\rho^* + \sigma(t)}.
\end{aligned}
$$

Our goal in this part is to give a proof of Theorem 1.2. The proof consists of the following two steps. One is local existence:

PROPOSITION 3.1. *If* $(\sigma, w)(0) \in \mathscr{H}^{3,3}$, *then there exists* $t_0 > 0$ *such that the initial value problem* (3.1)–(3.2) *with initial data* $(\sigma, w)(0)$ *admits a unique solution* $(\sigma, w)(t) \in \mathscr{C}(0, t_0; \mathscr{H}^{3,3})$. *Moreover*, $(\sigma, w)(t)$ *satisfies*

$$
\|(\sigma, w)(t)\|_{3,3}^2 \le 2\|(\sigma, w)(0)\|_{3,3}^2
$$

for any $t \in [0, t_0]$.

And the other is a priori estimate:

PROPOSITION 3.2. *Let* $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{3,3})$ *be a solution to* (3.1)–(3.2). *Then there exists* $\varepsilon_0 > 0$ *such that if* $\varepsilon \le \varepsilon_0$ *and* $\sup_{0 \le t \le t_1}\|(\sigma, w)(t)\|_{3,3}, \|(\rho^* - \bar{\rho}, v^*)\|_{\mathscr{I}^{4,5}} \le \varepsilon$, *then*

$$
\|(\sigma, w)(t)\|_{3,3}^2 + \int_0^t \|(\nabla\sigma, \nabla w, w_t)(s)\|_{2,3,2}^2\, ds \le C\|(\sigma, w)(0)\|_{3,3}^2 \qquad (3.5)
$$

for any $t \in [0, t_1]$, *where* $C > 0$ *is a constant depending only on* μ *and* μ'.

Concerning the local existence, we can apply the Matsumura-Nishida [**14**] method directly. So we shall devote the following sections to the proof of Proposition 3.2.

REMARK 3.1. In the following lemmas and these proofs, small number ε is at least taken in such a way that

$$
\|(\sigma, w)(t)\|_{3,3}, \|(\rho^* - \bar{\rho}, v^*)\|_{\mathscr{I}^{4,5}} \le \varepsilon < \frac{\bar{\rho}}{4},
$$

so that we have $3\bar{\rho}/4 \le \rho^* \le 5\bar{\rho}/4$, $\bar{\rho}/2 \le \rho^* + \sigma(t) \le 3\bar{\rho}/2$ etc.

3.1. Some estimates for $f(t)$ and its derivatives.

Lemma 3.1. *Let α be a multi-index with $0 \le |\alpha| \le 3$ and let us write $\partial_x^\alpha f(t)$ of the form*:

$$\partial_x^\alpha f(t) = -\frac{\sigma(t)}{\rho^*(\rho^* + \sigma(t))}[\mu\Delta\partial_x^\alpha w(t) + (\mu+\mu')\nabla(\nabla\cdot\partial_x^\alpha w(t))] + F_\alpha(t).$$

Then, there exists $\varepsilon > 0$ such that if $\|(\sigma, w)(t)\|_{3,3}, \|(\rho^* - \bar{\rho}, v^*)\|_{\mathscr{I}^{4,5}} \le \varepsilon$ then $F_\alpha(t)$ satisfies

$$|F_\alpha(t)| \le C\begin{cases} |\nabla v^*|\,|w(t)| + (|v^*| + |w(t)|)|\nabla w(t)| + (|\nabla\rho^*| + |\nabla^2 v^*|)|\sigma(t)| \\ \qquad \text{if } \alpha = 0, \\ |\nabla^{|\alpha|+1}v^*|\,|w(t)| + \sum_{\nu=1}^{|\alpha|+1}|\nabla^\nu w(t)| + \sum_{\nu=1}^{|\alpha|+1}(|\nabla^\nu\rho^*| + |\nabla^{\nu+1}v^*|)|\sigma(t)| \\ \qquad + \sum_{\nu=1}^{|\alpha|}|\nabla^\nu\sigma(t)| + R_{|\alpha|}(t) \quad \text{if } |\alpha| = 1,2,3. \end{cases} \tag{3.6}$$

Here, $C > 0$ is a constant depending only on μ, μ'; $R_1(t) = 0$ and $R_k(t)$ $(k = 2,3)$ satisfies the following estimates:

$$\|R_k(t)\|_{L_{3/2}} \le C\varepsilon\|(\nabla^2\sigma, \nabla^2 w)(t)\|_{k-2,k-2} \quad (k = 2,3),$$
$$\|R_2(t)\| \le C\varepsilon\|\nabla^3 w(t)\|.$$

Proof. By combination of the Leibniz rule and the Sobolev embedding: $H^2 \subset L_\infty$, we can easily check (3.6) with

$$R_k(t) = \begin{cases} 0 & \text{if } k = 1, \\ |\nabla^2 w(t)|\,|\nabla^2\sigma(t)| & \text{if } k = 2, \\ |\nabla^2 w(t)|\,|\nabla^3\sigma(t)| + (|\nabla^2 w(t)| + |\nabla^3 w(t)|)|\nabla^2\sigma(t)| & \\ \qquad + (|\nabla^3\rho^*| + |\nabla^4 v^*|)|\nabla\sigma(t)| + (|\nabla^3\rho^*| + |\nabla^2 w(t)|)|\nabla^2 w(t)| & \text{if } k = 3. \end{cases} \tag{3.7}$$

Using the Gagliard-Nirenberg inequality, we have

$$\|R_2(t)\|_{L_{3/2}} \le \|\nabla^2 w(t)\|_{L_6}\|\nabla^2\sigma(t)\| \le C\varepsilon\|\nabla^2\sigma(t)\|,$$

$$\begin{aligned}\|R_3(t)\|_{L_{3/2}} &\le \|\nabla^2 w(t)\|_{L_6}\|\nabla^3\sigma(t)\| + \|\nabla^2 w(t)\|_1\|\nabla^2\sigma(t)\|_{L_6} \\ &\qquad + \|(\nabla^3\rho^*, \nabla^4 v^*)\|\,\|\nabla\sigma(t)\|_{L_6} + \|(\nabla^3\rho^*, \nabla^2 w(t))\|\,\|\nabla^2 w(t)\|_{L_6} \\ &\le C\varepsilon\|(\nabla^2\sigma, \nabla^3 w)(t)\|_{1,0},\end{aligned}$$

moreover using the Sobolev inequality, we also have

$$\|R_2(t)\| \le \|\nabla^2 w(t)\|_{L_3}\|\nabla^2\sigma(t)\|_{L_6} \le C\varepsilon\|\nabla^2 w(t)\|_1.$$

This completes the proof of Lemma 3.1. □

3.2. Estimates for $\nabla w(t)$ and its derivatives up to $\nabla^4 w(t)$.

Lemma 3.2. *Let $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{3,3})$ be a solution to (3.1)–(3.2). Then, there exist $\varepsilon_0, \lambda_0 > 0$ and $\alpha_k > 0$ such that if $\varepsilon \le \varepsilon_0$ and $\|(\sigma, w)(t)\|_{3,3}, \|(\rho^* - \bar{\rho}, v^*)\|_{\mathscr{J}^{4,5}} \le \varepsilon$ then*

$$\frac{d}{dt}[\|\sigma(t)\|^2 + (B(t)w(t), w(t))] + \alpha_0 \|\nabla w(t)\|^2 \le C\varepsilon \|\nabla \sigma(t)\|^2, \tag{3.8}$$

and for $1 \le k \le 3$ and any λ with $0 < \lambda < \lambda_0$

$$\frac{d}{dt}[\|\nabla^k \sigma(t)\|^2 + (B(t)\nabla^k w(t), \nabla^k w(t))] + \alpha_k \|\nabla^{k+1} w(t)\|^2$$
$$\le C(\varepsilon + \lambda)\|(\nabla\sigma, w_t)(t)\|^2_{k-1,k-1} + C\lambda^{-1}\|\nabla w(t)\|^2_{k-1}, \tag{3.9}$$

where $C > 0$ is a constant depending only on μ, μ' and

$$B(t) = \frac{(\rho^* + \sigma(t))^2}{P'(\rho^* + \sigma(t))}.$$

Proof. Using the Friedrichs mollifier, we may assume that $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{\infty,\infty})$. For any multi-index α with $0 \le |\alpha| \le 3$, applying ∂_x^α to (3.1) and (3.2); multiplying the resultant equation by $\partial_x^\alpha \sigma(t)$ and $(\rho^* + \sigma(t))A(t)^{-1}\partial_x^\alpha w(t)$ respectively, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha \sigma(t)\|^2 - ((\rho^* + \sigma(t))\partial_x^\alpha w(t), \nabla\partial_x^\alpha \sigma(t)) = (-\partial_x^\alpha(v^*\sigma(t)) - I_\alpha(t), \nabla\partial_x^\alpha \sigma(t)),$$

$$(B(t)\partial_x^\alpha w_t(t), \partial_x^\alpha w(t)) - \left(\frac{B(t)}{\rho^*}\partial_x^\alpha\{\mu\Delta w(t) + (\mu + \mu')\nabla(\nabla \cdot w(t))\}, \partial_x^\alpha w(t)\right)$$
$$+ ((\rho^* + \sigma(t))\nabla\partial_x^\alpha \sigma(t), \partial_x^\alpha w(t)) = (\partial_x^\alpha f(t) + J_\alpha(t), B(t)\partial_x^\alpha w(t))$$

by integration with respect to x, where $I_\alpha(t)$ and $J_\alpha(t)$ are defined by

$$I_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}(\rho^* + \sigma(t)))\partial_x^\beta w(t),$$

$$J_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}\left[\left(\partial_x^{\alpha-\beta}\frac{1}{\rho^*}\right)\partial_x^\beta\{\mu\Delta w(t) + (\mu + \mu')\nabla(\nabla \cdot w(t))\}\right.$$
$$\left. + (\partial_x^{\alpha-\beta}A(t))\nabla\partial_x^\beta \sigma(t)\right].$$

Canceling the term of $((\rho^* + \sigma(t))\partial_x^\alpha w(t), \nabla\partial_x^\alpha \sigma(t))$ by the above two formulas and writing the first term of second formula as follows:

$$(B(t)\partial_x^\alpha w_t(t), \partial_x^\alpha w(t)) = \frac{1}{2}\frac{d}{dt}(B(t)\partial_x^\alpha(t), \partial_x^\alpha w(t)) - \frac{1}{2}(B_t(t)\partial_x^\alpha(t), \partial_x^\alpha w(t)),$$

and using integration by parts for the second term of second formula, we have

$$\frac{1}{2}\frac{d}{dt}\{\|\partial_x^\alpha\sigma(t)\|^2 + (B(t)\partial_x^\alpha w(t), \partial_x^\alpha w(t))\} + \frac{4B_0\mu}{5\bar{\rho}}\|\nabla\partial_x^\alpha w(t)\|^2$$

$$\leq |(\partial_x^\alpha(v^*\sigma(t)), \nabla\partial_x^\alpha\sigma(t))| + |(\partial_x^\alpha f(t), B(t)\partial_x^\alpha w(t))|$$

$$+ [|(I_\alpha(t), \nabla\partial_x^\alpha\sigma(t))| + |(J_\alpha(t), B(t)\partial_x^\alpha w(t))|] + \frac{1}{2}|(B_t(t)\partial_x^\alpha w(t), \partial_x^\alpha w(t))|$$

$$+ \left[\mu\left|\left(\nabla\partial_x^\alpha w(t), \left(\nabla\frac{B(t)}{\rho^*}\right)\partial_x^\alpha w(t)\right)\right| + (\mu+\mu')\left|\left(\left(\nabla\frac{B(t)}{\rho^*}\right)\nabla\cdot\partial_x^\alpha w(t), \partial_x^\alpha w(t)\right)\right|\right]$$

$$\equiv K_1 + K_2 + K_3 + K_4 + K_5, \tag{3.10}$$

where $B_0 = \min_{\bar{\rho}/2\leq s\leq 3\bar{\rho}/2} s^2/P'(s)$.

Now, we estimate the right hand side of (3.10), using the Sobolev inequality and the Gagliard-Nirenberg inequality. If $\alpha = 0$, employing the Hardy inequality, we have

$$K_1 \leq \|(1+|x|)v^*\|_{L_\infty}\left\|\frac{\sigma(t)}{|x|}\right\|\|\nabla\sigma(t)\| \leq C\varepsilon\|\nabla\sigma(t)\|^2. \tag{3.11}$$

If $1 \leq |\alpha| \leq 3$, by integration by parts, we can show that

$$K_1 \leq C\varepsilon\|\nabla\sigma(t)\|_{|\alpha|-1}^2. \tag{3.12}$$

To use Lemma 3.1, we divide K_2 into the following two parts:

$$K_2 \leq |(F_\alpha(t), \partial_x^\alpha w(t))| + \left|\left(\frac{\sigma(t)}{\rho^*(\rho^*+\sigma(t))}[\mu\Delta\partial_x^\alpha w(t) + (\mu+\mu')\nabla(\nabla\cdot\partial_x^\alpha w(t))], \partial_x^\alpha w(t)\right)\right|$$

$$\equiv K_{21} + K_{22}. \tag{3.13}$$

Concerning K_{22}, using integration by parts, we have

$$K_{22} \leq \mu\left|\left(\nabla\left\{\frac{B(t)\sigma(t)}{\rho^*(\rho^*+\sigma(t))}\partial_x^\alpha w(t)\right\}, \nabla\partial_x^\alpha w(t)\right)\right|$$

$$+ (\mu+\mu')\left|\left(\nabla\cdot\left\{\frac{B(t)\sigma(t)}{\rho^*(\rho^*+\sigma(t))}\partial_x^\alpha w(t)\right\}, \nabla\cdot\partial_x^\alpha w(t)\right)\right|$$

$$\leq C\{\|\sigma(t)\|_{L_\infty}\|\nabla\partial_x^\alpha w(t)\|^2 + \|(\nabla\rho^*, \nabla\sigma(t))\|_{L_3}\|\partial_x^\alpha w(t)\|_{L_6}\|\nabla\partial_x^\alpha w(t)\|\}$$

$$\leq C\varepsilon\|\nabla\partial_x^\alpha w(t)\|^2. \tag{3.14}$$

To estimate K_{21}, we use (3.6). If $\alpha = 0$,

$$K_{21} \leq C\left\{\|(1+|x|)^2\nabla v^*\|_{L\infty}\left\|\frac{w(t)}{|x|}\right\|^2\right.$$

$$+ \|(1+|x|)v^*\|_{L\infty}\|\nabla w(t)\|\left\|\frac{w(t)}{|x|}\right\| + \|w(t)\|_{L_3}\|\nabla w(t)\|\,\|w(t)\|_{L_6}$$

$$\left. + \|(1+|x|)(\nabla\rho^*, \nabla^2 v^*)\|_{L_3}\left\|\frac{\sigma(t)}{|x|}\right\|\,\|w(t)\|_{L_6}\right\}$$

$$\leq C\varepsilon\|(\nabla\sigma, \nabla w)(t)\|^2, \tag{3.15}$$

and if $1 \le |\alpha| \le 3$,

$$K_{21} \le C\Bigg\{ \|\nabla^{|\alpha|+1} v^*\| \, \|w(t)\|_{L_6} \|\nabla^{|\alpha|} w(t)\|_{L_3} + \sum_{\nu=1}^{|\alpha|+1} \|\nabla^{\nu} w(t)\| \, \|\nabla^{|\alpha|} w(t)\|$$

$$+ \sum_{\nu=1}^{|\alpha|+1} \|(\nabla^{\nu}\rho^*, \nabla^{\nu+1} v^*)\| \, \|\sigma(t)\|_{L_6} \|\nabla^{|\alpha|} w(t)\|_{L_3}$$

$$+ \sum_{\nu=1}^{|\alpha|} \|\nabla^{\nu}\sigma(t)\| \, \|\nabla^{|\alpha|} w(t)\| + \|R_{|\alpha|}(t)\|_{L_{3/2}} \|\nabla^{|\alpha|} w(t)\|_{L_3} \Bigg\}$$

$$\le C(\varepsilon + \lambda) \|(\nabla\sigma, \nabla w)(t)\|^2_{|\alpha|-1, |\alpha|} + C\lambda^{-1} \|\nabla^{|\alpha|} w(t)\|^2. \tag{3.16}$$

For $1 \le |\alpha| \le 2$, we can easily check that

$$K_3 \le C\varepsilon \|(\nabla\sigma, \nabla w)(t)\|^2_{|\alpha|-1, |\alpha|}. \tag{3.17}$$

It also turns out to be true for $|\alpha| = 3$, using the following inequality:

$$\left\| \frac{w(t)}{1+|x|} \right\|_{L_\infty} \le C \|\nabla w(t)\|_1,$$

which follows from combination of the Sobolev inequality and the Hardy inequality. In order to estimate K_4, we use (3.1). Then

$$2K_4 = |(\tilde{B}(t)\sigma_t(t)\partial_x^\alpha w(t), \partial_x^\alpha w(t))|$$

$$= |(\nabla \cdot \{(\rho^* + \sigma(t))w(t) + v^*\sigma(t)\}, \tilde{B}(t)\partial_x^\alpha w(t) \cdot \partial_x^\alpha w(t))|$$

$$\le C|(w(t) + v^*\sigma(t), \nabla\{\partial_x^\alpha w(t) \cdot \partial_x^\alpha w(t)\} + \{\nabla \tilde{B}(t)\}\partial_x^\alpha w(t) \cdot \partial_x^\alpha w(t))|$$

$$\le C\{(\|w(t)\|_{L_3} + \|v^*\|_{L_6} \|\sigma(t)\|_{L_6}) \|\nabla \partial_x^\alpha w(t)\| \, \|\partial_x^\alpha w(t)\|_{L_6}$$

$$+ \|(w, \sigma)(t)\|_{L_6} \|(\nabla\rho^*, \nabla\sigma(t))\| \, \|\partial_x^\alpha w(t)\|^2_{L_6}\}$$

$$\le C\varepsilon \|\nabla \partial_x^\alpha w(t)\|^2, \tag{3.18}$$

where $\tilde{B}(t)$ is defined by

$$\tilde{B}(t) = \frac{\rho^* + \sigma(t)}{P'(\rho^* + \sigma(t))} \left[2 - \frac{P''(\rho^* + \sigma(t))}{P'(\rho^* + \sigma(t))} (\rho^* + \sigma(t)) \right].$$

We also have

$$K_5 \le C \|(\nabla\rho^*, \nabla\sigma(t))\|_{L_3} \|\nabla \partial_x^\alpha w(t)\| \, \|\partial_x^\alpha w(t)\|_{L_6} \le C\varepsilon \|\nabla \partial_x^\alpha w(t)\|^2. \tag{3.19}$$

Combining (3.10)–(3.19), we obtain (3.8) and (3.9), if we choose $\varepsilon, \lambda > 0$ small enough. □

3.3. Estimates for $w_t(t)$ and its derivatives up to $\nabla^2 w_t(t)$.

Lemma 3.3. *Let $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{3,3})$ be a solution to (3.1)–(3.2). Then, there exist $\varepsilon_0 > 0$ and $\beta_k > 0$ such that if $\varepsilon \le \varepsilon_0$ and $\|(\sigma, w)(t)\|_{3,3}, \|(\rho^* - \bar\rho, v^*)\|_{\mathscr{J}^{4,5}} \le \varepsilon$ then we have*

$$\frac{d}{dt}(w(t), \nabla\sigma(t)) + \beta_1 \|w_t(t)\|^2 \le C\varepsilon\|\nabla\sigma(t)\|^2 + C\|\nabla w(t)\|_1^2, \tag{3.20}$$

$$\frac{d}{dt}(\nabla^{k-1} w(t), \nabla^k \sigma(t)) + \beta_k \|\nabla^{k-1} w_t(t)\|^2 \le C\|(\nabla\sigma, \nabla w, \nabla^{k-2} w_t)(t)\|_{k-2,k,0}^2 \tag{3.21}$$

for $2 \le k \le 3$, where $C > 0$ is a constant depending only on μ and μ'.

Proof. Using the Friedrichs mollifier, we may assume that $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{\infty,\infty})$. Multiplying (3.2) by $A(t)^{-1}$, we have

$$\frac{1}{A(t)} w_t(t) + \nabla\sigma(t) = \frac{1}{\rho^* A(t)}[\mu\Delta w(t) + (\mu + \mu')\nabla(\nabla \cdot w(t))] + \frac{1}{A(t)} f(t).$$

For any multi-index α with $0 \le |\alpha| \le 2$, applying ∂_x^α to this formula and multiplying the resultant equation by $\partial_x^\alpha w_t(t)$, we have

$$\begin{aligned}(\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha w_t(t)) &+ \left(\frac{1}{A(t)}\partial_x^\alpha w_t(t), \partial_x^\alpha w_t(t)\right)\\ &= \left(\partial_x^\alpha\left\{\frac{1}{\rho^* A(t)}[\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))] + \frac{1}{A(t)} f(t) - I_\alpha(t)\right\}, \partial_x^\alpha w_t(t)\right),\end{aligned}$$

where $I_\alpha(t)$ is defined by

$$I_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}\left(\partial_x^{\alpha-\beta}\frac{1}{A(t)}\right)\partial_x^\beta w_t(t).$$

The first term can be written in following form:

$$(\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha w_t(t)) = \frac{d}{dt}(\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha w(t)) + (\partial_x^\alpha\sigma_t(t), \nabla\cdot\partial_x^\alpha w(t)).$$

Therefore, putting $A_1 = \max_{\bar\rho/2 \le s \le 3\bar\rho/2} P'(s)/s$, we have

$$\begin{aligned}&\frac{d}{dt}(\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha w(t)) + \frac{1}{A_1}\|\partial_x^\alpha w_t(t)\|^2\\ &\quad\le \left|\left(\partial_x^\alpha\left\{\frac{1}{\rho^* A(t)}[\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))]\right\}, \partial_x^\alpha w_t(t)\right)\right| + |(I_\alpha(t), \partial_x^\alpha w_t(t))|\\ &\qquad+ \left|\left(\partial_x^\alpha\left\{\frac{1}{A(t)} f(t)\right\}, \partial_x^\alpha w_t(t)\right)\right| + |(\partial_x^\alpha\sigma_t(t), \nabla\cdot\partial_x^\alpha w(t))|\\ &\quad\equiv K_1 + K_2 + K_3 + K_4.\end{aligned} \tag{3.22}$$

Now we estimate the right hand side of (3.22), using the Sobolev inequality and the Gagliard-Nirenberg inequality. First, we can easily check that

$$\begin{aligned}K_1 &\le \lambda\|\nabla^{|\alpha|} w_t(t)\|^2 + C\lambda^{-1}\|\nabla^2 w(t)\|_{|\alpha|}^2,\\ K_2 &\le C\varepsilon\|\nabla w_t(t)\|_{|\alpha|-1}^2.\end{aligned} \tag{3.23}$$

We estimate K_3, using Lemma 3.1: If $\alpha = 0$,

$$\begin{aligned}K_3 &\le C\{\|\nabla v^*\|_{L_3}\|w(t)\|_{L_6} + \|(v^*, w(t))\|_{L_\infty}\|\nabla w(t)\| \\ &\quad + \|(\nabla\rho^*, \nabla^2 v^*)\|_{L_3}\|\sigma(t)\|_{L_6} + \|\sigma(t)\|_{L_\infty}\|\nabla^2 w(t)\|\}\|w_t(t)\| \\ &\le C\varepsilon\|(\nabla\sigma, \nabla w, w_t)(t)\|_{0,1,0}^2. \end{aligned} \tag{3.24}$$

If $1 \le |\alpha| \le 2$, we divide K_3 as

$$\begin{aligned}K_3 &\le \sum_{\beta<\alpha}\binom{\alpha}{\beta}|(\{\partial_x^{\alpha-\beta}A(t)^{-1}\}\partial_x^{\beta}f(t), \partial_x^{\alpha}w_t(t))| \\ &\quad + |(A(t)^{-1}\partial_x^{\alpha}f(t), \partial_x^{\alpha}w_t(t))| \equiv K_{31} + K_{32}. \end{aligned} \tag{3.25}$$

Then we have

$$\begin{aligned}K_{31} &\le C\sum_{\nu=1}^{|\alpha|}\|(\nabla^{\nu}\rho^*, \nabla^{\nu}\sigma(t))\|_{L_3}\sum_{\substack{0\le\nu_1\le|\alpha|-1\\0\le\nu_2\le|\alpha|+1}}\|(\nabla^{\nu_1}\sigma, \nabla^{\nu_2}w)(t)\|_{L_6}\|\nabla^{|\alpha|}w_t(t)\| \\ &\le C\varepsilon\|(\nabla\sigma, \nabla w, \nabla^{|\alpha|}w_t)(t)\|_{|\alpha|-1,|\alpha|+1,0}^2, \\ K_{32} &\le C\left\{\|\nabla v^*\|_{L_3}\|w(t)\|_{L_6} + \sum_{\nu=1}^{|\alpha|+1}\|(\nabla^{\nu}\rho^*, \nabla^{\nu+1}v^*)\|_{L_3}\|\sigma(t)\|_{L_6}\right. \\ &\quad \left. + \|(\nabla\sigma, \nabla w)(t)\|_{|\alpha|-1,|\alpha|+1} + \|R_{|\alpha|}(t)\|\right\}\|\nabla^{|\alpha|}w_t(t)\| \\ &\le \lambda\|\nabla^{|\alpha|}w_t(t)\|^2 + C\lambda^{-1}\|(\nabla\sigma, \nabla w)(t)\|_{|\alpha|-1,|\alpha|+1}^2. \end{aligned} \tag{3.26}$$

At last, in order to estimate K_4, we substitute (3.1) into σ_t as in (3.18): Indeed, if $\alpha = 0$,

$$\begin{aligned}K_4 &\le |(\nabla\cdot\{(\rho^* + \sigma(t))w(t)\}, \nabla\cdot w(t))| + |(v^*\sigma(t), \nabla(\nabla\cdot w(t)))| \\ &\le C\left\{\|(\nabla\rho^*, \nabla\sigma(t))\|_{L_3}\|w(t)\|_{L_6}\|\nabla w(t)\| + \|\nabla w(t)\|^2 + \|(1+|x|)v^*\|_{L_\infty}\left\|\frac{\sigma(t)}{|x|}\right\|\|\nabla^2 w(t)\|\right\} \\ &\le C\varepsilon\|(\nabla\sigma, \nabla^2 w)(t)\|^2 + C\|\nabla w(t)\|^2, \end{aligned} \tag{3.27}$$

and if $1 \le |\alpha| \le 2$,

$$\begin{aligned}K_4 &\le |(\partial_x^{\alpha}\{(\rho^* + \sigma(t))w(t) + v^*\sigma(t)\}, \nabla(\nabla\cdot\partial_x^{\alpha}w(t)))| \\ &\le C\sum_{\beta<\alpha}\{\|(\partial_x^{\alpha-\beta}\rho^*, \partial_x^{\alpha-\beta}\sigma(t))\|_{L_3}\|\partial_x^{\beta}w(t)\|_{L_6} + \|\partial_x^{\alpha}w(t)\| \\ &\quad + \|\partial_x^{\alpha-\beta}v^*\|_{L_3}\|\partial_x^{\beta}\sigma(t)\|_{L_6} + \|\partial_x^{\alpha}\sigma(t)\|\}\|\nabla^2\partial_x^{\alpha}w(t)\| \\ &\le C\|(\nabla\sigma, \nabla w)(t)\|_{|\alpha|-1,|\alpha|+1}^2. \end{aligned} \tag{3.28}$$

Insert (3.23)–(3.28) into (3.22). Then, we have (3.20) and (3.21) if we take $\varepsilon, \lambda > 0$ small enough. □

3.4. Estimates for $\nabla\sigma(t)$ and its derivatives up to $\nabla^3\sigma(t)$.

Lemma 3.4. *Let* $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{3,3})$ *be a solution to* (3.1)–(3.2). *Then, there exist* $\varepsilon_0 > 0$ *and* $\beta_k > 0$ *such that if* $\varepsilon \le \varepsilon_0$ *and* $\|(\sigma, w)(t)\|_{3,3}, \|(\rho^* - \bar\rho, v^*)\|_{\mathscr{I}^{4,5}} \le \varepsilon$ *then we have*

$$\|\nabla\sigma(t)\|^2 \le \|(\nabla w, w_t)(t)\|_{1,0}^2, \tag{3.29}$$

$$\|\nabla^k\sigma(t)\|^2 \le C\|(\nabla\sigma, \nabla w, \nabla^{k-1}w_t)(t)\|_{k-2,k,0}^2 \tag{3.30}$$

for $2 \le k \le 3$, *where* $C > 0$ *is a constant depending only on* μ *and* μ'.

Proof. Using the Friedrichs mollifier, we may assume that $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{\infty,\infty})$. For any multi-index α with $0 \le |\alpha| \le 2$, applying ∂_x^α to (3.2) and multiplying the resultant equation by $\nabla\partial_x^\alpha\sigma(t)$, we have

$$\begin{aligned} A_0\|\nabla\partial_x^\alpha\sigma(t)\|^2 &\le |(\partial_x^\alpha w_t(t), \nabla\partial_x^\alpha\sigma(t))| + \left|\left(\partial_x^\alpha\left\{\frac{1}{\rho^*}[\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))]\right\}, \nabla\partial_x^\alpha\sigma(t)\right)\right| \\ &\quad + |(I_\alpha(t), \nabla\partial_x^\alpha\sigma(t))| + |(\partial_x^\alpha f(t), \nabla\partial_x^\alpha\sigma(t))| \\ &\equiv K_1 + K_2 + K_3 + K_4, \end{aligned} \tag{3.31}$$

where $A_0 = \min_{\bar\rho/2 \le s \le 3\bar\rho/2} P'(s)/s$ and

$$I_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}A(t))\nabla\partial_x^\beta\sigma(t).$$

It immediately follows from the Sobolev inequality that

$$\begin{aligned} K_1 &\le \lambda\|\nabla^{|\alpha|+1}\sigma(t)\|^2 + C\lambda^{-1}\|\nabla^{|\alpha|}w_t(t)\|^2, \\ K_2 &\le \lambda\|\nabla^{|\alpha|+1}\sigma(t)\|^2 + C\lambda^{-1}\|\nabla^2 w(t)\|_{|\alpha|}^2, \\ K_3 &\le C\varepsilon\|\nabla\sigma(t)\|_{|\alpha|}^2. \end{aligned} \tag{3.32}$$

We employ Lemma 3.1 to estimate K_4. Using the Sobolev inequality and the Gagliard-Nirenberg inequality, we have

$$\begin{aligned} K_4 &\le C\Bigg\{\|\nabla v^*\|_{L_3}\|w(t)\|_{L_6} + \sum_{\nu=1}^{|\alpha|+1}\|(\nabla^\nu\rho^*, \nabla^{\nu+1}v^*)\|_{L_3}\|\sigma(t)\|_{L_6} \\ &\qquad + \|(\nabla\sigma, \nabla w)(t)\|_{|\alpha|-1,|\alpha|+1} + \|R_{|\alpha|}(t)\|\Bigg\}\|\nabla^{|\alpha|+1}\sigma(t)\| \\ &\le \lambda\|\nabla^{|\alpha|+1}\sigma(t)\|^2 + C\lambda^{-1}\|(\nabla\sigma, \nabla w)(t)\|_{|\alpha|-1,|\alpha|+1}^2 \end{aligned} \tag{3.33}$$

for $1 \le |\alpha| \le 2$. This calculation is also true for $\alpha = 0$, if we regard $R_0(t)$ and $\|\nabla\sigma(t)\|_{-1}$ as zero. Combining (3.31)–(3.33), we obtain (3.29) and (3.30) if we take $\varepsilon, \lambda > 0$ small enough. □

3.5. A Proof of Proposition 3.2.

Let $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{3,3})$ be a solution to (3.1)–(3.2) locally in time. Furthermore, we suppose that $\sup_{0\le t\le t_1}\|(\sigma, w)(t)\|_{3,3}, \|(\rho^* - \bar\rho, v^*)\|_{\mathscr{I}^{4,5}} \le \varepsilon$, where $\varepsilon > 0$ is small enough such that at least we can use Lemmas from 3.2 through 3.4. We use the notation:

$$[\sigma, w]_B(t) \equiv \|\sigma(t)\|^2 + (B(t)w(t), w(t)), \quad B(t) = \frac{(\rho^* + \sigma(t))^2}{P'(\rho^* + \sigma(t))}.$$

Summing up (3.8), (3.9) with $k = 1$, (3.20) and (3.29) (after multiplying (3.9), (3.20), (3.29) with small numbers respectively), we have

$$\frac{d}{dt}\left\{\sum_{v=0}^{1} \alpha_v [\nabla^v \sigma, \nabla^v w]_B + \beta_1 (w, \nabla\sigma)\right\} + \|(\nabla\sigma, \nabla w, w_t)\|_{0,1,0}^2 \leq 0, \tag{3.34}$$

if we take $\varepsilon, \lambda > 0$ sufficiently small. Here and hereafter, $\alpha_k, \beta_k > 0$ are constants depending only on μ and μ'. Similarly, summing up (3.9), (3.21), (3.30) with $k = 2$ and (3.34), we have

$$\frac{d}{dt}\left\{\sum_{v=0}^{2} \alpha_v [\nabla^v \sigma, \nabla^v w]_B + \sum_{v=1}^{2} \beta_v (\nabla^{v-1} w, \nabla^v \sigma)\right\} + \|(\nabla\sigma, \nabla w, w_t)\|_{1,2,1}^2 \leq 0. \tag{3.35}$$

Also, by (3.9), (3.21), (3.30) with $k = 3$ and (3.35), we obtain

$$\frac{d}{dt}\left\{\sum_{v=0}^{3} \alpha_v [\nabla^v \sigma, \nabla^v w]_B + \sum_{v=1}^{3} \beta_v (\nabla^{v-1} w, \nabla^v \sigma)\right\} + \|(\nabla\sigma, \nabla w, w_t)\|_{2,3,2}^2 \leq 0, \tag{3.36}$$

for any $t \in [0, t_1]$. Then, integration of (3.36) on $[0, t]$ implies that

$$N_B[\sigma, w](t) + \int_0^t \|(\nabla\sigma, \nabla w, w_t)(s)\|_{2,3,2}^2 \, ds \leq N_B[\sigma, w](0), \tag{3.37}$$

where $N_B[\sigma, w](s)$ is defined by

$$N_B[\sigma, w](s) \equiv \sum_{v=0}^{3} \alpha_v [\nabla^v \sigma, \nabla^v w]_B(s) + \sum_{v=1}^{3} \beta_v (\nabla^{v-1} w(s), \nabla^v \sigma(s))$$

for each $s \geq 0$.

Let us denote $B_0 = \min_{\bar\rho/2 \leq s \leq 3\bar\rho/2} \{s^2/P'(s), 1\}$ and $B_1 = \max_{\bar\rho/2 \leq s \leq 3\bar\rho/2} \{s^2/P'(s), 1\}$. Since we may assume without loss of generality that $\alpha_k \leq \alpha_{k-1}$ and $\beta_k \leq \alpha_k \min\{B_0, 1\}/4$ for $k = 1, 2, 3$, it follows from a simple calculation that

$$\frac{\alpha_3}{4} B_0 \|(\sigma, w)(s)\|_{3,3}^2 \leq N_B[\sigma, w](s) \leq 2B_1 \|(\sigma, w)(s)\|_{3,3}^2 \tag{3.38}$$

for each $s \in [0, t_1]$. Combining (3.37) and (3.38), we obtain (3.5), which completes the proof of Proposition 3.2. □

Hence, by Propositions 3.1 and 3.2, we finally arrive at the conclusion of Theorem 1.2.

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