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# On the Goursat problem in the Gevrey class for some second order equations

Dedicated to Professor Norio Shimakura on the occasion of his sixtieth birthday

By Yukiko Hasegawa

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**Abstract.** The Goursat problem for certain types of second order linear equations is considered. The Goursat problem for those second order equations is not $\mathscr{E}$-wellposed in general. For a certain type homogeneous equations, the Goursat problem is $\mathscr{E}$-wellposed. Necessary or sufficient conditions on lower order terms for $\mathscr{E}$-wellposedness are given. Wellposedness in Gevrey class is discussed.

## §1. Introduction.

We shall study the Goursat problem for the following second order linear partial differential operator:

$$P = \partial_t\partial_x - a(t)b(x)\partial_y^2 - F_1(t,x,y)\partial_y - F_2(t,x,y)\partial_t - F_3(t,x,y)\partial_x - F_4(t,x,y), \tag{1.1}$$

where $a(t)$, $b(x)$, and $F_i(t,x,y)$, $i = 1,2,3,4$ are real valued $C^\infty$-functions, $a(t) \geq 0$, $b(x) \geq 0$ and $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$.

For given $f, g, h$ with compatibility condition, the *Goursat problem* is to find a function $u(t,x,y)$ which satisfies

$$\begin{cases} Pu = f, \quad \text{in } \boldsymbol{R}_t^+ \times \boldsymbol{R}_x^+ \times \boldsymbol{R}_y, \\ u(0,x,y) = g(x,y), \\ u(t,0,y) = h(t,y), \\ g(0,y) = h(0,y). \end{cases} \tag{1.2}$$

By changing the unknown function, (1.2) can be reduced to

$$\begin{cases} Pu = f, \quad \text{in } \boldsymbol{R}_t^+ \times \boldsymbol{R}_x^+ \times \boldsymbol{R}_y, \\ u(0,x,y) = 0, \\ u(t,0,y) = 0. \end{cases} \tag{1.3}$$

In [**1**], we dealt with the Goursat problem:

$$\begin{cases} (\partial_t\partial_x - at^p x^q \partial_y^2)u = f, \quad \text{in } \boldsymbol{R}_t^+ \times \boldsymbol{R}_x^+ \times \boldsymbol{R}_y, \\ u(0,x,y) = 0, \\ u(t,0,y) = 0, \end{cases} \tag{1.4}$$

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where $a$ is a positive constant. This is a special case of (1.3), that is, the case that $a(t)b(x) = at^p x^q$ and $F_i = 0$, $i = 1,2,3,4$. In [**1**] we showed that the Goursat problem (1.4) has a unique solution in $C^\infty$ class. The Goursat problem:

$$(1.5)\qquad \begin{cases} (\partial_t\partial_x - a(t)b(x)\partial_y^2)u = f, & \text{in } \boldsymbol{R}_t^+ \times \boldsymbol{R}_x^+ \times \boldsymbol{R}_y, \\ u(0,x,y) = 0, \\ u(t,0,y) = 0, \end{cases}$$

is generalization of (1.4). We know that (1.5) has a unique solution in $C^\infty$ class. In this paper, we shall deal with lower order terms.

At first, let us consider a prototype $P = \partial_t\partial_x - \partial_y$, which is (1.1) with $a(t)b(x) = 0$, $F_1 = 1$ and $F_i = 0$, $i = 2,3,4$. Then, the Goursat problem

$$(1.6)\qquad \begin{cases} (\partial_t\partial_x - \partial_y)u = f, \\ u(0,x,y) = g(x,y), \\ u(t,0,y) = h(t,y), \\ g(0,y) = h(0,y), \end{cases}$$

is *not* $\mathscr{E}$-wellposed, which can be proved in a similar way to the proof of Theorem 2.3 in [**1**], p. 649. Hence, (1.2) cannot be $\mathscr{E}$-wellposed without any assumption on lower order terms. We shall study conditions on lower order terms that make (1.2) be solvable.

Now we introduce the *Gevrey class* of $s \geq 1$ with respect to $y$. Denote by $\Gamma^s_{(t,x)}$, the set of $C^\infty$ functions $f(t,x,y)$ such that for every compact set $K \subset \boldsymbol{R}_t^+ \times \boldsymbol{R}_x^+ \times \boldsymbol{R}_y$, there exist constants $M, C$ satisfying the following inequalities for fixed $i, j \in \boldsymbol{N} \cup \{0\}$, and for all $k \in \boldsymbol{N} \cup \{0\}$:

$$(1.7)\qquad \max_{(t,x,y)\in K} |\partial_t^i \partial_x^j \partial_y^k f| \leq MC^k (k!)^s.$$

Denote by $\Gamma^s_{(0,x)}$ (resp. $\Gamma^s_{(t,0)}$), the set of functions $f \in \Gamma^s_{(t,x)}$ which are independent of $t$ (resp. $x$).

The Goursat problem (1.2) is said to be *Gevrey-wellposed* if for any data $\{f,g,h\} \in \Gamma^s_{(t,x)} \times \Gamma^s_{(0,x)} \times \Gamma^s_{(t,0)}$ there exists a unique solution $u(t,x,y)$ of (1.2) belonging to $\Gamma^s_{(t,x)}$.

Our main result is the following:

Main Theorem. *If* $1 \leq s < 3/2$, *then* (1.2) *is Gevrey wellposed.*

Remark. In [**1**], we considered the Goursat problems in the half-spaces, that is, $\boldsymbol{R}_t^+ \times \boldsymbol{R}^2_{(x,y)}$ or $\boldsymbol{R}_t^- \times \boldsymbol{R}^2_{(x,y)}$. In this paper we discuss the Goursat problems in the quarter-space $\boldsymbol{R}_t^+ \times \boldsymbol{R}_x^+ \times \boldsymbol{R}_y$.

Other results will be given in the section 2 and those proofs will be given in the subsequent sections.

## §2. Other results.

In this section we give other results. Under extra conditions on lower order terms, we have more concrete results.

Theorem 2.1. *Assume that* $\partial_t F_2 = \partial_x F_3$, $F_i \in \Gamma^s_{(t,x)}$, $i = 1,2,3,4$ *and* $1 \leq s < 2$. *Then the Goursat problem* (1.2) *is Gevrey wellposed.*

Theorem 2.2. *Assume that* $F_i = 0$, $i = 2, 3$, $F_1(t, x, y) = a(t)b(x)q(y)$ *and* $q(y) \in C^\infty$, *then the Goursat problem* (1.2) *is* $\mathscr{E}$*-wellposed.*

Theorem 2.3. *Assume that* $P$ *has no lower order terms, namely* $F_i = 0$, $i = 1, 2, 3, 4$, $g(x, y) = 0$, $h(t, y) = 0$, *and* $f \in C^\infty$ (*resp.* $\Gamma^s_{(t,x)}$), *then the Goursat problem* (1.2) *has a unique solution* $u \in C^\infty$ (*resp.* $\Gamma^s_{(t,x)}$) *and* $u$ *can be represented in the form*:

$$u = \int_0^t \int_0^x \int_{-1}^1 \frac{f(\tau, \xi, y + \sigma\theta)}{\pi\sqrt{1 - \sigma^2}} d\sigma d\tau d\xi, \tag{2.1}$$

*where*

$$\theta = 2\sqrt{(A(t) - A(\tau))(B(x) - B(\xi))}, \quad A(t) = \int_0^t a(\tau)\, d\tau, \quad B(x) = \int_0^x b(\xi)\, d\xi. \tag{2.2}$$

*Moreover, the dependence domain of* $u(t, x, y)$ *is*

$$\begin{aligned} &D(t, x, y) \\ &\quad = \{(\tau, \xi, \eta) \mid 0 \le \tau \le t, 0 \le \xi \le x, |y - \eta| \le \theta = 2\sqrt{(A(t) - A(\tau))(B(x) - B(\xi))}\}. \end{aligned} \tag{2.3}$$

This dependence domain has the following property:

Corollary 2.1. *If* $(t, x, y) \in D(t_0, x_0, y_0)$, *then* $D(t, x, y) \subset D(t_0, x_0, y_0)$.

In the subsequent sections, we shall give proofs in the following way. First we prove Theorem 2.3. Secondly we prove Theorem 2.2. Thirdly we prove Theorem 2.1. At the last, we prove Main Theorem.

For the simplicity, after now we denote

$$\mathscr{L} = \partial_t \partial_x - a(t)b(x)\partial_y^2, \quad a(t) \ge 0, b(x) \ge 0. \tag{2.4}$$

## §3. Proof of Theorem 2.3.

By the assumptions in Theorem 2.3, (1.2) can be reduced to

$$\begin{cases} \mathscr{L}u = f, \quad \text{in } \boldsymbol{R}_t^+ \times \boldsymbol{R}_x^+ \times \boldsymbol{R}_y, \\ u(0, x, y) = 0, \\ u(t, 0, y) = 0. \end{cases} \tag{3.1}$$

Proposition 3.1. *Let define an operator* $K$ *by*

$$(Kf)(t, x, y) = \int_0^t \int_0^x \int_{-1}^1 \frac{f(\tau, \xi, y + \sigma\theta)}{\pi\sqrt{1 - \sigma^2}} d\sigma d\tau d\xi, \quad f(t, x, y) \in C^\infty, \tag{3.2}$$

*where* $\theta$ *is in* (2.2). *Then* $Kf \in C^\infty$ *and* $Kf$ *satisfies* (3.1).

Proposition 3.2. *If* $u \in C^\infty$ *satisfies* $\mathscr{L}u = 0$, $u(0, x, y) = 0$ *and* $u(t, 0, y) = 0$, *then* $u = 0$.

These propositions can be shown by the same way as in the proof of Theorem 2.2 (p. 646) in [**1**].

By these propositions, $Kf$ is the unique solution of (3.1). The dependence domain (2.3) follows from the right hand-side of (3.2).

## §4. Proof of Theorem 2.2.

When $F_2 = F_3 = 0$ and $F_1 = a(t)b(x)q(y)$, by changing the unknown function, (1.2) can be reduced to

$$\begin{cases} \mathscr{L}u = G(t,x,y)u + f(t,x,y), \\ u(0,x,y) = 0, \\ u(t,0,y) = 0. \end{cases} \tag{4.1}$$

Now, we fix $T, X, Y > 0$ and define $\Omega$ by

$$\Omega = \{(t,x,y) \mid (t,x,y) \in D(T,X,\eta), |\eta| < Y\}. \tag{4.2}$$

Hereafter we fix $\Omega$. Because of Corollary 2.1, we can study in $\Omega$.

Let us assume

$$|f(t,x,y)| < M, \quad |G(t,x,y)| < K, \quad (t,x,y) \in \Omega. \tag{4.3}$$

where $M$ and $K$ are positive constant. We constract a formal solution $\sum_{p=1}^{\infty} u_p(t,x,y)$ of (4.1) by the following recurrence equations:

$$\begin{cases} \mathscr{L}u_1 = f(t,x,y) \equiv f_0, \\ u_1(0,x,y) = 0, \\ u_1(t,0,y) = 0. \end{cases} \tag{4.4}$$

$$\begin{cases} \mathscr{L}u_p = Gu_{p-1} \equiv f_{p-1}, \\ u_p(0,x,y) = 0, \\ u_p(t,0,y) = 0, \quad p \geq 2. \end{cases} \tag{4.5}$$

We want to show that $\sum_{p=1}^{\infty} u_p(t,x,y)$ converges in $\Omega$. By Theorem 2.3, we have the following:

$$u_p = \int_0^t \int_0^x \int_{-1}^1 \frac{f_{p-1}(\tau,\xi,y+\sigma\theta)}{\pi\sqrt{1-\sigma^2}} d\sigma d\tau d\xi, \quad p \geq 1. \tag{4.6}$$

We have

$$|u_1| \leq \int_0^t \int_0^x \int_{-1}^1 \frac{M}{\pi\sqrt{1-\sigma^2}} d\sigma d\tau d\xi = txM. \tag{4.7}$$

Then by induction we obtain

$$|u_p| \leq MK^{p-1} \frac{(tx)^p}{(p!)^2}, \quad p \geq 1. \tag{4.8}$$

This shows that $\sum_{p=1}^{\infty} u_p$ converges absolutely and uniformly on $\Omega$.

We set $w = \sum_{p=1}^{\infty} u_p$. It is obvious that $w$ is the solution of (4.1) and $w \in C^{\infty}$.

Next, we shall show the uniqueness. Let $u$ be a solution of (4.1) with $f = 0$. And let

$$M_1 = \max_{(t,x,y) \in \Omega} |u(t,x,y)|. \tag{4.9}$$

By Theorem 2.3, we have

$$u = \int_0^t \int_0^x \int_{-1}^1 \frac{G(\tau, \xi, y + \sigma\theta)u(\tau, \xi, y + \sigma\theta)}{\pi\sqrt{1 - \sigma^2}} d\sigma d\tau d\xi. \tag{4.10}$$

Therefore

$$|u| \leq M_1 K t x. \tag{4.11}$$

By induction we have

$$|u| \leq M_1 K^p \frac{(tx)^p}{(p!)^2}, \quad \text{for } \forall p \geq 1. \tag{4.12}$$

Then $u \equiv 0$ in $\Omega$.

Thus we complete the proof of Theorem 2.2.

## §5. Proof of Theorem 2.1.

When $\partial_t F_2 = \partial_x F_3$, by changing the unknown function, (1.2) can be reduced to

$$\begin{cases} \mathscr{L}u = F(t, x, y)\partial_y u + G(t, x, y)u + f(t, x, y), \\ u(0, x, y) = 0, \\ u(t, 0, y) = 0. \end{cases} \tag{5.1}$$

Hereafter we fix $\Omega$ (defined in §4) and assume

$$|\partial_y^i F|, |\partial_y^i G| < M^{i+1}(i!)^s, \quad |\partial_y^j f| < (3M)^{j+1}(j!)^s, \tag{5.2}$$

where $M$ is positive constant and $i, j \in \boldsymbol{N} \cup \{0\}$.

We construct a formal solution $\sum_{p=1}^{\infty} u_p(t, x, y)$ of (5.1) by the following recurrence equations:

$$\begin{cases} \mathscr{L}u_1 = f(t, x, y) \equiv f_0, \\ u_1(0, x, y) = 0, \\ u_1(t, 0, y) = 0. \end{cases} \tag{5.3}$$

$$\begin{cases} \mathscr{L}u_p = (F\partial_y + G)u_{p-1} \equiv f_{p-1}, \\ u_p(0, x, y) = 0, \\ u_p(t, 0, y) = 0, \quad p \geq 2. \end{cases} \tag{5.4}$$

We want to show that $\sum_{p=1}^{\infty} u_p(t, x, y)$ converges in $\Omega$. From (5.2) we have the following lemma.

Lemma 5.1. *If $1 \leq s$, then for any $k \in \boldsymbol{N} \cup \{0\}$, we have*

$$|\partial_y^k u_p| \leq \frac{(tx)^p}{(p!)^2}(3M)^{k+2p-1}((k + p - 1)!)^s, \quad p \geq 1. \tag{5.5}$$

We can prove this lemma by induction. Lemma 5.1 implies the following proposition.

Proposition 5.1. *If $1 \leq s < 2$, then the series $\sum_{p=1}^{\infty} \partial_y^k u_p$ converges uniformly on $\Omega$ for any $k \in \boldsymbol{N} \cup \{0\}$.*

We set

$$w = \sum_{p=1}^{\infty} u_p(t,x,y). \tag{5.6}$$

It is obvious that $w$ is the solution of (5.1) and $w \in \Gamma^s_{(t,x)}$. Next, we shall show the uniqueness.

Proposition 5.2. *If* $u \in \Gamma^s_{(t,x)}$ *satisfies* $\mathscr{L}u = F\partial_y u + Gu$, $u(0,x,y) = 0$, $u(t,0,y) = 0$, *then* $u = 0$.

Proof. Let $u \in \Gamma^s_{(t,x)}$ be a solution of (5.1) with $f = 0$. Setting $\tilde{f} = F(t,x,y)\partial_y u + Gu$, we can assume

$$\begin{cases} |\partial_y^i F|, |\partial_y^i G| < M^{i+1}(i!)^s, \\ |\partial_y^i \tilde{f}| < (3M)^{i+1}(i!)^s, \end{cases} \tag{5.7}$$

where $M$ is a positive constant and $i \in \boldsymbol{N} \cup \{0\}$. By the same way as in Lemma 5.1, we have

$$|u| \le \frac{(tx)^p}{(p!)^2}(3M)^{2p-1}((p-1)!)^s, \quad (t,x,y) \in \Omega, p \ge 1. \tag{5.8}$$

By the assumption $s < 2$, this implies that $u \equiv 0$ in $\Omega$.

Thus we complete the proof of Theorem 2.1.

## §6. Proof of Main Theorem.

Recall that $\Omega$ is fixed. By the change of unknown function, (1.2) can be reduced to

$$\begin{cases} (\mathscr{L} - F(t,x,y)\partial_t - G(t,x,y)\partial_y - H(t,x,y))u = f, \\ u(0,x,y) = 0, \\ u(t,0,y) = 0. \end{cases} \tag{6.1}$$

We assume that for any $k \in \boldsymbol{N} \cup \{0\}$

$$\begin{cases} |\partial_y^k F|, |\partial_y^k G|, |\partial_y^k H| \le N^k(k!)^s, \\ |\partial_y^k f| \le N^{2k}(k!)^s, \\ 0 \le a(t)b(x) \le M, \quad (t,x,y) \in \Omega, \end{cases} \tag{6.2}$$

where $M$ and $N$ are positive constants. We construct a formal solution $\sum_{p=1}^{\infty} u_p(t,x,y)$ of (6.1) by the following recurrence equations:

$$\begin{cases} \mathscr{L}u_1 = f \equiv f_0, \\ u_1(0,x,y) = 0, \\ u_1(t,0,y) = 0. \end{cases} \tag{6.3}$$

$$\begin{cases} \mathscr{L}u_{p+1} = (F\partial_t + G\partial_y + H)u_p \equiv f_p, \\ u_{p+1}(0,x,y) = 0, \\ u_{p+1}(t,0,y) = 0, \quad p \ge 1. \end{cases} \tag{6.4}$$

By (6.2), we have the following estimate:

Lemma 6.1. *If* $1 \le s$, *then for any* $k$ *we have*

$$|\partial_y^k f_p| \le 2^p \sum_{|r|=p} \frac{p!}{r!} M^{r_1} \frac{t^{p-r_2}}{(p-r_2)!} \frac{x^{p+r_1}}{(p+r_1)!} (N^2)^{2r_1+r_3+k}((2r_1+r_3+k)!)^s, \quad p \ge 1, \tag{6.5}$$

*where* $r = (r_1, r_2, r_3, r_4)$.

We shall prove this lemma in the section 7. The solution of (6.4) is

$$u_{p+1} = \int_0^t \int_0^x \int_{-1}^1 \frac{f_p(\tau, \xi, y + \sigma\theta)}{\pi\sqrt{1-\sigma^2}} d\sigma d\tau d\xi, \tag{6.6}$$

where $\theta = 2\sqrt{(A(t) - A(\tau))(B(x) - B(\xi))}$, $A(t) = \int_0^t a(\tau)\, d\tau$, $B(x) = \int_0^x b(\xi)\, d\xi$.
Lemma 6.1 implies the following lemma.

Lemma 6.2. *If* $1 \le s$, *then for any* $k \in \boldsymbol{N} \bigcup \{0\}$ *we have*

$$|\partial_y^k u_{p+1}| \le C_1^p C_2^k \frac{((2p)!)^s (k!)^s}{p!(2p)!}, \quad \text{for } (t, x, y) \in \Omega, \tag{6.7}$$

*where* $C_1, C_2$ *are positive constants.*

By this lemma, we have the following:

Proposition 6.1. *If* $1 \le s < 3/2$, *then the series* $\sum_{p=1}^{\infty} \partial_y^k u_p(t, x, y)$ *converges uniformly on* $\Omega$.

We set

$$w = \sum_{p=1}^{\infty} u_p(t, x, y). \tag{6.8}$$

It is obvious that $w$ belongs to $\Gamma_{(t,x)}^s$ and satisfies (6.1). The uniqueness in $\Gamma_{(t,x)}^s$ can be shown by the same way as in the proof of Theorem 2.1. Thus we complete the proof of Main Theorem.

## §7. Proof of Lemma 6.1.

We prove this lemma by induction. Let us recall the definition of $u_p$ and $f_p$.

$$f_p = (F\partial_t + G\partial_y + H)u_p, \tag{7.1}$$

$$\begin{cases} \partial_t \partial_x u_p = a(t)b(x)\partial_y^2 u_p + f_{p-1}, \\ u_p(0, x, y) = 0, \\ u_p(t, 0, y) = 0, \quad p \ge 1. \end{cases} \tag{7.2}$$

For simplicity, we use the notations:

$$D_x^{-1} u = \int_0^x u(t, \xi, y)\, d\xi, \quad D_t^{-1} u = \int_0^t u(\tau, x, y)\, d\tau, \quad u \in C^\infty. \tag{7.3}$$

By (6.2) and (7.2), we obtain

$$\partial_t u_p = D_x^{-1}(a(t)b(x)\partial_y^2 u_p + f_{p-1}), \tag{7.4}$$

$$|\partial_y^k(\partial_t u_p)| \le M|D_x^{-1}(\partial_y^{k+2}u_p)| + |D_x^{-1}(\partial_y^k f_{p-1})|. \tag{7.5}$$

By Theorem 2.3, we have

$$u_p = \int_0^t\int_0^x\int_{-1}^1 \frac{f_{p-1}(\tau,\xi,y+\sigma\theta)}{\pi\sqrt{1-\sigma^2}}\,d\sigma d\tau d\xi,$$

$$|\partial_y^k u_p| \le D_t^{-1}D_x^{-1}|\partial_y^k f_{p-1}|. \tag{7.6}$$

By (7.6), (7.5) implies

$$|\partial_y^k(\partial_t u_p)| \le MD_t^{-1}(D_x^{-1})^2|\partial_y^{k+2}f_{p-1}| + D_x^{-1}|\partial_y^k f_{p-1}|. \tag{7.7}$$

Using (7.1) we have

$$\partial_y^k f_p = \sum_{l=0}^k \binom{k}{l}(\partial_y^l F)(\partial_y^{k-l}\partial_t u_p) + \sum_{l=0}^k\binom{k}{l}(\partial_y^l G)(\partial_y^{k-l+1}u_p) + \sum_{l=0}^k\binom{k}{l}(\partial_y^l H)(\partial_y^{k-l}u_p). \tag{7.8}$$

By (7.6), (7.7) and (7.8), we have

$$\begin{aligned}|\partial_y^k f_p| \le{}& \sum_{l=0}^k\binom{k}{l}|\partial_y^l F|\{MD_t^{-1}(D_x^{-1})^2|\partial_y^{k-l+2}f_{p-1}| + D_x^{-1}|\partial_y^{k-l}f_{p-1}|\}\\ &+ \sum_{l=0}^k\binom{k}{l}|\partial_y^l G|D_t^{-1}D_x^{-1}|\partial_y^{k-l+1}f_{p-1}|\\ &+ \sum_{l=0}^k\binom{k}{l}|\partial_y^l H|D_t^{-1}D_x^{-1}|\partial_y^{k-l}f_{p-1}|.\end{aligned} \tag{7.9}$$

Let us consider $p=1$ in (7.9). By (6.2), (6.3) and (7.9), we have

$$\begin{aligned}|\partial_y^k f_1| \le{}& \sum_{l=0}^k\binom{k}{l}N^l(l!)^s\Big\{Mt\frac{x^2}{2!}(N^2)^{k-l+2}((k-l+2)!)^s + x(N^2)^{k-l}((k-l)!)^s\\ &+ tx(N^2)^{k-l+1}((k-l+1)!)^s + tx(N^2)^{k-l}((k-l)!)^s\Big\}\\ ={}& \sum_{l=0}^k\Bigg[\binom{k}{l}\left(\frac{l!(k+2-l)!}{(k+2)!}\right)^s N^{-l}Mt\frac{x^2}{2!}(N^2)^{k+2}((k+2)!)^s\\ &+\binom{k}{l}\left(\frac{l!(k-l)!}{k!}\right)^s N^{-l}x(N^2)^k(k!)^s\end{aligned} \tag{7.10}$$

$$+\binom{k}{l}\left(\frac{l!(k+1-l)!}{(k+1)!}\right)^s N^{-l}tx(N^2)^{k+1}((k+1)!)^s$$

$$\left.+\binom{k}{l}\left(\frac{l!(k-l)!}{k!}\right)^s N^{-l}tx(N^2)^k(k!)^s\right].$$

Assuming $s \geq 1$ and $N > 2$, we have (7.11) and (7.12) for any $i \in \boldsymbol{N} \cup \{0\}$.

$$\binom{k}{l}\left(\frac{l!(k+i-l)!}{(k+i)!}\right)^s < \binom{k}{l}\left(\frac{l!(k-l)!}{k!}\right)^s = \left(\frac{l!(k-l)!}{k!}\right)^{s-1} < 1, \tag{7.11}$$

$$\sum_{l=0}^{k} N^{-l} < \frac{1}{1-1/N} = \frac{N}{N-1} < 2. \tag{7.12}$$

Then (7.10) implies

(7.13)
$$|\partial_y^k f_1| \leq 2\left(Mt\frac{x^2}{2!}(N^2)^{k+2}((k+2)!)^s + x(N^2)^k(k!)^s + tx(N^2)^{k+1}((k+1)!)^s + tx(N^2)^k(k!)^s\right).$$

This shows that (6.5) holds for $p = 1$. Let us assume that (6.5) holds for $p$ and any $k$. By (7.9) and the assumption of induction, we have

$$\begin{aligned}
|\partial_y^k f_{p+1}| \leq \sum_{l=0}^{k}\binom{k}{l}(l!)^s N^l \\
&\times\left(2^p M \sum_{|r|=p}\frac{p!}{r!}M^{r_1}\frac{t^{p+1-r_2}}{(p+1-r_2)!}\frac{x^{p+2+r_1}}{(p+2+r_1)!}\right. \\
&\times (N^2)^{2r_1+r_3+k+2-l}((2r_1+r_3+k+2-l)!)^s \\
&+ 2^p \sum_{|r|=p}\frac{p!}{r!}M^{r_1}\frac{t^{p-r_2}}{(p-r_2)!}\frac{x^{p+1+r_1}}{(p+1+r_1)!} \\
&\times (N^2)^{2r_1+r_3+k-l}((2r_1+r_3+k-l)!)^s \\
&+ 2^p \sum_{|r|=p}\frac{p!}{r!}M^{r_1}\frac{t^{p+1-r_2}}{(p+1-r_2)!}\frac{x^{p+1+r_1}}{(p+1+r_1)!} \\
&\times (N^2)^{2r_1+r_3+k-l+1}((2r_1+r_3+k-l+1)!)^s \\
&+ 2^p \sum_{|r|=p}\frac{p!}{r!}M^{r_1}\frac{t^{p+1-r_2}}{(p+1-r_2)!}\frac{x^{p+1+r_1}}{(p+1+r_1)!} \\
&\left.\times (N^2)^{2r_1+r_3+k-l}((2r_1+r_3+k-l)!)^s\right).
\end{aligned} \tag{7.14}$$

By (7.11) and (7.12), (7.14) implies

$$(7.15)\qquad |\partial_y^k f_{p+1}| \le 2^{p+1}\Bigg(\sum_{|r|=p}\frac{p!}{r!}M^{r_1+1}\frac{t^{p+1-r_2}}{(p+1-r_2)!}\frac{x^{p+2+r_1}}{(p+2+r_1)!}$$
$$\times (N^2)^{2r_1+r_3+k+2}((2r_1+r_3+k+2)!)^s$$
$$+\sum_{|r|=p}\frac{p!}{r!}M^{r_1}\frac{t^{p-r_2}}{(p-r_2)!}\frac{x^{p+1+r_1}}{(p+1+r_1)!}$$
$$\times (N^2)^{2r_1+r_3+k}((2r_1+r_3+k)!)^s$$
$$+\sum_{|r|=p}\frac{p!}{r!}M^{r_1}\frac{t^{p+1-r_2}}{(p+1-r_2)!}\frac{x^{p+1+r_1}}{(p+1+r_1)!}$$
$$\times (N^2)^{2r_1+r_3+k+1}((2r_1+r_3+k+1)!)^s$$
$$+\sum_{|r|=p}\frac{p!}{r!}M^{r_1}\frac{t^{p+1-r_2}}{(p+1-r_2)!}\frac{x^{p+1+r_1}}{(p+1+r_1)!}$$
$$\times (N^2)^{2r_1+r_3+k}((2r_1+r_3+k)!)^s\Bigg).$$

Let us consider the first term in the right-hand side of (7.15) and denote it by

$$(7.16)\qquad Q=\sum_{|r|=p}\frac{p!}{r!}M^{r_1+1}\frac{t^{p+1-r_2}}{(p+1-r_2)!}\frac{x^{p+2+r_1}}{(p+2+r_1)!}(N^2)^{2r_1+r_3+k+2}((2r_1+r_3+k+2)!)^s,$$

and use the notation $r_1+1=r_1'$, $r'=(r_1',r_2,r_3,r_4)$. If $|r|=p$, then $|r'|=p+1$. Therefore, (7.16) implies

(7.17)
$$Q=\sum_{|r'|=p+1,\,r_1'\ge 1}\frac{p!}{(r_1'-1)!r_2!r_3!r_4!}M^{r_1'}\frac{t^{p+1-r_2}}{(p+1-r_2)!}\frac{x^{p+1+r_1'}}{(p+1+r_1')!}(N^2)^{2r_1'+r_3+k}((2r_1'+r_3+k)!)^s.$$

Considering that $r_1'/(r_1'!)=0$ for $r_1'=0$, (7.17) implies

(7.18)
$$Q=\sum_{|r'|=p+1}\frac{p!r_1'}{(r_1')!r_2!r_3!r_4!}M^{r_1'}\frac{t^{p+1-r_2}}{(p+1-r_2)!}\frac{x^{p+1+r_1'}}{(p+1+r_1')!}(N^2)^{2r_1'+r_3+k}((2r_1'+r_3+k)!)^s.$$

Replacing $r_1'$ by $r_1$ and $r'$ by $r$, (7.18) implies

$$(7.19)\qquad Q=\sum_{|r|=p+1}\frac{p!r_1}{r!}M^{r_1}\frac{t^{p+1-r_2}}{(p+1-r_2)!}\frac{x^{p+1+r_1}}{(p+1+r_1)!}(N^2)^{2r_1+r_3+k}((2r_1+r_3+k)!)^s.$$

The other terms in the right-hand side of (7.15) can be treated with the same way as the above, (7.15) implies

$$(7.20) \qquad |\partial_y^k f_{p+1}| \le 2^{p+1}\left(\sum_{|r|=p+1} \frac{p!(r_1+r_2+r_3+r_4)}{r!} M^{r_1} \right.$$
$$\left. \times \frac{t^{p+1-r_2}}{(p+1-r_2)!} \frac{x^{p+1+r_1}}{(p+1+r_1)!} (N^2)^{2r_1+r_3+k}((2r_1+r_3+k)!)^s \right)$$
$$= 2^{p+1}\left(\sum_{|r|=p+1} \frac{(p+1)!}{r!} M^{r_1} \right.$$
$$\left. \times \frac{t^{p+1-r_2}}{(p+1-r_2)!} \frac{x^{p+1+r_1}}{(p+1+r_1)!} (N^2)^{2r_1+r_3+k}((2r_1+r_3+k)!)^s \right).$$

This shows that (6.5) holds for $p+1$. Thus we complete the proof of Lemma 6.1.

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## References

[1] Y. Hasegawa, On the $C^\infty$ Goursat problem for some second order equations with variable coefficients, J. Math. Soc. Japan, **50**, No. 3 (1998), 639–662.

Yukiko HASEGAWA
R-106, toriimae 8-5, Enmyouji
Ooyamazaki-cho, Otokunigun
Kyoto, 618-0091
Japan