J. Math. Soc. Japan
Vol. 55, No. 3, 2003

Variety of nets of degree $g-1$ on smooth curves of low genus

Dedicated to Professor Makoto Namba on the occasion of his sixtieth birthday

By Kyung-Hye Cho, Changho Keem[†] and Akira Ohbuchi[‡]

(Received Mar. 14, 2001)
(Revised Dec. 10, 2001)

Abstract. We classify smooth complex projective algebraic curves C of low genus $7 \le g \le 10$ such that the variety of nets $W^2_{g-1}(C)$ has dimension $g-7$. We show that $\dim W^2_{g-1}(C) = g-7$ is equivalent to the following conditions according to the values of the genus g. (i) C is either trigonal, a double covering of a curve of genus 2 or a smooth plane curve degree 6 for $g=10$. (ii) C is either trigonal, a double covering of a curve of genus 2, a tetragonal curve with a smooth model of degree 8 in $\boldsymbol{P}^3$ or a tetragonal curve with a plane model of degree 6 for $g=9$. (iii) C is either trigonal or has a birationally very ample g^2_6 for $g=8$ or $g=7$.

1. Introduction and motivation.

Let C be a smooth projective algebraic curve of genus g over the field of complex numbers. We denote by $W^r_d(C)$ the locus in the Jacobian variety $J(C)$ corresponding to those line bundles of degree d with $r+1$ or more independent global sections. Then $W^r_d(C)$ is a subvariety of $J(C)$ and can equivalently be viewed as the subvariety consisting of all effective divisor classes of degree d which move in a linear system of projective dimension at least r.

By a well known thorem of Kleiman-Laksov [**KL**], if $d \le g+r-2$, the dimension of $W^r_d(C)$ is greater than or equal to the Brill-Noether number

$$\rho(d,g,r) := g-(r+1)(g-d+r)$$

for any curve C. Furthermore, by a theorem of Griffiths-Harris [**GH**], the dimension of $W^r_d(C)$ is equal to $\rho(d,g,r)$ for a general curve C; whereas the dimension of $W^r_d(C)$ might be greater than $\rho(d,g,r)$ for some special curves C.

On the other hand, the upper bound on the dimension of $W^r_d(C)$ and the description of those special (in the sense of moduli) curves C such that $W^r_d(C)$ has dimension more than the expected value $\rho(d,g,r)$ were given by H. Martens and D. Mumford, which can be stated as follows; cf. [**Ma**], [**Mu**], or [**ACGH**].

2000 *Mathematics Subject Classification*. 14H45, 14H10, 14C20.
Key Words and Phrases. algebraic curves, linear series, branched covering.

[†] During the period when this manuscript was prepared for publication, the second named author was visiting the Mathematics Department of University of Notre Dame with a support from Seoam Scholarship Foundation. He is currently affiliated with the Research Institute of Mathematics at Seoul National University as a research fellow.
[‡] The third named author was partially supported by Grant-in-Aid for Scientific Research (#11640032), Japan Society for the Promotion of Science.

Theorem 1.1 (H. Martens). *Let d and r be integers such that*

$$d - 2r > \rho(d, g, r), \quad r \geq 1.$$

Then

$$\dim W_d^r(C) \leq d - 2r$$

and the equality holds if and only if C is hyperelliptic.

Theorem 1.2 (Mumford). *Let d and r be integers such that*

$$d - 2r - 1 > \rho(d, g, r), \quad r \geq 1.$$

Suppose that

$$\dim W_d^r(C) = d - 2r - 1.$$

Then C is either trigonal, bi-elliptic or a smooth plane quintic.

There have been several partial extensions of the above two theorems due to many authors; cf. [**BKMO**], [**C**], [**K**] and [**Muk**]). Furthermore, by a recent progress made by the authors in [**CKO**], the next extension of H. Martens-Mumford theorem on dimensions of $W_d^r(C)$ for a smooth curve C has been finished off and therefore one knows that the following statement holds; [**CKO**; Theorem 1.5].

Theorem 1.3. *Let C be a smooth algebraic curve of genus g. Let d and r be integers such that*

$$d - 2r - 2 > \rho(d, g, r), \quad r \geq 1.$$

If

$$\dim W_d^r(C) \geq d - 2r - 2 \geq 0$$

then C is either hyperelliptic, trigonal, bi-elliptic, tetragonal, a smooth plane sextic or a double covering of a curve of genus 2.

Indeed, [**CKO**; Theorem 1.4] gives necessary conditions for C satisfying

$$\dim W_{g-1}^2 = g - 7,$$

which was the only case left out in previous extensions of H. Martens-Mumford's Theorem. Furthermore, in the range of the genus $g \geq 11$, [**CKO**; Theorem 1.4] has been pushed forward and it has been shown that $\dim W_{g-1}^2 = g - 7$ **if and only if** C is either trigonal or a double covering of a curve of genus 2, eliminating the possibility for C being tetragonal other than a two sheeted covering over a curve of genus 2; cf. [**CKO**; Theorem 1.7]. However, [**CKO**] did not treat curves of low genus with $\dim W_{g-1}^2 = g - 7$, namely in the genus range $7 \leq g \leq 10$, in the same way as higher genus curves were treated. The aim of this paper is to pursue a complete description of those special curves and to come up with a necessary and sufficient condition for C having $\dim W_{g-1}^2 = g - 7$ when the genus of the curve C is low. Our main results are:

THEOREM I. *Let C be a smooth projective algebraic curve of genus $g = 10$. Then* $\dim W^2_{g-1}(C) = g - 7$ *if and only if C is either trigonal, a double covering of a curve of genus* 2 *or a smooth plane curve degree* 6.

THEOREM II. *Let C be a smooth projective algebraic curve of genus $g = 9$. Then* $\dim W^2_{g-1}(C) = g - 7$ *if and only if C is either trigonal, a double covering of a curve of genus* 2, *a tetragonal curve with a smooth model of degree* 8 *in $\boldsymbol{P}^3$ or a tetragonal curve with a plane model of degree* 6.

THEOREM III. *Let C be a smooth algebraic curve of genus $g = 8$ or $g = 7$. Then* $\dim W^2_{g-1}(C) = g - 7$ *if and only if C is either trigonal or has a birationally very ample* g^2_6.

One notes immediately that nearly (but not exactly) the same statements as [**CKO**; Theorem 1.7] hold. However, unlike the case $g \geq 11$, there appear smooth plane sextics and some particular tetragonal curves C with $\dim W^2_{g-1}(C) = g - 7$ other than double coverings of genus two curves or trigonal curves. On the technical side, some of the lemmas which were used to prove [**CKO**; Theorem 1.6] and [**CKO**; Theorem 1.7]—e.g. [**CKO**; Lemma 3.4] which describes the component of $W^1_{g-3}(C)$ of maximal dimension on a tetragonal curve—still have to be verified for curves of low genus and this will require preparing several relevant results on $W^r_d(C)$ for curves of low genus whose proof we could not locate in any of the literature.

The organization of this paper is as follows. In Section 2, we collect several results obtained in [**CKO**] which we will be using in this paper. In section 3, we prove Theorem I after finding a proper description of a component of $W^1_{g-3}(C)$ of maximal dimension on a tetragonal curve of low genus. In section 4, we prove Theorem II. In section 5, after proving Theorem III, we discuss related results on $W^2_{g-1}(C)$ for a double coverings of a curve of genus 2.

For notations and conventions, we adopt those from [**ACGH**]. Specifically, C always denotes a smooth irreducible complex projective curve and g^r_d is a possibly incomplete r-dimensional linear system of degree d on C. A g^r_d is said to be birationally very ample if the induced morphism $C \to \boldsymbol{P}^r$ given by the base-point-free part of g^r_d is birational onto its image. We also say that a line bundle $\mathscr{L} \in \mathrm{Pic}^d(C)$ is birationally very ample if the corresponding complete linear system g^r_d is birationally very ample. The set of all effective divisors of degree d on C is denoted by C_d. K_C and ω_C denote a canonical divisor and the canonical bundle on C respectively. A curve C is called k-gonal if C has a g^1_k but no g^1_{k-1}.

2. Preliminary results.

We first collect several elementary results regarding $W^2_{g-1}(C)$ which have been observed in [**CKO**] already; cf. [**CKO**; Remark 2.1, Proposition 2.2 and Corollary 2.3].

REMARK 2.1. Let C be a smooth algebraic curve of genus g.

(i) $\dim W^2_{g-1}(C) = g - 5 \geq 0$ if and only if C is hyperelliptic.

(ii) If $g \geq 7$, $\dim W^2_{g-1}(C) = g - 6$ if and only if C is bi-elliptic.

(iii) For $g = 6$, $\dim W^2_{g-1}(C) = g - 6$ if and only if C is a smooth plane quintic.

(iv) For a bi-elliptic curve of genus 6, $W^2_{g-1}(C) = \emptyset$.
(v) For a trigonal curve of genus $g \geq 7$, $\dim W^2_{g-1}(C) = g - 7$.
(vi) If C is a double covering of a curve of genus 2 and $g \geq 9$, $\dim W^2_{g-1}(C) = g - 7$.

We will make use of the following lemmas which also have been proved in [**CKO**]; cf. [**CKO**; Lemma 2.5, Lemma 2.6 and Lemma 3.1].

Lemma 2.2. *Let C be a smooth algebraic curve of genus $g \geq 7$ which is neither a double covering of a curve of genus $h \leq 2$ nor a trigonal curve. Assume* $\dim W^2_{g-1}(C) = g - 7$ *and let X be a component of $W^2_{g-1}(C)$ of maximal dimension. If every component of $W^2_{g-1}(C)$ of maximal dimension is generically base-point-free, then for a general element $\mathscr{L} \in X$, both $\mathscr{L}$ and $\omega_C \otimes \mathscr{L}^{-1}$ are complete base-point-free birationally very ample nets.*

Lemma 2.3. *Let C be a smooth algebraic curve of genus $g \geq 7$ such that* $\dim W^2_{g-1}(C) = g - 7$. *Suppose that every component of $W^2_{g-1}(C)$ of maximal dimension is generically base-point-free. Assume further that for a general member $\mathscr{L} \in X$—where $X \subset W^2_{g-1}(C)$ is a component of maximal dimension—both $\mathscr{L}$ and $\omega_C \otimes \mathscr{L}^{-1}$ are birationally very ample. Then the following statements hold.*
(i) $\dim W^1_{g-3}(C) \geq g - 7$.
(ii) *If* $\dim W^1_{g-3}(C) = g - 7$, *then there is a component $T \subset W^1_{g-3}(C)$ with* $\dim T = g - 7$ *such that every $\mathscr{L} \in X$ is of the form*

$$\mathscr{L} = \mathscr{M} \otimes \mathcal{O}_C(P + Q)$$

for some $\mathscr{M} \in T$ and some $P, Q \in C$.

Lemma 2.4. *Let C a smooth tetragonal curve of genus $g \geq 7$ with* $\dim W^2_{g-1}(C) = g - 7$. *We fix a g^1_4 on C. Suppose that $h^0(C, \mathcal{O}_C(2g^1_4)) = 3$. Let $X \subset W^2_{g-1}(C)$ be a component of maximal dimension and set*

$$\mathscr{E}_X := \{D \in C_{g-5} \mid |g^1_4 + D| \in X\}.$$

Then for any $D \in \mathscr{E}_X$, $|K_C - 2g^1_4 - D| \neq \emptyset$ and

$$|K_C - 2g^1_4| = \bigcup_{D \in \mathscr{E}_X} D + |K_C - 2g^1_4 - D|,$$

where the locus $D + |K_C - 2g^1_4 - D| \subset C_{2g-10}$ is considered as a subset of $|K_C - 2g^1_4|$.

Let's briefly recall basic notions of scrollar invariants of an algebraic curve with a pencil g^1_d. For a smooth algebraic curve C with a complete base-point-free pencil g^1_d, we set

$$F_i = H^0(C, \omega_C \otimes \mathcal{O}_C(-ig^1_d)).$$

The vector spaces F_i $(i = 1, 2, \dots)$ give a filtration,

$$F_0 \supset F_1 \supset \cdots \supset F_n \supset \cdots$$

and we define the scrollar invariants $e_i = e_i(g^1_d)$ $(i = 1, 2, \dots, d-1)$ by

$$e_i = e_i(g^1_d) = \#\{j \in \mathbf{N}; \dim(F_{j-1}/F_j) \geq i\} - 1 \quad (i = 1, 2, \dots, d-1).$$

One can easily show that

$$e_1 + \cdots + e_{d-1} = g - d + 1 \quad \text{and} \quad e_{d-1} \le \cdots \le e_1$$

hold; cf. [**KO**] for further details.

The following lemma, which may seem to be a little bit technical, however plays an important role as it did in [**CKO**]; cf. [**CKO**; Lemma 3.4].

LEMMA 2.5. *Let C be a smooth tetragonal curve with a unique g^1_4 of genus $g \ge 8$. We assume that the following conditions hold on C:*

(i) $\dim W^2_{g-1}(C) = g - 7$.

(ii) *C has no g^2_6.*

(iii) *C is not a double covering of a curve of genus 2 in case $g \ge 9$.*

(iv) *For a general $\mathscr{L} \in X$—where $X \subset W^2_{g-1}(C)$ is a component of maximal dimension—both $\mathscr{L}$ and $\omega_C \otimes \mathscr{L}^{-1}$ are base-point-free, birationally very ample and $|\mathscr{L} - g^1_4| \ne \emptyset$, $|\omega_C \otimes \mathscr{L}^{-1} - g^1_4| \ne \emptyset$.*

(v) *For $g = 9$, $e_3 \ge 1$ and $(e_2, e_3) \ne (1, 1)$.*

(vi) *For $g = 8$, $e_3 \ge 1$.*

Let $\psi_{\mathscr{L}} : C \to C_{\mathscr{L}} \subset \boldsymbol{P}^2$ be the morphism defined by $\mathscr{L} \in X$ and let $\tilde{P} \in C_{\mathscr{L}}$ be the $(g-5)$-fold singular point corresponding to g^1_4, i.e. the image of points in the support of $|\mathscr{L} - g^1_4|$. Then $\tilde{P} \in C_{\mathscr{L}}$ is an ordinary singular point if $\mathscr{L} \in X$ is general.

We close this section by recalling the well-known Riemann-Hurwitz relation for double coverings. Let E be a curve of genus h and let $\pi : C \to E$ be a double covering. Let $R \subset E$ be a branch locus of π. Then we have

$$\pi_*(\mathscr{O}_C) \cong \mathscr{O}_E \oplus \mathscr{S} \quad \text{and} \quad \mathscr{S}^{\otimes 2} \cong \mathscr{O}_E(-R). \tag{2.A}$$

Also the algebra structure of $\pi_*(\mathscr{O}_C)$ is given by the isomorphism

$$\psi : \mathscr{S}^{\otimes 2} \cong \mathscr{O}_E(-R) \subset \mathscr{O}_E. \tag{2.B}$$

3. Curves of genus ten.

In this section we mainly treat curves of genus $g = 10$ with $\dim W^2_{g-1}(C) = g - 7$. As was mentioned earlier in the introduction, we have new classes of curves in Theorems I and II such as smooth plane sextics or some special tetragonal curves C with $\dim W^2_{g-1}(C) = g - 7$ besides double coverings of genus two curves or trigonal curves. Specifically, some of these curves emerge fairly naturally in the course of the proofs of the lemmas which describe the component of $W^1_{g-3}(C)$ of maximal dimension on a tetragonal curve of genus $g = 9, 10$.

For trigonal curves or double covering of genus two curves, we already have $\dim W^2_{g-1}(C) = g - 7$; cf. Remark 2.1 (v) and (vi). For the other curves C which newly appear in Theorems I and II, one can show easily that $\dim W^2_{g-1}(C) = g - 7$ as follows.

PROPOSITION 3.1. *Let C be a smooth algebraic curve of genus $g \ge 7$ with a birationally very ample g^2_6. Then $\dim W^2_{g-1}(C) = g - 7$.*

PROOF. C always has a base-point-free and complete g^1_5 cut out by lines through a general point of the plane model induced by g^2_6. We note that $\{g^2_6\} + W_{g-7}(C)$

is an irreducible subvariety of $W^2_{g-1}(C)$, therefore $\dim W^2_{g-1}(C) \geq g-7$. Suppose that $\dim W^2_{g-1}(C) > g-7$. By Remark 2.1 (i) and (ii), C must be either hyperelliptic or bi-elliptic, in which cases C cannot have a base-point-free and complete g^1_5 by the Castelnuovo-Severi inequality; cf. [**A**; Theorem 3.5]. □

Remark 3.2. (i) For a tetragonal curve C of genus $g = 9$ with a smooth model of degree 8 in $\boldsymbol{P}^3$, the locus

$$\{\mathcal{O}_C(g^3_8 - P + Q_1 + \cdots + Q_{g-8}) \mid P, Q_1, \ldots, Q_{g-8} \in C\}$$

is an irreducible subvariety of $W^2_{g-1}(C)$ of dimension $g-7$. Since C is an extremal space curve of degree 8, C lies on a quadric surface $S \subset \boldsymbol{P}^3$ and hence C is a complete intersection of a quadric and a quartic. By using the adjunction formula for curves on a smooth quadric surface (or on the ruled surface $\boldsymbol{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-2))$ in case S is a quadric cone), one can deduce that the canonical linear system $|K_C|$ is cut out by quadrics. A divisor $D \in g^1_4$ must be collinear in $\boldsymbol{P}^3$ since D fails to impose independent conditions on $|K_C|$ which is cut out by quadrics, whence a g^1_4 is cut out by the rulings of S and $\dim W^1_4(C) = 0$; cf. [**ACGH**; Exercise F-2, page 199]. Thus C is not bi-elliptic and hence $\dim W^2_{g-1}(C) = g-7$ by Remark 2.1 (ii).

(ii) For a tetragonal curve of genus $g = 9$, it is easy to verify that there exists a smooth model of degree 8 in $\boldsymbol{P}^3$ if and only if either $e_3 = 0$ (with a unique g^1_4) or there exist two g^1_4's.

(iii) It is worthwhile to note that curves which newly appeared in Theorem I and Theorem II—i.e. a smooth plane sextic, a curve with a plane model of degree 6 or a tetragonal curve with a smooth space model of degree 8—are neither trigonal nor a double covering of a curve of genus two. It is clear that a smooth plane sextic (or a curve with a plane model of degree 6) is not a double covering of a curve of genus 2; a plane sextic has a base-point-free g^1_5, whereas a double covering of a curve of genus 2 does not have a base-point-free g^1_5 by Castelnuovo-Severi inequality.

Let $\pi : C \to E$ be a double covering of a curve of genus $h = 2$ and $g(C) = 9$. By (2.A), $\deg \mathscr{S} = -6$ and

$$h^0(C, \mathcal{O}_C(2\pi^* g^1_2)) = h^0(E, \mathcal{O}_E(2g^1_2)) + h^0(E, \mathcal{O}_E(2g^1_2) \otimes \mathscr{S}) = 3,$$

which implies $e_3 \neq 0$. Furthermore, C has only one $g^1_4 = \pi^*(g^1_2)$ by Castelnuovo-Severi inequality. Therefore it follows that C cannot have a smooth space model of degree 8 by (ii).

(iv) For the proofs of Theorem I and Theorem II, we will use Lemma 2.5 regarding a tetragonal curve having a plane model of degree $g-1$ with an ordinary singular point of high multiplicity. Recall that Lemma 2.5 holds for a tetragonal curve of genus $g = 9$ under the assumption $(e_2, e_3) \neq (1, 1)$. For tetragonal of curve of genus $g = 9$ with $(e_2, e_3) = (1, 1)$, we have the following result.

Lemma 3.3. *Let C be a tetragonal curve of genus $g = 9$ with a unique g^1_4 such that $e_3 = e_2 = 1$ i.e. $h^0(C, \mathcal{O}_C(2g^1_4)) = 3$, $h^0(C, \mathcal{O}_C(3g^1_4)) = 6$. Suppose that C does not have a g^2_6 and* $\dim W^2_{g-1}(C) = g-7$. *Then C is a double covering of a curve of genus* 2.

Proof. We first claim that $W^2_8(C) \subset \{g^1_4\} + W_4(C)$. For $\mathscr{L} \in W^2_8(C)$ which is of the form $\mathscr{L} = 2g^1_4$, we clearly have $\mathscr{L} \in \{g^1_4\} + W_4(C)$. Therefore we assume $\mathscr{L} \neq 2g^1_4$. Note that we have the following exact sequence by the base-point-free pencil trick;

(3.3.1) $\quad 0 \to H^0(C, \mathscr{L} \otimes \mathscr{O}(-g^1_4)) \to H^0(C, \mathscr{L}) \otimes H^0(C, \mathscr{O}_C(g^1_4)) \to H^0(C, \mathscr{L} \otimes \mathscr{O}_C(g^1_4)).$

Recall that $e_1 + e_2 + e_3 = g - 3$ and hence $e_1 = 4$, which in turn implies $\omega_C \cong \mathscr{O}_C(4g^1_4)$. If $h^0(C, \mathscr{L} \otimes \mathscr{O}_C(g^1_4)) \geq 6$, then $\omega_C \otimes \mathscr{L}^{-1} \otimes \mathscr{O}_C(-g^1_4) \cong \mathscr{O}_C(g^1_4)$ by Riemann-Roch formula and the uniqueness of g^1_4, hence $\mathscr{L} = 2g^1_4$ which is a contradiction. Therefore we have $h^0(C, \mathscr{L} \otimes \mathscr{O}_C(g^1_4)) \leq 5$ which implies $h^0(C, \mathscr{L} \otimes \mathscr{O}_C(-g^1_4)) = 1$ by (3.3.1), finishing the claim.

From now on, we assume that C is not a double covering of a curve of genus 2.

(3.3.2) Claim: For any $P + Q \in C_2$, there exists $A + B \in C_2$ such that $\mathscr{O}_C(g^1_4 + P + Q + A + B) \in W^2_8(C)$.

We consider the locus

$$\Sigma := \{(\mathscr{L}, \mathscr{O}_C(P+Q)) \mid P + Q \leq D \text{ for some } D \in |\mathscr{L} - g^1_4|\} \subset W^2_8(C) \times W_2(C)$$

and the projection map $\Sigma \xrightarrow{\phi} W_2(C)$ to the second factor. Take an element $\mathscr{O}_C(P+Q) \in \phi(\Sigma) \subset W_2(C)$. By the non-existence of g^2_6, $|K_C - g^1_4|$ is very ample. Suppose that $|K_C - g^1_4 - P - Q| = g^3_{10}$ is not base-point-free with a base point $T \in C$. Then $|K_C - g^1_4 - P - Q - T| = g^3_9$ and therefore $|K_C - g^1_4 - P - Q - T - S| \in W^2_8(C)$ for general $S \in C$. By the previous claim,

$$|K_C - 2g^1_4 - P - Q - T - S| = |2g^1_4 - P - Q - T - S| \neq \varnothing.$$

Since $S \in C$ is general, $|2g^1_4 - P - Q - T|$ is a pencil of degree 5. Therefore, by $h^0(C, \mathscr{O}_C(2g^1_4)) = 3$, there exists $R \in C$ such that $P + Q + T + R \in g^1_4$. But then $g^2_7 = |g^1_4 + P + Q + T| = |2g^1_4 - R|$ and R is a base point of $2g^1_4$ which is a contradiction. Therefore $|K_C - g^1_4 - P - Q|$ must be base-point-free and birationally very ample since C is not a double covering of a curve of genus 2. Note that $\mathscr{O}_C(g^1_4 + P + Q + A + B) \in W^2_8(C)$ for some $A + B \in C_2$ if and only if $A + B$ maps to a singular point of the model induced by the birationally very ample base-point-free $|K_C - g^1_4 - P - Q|$. Therefore ϕ has a finite fiber over a general point in the image. Furthermore since $\dim \Sigma = 2$, ϕ is in fact surjective and this finishes the proof of (3.3.2).

We take general $P_1 \in C$ such that P_1 is contained in a reduced member $P_1 + P_2 + P_3 + P_4 \in g^1_4$.

(3.3.3) Claim: $|3g^1_4 - P_i - P_j|$ is base-point-free g^3_{10} for any $i \neq j \in \{1, 2, 3, 4\}$.

Without loss of generality, we assume that $i = 1$, $j = 2$. Note that $\mathrm{Bs}|3g^1_4 - P_1 - P_2| \subset \{P_3, P_4\}$ since $|3g^1_4 - P_1 - P_2| = |2g^1_4 + P_3 + P_4|$. Assume that the base locus is not empty, say $P_3 \in \mathrm{Bs}|3g^1_4 - P_1 - P_2|$. Then $h^0(C, \mathscr{O}_C(3g^1_4 - P_1 - P_2 - P_3)) = 4$ and by taking the dual series it follows that $h^0(C, \mathscr{O}_C(g^1_4 + P_1 + P_2 + P_3)) = 3$, contradicting $|2g^1_4|$ being a base-point-free g^2_8.

We now take $Q_1 \in C$ such that $Q_1 \notin \mathrm{Supp}(P_1 + P_2 + P_3 + P_4)$ and let $Q_1 + Q_2 +$

$Q_3 + Q_4 \in g_4^1$. By Claim (3.3.2) we already know that $\mathcal{O}_C(g_4^1 + P_1 + Q_1 + A + B) \in W_8^2(C)$ for some $A, B \in C$. By Lemma 2.4,

$$|K_C - 2g_4^1 - P_1 - Q_1 - A - B| = |2g_4^1 - P_1 - Q_1 - A - B| = \left|\sum_{i=2}^{4}(P_i + Q_i) - A - B\right| \neq \emptyset$$

and $h^0(C, \mathcal{O}_C(2g_4^1 - P_1 - Q_1)) = 1$ by the general choice of P_1 and Q_1 therefore one finds that $A + B < \sum_{i=2}^{4}(P_i + Q_i)$. Assume that $A, B \in \{P_2, P_3, P_4\}$, say $A = P_2$, $B = P_3$ and hence $h^0(C, \mathcal{O}_C(g_4^1 + P_1 + P_2 + P_3 + Q_1)) = 3$. But this is a contradiction; since $|2g_4^1 + Q_1|$ is a g_9^2 with the unique base point Q_1, P_4 cannot be a base point of $|2g_4^1 + Q_1|$. The case $A, B \in \{Q_2, Q_3, Q_4\}$ does not occur by the same reason. Therefore we must have $A = P_2$, $B = Q_2$ and $h^0(C, \mathcal{O}_C(g_4^1 + \sum_{i=1}^{2}(P_i + Q_i))) = 3$. By taking the dual series,

$$h^0\left(C, \mathcal{O}_C\left(K_C - g_4^1 - \sum_{i=1}^{2}(P_i + Q_i)\right)\right) = h^0\left(C, \mathcal{O}_C\left(3g_4^1 - \sum_{i=1}^{2}(P_i + Q_i)\right)\right) = 3.$$

As we vary $Q_1 \in C$, it follows that the base-point-free $g_{10}^3 = |3g_4^1 - P_1 - P_2|$ induces a degree 2 morphism onto a curve of degree 5 in $\boldsymbol{P}^3$. Therefore C becomes a double covering of a curve of genus 2, contrary to the assumption which we made before (3.3.2). □

For curves of genus $g = 9$ or 10, we would like to have a proper description of components of $W_{g-3}^1(C)$ of maximal dimension; cf. [**CKO**; Lemma 3.4] in higher genus cases. For this purpose, we begin with the following which is due to M. Coppens.

Lemma 3.4. *Let C be a smooth algebraic curve of genus g. If $g = 10$ and* $\dim W_7^1(C) = 3$, *then* $\dim W_6^1(C) = 2$. *If $g = 9$ and* $\dim W_6^1(C) = 2$, *then* $\dim W_5^1(C) = 1$.

Proof. See [**K**; Lemma 2.6] or [**C**; Proposition 12 and Proposition 13]. □

Lemma 3.5. (i) *Let $A \subset W_d^r(C)$ be an irreducible closed subset satisfying*

$$\dim A \geq r + 1.$$

Then for any $P \in C$,

$$\dim[W_{d-1}^r(C) \otimes \mathcal{O}_C(P)] \cap A \geq \dim A - (r + 1).$$

(ii) *Let C be a tetragonal curve without g_6^2. Then for a component $A \subset W_6^1(C)$ of dimension* 2,

$$\dim A \cap [W_5^1(C) + W_1(C)] \geq 1.$$

(iii) *Let C be a tetragonal curve with a unique g_4^1 of genus $g = 9$ or 10. Assume that $W_6^2(C) = \emptyset$, $\dim W_{g-3}^1(C) = g - 7$ and $W_{g-4}^1(C)$ has only one component $\{g_4^1\} + W_{g-8}(C)$ of maximal dimension. Let $A \subset W_{g-3}^1(C)$ be a component of maximal dimension. If $A \supset T + W_1(C)$ for some closed irreducible nonempty subset $T \subset W_{g-4}^1(C)$, then $A = \{g_4^1\} + W_{g-7}(C)$.*

Proof. (i) is also due to M. Coppens which follows easily from excess linear series result of Fulton-Harris-Lazarsfeld [**FHL**]; cf. [**C**; Proposition 1].

(ii) The proof is a minor modification of the proof of [**C**; Theorem 2]. We consider the diagram:

$$\begin{array}{ccccc} W_1(C) & \xleftarrow{p_1} & Z \subset W_5^1(C) \times W_1(C) & \ni & (\mathscr{L}, \mathscr{O}_C(P)) \\ & & \downarrow p_2 \qquad \downarrow p_2 & & \downarrow \\ & & A \subset \quad W_6^1(C) & \ni & \mathscr{L} \otimes \mathscr{O}_C(P) \end{array}$$

where $Z = p_2^{-1}(A)$ and p_1 is the projection map onto the second factor. By (i), p_1 is surjective and hence there exists an irreducible component $\tilde{Z}$ of Z dominating $W_1(C)$ with $\dim(\tilde{Z}) \geq 1$. If $\dim(p_2(\tilde{Z})) = \dim(\tilde{Z})$, then we are done. Suppose $\dim(p_2(\tilde{Z})) < \dim(\tilde{Z})$. Note that p_2 is injective on the fibers of p_1. Therefore it follows that $\dim(p_2(\tilde{Z})) = \dim(\tilde{Z}) - 1$. Furthermore, for each $\mathscr{L} \in p_2(\tilde{Z})$ and for each $P \in C$ there exists $\mathscr{M} \in W_5^1(C)$ such that $\mathscr{L} \cong \mathscr{M} \otimes \mathscr{O}_C(P)$. Since there is no g_6^2 this would imply that every point in C is base point of $\mathscr{L}$ which is an absurdity.

(iii) For $A \neq \{g_4^1\} + W_{g-7}(C)$, a general element of A has no base point by the assumption on $W_{g-4}^1(C)$. Suppose $A \supset T + W_1(C)$. Since a general $\mathscr{L} \in A$ is base-point-free, we have $h^0(C, \omega_C \otimes \mathscr{L}^{-2}) \geq 1$ by the description of the Zariski tangent spaces to the scheme $W_{g-3}^1(C)$ and by the base-point-free pencil trick; cf. [**ACGH**; Propostion 4.2, page 189]. Hence $h^0(C, \omega_C \otimes \mathscr{L}_0^{-2} \otimes \mathscr{O}_C(-2P)) \geq 1$ for any $\mathscr{L}_0 \in T$ and for any $P \in C$ by semi-continuity. Therefore $h^0(C, \omega_C \otimes \mathscr{L}_0^{-2}) \geq 3$ for $\mathscr{L}_0 \in T$ and $|\omega_C \otimes \mathscr{L}_0^{-2}| = g_6^2$, a contradiction. □

Lemma 3.6. *Let C be a tetragonal curve of genus $g = 10$ without g_6^2 and assume that C is not a double covering of a curve of genus 2. If $\dim W_7^1(C) = 3$, then $\{g_4^1\} + W_3(C)$ is the only component of $W_7^1(C)$ of maximal dimension.*

Proof. We first note that there is a unique g_4^1 and $h^0(C, \mathscr{O}_C(2g_4^1)) = 3$. Otherwise there exists a base-point-free g_8^3 which is either birationally very ample or compounded inducing a morphism of degree two onto a curve of degree 4 in $\boldsymbol{P}^3$; but both cases cannot occur by Castelnuovo genus bound or C not being bi-elliptic (by the non-existence of g_6^2). Note that $\dim W_6^1(C) = 2$ by Lemma 3.4, and hence $\dim W_5^1(C) = 1$. We further remark that a component of $W_5^1(C)$ of dimension one is of the form $W_4^1(C) + W_1(C)$. If not, there is a component of $W_5^1(C)$ of dimension one whose general element is base-point-free, and by the same argument as in [**K**]—especially [**K**; Theorem 2.3 and the case (a) in the proof]—one concludes that C is a smooth plane sextic contrary to the hypothesis.

(3.6.1) Claim: $W_4^1(C) + W_2(C)$ is the only component of $W_6^1(C)$ of maximal dimension.

Suppose that a general element of a component $Y \subset W_6^1(C)$ of maximal dimension has a base point. Then it follows that $Y = Z + W_1(C)$—where $Z \subset W_5^1(C)$ is a component of dimension one—hence $Z = W_4^1(C) + W_1(C)$ and $Y = W_4^1(C) + W_2(C)$.

Suppose that a general element of a component $Y \subset W_6^1(C)$ of maximal dimension is base-point-free. By Lemma 3.5 (ii), we have

$$\dim(Y \cap [W_5^1(C) + W_1(C)]) \geq 1.$$

Assume that $Y \supset Z + W_1(C)$ where $Z \subset W_5^1(C)$ is a closed subset. By the description of the Zariski tangent space to the scheme $W_6^1(C)$, $h^0(C, \omega_C \otimes \mathscr{L}^{-2}) \geq 2$ for a general $\mathscr{L} \in Y$ and hence $h^0(C, \omega_C \otimes \mathscr{L}^{-2}) \geq 2$ for any $\mathscr{L} \in Y$ by semi-continuity. Therefore $h^0(C, \omega_C \otimes \mathscr{L}_0^{-2} \otimes \mathscr{O}_C(-2P)) \geq 2$ for any choices of $\mathscr{L}_0 \in Z$ and $P \in C$, which implies $h^0(C, \omega_C \otimes \mathscr{L}_0^{-2}) \geq 4$ and hence $|K_C - 2\mathscr{L}_0| = g_8^3$. Since C is a curve without g_6^2, g_8^3 must be base-point-free. By Castelnuovo genus bound, g_8^3 cannot be birationally very ample and therefore induces a morphism of degree 2 onto an elliptic curve or a rational curve. This is again a contradiction to the fact that C is a curve without g_6^2. Therefore we have $Y \not\supset Z + W_1(C)$ for any closed subset $Z \subset W_5^1(C)$ and hence

$$\dim(Y \cap [W_4^1(C) + W_2(C)]) \geq 1.$$

Let Σ be a one-dimensional component of $Y \cap [W_4^1(C) + W_2(C)]$ and let $\sigma \subset W_2(C)$ be a one-dimensional locus such that

$$\Sigma = \{\mathscr{O}_C(g_4^1 + P + Q) \mid \mathscr{O}_C(P + Q) \in \sigma\}.$$

For $\mathscr{O}_C(P + Q) \in \sigma$, $\mathscr{O}_C(g_4^1 + P + Q) \in Y \cap [W_4^1(C) + W_2(C)] \subset W_6^1(C)$ is a singular point and hence $h^0(C, \mathscr{O}_C(K_C - 2g_4^1 - P - Q)) \geq 3$ by the description of the Zariski tangent space to the scheme $W_6^1(C)$. Assume that $\mathrm{Bs}|K_C - 2g_4^1| = \emptyset$. Recalling $h^0(C, \mathscr{O}_C(K_C - 2g_4^1)) = 4$, we see that $\mathscr{O}_C(P + Q) \in \sigma$ maps to one point by the morphism induced by $|K_C - 2g_4^1|$. Therefore $|K_C - 2g_4^1|$ is not birationally very ample and C should be a double covering of a curve of genus $h \leq 2$ contrary to the hypothesis. Assume $\mathrm{Bs}|K_C - 2g_4^1| = \Delta \neq \emptyset$, say $R_0 \in \mathrm{Bs}|K_C - 2g_4^1|$, then $R_0 \not\leq P + Q$ for general $\mathscr{O}_C(P + Q) \in \sigma$; for if $\sigma = \{\mathscr{O}_C(R_0 + P) \mid P \in C\}$ then $\{\mathscr{O}_C(g_4^1 + R_0)\} + W_1(C) \subset Y$, which we avoided already. Therefore $|K_C - 2g_4^1 - \Delta|$ is not birationally very ample. Hence C is either hyperelliptic, bi-elliptic or trigonal and this finishes the proof of Claim (3.6.1).

(3.6.2) Claim: For a component $A \subset W_7^1(C)$ of maximal dimension,

$$\dim[A \cap (W_6^1(C) + W_1(C)] \geq 2.$$

The proof for this claim is very much similar to the proof of Lemma 3.5 (ii). We again consider the diagram:

$$\begin{array}{ccccc}
W_1(C) & \xleftarrow{p_1} & Z \subset W_6^1(C) \times W_1(C) & \ni & (\mathscr{L}, \mathscr{O}_C(P)) \\
 & & \downarrow p_2 \qquad \downarrow p_2 & & \downarrow \\
 & & A \subset \quad W_7^1(C) & \ni & \mathscr{L} \otimes \mathscr{O}_C(P)
\end{array}$$

where $Z = p_2^{-1}(A)$ and p_1 is a projection map. By Lemma 3.5 (i), p_1 is surjective and each fiber of p_1 contains an irreducible component of dimension at least 1. Hence there exists an irreducible component $\tilde{Z}$ of Z dominating $W_1(C)$ with $\dim(\tilde{Z}) \geq 2$. If $\dim(p_2(\tilde{Z})) = \dim(\tilde{Z})$, then we are done. Suppose $\dim(p_2(\tilde{Z})) < \dim(\tilde{Z})$. Note that

p_2 is injective on the fibers of p_1. Therefore it follows that $\dim(p_2(\tilde{Z})) = \dim(\tilde{Z}) - 1$. Furthermore for each $\mathscr{L} \in p_2(\tilde{Z})$ and for each $P \in C$ there exists $\mathscr{M} \in W_6^1(C)$ such that $\mathscr{L} \cong \mathscr{M} \otimes \mathscr{O}_C(P)$. If $\mathscr{L} \notin W_7^2(C)$, then every point $P \in C$ would be a base point of $\mathscr{L}$. This is a contradiction. Hence $p_2(\tilde{Z}) \subset W_7^2(C)$. By the non-existence of g_6^2, $\mathscr{L} \in p_2(\tilde{Z}) \subset W_7^2(C)$ is base-point-free and birationally very ample. Therefore it follows that $p_2(\tilde{Z}) - W_1(C) \subset W_6^1(C)$ is a component of dimension 2 with a base-point-free general element, contradicting (3.6.1) and this finishes the proof of (3.6.2).

We now suppose that there is a component $A \subset W_7^1(C)$ of dimension 3 other than $W_4^1(C) + W_3(C)$. By (3.6.1), a general element of A is base-point-free. We also have the inequality

$$\dim A \cap [W_4^1(C) + W_3(C)] \geq 2,$$

by (3.6.2) and Lemma 3.5 (iii). Let Σ be a component of $A \cap [W_4^1(C) + W_3(C)]$ of dimension two and consider the morphism $\pi : \Sigma \to W_3(C)$ where $\pi(\mathscr{L}) = |\mathscr{L} - g_4^1|$ and put $\sigma := \pi(\Sigma)$, which is a two dimensional family of effective divisor classes $\mathscr{O}_C(P+Q+R)$ such that $\mathscr{O}_C(g_4^1 + P + Q + R)$ are also contained in the component A. Therefore $\mathscr{O}_C(g_4^1 + P + Q + R)$ is a singular point of $W_7^1(C)$ for every $\mathscr{O}_C(P+Q+R) \in \sigma$. Hence by the description of the Zariski tangent spaces to the scheme $W_7^1(C)$ and by the base-point-free pencil trick, one has

(3.6.3) $\quad h^0(C, \mathscr{O}_C(K_C - 2g_4^1 - P - Q - R)) \geq 2 \quad$ for every $\mathscr{O}_C(P+Q+R) \in \sigma$.

We remark that $\dim|K_C - 2g_4^1| = 3$ and the morphism defined by $|K_C - 2g_4^1|$ is birationally very ample (even when it has a base point) by the hypothesis that C is not a double covering of a curve of genus $h \leq 2$. Let $\phi : C \to \boldsymbol{P}^3$ be the morphism defined by $|K_C - 2g_4^1|$ and we consider the following two possibilities.

(a) Suppose $|K_C - 2g_4^1|$ is base-point-free. For every $\mathscr{O}_C(P+Q+R) \in \sigma$, $\mathscr{O}_C(P+Q+R)$ fails to impose independent conditions on the linear system $|K_C - 2g_4^1|$ by (3.6.3), and hence the linear span of the image $\phi(P+Q+R)$ is a trisecant line to $\phi(C)$. Therefore the non-degenerate curve $\phi(C) \subset \boldsymbol{P}^3$ has a two dimensional family of tri-secant lines which contradicts the general position theorem; [**ACGH**, page 109].

(b) Suppose $|K_C - 2g_4^1|$ has nonempty base locus. We remark that there is only one base point, otherwise there exists g_8^3 which would imply $g \leq 9$ by Castelnuovo bound or C is bi-elliptic. Let $R_0 \in C$ be the base point and consider the morphism $\phi : C \to \boldsymbol{P}^3$ defined by $|K_C - 2g_4^1 - R_0| = g_9^3$. In case $R_0 \notin \operatorname{Supp} \mathscr{O}_C(P+Q+R)$ for general $\mathscr{O}_C(P+Q+R) \in \sigma$, the same argument as in (a) applies to get a contradiction. We now suppose that $R_0 \in \operatorname{Supp} \mathscr{O}_C(P+Q+R)$ for general $\mathscr{O}_C(P+Q+R) \in \sigma$. Since $R_0 \in \operatorname{Supp} \mathscr{O}_C(P+Q+R)$ is a closed condition and σ is irreducible, we have $R_0 \in \operatorname{Supp} \mathscr{O}_C(P+Q+R)$ for all $\mathscr{O}_C(P+Q+R) \in \sigma$. Consider the morphism $\alpha : \sigma \to W_2(C)$, where $\alpha(\mathscr{O}_C(P+Q+R)) = \mathscr{O}_C(P+Q+R-R_0)$. Again by the irreducibility of σ, $\alpha(\sigma) = W_2(C)$ which in turn implies

$$\sigma = \{\mathscr{O}_C(P+Q+R_0) : P+Q \in C_2\}.$$

Therefore $\Sigma = \{g_4^1\} + \sigma = \mathscr{O}_C(g_4^1 + R_0) + W_2(C) \subset A$ and $A = \{g_4^1\} + W_3(C)$ by Lemma 3.5 (iii). $\square$

Theorem 3.7. *Let C be a tetragonal curve of genus $g = 10$ which is not a double covering of a curve of genus $h \leq 2$. Then*

$$\dim W^2_{g-1}(C) \lneq g - 7.$$

Proof. We assume that there is a component $X \subset W^2_{g-1}(C)$ with $\dim X = g - 7$. By exactly the same argument as in the proof of [**CKO**; Theorem 1.6], one can easily show that a general element of X is base-point-free. Therefore can apply Lemma 2.3 as well as Lemma 2.2 to our present situation. Recall the following diagram encountered in the proof of [**CKO**; Lemma 2.6];

$$\begin{array}{ccc} W^1_{g-3}(C) \times W_2(C) \supset q^{-1}(X) & \xrightarrow{\;q\;} & X \subset W^2_{g-1}(C) \\ \Big\downarrow{\scriptstyle p} & & \\ W^1_{g-3}(C). & & \end{array}$$

Let

$$q^{-1}(X) = Z_0 \cup \cdots \cup Z_\alpha \cup Y_1 \cup \cdots \cup Y_\beta,$$

where $Z_0, \dots, Z_\alpha$ are components of $q^{-1}(X)$ dominating X and $Y_1, \dots, Y_\beta$ are those which do not dominate X. Since $\dim p(Z_i) = g - 7$ for every i by Lemma 2.3 (ii) (more precisely by the proof of [**CKO**; Lemma 2.6 (ii)]), we apply Lemma 3.6 to have

$$p(Z_i) = g^1_4 + W_{g-7}(C) \tag{3.7.1}$$

for every component $Z_i \subset q^{-1}(X)$ dominating X.

We take a general $\mathscr{L} \in X \backslash (q(Y_1) \cup \cdots \cup q(Y_\beta))$ and let $(\mathscr{M}, \mathscr{O}_C(A + B)) \in q^{-1}(\mathscr{L})$ and fix i, say $i = 0$ such that $q^{-1}(\mathscr{L}) \cap Z_0 \neq \varnothing$. Since $\mathscr{L}$ is base-point-free, $\mathscr{M} \in p(Z_0)$ is a complete pencil of degree $g - 3$ and hence $\mathscr{M} = \mathscr{O}_C(g^1_4 + P_1 + \cdots + P_{g-7})$ by (3.7.1). Since $\mathscr{L} = \mathscr{O}_C(g^1_4 + P_1 + \cdots + P_{g-7} + A + B)$ is birationally very ample, the plane curve $C_{\mathscr{L}}$—the image of the morphism $\psi_{\mathscr{L}}$ induced by $\mathscr{L}$—has an ordinary singular point of multiplicity $g - 5$ corresponding to the divisor $P_1 + \cdots + P_{g-7} + A + B$ by Lemma 2.5. Since

$$p_a(C_{\mathscr{L}}) = \frac{(g-2)(g-3)}{2} > g + \frac{(g-5)(g-6)}{2},$$

it follows that there exists at least one extra singular point on the curve $C_{\mathscr{L}}$. Let μ be the multiplicity of an extra singular point of the plane curve $C_{\mathscr{L}}$. Then it follows that there is a complete base-point-free pencil $h^1_{g-1-\mu}$ such that $\mathscr{L} = h^1_{g-1-\mu} \otimes \mathscr{O}_C(Q_1 + \cdots + Q_\mu)$ for some $Q_1, \dots, Q_\mu \in C$; $h^1_{g-1-\mu}$ is cut out by lines through extra singular point. On the other hand, by the choice of $\mathscr{L} \in X \backslash (q(Y_1) \cup \cdots \cup q(Y_\beta))$, we have

$$(h^1_{g-1-\mu} \otimes \mathscr{O}_C(Q_1 + \cdots + Q_{\mu-2}), \mathscr{O}_C(Q_{\mu-1} + Q_\mu)) \in q^{-1}(\mathscr{L}) \cap Z_i$$

for some component $Z_i \subset q^{-1}(X)$ dominating X.

If $\mu < g - 5$ (i.e. $\mu - 2 < g - 7$), then this leads to a contradiction to the assertion (3.7.1) that every complete pencil of degree $g - 3$ in $W^1_{g-3}(C)$ which is in $p(Z_i)$, must have $g-7$ base points. Therefore $\mu = g-5$ and we have at least two singular points on $C_{\mathscr{L}}$ with multiplicities $g-5$, hence

$$g \le p_a(C_{\mathscr{L}}) - 2\cdot\frac{(g-5)(g-6)}{2} = \frac{(g-2)(g-3)}{2} - 2\cdot\frac{(g-5)(g-6)}{2}.$$

But this is impossible for $g = 10$. □

Proof of Theorem I. For a trigonal curve, a double covering of a curve of genus 2 or a smooth plane sextic, $\dim W^2_{g-1} = g-7$ by Remark 2.1 (v), Remark 2.1 (vi) and Proposition 3.1. The converse holds by [**CKO**; Theorem 1.4] and Theorem 3.7. □

4. Curves of genus nine.

For curves of genus $g = 9$, we also need to describe precisely the component of $W^1_{g-3}(C)$ of maximal dimension as we did in Lemma 3.6 for curves of genus $g = 10$. Unfortunately, in the course of the proof of Lemma 4.1, there emerges another class of curves of genus $g = 9$ which needs to be examined carefully; namely, the double coverings of a smooth plane quartic. In fact, it turns out that such a curve C does not satisfy $\dim W^2_{g-1}(C) = g-7$, whose proof is rather lengthy and technical. Therefore we provide a proof of the result regarding double coverings of a smooth plane quartic in the Appendix.

Lemma 4.1. *Let C be a tetragonal curve of genus $g = 9$ without g^2_6. We assume that* $\dim W^2_{g-1}(C) = g-7$ *and that C is neither a double covering of a curve of genus 2 nor a curve with a smooth model of degree 8 in $\boldsymbol{P}^3$; hence C has only one g^1_4 and $h^0(C, \mathcal{O}_C(2g^1_4)) = 3$. We assume further that $(e_2, e_3) \neq (1,1)$, i.e. $h^0(C, \mathcal{O}_C(3g^1_4)) \le 5$. If* $\dim W^1_{g-3}(C) = g-7$, *then $\{g^1_4\} + W_2(C)$ is the only component of $W^1_6(C)$ of maximal dimension.*

Proof. Note that C is not bi-elliptic by the non-existence of g^2_6 and $\dim W^1_5(C) = 1$ by Lemma 3.4.

(4.1.1) Claim: $W^1_4(C) + W_1(C)$ is the only component of $W^1_5(C)$ of maximal dimension; this follows easily from [**K**; Theorem 2.3 and the case (b) in the proof].

We suppose that there is a component $A \subset W^1_6(C)$ of dimension 2 other than $W^1_4(C) + W_2(C)$. Then a general element of A is base-point-free by (4.1.1). By Lemma 3.5 (ii), we have

$$\dim A \cap [W^1_5(C) + W_1(C)] \ge 1. \tag{4.1.2}$$

Since $W^1_4(C) + W_1(C)$ is the only component of $W^1_5(C)$ of maximal dimension, we have the following inequality by (4.1.2) and Lemma 3.5 (iii),

$$\dim A \cap [W^1_4(C) + W_2(C)] \ge 1. \tag{4.1.3}$$

Let Σ be a component of $A \cap [W^1_4(C) + W_2(C)]$ of dimension one and consider the morphism $\pi : \Sigma \to W_2(C)$ where $\pi(\mathscr{L}) = |\mathscr{L} - g^1_4|$. Put $\sigma := \pi(\Sigma)$. Note that

$\mathcal{O}_C(g_4^1 + P + Q)$ is a singular point of $W_6^1(C)$ for every $\mathcal{O}_C(P+Q) \in \sigma$. Hence by the description of the Zariski tangent spaces to the scheme $W_6^1(C)$ and by the base-point-free pencil trick, one has

(4.1.4) $$h^0(C, \mathcal{O}_C(K_C - 2g_4^1 - P - Q)) \geq 2 \quad \text{for all } \mathcal{O}_C(P+Q) \in \sigma.$$

Let $\phi : C \to \boldsymbol{P}^2$ be the morphism induced by $|K_C - 2g_4^1| = g_8^2$ and we consider the following two cases.

(a) Suppose $|K_C - 2g_4^1|$ has a nonempty base locus. We note that there is only one base point, otherwise there exists g_6^2. Let $R_0 \in C$ be the base point. Then morphism $\phi : C \to \boldsymbol{P}^2$ is indeed induced by $|K_C - 2g_4^1 - R_0| = g_7^2$ which must be birationally very ample. If $R_0 \notin \operatorname{Supp} \mathcal{O}_C(P+Q)$ for general $\mathcal{O}_C(P+Q) \in \sigma$, (4.1.4) implies $\deg \phi \geq 2$ which is a contradiction. We now suppose that $R_0 \in \operatorname{Supp} \mathcal{O}_C(P+Q)$ for general $\mathcal{O}_C(P+Q) \in \sigma$. Since $R_0 \in \operatorname{Supp} \mathcal{O}_C(P+Q)$ is a closed condition and σ is irreducible, we have $R_0 \in \operatorname{Supp} \mathcal{O}_C(P+Q)$ for all $\mathcal{O}_C(P+Q) \in \sigma$. Consider a morphism $\alpha : \sigma \to W_1(C)$, where $\alpha(\mathcal{O}_C(P+Q)) = \mathcal{O}_C(P+Q-R_0)$. By the irreducibility of σ, $\alpha(\sigma) = W_1(C)$ which in turn implies

(4.1.5) $$\sigma = \{\mathcal{O}_C(P+R_0) : P \in C\}.$$

Therefore

$$\Sigma = \{g_4^1\} + \sigma = \mathcal{O}_C(g_4^1 + R_0) + W_1 \subset A$$

and hence $A = \{g_4^1\} + W_2(C)$ by Lemma 3.5 (iii).

(b) Suppose $|K_C - 2g_4^1|$ is base-point-free. If $\deg \phi = 4$, then $|K_C - 2g_4^1| = |2g_4^1|$ and hence $h^0(C, \mathcal{O}_C(3g_4^1)) \geq 6$, contrary to the hypothesis $(e_2, e_3) \neq (1, 1)$. We also note that $\deg \phi \neq 1$ by (4.1.4); for one dimensional family of effective divisors $\mathcal{O}_C(P+Q) \in \sigma$, P and Q have the same image under ϕ.

Finally we assume $\deg \phi = 2$ and let $\phi(C) = E$ which is a plane curve of degree 4. If E is singular, then E is a curve of genus $h \leq 2$, contrary to the hypothesis. Hence E is a smooth non-hyperelliptic curve of genus 3 and $|K_C - 2g_4^1| = |\phi^* K_E|$. Let $\iota : C \to C$ be an involution induced by the double covering ϕ. By Riemann-Hurwitz relation (2.A), $\phi_* \mathcal{O}_C = \mathcal{O}_E \oplus \mathcal{S}$ and $\deg \mathcal{S}^{-1} = 4$. Hence

$$\mathcal{S}^{-1} \cong \mathcal{O}_E(K_E) \quad \text{or} \quad \mathcal{S}^{-1} \cong \mathcal{O}_E(h_4^1)$$

where h_4^1 is a complete pencil of degree 4 on E. Assume $\mathcal{S}^{-1} \cong \mathcal{O}_E(K_E)$. Then

$$\begin{aligned} H^0(C, \mathcal{O}_C(K_C - 2g_4^1)) &\cong H^0(C, \mathcal{O}_C(\phi^* K_E)) \\ &\cong H^0(E, \mathcal{O}_E(K_E)) \oplus H^0(E, \mathcal{S} \otimes \mathcal{O}_E(K_E)), \end{aligned}$$

hence $h^0(C, \mathcal{O}_C(K_C - 2g_4^1)) = 4$, contrary to the assumption $h^0(C, \mathcal{O}_C(2g_4^1)) = 3$. Therefore we have $\mathcal{S}^{-1} \cong \mathcal{O}_E(h_4^1)$. Since g_4^1 is unique, $\iota^* g_4^1 = g_4^1$ and it follows that $\mathcal{O}_C(2g_4^1) \cong \phi^* \mathcal{M}$ for some line bundle $\mathcal{M}$ of degree 4 on E. Therefore

$$H^0(C, \mathcal{O}_C(2g_4^1)) \cong H^0(E, \mathcal{M}) \oplus H^0(E, \mathcal{O}_E(-h_4^1) \otimes \mathcal{M}).$$

Since $h^0(C, \mathcal{O}_C(2g_4^1)) = 3$, we have either $\mathcal{M} \cong \mathcal{O}_E(h_4^1)$ or $\mathcal{M} \cong \mathcal{O}_E(K_E)$. If $\mathcal{M} \cong \mathcal{O}_E(K_E)$, then the fact $\mathcal{O}_C(2g_4^1) \cong \phi^* \mathcal{M}$ together with $|K_C - 2g_4^1| = |\phi^* K_E|$ imply

$|K_C| = |4g_4^1|$ which is contradictory to the hypothesis $(e_2, e_3) \neq (1,1)$. Therefore we have reached to the following special situation:

$$(4.1.6)\qquad \begin{cases} \text{(i)} & \text{There is a degree two map } \phi : C \to E, \\ & E \text{ is a non-hyperelliptic curve of genus 3.} \\ \text{(ii)} & K_C \sim \phi^*(K_E) + 2g_4^1 \\ \text{(iii)} & h^0(C, \mathcal{O}_C(2g_4^1)) = 3 \\ \text{(iv)} & 2g_4^1 = \phi^*(h_4^1) \text{ for some } h_4^1 \text{ on } E \\ \text{(v)} & \mathcal{O}_E(h_4^1) \not\cong \mathcal{O}_E(K_E) \\ \text{(vi)} & \phi_*\mathcal{O}_C \cong \mathcal{O}_E \oplus \mathcal{O}_E(-h_4^1) \\ \text{(vii)} & C \text{ is not a double covering of a curve of genus } h \leq 2. \end{cases}$$

By Proposition A.0 in the Appendix, for a curve C of genus $g = 9$ satisfying the conditions (4.1.6), one has $\dim W_8^2(C) < 2$ and this finishes the proof. □

We are now ready to prove the following theorem which is a genus nine version of Theorem 3.7.

Theorem 4.2. *Let C be a tetragonal curve of genus $g = 9$ without g_6^2 which is neither a double covering of a curve of genus $h \leq 2$ nor a curve with a smooth model of degree* 8 *in $\boldsymbol{P}^3$, then*

$$\dim W_{g-1}^2(C) \lneq g - 7.$$

Proof. By Remark 3.2 (ii), we may assume that C is a tetragonal curve with a unique g_4^1. By Lemma 3.3, we may further assume that $(e_2, e_3) \neq (1,1)$. In order to save ink, we avoid repeating same argument which already appeared in the proof of Theorem 3.7. Since we now have Lemma 4.1 for $g = 9$ which is a variation of Lemma 3.6, one can argue as in the proof of Theorem 3.7 that a plane model $C_{\mathscr{L}}$ defined by a general $\mathscr{L} \in X$ has at least two singular points of multiplicities 4; at least one of them being an ordinary 4-fold point by Lemma 2.5. Accordingly

$$p_a(C_{\mathscr{L}}) = 21 = \frac{(g-2)(g-3)}{2} \geq g + 2 \cdot \frac{(g-5)(g-6)}{2} = 21$$

and hence $C_{\mathscr{L}} \subset \boldsymbol{P}^2$ has exactly two singular points of multiplicities 4 as its only singularities, contrary to the assumption for g_4^1 being unique. □

Proof of Theorem II. If C is either trigonal, a double covering of a curve of genus 2, a tetragonal curve with a smooth model of degree 8 in $\boldsymbol{P}^3$ or a curve with a plane model of degree 6, then $\dim W_{g-1}^2(C) = g - 7$ by Remark 2.1 (v), Remark 2.1 (vi), Remark 3.2 (i) and Proposition 3.1. The converse holds by [**CKO**; Theorem 1.4] and Theorem 4.2. □

5. Curves of genus seven and eight.

In this section, we prove Theorem III. We also estimate the dimension of $W_{g-1}^2(C)$ for a double covering of a curve of genus 2 which was left out in [**CKO**; Corollary 2.3] for the cases $g = 7, 8$; cf. [**CKO**; Remarks 2.4 (i)].

Proof of Theorem III. Suppose C is either trigonal or has a birationally very ample g_6^2. Then $\dim W_{g-1}^2(C) = g-7$ by Remark 2.1 (v) and Proposition 3.1.

If $\dim W_{g-1}^2(C) = g-7$ and $g = 7$, then $g_6^2 \in W_6^2(C)$ must be either birationally very ample or of the form $g_6^2 = 2g_3^1$ by Remark 2.1 (ii).

Suppose $\dim W_{g-1}^2(C) = g-7$ and $g = 8$. If there exists $g_7^2 \in W_7^2(C)$ with nonempty base locus, then C must be either trigonal or a curve with a birationally very ample g_6^2. Therefore we may assume that every $g_7^2 \in W_7^2(C)$ is base-point-free, birationally very ample and that C is neither trigonal nor a curve with a birationally very ample g_6^2. From Lemma 2.3, Theorem 1.2 and Remark 2.1, it follows that $\dim W_5^1(C) = 1$. Therefore by [**BKMO**; Theorem 1], C must be a tetragonal curve. Furthermore every component of $\dim W_5^1(C)$ of maximal dimension is of the form

$$\{g_4^1\} + W_1(C) \tag{5.I.1}$$

by [**BKMO**; (3.2.1) Corollary 2]. Note that C has only one g_4^1; otherwise we have a g_8^3 which is birationally very ample but not very ample, in which case there exists a g_6^2 cut out by hyperplanes through a singular point. Since every element of $W_7^2(C)$ is base-point-free and birationally very ample, C has a plane model $C_{\mathscr{L}}$ of degree 7 with an ordinary triple point by Lemma 2.5. Since $g = 8 < p_a(C_{\mathscr{L}}) - 3$, there exists another singular point on $C_{\mathscr{L}}$ whose multiplicity must be 3 by (5.I.1) inducing another base-point-free g_4^1; the arguments in these lines are almost parallel to the latter part of the proof of Theorem I or Theorem II. And this is contradictory to the uniqueness of g_4^1. □

For the rest of this section, we would like to concentrate on the dimension estimate of $W_{g-1}^2(C)$ for curves of genus $g = 7, 8$ which are double coverings of a curve of genus 2. As we saw in Theorem III, the necessary and sufficient conditions for $W_{g-1}^2(C)$ being of dimension $g-7$ for these low genus curves are slightly different from those of higher genus; doubling coverings of curves of genus 2 suddenly disappear from the list of curves with $\dim W_{g-1}^2(C) = g-7$. Since Theorem III does not provide any direct information on the dimension of $W_{g-1}^2(C)$ for double coverings of curves of genus 2, it seems to be worthwhile to estimate the dimension of $W_{g-1}^2(C)$.

Proposition 5.1. *Let C be curve of genus $g = 8$ which is a double covering of a curve of genus 2. Then C does not have a birationally very ample g_6^2 and* $\dim W_{g-1}^2(C) \leq g-8$.

Proof. For a double cover $\pi : C \to E$ of a curve of genus 2, one notes that $g_4^1 = \pi^*(g_2^1)$ is a unique base-point-free g_4^1 by Castelnuovo-Severi inequality.

Claim: $|K_C - g_4^1| = g_{10}^4$ is very ample.

Let $P, Q \in C$ and consider $\mathscr{O}_C(g_4^1 + P + Q)$. Recall that $\pi_*(\mathscr{O}_C) \cong \mathscr{O}_E \oplus \mathscr{S}$, $\deg \mathscr{S} = -5$ by the Riemann-Hurwitz relation (2.A). If $P + Q = \pi^*(p)$ for some $p \in E$, then

$$\begin{aligned} h^0(C, \mathscr{O}_C(g_4^1 + P + Q)) &= h^0(E, \pi_*\mathscr{O}_C(\pi^*(g_2^1 + p))) \\ &= h^0(E, \mathscr{O}_E(g_2^1 + p)) + h^0(E, \mathscr{O}_E(g_2^1 + p) \otimes \mathscr{S}) \\ &= h^0(E, \mathscr{O}_E(g_2^1 + p)) = 2. \end{aligned}$$

If $P+Q \neq \pi^*(p)$ for any $p \in E$, we take $P', Q' \in C$ which are conjugate points of P, Q with respect to π. Set $p = \pi(P)$, $q = \pi(Q)$. Again by (2.A) and the projection formula, we have

$$\begin{aligned} h^0(C, \mathcal{O}_C(g_4^1 + P + P' + Q + Q')) &= h^0(C, \mathcal{O}_C(\pi^*(g_2^1 + p + q))) \\ &= h^0(E, \pi_* \mathcal{O}_C(\pi^*(g_2^1 + p + q))) \\ &= h^0(E, \mathcal{O}_E(g_2^1 + p + q)) + h^0(E, \mathcal{O}_E(g_2^1 + p + q) \otimes \mathcal{S}) \\ &= 3. \end{aligned}$$

Since $\pi^*(g_2^1 + p + q)$ is base-point-free, $h^0(C, \mathcal{O}_C(g_4^1 + P + Q)) = 2$ and this finishes the proof of the claim.

Now we assume that C has a birationally very ample $\mathcal{L} = g_6^2$ with the induced plane model $C_{\mathcal{L}}$, which must be singular since $g = 8 < p_a(C_{\mathcal{L}}) = 10$. Since C cannot be trigonal by Castelnuovo-Severi inequality, $C_{\mathcal{L}}$ has only double points and hence there exist $P, Q \in C$ (corresponding to a double point) such that $|\mathcal{L} - P - Q| = \pi^*(g_2^1)$. On the other hand, it follows from the Claim that

$$h^0(C, \omega_C \otimes \mathcal{O}_C(-g_4^1 - P - Q)) = h^0(C, \omega_C \otimes \mathcal{L}^{-1}) = h^0(C, \omega_C \otimes \mathcal{O}_C(-g_4^1)) - 2 = 3$$

which in turn implies $h^0(C, \mathcal{L}) = 2$ by the Riemann-Roch formula and this is a contradiction. Therefore C cannot have a birationally very ample g_6^2 and $\dim W_7^2(C) \leq 0$ by Theorem III. □

For a curve of genus $g = 7$ which is a double covering of a curve of genus 2, it may happen that $\dim W_{g-1}^2(C) = g - 7$ or $W_{g-1}^2(C) = \varnothing$.

Proposition 5.2. *For $g = 7$,*

(i) *there exists a double covering C of a curve of genus 2 such that* $\dim W_6^2(C) = 0$.
(ii) *There also exists a double covering C of a curve of genus 2 such that $W_6^2(C) = \varnothing$.*

A proof of Proposition 5.2 requires several supplementary lemmas.

Lemma 5.3. *For a tetragonal curve C of genus $g = 7$, C has a birationally very ample g_6^2 if and only if either $2 \leq \operatorname{Card} W_4^1(C) < \infty$ or $\operatorname{Card} W_4^1(C) = 1$ and $h^0(C, \mathcal{O}_C(2g_4^1)) = 4$.*

Proof. Let C be a curve with a birationally very ample $\mathcal{L} = g_6^2$. We note that C is not bi-elliptic; a curve with a birationally very ample g_6^2 has a base-point-free g_5^1 whereas a bi-elliptic curve cannot have a base-point-free g_5^1 by Castelnuovo-Severi inequality. Therefore $\dim W_4^1(C) = 0$ by Theorem 1.2. Let $\psi_{\mathcal{L}} : C \to \boldsymbol{P}^2$ be the morphism induced by $\mathcal{L}$ and assume $\operatorname{Card} W_4^1(C) = 1$. Since $g = 7 < p_a(\psi_{\mathcal{L}}(C)) = 10$, the unique g_4^1 on C is induced by a unique double point $P \in \psi_{\mathcal{L}}(C)$. Moreover we have two infinitely near singular points Q and R. Considering the linear system on C induced by the linear system $|\mathcal{O}_{\boldsymbol{P}^2}(3) - P - Q - R|$ of cubics in $\boldsymbol{P}^2$ with assigned base points P, Q and R, one has

$$h^0(C, \mathcal{O}_C(3g_4^1)) \geq h^0(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(3) - P - Q - R) \geq 7.$$

Therefore $\mathcal{O}_C(3g_4^1) \cong \omega_C$ and hence $h^0(C, \mathcal{O}_C(2g_4^1)) = 4$.

Assume that either $2 \le \operatorname{Card} W_4^1(C) < \infty$ or $\operatorname{Card} W_4^1(C) = 1$ and $h^0(C, \mathcal{O}(2g_4^1)) = 4$ holds. In both cases, we have a base-point-free $\mathcal{D} = g_8^3$ which is of the form $|g_4^1 + h_4^1|$, $g_4^1 \neq h_4^1$ or $|2g_4^1|$. We note that C cannot be bi-elliptic; for a bi-elliptic curve C, $\dim W_4^1(C) = 1$. Therefore $\mathcal{D}$ induces a birational morphism $\psi_{\mathcal{D}}$ onto $C_0 = \psi_{\mathcal{D}}(C) \subset \boldsymbol{P}^3$ which lies either on a smooth quadric if $\mathcal{D} = |g_4^1 + h_4^1|$ or on a quadric cone if $\mathcal{D} = |2g_4^1|$. Since $p_a(C_0) = 9$, C_0 has a singular point P with $\operatorname{mult}_P(C_0) = 2$ and hence hyperplanes through P cut out a birationally very ample g_6^2. □

Let $C \xrightarrow{\pi} E$ be a double covering of a curve E with $g(E) = 2$ and $g(C) = 7$. Let $R \subset E$ be the branch locus of π. By (2.A) and base-point-free pencil trick, we have $\mathcal{S} \cong \mathcal{O}_E(-g_2^1 - p - q)$. Let $s \in H^0(E, \mathcal{O}_E(2g_2^1 + 2p + 2q))$ be a section with the zero locus $(s)_0 = R$. Let V be a subspace of $H^0(E, \mathcal{O}_E(2g_2^1 + 2p + 2q))$ which is the image of the cup product map

$$H^0(E, \mathcal{O}_E(g_2^1 + p + q))^{\otimes 2} \xrightarrow{\delta} H^0(E, \mathcal{O}_E(2g_2^1 + 2p + 2q))$$

and consider a natural morphism

$$H^0(C, \pi^*\mathcal{O}_E(g_2^1 + p + q))^{\otimes 2} \xrightarrow{\alpha} H^0(C, \pi^*\mathcal{O}_E(2g_2^1 + 2p + 2q)).$$

Using the identifications

$$\begin{aligned} H^0(C, \mathcal{O}_C(\pi^*(g_2^1 + p + q))) &\cong H^0(E, \pi_*\pi^*\mathcal{O}_E(g_2^1 + p + q)) \\ &\cong H^0(E, \mathcal{O}_E(g_2^1 + p + q)) \oplus H^0(E, \mathcal{S} \otimes \mathcal{O}_E(g_2^1 + p + q)) \\ &\cong H^0(E, \mathcal{O}_E(g_2^1 + p + q)) \oplus H^0(E, \mathcal{O}_E) \end{aligned}$$

and

$$\begin{aligned} \text{(5.A)} \qquad & H^0(C, \mathcal{O}_C(\pi^*(2g_2^1 + 2p + 2q))) \\ &\cong H^0(E, \mathcal{O}_E(2g_2^1 + 2p + 2q)) \oplus H^0(E, \mathcal{O}_E(g_2^1 + p + q)), \end{aligned}$$

we see that the morphism α induces the map

$$\psi : H^0(E, \mathcal{O}_E) \otimes H^0(E, \mathcal{O}_E) \cong k \otimes k \to H^0(E, \mathcal{O}_E(2g_2^1 + 2p + 2q))$$

with $\psi(1) = s$ by the algebra structure of $\pi_*(\mathcal{O}_C)$. We also see that

$$\text{(5.B)} \qquad \operatorname{Im} \alpha = \operatorname{Span}\{V, s\} \oplus H^0(E, \mathcal{O}_E(g_2^1 + p + q)),$$

under the identification (5.A).

Lemma 5.4. *Let $C \xrightarrow{\pi} E$ be a double covering with $g(E) = 2$ and $g(C) = 7$. We assume that C is not bi-elliptic and put $g_4^1 = \pi^*(g_2^1)$.*

(i) *$h^0(C, \mathcal{O}_C(2g_4^1)) \ge 4$ if and only if $\mathcal{S} \cong \mathcal{O}_E(-2g_2^1)$.*

(ii) *Suppose $\mathcal{S} \not\cong \mathcal{O}_E(-2g_2^1)$. Then C has at least two complete pencil of degree 4 if and only if $s \in V$.*

Proof. (i) follows directly from the following equality;

$$\begin{aligned} h^0(C, \mathcal{O}_C(2g_4^1)) = h^0(C, \mathcal{O}_C(\pi^*(2g_2^1))) &= h^0(E, \pi_*\mathcal{O}_C(\pi^*(2g_2^1))) \\ &= h^0(E, \mathcal{O}_E(2g_2^1)) + h^0(E, \mathcal{O}_E(2g_2^1) \otimes \mathcal{S}). \end{aligned}$$

(ii) Since $\mathscr{S} \not\cong \mathcal{O}_E(-2g_2^1)$, $\mathscr{S} \cong \mathcal{O}_E(-g_2^1 - p - q)$ for $p + q \notin g_2^1$ and hence the morphism $E \to \boldsymbol{P}^2$ induced by $|g_2^1 + p + q|$ is birationally very ample. Thus we have an exact sequence

$$0 \to \operatorname{Sym}^2 H^0(E, \mathcal{O}_E(g_2^1 + p + q)) \xrightarrow{\tilde{\delta}} H^0(E, \mathcal{O}_E(2g_2^1 + 2p + 2q)),$$

and we see that

$$\dim V = \dim \operatorname{Im} \delta = \dim \operatorname{Im} \tilde{\delta} = \dim \operatorname{Sym}^2 H^0(E, \mathcal{O}_E(g_2^1 + p + q)) = 6.$$

Therefore it follows that α is surjective if and only if $s \notin V$ by (5.B).

Assume that C has a pencil $h_4^1 \neq g_4^1$ and put $k_4^1 := |K_C - g_4^1 - h_4^1|$. Since C is not bi-elliptic, the morphism $\zeta : C \to \boldsymbol{P}^3$ induced by $|K_C - g_4^1| = |h_4^1 + k_4^1|$ birational onto its image and $\zeta(C) \subset \boldsymbol{P}^3$ lies on a quadric surface. By the Riemann-Hurwitz relation $|K_C| = |\pi^*(K_E + \mathscr{S}^{-1})|$, we have $|\pi^*(g_2^1 + p + q)| = |h_4^1 + k_4^1|$. Since

$$\dim \operatorname{Sym}^2 H^0(C, \pi^*\mathcal{O}_E(g_2^1 + p + q)) = h^0(C, \pi^*\mathcal{O}_E(2g_2^1 + 2p + 2q)) = 10,$$

it follows that the map

$$\operatorname{Sym}^2 H^0(C, \pi^*\mathcal{O}_E(g_2^1 + p + q)) \to H^0(C, \pi^*\mathcal{O}_E(2g_2^1 + 2p + 2q))$$

is not surjective. Hence the map α is not surjective and we have $s \in V$.

Conversely, assume that $s \in V$. Then the morphism α is not surjective and hence the image of the morphism $C \to \boldsymbol{P}^3$ induced by the birationally very ample $|\pi^*(g_2^1 + p + q)|$ is contained in a quadric surface $S \subset \boldsymbol{P}^3$. In case S is a cone, there is a pencil h_4^1 such that $h^0(C, \mathcal{O}_C(2h_4^1)) = 4$. From the assumption $\mathscr{S} \not\cong \mathcal{O}_E(-2g_2^1)$, it follows that $h^0(C, \mathcal{O}_C(2g_4^1)) = 3$ by (i) and hence $g_4^1 \neq h_4^1$. In case S is a non-singular quadric, we also have two pencils of degree 4 corresponding to the rulings of S. □

A curve C of genus $g \leq 7$ which is a double covering of a curve of genus 2 may be also bi-elliptic, whereas a curve of genus $g \geq 8$ cannot be both bi-elliptic and a double covering of a curve of genus 2 by Castelnuovo-Severi inequality. The following lemma provides a simple criteria for a double covering of a curve of genus 2 being bi-elliptic.

Lemma 5.5. *Let C be a curve of genus $g = 7$ which is a double covering of a curve of genus* 2 *with an involution ι induced by the covering. If C is also bi-elliptic, then the bi-elliptic involution τ commutes with ι; i.e., $\iota\tau = \tau\iota$.*

Proof. Let $\pi_1 : C \to E_1$ be a double covering, where E_1 is an elliptic curve and let τ be the involution induced by π_1. Consider the double covering $\pi_2 : C \to E_2$ induced by $\iota^{-1}\tau\iota$. We remark that $Q \in C$ is invariant under $\iota^{-1}\tau\iota$ if and only if $\iota(Q)$ is invariant under τ. Thus, if R is the ramification locus of π_1 then $\iota(R)$ is the ramification locus of π_2. It follows that $\iota^{-1}\tau\iota$ is also a bi-elliptic involution by the Riemann-Hurwitz formula. By Castelnuovo-Severi inequality, bi-elliptic involution of C is unique and hence $\iota\tau = \tau\iota$. □

Remark 5.6. From Lemma 5.5, it follows easily that if a double covering $C \xrightarrow{\pi} E$ is also bi-elliptic then there is an automorphism on E which lifts to the bi-elliptic involution τ via π.

Proof of Proposition 5.2. We take a curve E of genus 2 such that $\mathrm{Aut}(E) = \{\sigma, 1_E\}$ where σ is the hyperelliptic involution on E. Let

$$C = \mathbf{Spec}(\mathcal{O}_E \oplus \mathcal{O}_E(-g_2^1 - p - q))$$

with $p + q \notin g_2^1$ where $p, q \in E$ are not fixed points of σ. If C is bi-elliptic with a bi-elliptic involution τ, then σ lifts to τ by Remark 5.6; note that 1_E does not lift to τ via π. Then it follows that $\sigma^*\mathcal{O}_E(-g_2^1 - p - q) \cong \mathcal{O}_E(-g_2^1 - p - q)$, which implies $\sigma(p) + \sigma(q) = p + q$ and hence $\sigma(p) = p$, $\sigma(q) = q$ contrary to the choice of $p, q \in E$ as non-fixed points of σ. Therefore C cannot be bi-elliptic.

We now prove (i) and use the same notations we used in Lemma 5.4. Choose $t_1, t_2 \in H^0(E, \mathcal{O}_E(g_2^1 + p + q))$ such that $(t_1)_0$, $(t_2)_0$ are reduced and $(t_1)_0 \cap (t_2)_0 = \varnothing$. Then $R = (s)_0$, where $s = t_1 t_2$, is also reduced. We note that $s \in V$. Therefore the curve $C = \mathbf{Spec}(\mathcal{O}_E \oplus \mathcal{O}_E(-g_2^1 - p - q))$ with the branch locus R is nonsingular, which has g_6^2 by Lemma 5.3 and Lemma 5.4. For (ii), recall that $V \subsetneq H^0(E, \mathcal{O}_E(2g_2^1 + 2p + 2q))$ and take $s \notin V$ such that $R = (s)_0$ is reduced. Then the curve C with branch locus R has no g_6^2 by Lemma 5.3 and Lemma 5.4. □

Appendix.

We prove the following proposition which was left out in the proof of Lemma 4.1. Throughout we assume that C is a tetragonal curve of genus $g = 9$ with a unique g_4^1 satisfying the condition (4.1.6). In particular, C admits a degree 2 morphism $\phi : C \to E$ induced by $|K_C - 2g_4^1|$, where E is a smooth plane quartic. We also assume that C has no g_6^2.

Proposition A.0. $\dim W_8^2(C) \lneq 2$.

Lemma A.1. *C does not have a base-point-free g_5^1.*

Proof. Assume that there is a base-point-free g_5^1. By the base-point-free pencil trick, we have $h^0(C, \mathcal{O}_C(\phi^* K_E - g_5^1)) \geq 1$ and hence $|\phi^* K_E| = |D + g_5^1|$ for some effective divisor D of degree 3. Since ϕ is a morphism of degree 2, this is a contradiction. □

Lemma A.2. *Let $g_6^1 \in W_6^1(C)$ be a base-point-free pencil. Then $|K_C - g_4^1 - g_6^1|$ is a pencil of degree 6. Moreover $|K_C - g_4^1 - g_6^1|$ is base-point-free if and only if $g_6^1 \notin \phi^* W_3^1(E)$.*

Proof. Since there is no g_6^2, a base-point-free g_6^1 is complete and $h^0(C, \mathcal{O}(g_4^1 + g_6^1)) \geq 4$ by the base-point-free pencil trick. Therefore $h^0(C, \mathcal{O}_C(K_C - g_4^1 - g_6^1)) \geq 2$ and hence $|K_C - g_4^1 - g_6^1|$ is a pencil of degree 6. Put $h_6^1 := |K_C - g_4^1 - g_6^1|$ and assume $\mathrm{Bs}\, h_6^1 \neq \varnothing$. Then $h_6^1 = |g_4^1 + P + Q|$ for some $P, Q \in C$ by Lemma A.1. Therefore $g_6^1 = |K_C - 2g_4^1 - P - Q| = |\phi^*(K_E) - P - Q|$. Since

$$h^0(C, \mathcal{O}_C(\phi^*(K_E) - P - Q)) = h^0(C, \mathcal{O}_C(g_6^1)) = 2,$$

we have $P + Q = \phi^*(r)$ for some $r \in E$ and therefore $g_6^1 \in \phi^* W_3^1(E)$. Conversely, assume that $g_6^1 \in \phi^* W_3^1(E)$. Then $g_6^1 = \phi^*(g_3^1) = \phi^*(|K_E - r|)$ for some $r \in E$. Hence $h_6^1 = |K_C - g_4^1 - g_6^1| = |g_4^1 + \phi^*(r)|$ has non-empty base locus. □

From now on, we assume $\dim W_8^2(C) = 2$ and let $A \subset W_8^2(C)$ be a component of dimension 2.

Lemma A.3. *For a general $\mathscr{L} \in A$, $\mathscr{L}$ and $|K_C - \mathscr{L}|$ are base-point-free and birationally very ample.*

Proof. Assume that a general $\mathscr{L} \in A$ has a base point. Then there is a component $B \subset W_7^2(C)$ such that $A = B + W_1(C)$ and $\dim B = 1$. Since C does not have a g_6^2, every element $\mathscr{M} \in B$ is base-point-free. We put

$$Z = B - W_1(C) = \{\mathscr{M} \otimes \mathscr{O}_C(-P) \mid \mathscr{M} \in B, P \in C\} \subset W_6^1(C),$$

which is a component of dimension 2. Note that $W_3^1(E) = \{\mathscr{O}_E(K_E - r) \mid r \in E\}$ and hence $\dim \phi^*(W_3^1(E)) = 1$. We take a general

$$g_6^1 = \mathscr{M} \otimes \mathscr{O}_C(-P) \in Z \backslash \phi^*(W_3^1(E)),$$

which may be assumed to be base-point-free; recall $\mathscr{M} \in B$ is base-point-free. Therefore $|K_C - g_4^1 - g_6^1| = h_6^1$ is base-point-free by Lemma A.2 and hence $|K_C - g_6^1| = |h_6^1 + g_4^1|$ is also base-point-free. On the other hand, $|g_6^1 + P| = |\mathscr{M}| = g_7^2$ which implies that P is a base point of $|K_C - g_6^1|$. And this contradiction shows that a general $\mathscr{L} \in A$ has no base point. In the same way, one can easily check that $|K_C - \mathscr{L}|$ has no base point. Finally both $\mathscr{L}$ and $|K_C - \mathscr{L}|$ are birationally very ample for a general $\mathscr{L} \in A$ by Lemma 2.2. □

Lemma A.4. *For every $\mathscr{L} \in A$, $|\mathscr{L} - g_4^1| \neq \emptyset$ and $|K_C - \mathscr{L} - g_4^1| \neq \emptyset$.*

Proof. This is clear by the base-point-free pencil trick. □

Let $\psi_{\mathscr{L}} : C \to C_{\mathscr{L}} \subset \boldsymbol{P}^2$ be the morphism defined by a general $\mathscr{L} \in A$. By Lemma A.3, Lemma A.4, Lemma 3.3 and the condition (4.1.6) on C, every assumption in Lemma 2.5 is satisfied and hence there is an ordinary 4-fold singular point $P_1 \in C_{\mathscr{L}}$. Since $p_a(C_{\mathscr{L}}) = 21 > 9 + (4-1)(4-2)/2$, $C_{\mathscr{L}}$ has another singular point Q. If $\mathrm{mult}_Q\, C_{\mathscr{L}} \geq 4$, then C has two g_4^1's, contrary to the uniqueness of g_4^1. If $\mathrm{mult}_Q\, C_{\mathscr{L}} = 3$, then C has a base-point-free g_5^1, contradicting Lemma A.1. Therefore $\mathrm{mult}_Q\, C_{\mathscr{L}} = 2$ for any singular point $Q \in C_{\mathscr{L}}$ other than P_1. Since $p_a(C_{\mathscr{L}}) - (9 + (4-1)(4-2)/2) = 6$, $C_{\mathscr{L}}$ has 6 double points $P_2, \dots, P_7$ where some of $P_2, \dots, P_7$ may possibly be infinitely near singular points, i.e. singular points appearing in the blowing-ups of $\boldsymbol{P}^2$.

Let $\pi : S_{\mathscr{L}} \to \boldsymbol{P}^2$ be the blowing-up at $P_1, \dots, P_7$ and let e_i be the total transform of the exceptional divisor corresponding to P_i. Then

$$\mathrm{Pic}(S_{\mathscr{L}}) = \boldsymbol{Z}l \oplus \boldsymbol{Z}e_1 \oplus \cdots \oplus \boldsymbol{Z}e_7 \quad \text{and} \quad C \sim 8l - 4e_1 - 2e_2 - \cdots - 2e_7$$

where $l = \pi^* \mathscr{O}_{\boldsymbol{P}^2}(1)$.

We put $S_0 = \boldsymbol{P}^2$, $S_1 =$ a blowing-up of S_0 at P_1 and let $E_1 \subset S_1$ be the exceptional divisor. Let $C_1 \subset S_1$ be the proper transform of $C_{\mathscr{L}}$. Since P_1 is an ordinary singular point, any singular point of C_1 lies outside E_1. Let P_2 be one of singular points of C_1 and let S_2 be the blowing up at P_2 with the exceptional divisor $E_2 \subset S_2$ corresponding to P_2. Let $C_2 \subset S_2$ be the proper transform of C_1. If all the singular points of C_2 lie outside E_2, then we take P_3 as one of singular points of C_2. In case a singular point of C_2 is in E_2, we remark that such a singular point is unique: If there were more than one singular points of C_2 in E_2, we have $(E_2.C_2) \geq 4$. On the other hand, since any singular point of C_1 is a double point we also have $(E_2.C_2) = 2$, a contradiction. Therefore, if a singular point of C_2 lies on E_2, we take P_3 as the unique singular point lying on E_2. We continue this process and finally we get $\boldsymbol{P}^2 = S_0, S_1, S_2, \dots, S_{\mathscr{L}} = S_7$, $P_1 \in S_0$, $P_2 \in S_1, \dots, P_7 \in S_6$ and $E_i \subset S_i$ $(i = 1, \dots, 7)$ which are the exceptional divisors corresponding to P_i. In our situation, we always regard e_i, which is the total transform of E_i, sitting inside $S_{\mathscr{L}} = S_7$. By [**De**; page 36, (1)], an irreducible component of a support of $e_1 + \cdots + e_7$ is one of the following form:

(A.4.1) $$e_1, e_2, \dots, e_7 \quad \text{or} \quad e_i - e_{i+1} - \cdots - e_t \quad (2 \leq i < t).$$

We denote by $\hat{E}_i$ the proper transform of $E_i \subset S_i$ in each steps, i.e. in $S_{i+1}, S_{i+2}, \dots$ and let $\Sigma = \{P_1, \dots, P_r\}$; $r = 7$ in our situation. We use the following notion and result due to Demazure; cf. [**De**; pp. 38–39, p. 38 a) and p. 39 Définition 1].

Definition A.5. We say that Σ is in almost general position if:
(1) For any $i = 1, \dots, r$, $P_i \notin \hat{E}_1, \dots, \hat{E}_{i-2}$.
(2) There is no line which pass through 4 points of Σ.
(3) There is no conic which pass through all the points of Σ.

By definition, the condition (1) is equivalent to the following condition.
$(1)^\star$ For any $i = 1, \dots, r$, $\hat{E}_i = e_i - \varepsilon e_{i+1}$ where $\varepsilon = 1$ or 0.

Lemma A.6 (Demazure; [**De**, Théorème 1]). *If Σ is in almost general position, then $-K_{S_{\mathscr{L}}}$ is nef.*

We also use the following result which is called Reider's method.

Lemma A.7 (Reider; [**R**, Theorem 1 (i)]). *Let S be a smooth algebraic surface over $\boldsymbol{C}$ and let $\mathscr{L}$ be a nef line bundle. If $(\mathscr{L}^2) \geq 5$ and p is a base point of $|K_S + \mathscr{L}|$, then there exists an effective divisor E passing through p such that*

$$\text{either } (\mathscr{L}.E) = 0, (E^2) = -1 \quad \text{or} \quad (\mathscr{L}.E) = 1, (E^2) = 0.$$

Now we consider linear systems $|l - e_1|$ and $|3l - e_1 - e_2 - \cdots - e_7|$ on the smooth rational surface $S_{\mathscr{L}}$.

Lemma A.8. $\dim|l - e_1| = 1$, $\dim|3l - e_1 - e_2 - \cdots - e_7| = 2$, $|l - e_1|_{|C} = |g_4^1|$ *and* $|3l - e_1 - e_2 - \cdots - e_7|_{|C} = |K_C - 2g_4^1|$.

Proof. It is clear that $\dim|l - e_1| = 1$, $|l - e_1|_{|C} = |g_4^1|$ and $\mathrm{Bs}|l - e_1| = \varnothing$. We note that

$$K_{S_{\mathscr{L}}} \sim -3l + e_1 + \cdots + e_7 \quad \text{and} \quad K_{S_{\mathscr{L}}} + C - 2(l - e_1) \sim 3l - e_1 - e_2 - \cdots - e_7,$$

therefore it follows that

$$\mathcal{O}_C(3l - e_1 - e_2 - \cdots - e_7) \cong \mathcal{O}_C(K_C - 2g_4^1).$$

Since

$$3l - e_1 - e_2 - \cdots - e_7 - C \sim -5l + 3e_1 + e_2 + \cdots + e_7$$

and

$$h^0(S_{\mathscr{L}}, \mathcal{O}_{S_{\mathscr{L}}}(5l - 3e_1 - e_2 - \cdots - e_7)) > 0,$$

we have $h^0(S_{\mathscr{L}}, \mathcal{O}_{S_{\mathscr{L}}}(3l - e_1 - e_2 - \cdots - e_7 - C)) = 0$. Therefore, from the exact sequence

$$0 \to \mathcal{O}_{S_{\mathscr{L}}}(3l - e_1 - \cdots - e_7 - C) \to \mathcal{O}_{S_{\mathscr{L}}}(3l - e_1 - \cdots - e_7) \to \mathcal{O}_C(3l - e_1 - \cdots - e_7) \to 0,$$

we deduce that

$$h^0(S_{\mathscr{L}}, \mathcal{O}_{S_{\mathscr{L}}}(3l - e_1 - e_2 - \cdots - e_7)) \leq h^0(C, \mathcal{O}_C(K_C - 2g_4^1)) = 3.$$

On the other hand, $h^0(S_{\mathscr{L}}, \mathcal{O}_{S_{\mathscr{L}}}(3l - e_1 - e_2 - \cdots - e_7)) \geq h^0(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(3)) - 7 = 3$, and hence $\dim|3l - e_1 - e_2 - \cdots - e_7| = 2$. □

Lemma A.9. *$-K_{S_{\mathscr{L}}}$ is nef.*

Proof. By Lemma A.6, we only need to check that Σ is in almost general position. Note that $(C^2) > 0$ since $C \sim 8l - 4e_1 - 2e_2 - \cdots - 2e_7$. Hence C is nef by the irreducibility of C. Let F be an irreducible component of $e_1 + \cdots + e_7$ which should be of the form

$$e_1, \ldots, e_7 \text{ or } e_i - \cdots - e_t \quad \text{where } 2 \leq i < t$$

by (A.4.1). Since C is nef, $(C.F) \geq 0$ which in turn implies $t = i + 1$. Therefore, on $S_{\mathscr{L}}$, the condition $(1)^\star$ of the Definition A.5 is satisfied.

We now check the condition (2). Assume that (2) does not hold, then $l - e_{i_4} - \cdots - e_{i_7}$ is linearly equivalent to an effective divisor G for some four distinct indices $i_4, \ldots, i_7 \in \{1, \ldots, 7\} = \{i_1, \ldots, i_7\}$. Since C is nef, $(C.G) \geq 0$ and we have $i_4, \ldots, i_7 \in \{2, \ldots, 7\}$. From the exact sequence

$$0 \to \mathcal{O}_{S_{\mathscr{L}}}(2l - e_{i_1} - e_{i_2} - e_{i_3}) \to \mathcal{O}_{S_{\mathscr{L}}}(3l - e_1 - \cdots - e_7) \to \mathcal{O}_G(3l - e_1 - \cdots - e_7) \to 0,$$

and

$$(G.3l - e_1 - \cdots - e_7) = (l - e_{i_4} - \cdots - e_{i_7}.3l - e_1 - \cdots - e_7) = -1 < 0,$$

we have $H^0(G, \mathcal{O}_G(3l - e_1 - \cdots - e_7)) = 0$. Therefore G is in a fixed component of $|3l - e_1 - \cdots - e_7|$.

A divisor $D \in |2l - e_{i_1} - e_{i_2} - e_{i_3}|$ corresponds a conic in $\boldsymbol{P}^2$ which passes through $P_{i_1}, P_{i_2}, P_{i_3}$. If D is irreducible, then $|2l - e_{i_1} - e_{i_2} - e_{i_3}|$ is fixed component free and it follows that $|2l - e_{i_1} - e_{i_2} - e_{i_3}|$ is base-point-free. Since $((2l - e_{i_1} - e_{i_2} - e_{i_3})^2) = 1$, $|2l - e_{i_1} - e_{i_2} - e_{i_3}|$ defines a birational morphism $S_{\mathscr{L}} \to \boldsymbol{P}^2$ which is bijective outside the locus T of (-1)-curves or total transform of (-1)-curves. Hence $T \not\supset C$ implies that the linear system $|3l - e_1 - e_2 - \cdots - e_7 - G|_{|C} = |2l - e_{i_1} - e_{i_2} - e_{i_3}|_{|C}$ defines a birational morphism on C which is contradictory to the fact that

$$|2l - e_{i_1} - e_{i_2} - e_{i_3}|_{|C} = |K_C - 2g_4^1|$$

is not birationally very ample. If any member of $|2l - e_{i_1} - e_{i_2} - e_{i_3}|$ is a union of two lines, then moving part should be $|l|$ and $l - e_{i_1} - e_{i_2} - e_{i_3}$ is linearly equivalent to some effective divisor F'. This implies 7 points are in a conic (i.e. union of two lines), which will be considered in the next case.

Finally we check the condition (3). If the condition (3) does not hold, then $|2l - e_1 - \cdots - e_7| \neq \emptyset$, which implies $2l - e_1 - \cdots - e_7$ should be a fixed part of $|3l - e_1 - \cdots - e_7|$ whereas $|l|$ is the moving part. As $|3l - e_1 - \cdots - e_7|_{|_C} = |K_C - 2g_4^1|$ is not birationally very ample and $|l|_{|_C}$ is birationally very ample, this is a contradiction. □

Lemma A.10. *$|-K_{S_{\mathscr{L}}}| = |3l - e_1 - \cdots - e_7|$ is base-point-free.*

Proof. We apply Reider's method (Lemma A.7) to $-2K_{S_{\mathscr{L}}}$. Note that $-2K_{S_{\mathscr{L}}}$ is also nef by Lemma A.9 and $((-2K_{S_{\mathscr{L}}})^2) = 8 \geq 5$. Let $E \sim al - b_1e_1 - \cdots - b_7e_7$ be an effective divisor such that $(-2K_{S_{\mathscr{L}}}.E) = 0$ and $(E^2) = -1$. Then we have $a^2 - b_1^2 - \cdots - b_7^2 = -1$ and $3a - b_1 - \cdots - b_7 = 0$. By Schwartz's inequality $(b_1 + \cdots + b_7)^2 \leq 7(b_1^2 + \cdots + b_7^2)$, it follows that $(3a)^2 \leq 7(a^2 + 1)$ and hence $a = 0, 1, -1$. But for these values of a, the equations $a^2 - b_1^2 - \cdots - b_7^2 = -1$ and $3a - b_1 - \cdots - b_7 = 0$ have no integral solutions. Therefore there is no effective divisor E such that $(-2K_{S_{\mathscr{L}}}.E) = 0$ and $(E^2) = -1$. Since it is clear that there is no effective divisor E such that $(-2K_{S_{\mathscr{L}}}.E) = 1$ and $(E^2) = 0$, we conclude that $|-K_{S_{\mathscr{L}}}|$ is base-point-free by Lemma A.7. □

Let $f: C \to \boldsymbol{P}^1 \times \boldsymbol{P}^2$ be the morphism defined by $(|g_4^1|, |K - 2g_4^1|)$. By Lemmas A.8 and A.10, the morphism $f_{\mathscr{L}}: S_{\mathscr{L}} \to \boldsymbol{P}^1 \times \boldsymbol{P}^2$ induced by $(|l - e_1|, |3l - e_1 - e_2 - \cdots - e_7|)$ is an extension of the morphism f for a general $\mathscr{L} \in A$.

Lemma A.11. *$f_{\mathscr{L}}(S_{\mathscr{L}}) \sim 2\{\mathrm{pt}\} \times \boldsymbol{P}^2 + 2\boldsymbol{P}^1 \times H$ where H is a hyperplane in $\boldsymbol{P}^2$.*

Proof. Let $f_{\mathscr{L}}(S_{\mathscr{L}}) \sim a\{\mathrm{pt}\} \times \boldsymbol{P}^2 + b\boldsymbol{P}^1 \times H$. Then

$$2 = ((3l - e_1 - \cdots - e_7)^2) = (f_{\mathscr{L}}^*(\boldsymbol{P}^1 \times H)^2)$$
$$= \deg f_{\mathscr{L}} \cdot ((\boldsymbol{P}^1 \times H)^2.f_{\mathscr{L}}(S_{\mathscr{L}})) = a \cdot \deg f_{\mathscr{L}},$$

and

$$2 = (l - e_1.3l - e_1 - \cdots - e_7) = (f_{\mathscr{L}}^*(\{\mathrm{pt}\} \times \boldsymbol{P}^2).f_{\mathscr{L}}^*(\boldsymbol{P}^1 \times H))$$
$$= \deg f_{\mathscr{L}} \cdot (\{\mathrm{pt}\} \times \boldsymbol{P}^2.\boldsymbol{P}^1 \times H.f_{\mathscr{L}}(S_{\mathscr{L}})) = b \cdot \deg f_{\mathscr{L}}.$$

If $\deg f_{\mathscr{L}} = 2$, then $f_{\mathscr{L}}(S_{\mathscr{L}}) \sim \{\mathrm{pt}\} \times \boldsymbol{P}^2 + \boldsymbol{P}^1 \times H$. Therefore $f_{\mathscr{L}}(S_{\mathscr{L}})$ is in a hyperplane in $\boldsymbol{P}^5$ by the Segre map $\boldsymbol{P}^1 \times \boldsymbol{P}^2 \hookrightarrow \boldsymbol{P}^5$ and hence C also lies on a hyperplane in $\boldsymbol{P}^5$. But the morphism $C \to \boldsymbol{P}^5$ which is defined by $|K_C - 2g_4^1 + g_4^1| = |K_C - g_4^1|$ is non-degenerate. This contradiction shows that $\deg f_{\mathscr{L}} = 1$, and hence $a = b = 2$. □

Lemma A.12. *For a general $\mathscr{L} \in A$, $\{\mathscr{M} \in A \mid f_{\mathscr{L}}(S_{\mathscr{L}}) = f_{\mathscr{M}}(S_{\mathscr{M}})\}$ is a finite set.*

Proof. By a simple computation using Schwartz inequality, one finds easily that $(E.4l - 2e_1 - e_2 - \cdots - e_7) \neq 0$ for any (-1)-curve E on $S_{\mathscr{L}}$. Therefore $S_{\mathscr{L}}$ is the minimal resolution of the normalization of $f_{\mathscr{L}}(S_{\mathscr{L}})$. Hence $f_{\mathscr{L}}(S_{\mathscr{L}}) = f_{\mathscr{M}}(S_{\mathscr{M}})$ implies $S_{\mathscr{L}} \overset{\alpha}{\cong} S_{\mathscr{M}}$. We note that the composition morphism $\pi \circ \alpha : S_{\mathscr{L}} \xrightarrow{\alpha} S_{\mathscr{M}} \xrightarrow{\pi} \boldsymbol{P}^2$ is completely determined by 7 divisors $\varepsilon_1, \ldots, \varepsilon_7 \subset S_{\mathscr{L}}$ such that $(K_{S_{\mathscr{L}}} + \varepsilon_i.\varepsilon_i) = -2$, $(\varepsilon_i^2) = -1$; here π is the blowing up of $\boldsymbol{P}^2$ at the singular points of $\phi_{\mathscr{M}}(C) \subset \boldsymbol{P}^2$.

Since $S_{\mathscr{L}}$ is a 7-points blowing-up of $\boldsymbol{P}^2$, there exist only finitely many divisors ε such that $(K_{S_{\mathscr{L}}} + \varepsilon.\varepsilon) = -2$, $(\varepsilon^2) = -1$. Hence we have only finitely many possibilities of such morphisms $S_{\mathscr{L}} \xrightarrow{\alpha} S_{\mathscr{M}} \xrightarrow{\pi} \boldsymbol{P}^2$ and therefore finitely many possibilities for $\mathscr{M}$ with $f_{\mathscr{L}}(S_{\mathscr{L}}) = f_{\mathscr{M}}(S_{\mathscr{M}})$. □

Proof of Proposition A.0. Consider the following exact sequence coming from the morphism $f : C \to \boldsymbol{P}^1 \times \boldsymbol{P}^2$ induced by $(|g_4^1|, |K - 2g_4^1|)$;

(A.0.1) $$0 \to \mathscr{I} \to \mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^2} \to \mathscr{O}_C \to 0.$$

We put $\mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^2}(n) = \mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^2}(\{\mathrm{pt}\} \times \boldsymbol{P}^2 + \boldsymbol{P}^1 \times H)^{\otimes n}$ and $\mathscr{I}(n) = \mathscr{I} \otimes \mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^2}(n)$. By Lemma A.11,

$$f_{\mathscr{L}}(S_{\mathscr{L}}) \in \boldsymbol{P}H^0(\boldsymbol{P}^1 \times \boldsymbol{P}^2, \mathscr{I}(2))$$

and by Lemma A.12 we have a generically finite morphism

$$\pi : A \to \boldsymbol{P}H^0(\boldsymbol{P}^1 \times \boldsymbol{P}^2, \mathscr{I}(2))$$

defined by $\pi(\mathscr{L}) = f_{\mathscr{L}}(S_{\mathscr{L}})$. From the long exact sequence on cohomology induced by the short exact sequence (A.0.1), we have

$$h^0(\boldsymbol{P}^1 \times \boldsymbol{P}^2, \mathscr{I}(2)) \geq 2.$$

Let $Q_1, Q_2 \in H^0(\boldsymbol{P}^1 \times \boldsymbol{P}^2, \mathscr{I}(2))$ be two independent quadrics. Since

$$(Q_1.Q_2.\{\mathrm{pt}\} \times \boldsymbol{P}^2 + \boldsymbol{P}^1 \times H)_{\boldsymbol{P}^1 \times \boldsymbol{P}^2} = 12,$$

the scheme theoretic intersection $Q_1 \cap Q_2 \subset \boldsymbol{P}^1 \times \boldsymbol{P}^2 \subset \boldsymbol{P}^5$ has degree 12. Note that

$$\deg(C) = \deg(K_C - g_4^1) = 12 \quad \text{and hence} \quad C = Q_1 \cap Q_2.$$

Therefore the ideal of C $(\subset \mathscr{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^2})$ is generated by the two quadrics Q_1 and Q_2 in $\boldsymbol{P}^1 \times \boldsymbol{P}^2$; i.e. $h^0(\boldsymbol{P}^1 \times \boldsymbol{P}^2, \mathscr{I}(2)) = 2$. Since

$$f_{\mathscr{L}}(S_{\mathscr{L}}) \in \boldsymbol{P}H^0(\boldsymbol{P}^1 \times \boldsymbol{P}^2, \mathscr{I}(2)) \quad \text{and} \quad \dim \boldsymbol{P}H^0(\boldsymbol{P}^1 \times \boldsymbol{P}^2, \mathscr{I}(2)) = 1,$$

$\pi : A \to \boldsymbol{P}H^0(\boldsymbol{P}^1 \times \boldsymbol{P}^2, \mathscr{I}(2))$ being a generically finite morphism is contradictory to the assumption $\dim A = 2$. □

References

[A] R. D. M. Accola, Topics in the theory of Riemann surfaces, Lecture Notes in Math., vol. 1595, Springer-Verlag, 1994.

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, Geometry of Algebraic Curves I, Springer-Verlag, 1985.

[BKMO] E. Ballico, C. Keem, G. Martens and A. Ohbuchi, On curves of genus eight, Math. Z., **227** (1998), 543–554.

[C] M. Coppens, Some remarks on the scheme W_d^r, Ann. Mat. Pura Appl. (4), **157** (1990), 183–197.

[CKO] K. Cho, C. Keem and A. Ohbuchi, On the variety of special linear systems of degree $g-1$ on smooth algebraic curves, International Journal of Mathematics, Vol. 13, No. 1 (2002), 11–29.

[De] M. Demazure, Surface de Del Pezzo—III, Positions presque générales., Séminaire sur les Singularités des Surfaces (Lecture Note in Math., **777**) (A. Dold, B. Eckmann, eds.), Springer-Verlag, 1980, pp. 36–49.

[FHL] W. Fulton, R. Lazarsfeld and J. Harris, Excess linear series on an algebraic curve, Proc. Amer. Math. Soc., **92** (1984), 320–322.

[GH] Griffiths and Harris, The dimension of the variety of special linear systems on a general curve, Duke Math. J., **47** (1980), 233–272.

[K] C. Keem, On the variety of special linear systems on an algebraic curve, Math. Ann., **288** (1990), 309–322.

[KO] T. Kato and A. Ohbuchi, Very ampleness of multiple of tetragonal linear systems, Comm. Algebra, **21** (1993), 4587–4597.

[KL] S. Kleiman and D. Laksov, On the existence of special divisors, Amer. J. Math., **94** (1972), 431–436.

[Ma] H. Martens, On the varieties of special divisors on a curve, J. Reine Angew. Math., **227** (1967), 111–120.

[Mu] D. Mumford, Prym varieties, Contributions to analysis (L. Ahlfors, I. Kra, B. Maskit and L. Nirenberg, eds.), Academic Press, 1974, pp. 325–350.

[Muk] S. Mukai, Curves and Grassmannians, Algebraic Geometry and Related Topics; Incheon-Korea, International Press, 1993, pp. 19–49.

[R] I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. of Math., **127** (1988), 309–316.

Kyung-Hye Cho
Department of Mathematics
Seoul National University
Seoul 151-742
Korea
E-mail: khcho@math.snu.ac.kr

Changho Keem
Department of Mathematics
Seoul National University
Seoul 151-742
Korea
E-mail: ckeem@math.snu.ac.kr

Akira Ohbuchi
Department of Mathematics
Faculty of Integrated Arts and Sciences Tokushima University
Tokushima 770-8502
Japan
E-mail: ohbuchi@ias.tokushima-u.ac.jp