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Estimates of the Besov norms on the fractal boundary and applications

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Abstract. Let u be a λ-Hölder continuous function on the closure of a bounded domain D with fractal boundary ∂D. We estimate the Besov norm of the restriction of u to ∂D by the $L^p(D)$-norm of $|\nabla u(y)| \operatorname{dist}(y, \partial D)^{\lambda}$ for an adequate $\lambda > 0$. We apply it to the boundedness of operators related to the double layer potentials on the Besov spaces on ∂D.

1. Introduction.

Let D be a bounded Lipschitz domain in $\boldsymbol{R}^d$ $(d \geq 3)$. For some operators on $L^p(\partial D)$ we need to consider singular integrals and some techniques to prove the $L^p(\partial D)$-boundedness of them. One of them is the following operator K:

$$Kf(z) := -\int f(y)\langle \nabla_y N(z-y), n_y \rangle \, d\sigma(y) \tag{1.1}$$

for $f \in L^p(\partial D)$ and $z \in \partial D$, where $\langle \ \rangle$ is the inner product, σ is the surface measure on ∂D and $N(x-y)$ is the Newton kernel, i.e.,

$$N(x-y) = \frac{1}{\omega_d (d-2)|x-y|^{d-2}}$$

and ω_d stands for the surface area of the unit ball in $\boldsymbol{R}^d$.

In [**CMM**] R. Coifman, A. McIntosh and Y. Meyer established the L^p-boundedness of the Cauchy integral on curves with arbitrary large Lipschitz norms and their theory played important roles to consider boundary value problems, especially, the Dirichlet problem and the Neumann problem in a Lipschitz domain (cf. [**Ve**], [**Ke**], [**DK**]).

The function defined by the right-hand side of (1.1) for $z \in \boldsymbol{R}^d \setminus \partial D$ is called the double layer potential for f and the L^p-boundedness of K is necessary to solve the Dirichlet problem and the Neumann problem in a Lipschitz domain D by using layer potentials.

In this paper we consider the boundedness of the operator K on the Besov space on the fractal boundary. More precisely, let D be a bounded domain in $\boldsymbol{R}^d$ $(d \geq 2)$ and assume that ∂D is a β-set $(d-1 \leq \beta < d)$, i.e., there exist a positive Radon measure μ on ∂D and positive real numbers b_1, b_2, r_0 such that

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$$b_1 r^\beta \le \mu(B(x,r) \cap \partial D) \le b_2 r^\beta \tag{1.2}$$

for all $z \in \partial D$ and all $r \le r_0$, where $B(z,r)$ stands for the open ball in $\boldsymbol{R}^d$ with center z and radius r. Such a measure μ is called a β-measure.

We give examples.

1. If D is a bounded Lipschitz domain in $\boldsymbol{R}^d$, then ∂D is a $(d-1)$-set and the surface measure is a $(d-1)$-measure.

2. If ∂D consists of a finite number of self-similar sets, which satisfies the open set condition, and whose similarity dimensions are β, then ∂D is a β-set and the β-dimensional Hausdorff measure restricted to ∂D is a β-measure.

By the same method as in [**JW**] we constructed an extention operator $\mathscr{E}$ in [**W3**] and defined the double layer potential Φf for $f \in L^p(\mu)$ by

$$\Phi f(x) = \int_{\boldsymbol{R}^d \setminus \bar{D}} \langle \nabla \mathscr{E}(f)(y), \nabla_y N(x-y) \rangle \, dy \tag{1.3}$$

if $x \in D$ and

$$\Phi f(x) = -\int_D \langle \nabla \mathscr{E}(f)(y), \nabla_y N(x-y) \rangle \, dy \tag{1.4}$$

if $x \in \boldsymbol{R}^d \setminus \bar{D}$. Here $N(x-y)$ stands for the Newton kernel if $d \ge 3$ and the logarithmic kernel if $n = 2$, respectively.

But the integral of the right-hand side of (1.3) or (1.4) does not always converge for $f \in L^p(\mu)$. So we consider Besov spaces on a β-set ∂D. In general, let F be a closed β-set in $\boldsymbol{R}^d$ and μ be a β-measure on F. Let $0 < \alpha \le 1$. We define a Besov space $\Lambda_\alpha^p(F)$ by the Banach space of all functions $f \in L^p(\mu)$ such that

$$\iint \frac{|f(x) - f(z)|^p}{|x-z|^{\beta + p\alpha}} d\mu(x) d\mu(z) < \infty$$

with norm

$$\|f\|_{\alpha,p} = \left(\int |f(x)|^p \, d\mu(x) \right)^{1/p} + \left(\iint \frac{|f(x) - f(z)|^p}{|x-z|^{\beta + p\alpha}} d\mu(x) d\mu(z) \right)^{1/p}.$$

If $0 \le \beta - (d-1) < \alpha \le 1$ and $f \in \Lambda_\alpha^p(\partial D)$, then the integral defined by the right-hand side of (1.3) (resp. (1.4)) always converges and Φf is harmonic in $\boldsymbol{R}^d \setminus \partial D$. Furthermore Φf converges 'non-tangentially' to $K_1 f(z)$ (resp. $K_2 f(z)$) for μ-*a.e.* $z \in \partial D$, where

$$K_1 f(z) = \int_{\boldsymbol{R}^d \setminus \bar{D}} \langle \nabla_y \mathscr{E}(f)(y), \nabla_y N(z-y) \rangle \, dy \tag{1.5}$$

if it is well-defined and $K_1 f(z) = 0$ otherwise, and

$$K_2 f(z) = -\int_D \langle \nabla_y \mathscr{E}(f)(y), \nabla_y N(z-y) \rangle \, dy \tag{1.6}$$

if it is well-defined and $K_2 f(z) = 0$ otherwise (cf. [**W3**]). We also define, for $f \in \Lambda_\alpha^p(\partial D)$,

$$Kf(z) = \frac{1}{2}(K_1 f(z) + K_2 f(z)). \tag{1.7}$$

This operator is a generalization of K defined by (1.1) and we see that, if both of K_1 and K_2 are bounded operators from $\Lambda_\alpha^p(\partial D)$ to $\Lambda_\alpha^p(\partial D)$, so is K.

Hereafter we shall fix a β-measure μ on ∂D and suppose $\bar{D} \subset B(0, R/2)$ with $R \geq 1$.

To prove that K_1 is bounded on $\Lambda_\alpha^p(\partial D)$, we consider an estimate of the Besov norm of a function u, which is Hölder continuous on $\bar{D}$ and a C_1-function in D, by the $L^p(D)$-norm of $|\nabla u|\delta(y)^\lambda$ for an adequate $\lambda > 0$. Here $\delta(y)$ stands for the distance from y to ∂D.

We now consider a uniform domain by O. Martio and J. Sarvas (cf. [**MS**]). Recall that D is called a uniform domain if there exist constants a and b such that each pair of points $x_0, y_0 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$\begin{aligned} &l(\gamma) \leq a|x_0 - y_0|, \\ &\min\{l(\gamma(x_0, x)), l(\gamma(x, y_0))\} \leq b\,\mathrm{dist}(x, \partial D) \quad \text{for all } x \in \gamma. \end{aligned} \tag{1.8}$$

Here $l(\gamma)$ (resp. $\gamma(x_0, x)$) stands for the euclidean length of γ (resp. the part of γ between x_0 and x).

Note that an (ε, ∞) domain, which was introduced by P. W. Jones in [**Jo**], is a uniform one (cf. [**Vâ**]). Therefore a Lipschitz domain is uniform and the snow flake domain is also uniform.

In §4 we shall prove the following theorem.

THEOREM 1. *Assume that D is a bounded uniform domain such that ∂D is a β-set $(d-1 \leq \beta < d)$. Let $1 < p < \infty$. If $1-(d-\beta) < \alpha < 1-(d-\beta)/p$, $\alpha + (d-\beta)/p < \lambda < 1$ and, u is λ-Hölder continuous on $\bar{D}$ and of C^1 in D, then*

$$\iint \frac{|u(x) - u(z)|^p}{|x-z|^{\beta+p\alpha}}\, d\mu(x) d\mu(z) \leq c \int_D |\nabla u(y)|^p \delta(y)^{p-p\alpha-d+\beta}\, dy,$$

where c is a constant independent of u.

To prove Theorem 1, we use a similar covering argument to that in [**Ko**].

In §5 we shall show that the following theorem on the extension operator $\mathscr{E}$.

THEOREM 2. *Let $1 < p < \infty$, $1-(d-\beta) < \alpha < 1-(d-\beta)/p$ and $f \in \Lambda_\alpha^p(\partial D)$. Then*

$$\int_{\boldsymbol{R}^d} |\nabla \mathscr{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta}\, dy \leq c\|f\|_{p,\alpha}^p,$$

where c is a constant independent of f.

Further we consider a weaker assumption for D. We say that a set G satisfies the condition (b) if there exist a constant c and $r_1 > 0$ such that

$$|B(z, r) \cap G| \geq cr^d$$

for each point $z \in \partial G$ and each positive real number $r \leq r_1$.

With the aid of Theorem 1 and Theorem 2 we see that, to prove the boundedness of the operator K_1 from $\Lambda_\alpha^p(\partial D)$ to $\Lambda_\alpha^p(\partial D)$, it suffices to show the following theorem, which will be proved in §6.

THEOREM 3. *Assume that* $\boldsymbol{R}^d \backslash \bar{D}$ *is connected and satisfies the condition* (b). *Let* $p > 1$, $1 - (d - \beta) < \alpha < 1 - (d - \beta)/p$ *and* $f \in \Lambda_\alpha^p(\partial D)$. *Then*

$$\int_D |\nabla_x \Phi f(x)|^p \delta(x)^{p-p\alpha-d+\beta}\, dx \le c \int_{\boldsymbol{R}^d \backslash \bar{D}} |\nabla_x \mathscr{E}(f)(x)|^p \delta(x)^{p-p\alpha-d+\beta}\, dx,$$

where c *is a constant independent of* f.

Thus it suffices to compare volume integrals over disjoint domains of two functions instead of considering directly the singular integral.

We introduce three kinds of maximal functions relative to two disjoint spaces to prove Theorem 1, Theorem 2 and Theorem 3.

Using above theorems, we shall give the proof of our main theorems in §6.

THEOREM 4. *Assume that* D *is a bounded uniform domain in* R^d $(d \ge 2)$ *and* ∂D *is a* β*-set* $(d - 1 \le \beta < d)$. *Further assume that* $\boldsymbol{R}^d \backslash D$ *is connected and satisfies the condition* (b). *Let* $1 < p < \infty$, $1 - (d - \beta) < \alpha < 1 - (d - \beta)/p$. *Then operator* K_1 *is bounded from* $\Lambda_\alpha^p(\partial D)$ *to* $\Lambda_\alpha^p(\partial D)$.

THEOREM 5. *Assume that* D *is a bounded uniform domain in* R^d $(d \ge 2)$ *and* ∂D *is a* β*-set* $(d - 1 \le \beta < d)$. *Further assume that* $\boldsymbol{R}^d \backslash D$ *is connected and each pair of points* x_0, y_0 *of the set*

$$F_t(\boldsymbol{R}^d \backslash \bar{D}) = \{y \in \boldsymbol{R}^d \backslash \bar{D}; \delta(y) < t\}$$

is joined by a rectifiable arc $\gamma \subset \boldsymbol{R}^d \backslash \bar{D}$ *satisfying* (1.8) *for every* $t \le t_0$ *for some* t_0. *Let* $1 < p < \infty$, $1 - (d - \beta) < \alpha < 1 - (d - \beta)/p$. *Then the operator* K *defined by* (1.7) *is bounded from* $\Lambda_\alpha^p(\partial D)$ *to* $\Lambda_\alpha^p(\partial D)$.

It seems that our methods used in this paper are also useful to prove the boundedness of other operators on $\Lambda_\alpha^p(\partial D)$ for a bounded domain D with fractal boundary.

2. Properties of a uniform domain.

In this section we prepare the properties of a uniform domain, which are used to prove Theorem 1.

Let us begin with the following lemma.

LEMMA 2.1. *Let* G *be a domain in* $\boldsymbol{R}^d$ *such that* ∂G *is compact. Assume that each pair of points* x_0, y_0 *of* $F_t(G)$ *for every* $t \le t_0$ *for some* t_0 *is joined by a rectifiable arc* $\gamma \subset G$ *satisfying* (1.8). *Then there exist positive real numbers* b' *and* r_2 *such that for each* $z \in \partial G$ *and each* r $(r \le r_2)$ *we can find a point* $y_1 \in G$ *satisfying*

$$B(y_1, b'r) \subset B(z, r) \cap G.$$

Therefore G *satisfies the condition* (b).

PROOF. Choose $y_0 \in G$ satisfying $\delta(y_0) = \max_{y \in \overline{F_{t_0/2}(G)}} \delta(y)$ and put $r_2 = \delta(y_0)/4$. Let $z \in \partial G$ and $0 < r \le r_2$. Then $|z - y_0| \ge \delta(y_0)$. Pick $x_1 \in G$ such that $|z - x_1| < r/8$. Then

$$|y_0 - x_1| \ge |y_0 - z| - |z - x_1| \ge 4r - \frac{r}{8} = \frac{31r}{8}.$$

By the assumption there is an arc $\gamma \subset G$ joining x_1 to y_0 such that

$$l(\gamma) \le a|x_1 - y_0|$$

and

$$\min\{l(\gamma(x_1, x)), l(\gamma(x, y_0))\} \le b\delta(x)$$

for every $x \in \gamma$. We may assume that $b \ge 1$. Take the first point $y_1 \in \gamma$ such that $|x_1 - y_1| = r/8$. Then

$$\delta(y_1) \ge \frac{1}{b}\min\{|x_1 - y_1|, |y_1 - y_0|\} \ge \frac{r}{8b}.$$

Hence $B(y_1, r/(16b)) \subset G$.

We also get $B(y_1, r/(16b)) \subset B(z, r)$. Indeed, for $x \in B(y_1, r/(16b))$,

$$|x - z| \le |x - y_1| + |y_1 - x_1| + |x_1 - z| < r,$$

whence

$$B\left(y_1, \frac{r}{16b}\right) \subset B(z, r) \cap G.$$

Thus G satisfies the condition (b). □

LEMMA 2.2. *Let G be a domain in $\boldsymbol{R}^d$ such that ∂G is compact. Assume that two points $x_0, y_0 \in G$ can be joined by a rectifiable arc $\gamma \subset G$ satisfying* (1.8). *If $c_1\delta(x_0) \le \delta(y_0) \le c_2\delta(x_0)$ and*

$$2^j\delta(x_0) \le |x_0 - y_0| < 2^{j+1}\delta(x_0),$$

then there exist balls $B_k = B(z_k, r_k)$ $(k = 0, 1, \dots, m)$, $B'_k = B(z'_k, r'_k)$ $(k = 0, 1, \dots, n)$ having the following properties:

(i) $z_0 = x_0$, $r_0 = \delta(x_0)/(4b)$, $z'_0 = y_0$, $r'_0 = \delta(y_0)/(4b)$, $z_m = z'_n$ *and* $r_m = r'_n$, $r_k < r_{k+1} \le (1 + 1/(4b))r_k$, $r'_k < r'_{k+1} \le (1 + 1/(4b))r'_k$,

(ii) $m \le c_3 j$ *and* $n \le c_4 j$, *where* c_i $(i = 3, 4)$ *are constants independent of* x_0, y_0 *and* j,

(iii) $\mathrm{dist}(\overline{B_k}, \partial G) \ge 3r_k$ *and* $\mathrm{dist}(\overline{B'_k}, \partial G) \ge 3r'_k$,

(iv) $x_0 \in B(z_k, 5br_k)$ *and* $y_0 \in B(z'_k, 5br'_k)$,

(v) $B(y_k, r_{k/2}) \subset B_k \cap B_{k+1}$, $B(y'_k, r'_{k/2}) \subset B'_k \cap B'_{k+1}$,

(vi) $y \in B(z_k, r_k)$ *implies* $\delta(y) < (8b + 1)r_k$ *and* $y \in B(z'_k, r'_k)$ *implies* $\delta(y) < (8b + 1)r'_k$,

(vii) $r_k \le (a/(8b))|x_0 - y_0|$, *and* $r'_k \le (a/(8b))|x_0 - y_0|$.

PROOF. By the assumption there are positive real numbers a, b and a rectifiable arc γ such that

$$l(\gamma) \le a|x_0 - y_0| \tag{2.1}$$

and

$$\min\{l(\gamma(x_0, z)), l(\gamma(z, y_0))\} \le b\delta(z) \quad \text{for all } z \in \gamma.$$

We may assume that $a, b \ge 1$. Take $z' \in \gamma$ satisfying $l(\gamma(x_0, z')) = l(\gamma(z', y_0))$. Further we choose points $z_0, z_1, z_2, \dots, z_m$ on $\gamma(x_0, z')$ inductively as follows.

Put $z_0 = x_0$ and let x_1 be the point satisfying $l(\gamma(z_0, x_1)) = \delta(x_0)/4b$. If $l(\gamma(x_0, x_1)) \ge l(\gamma(x_0, z'))$, we set $z_1 = z'$ and stop the process. If $l(\gamma(x_0, x_1)) < l(\gamma(x_0, z'))$, we set $z_1 = x_1$. Let us now suppose that $z_1, z_2, \dots, z_{k-1}$ have already been chosen on $\gamma(x_0, z')$. Choose a point $x_k \in \gamma$ satisfying $l(\gamma(z_{k-1}, x_k)) = l(\gamma(x_0, z_{k-1}))/(4b)$. If $l(\gamma(x_0, x_k)) \ge l(\gamma(x_0, z'))$, set $z_k = z'$ and stop the process. If $l(\gamma(x_0, x_k)) < l(\gamma(x_0, z'))$, we set $z_k = x_k$.

Put $\rho = \delta(x_0)/(4b)$. Then

$$\begin{aligned} &l(\gamma(z_0, z_1)) = \rho, \\ &l(\gamma(z_0, z_2)) = \rho + \frac{\rho}{4b} = \rho\left(1 + \frac{1}{4b}\right), \\ &l(\gamma(z_0, z_3)) = l(\gamma(z_0, z_2)) + \frac{l(\gamma(z_0, z_2))}{4b} = \rho\left(1 + \frac{1}{4b}\right)^2. \end{aligned}$$

Hence we see inductively that

$$l(\gamma(z_0, z_k)) = \rho\left(1 + \frac{1}{4b}\right)^{k-1} \quad \text{for } k = 1, 2, \dots, m-1. \tag{2.2}$$

Thus we have, by (2.1) and the assumption,

$$\rho\left(1 + \frac{1}{4b}\right)^{m-2} = l(\gamma(z_0, z_{m-1})) \le \frac{l(\gamma)}{2} \le \frac{a}{2}|x_0 - y_0| \le 2^j a\delta(x_0),$$

whence

$$\left(1 + \frac{1}{4b}\right)^{m-2} \le 4ab2^j.$$

This implies

$$m \le 2 + \frac{\log(4ab)}{\log(1 + 1/(4b))} + \frac{j\log 2}{\log(1 + 1/(4b))},$$

which leads to the first inequality of (ii).

We next put

$$r_0 = \frac{\delta(x_0)}{4b},$$

$$r_k = \frac{l(\gamma(x_0, z_k))}{4b} \quad (k = 1, 2, \ldots, m)$$

and

$$B_k = B(z_k, r_k) \quad (k = 0, 1, \ldots, m).$$

Then

$$r_k < r_{k+1} \le \left(1 + \frac{1}{4b}\right) r_k \quad (k = 0, 1, \ldots, m-1)$$

and

$$B\left(y_k, \frac{r_k}{2}\right) \subset B_k \cap B_{k+1} \quad (k = 0, 1, \ldots, m-1)$$

for some $y_k \in G$. Hence the first parts on $\{B_k\}$ of (i) and (v) hold.

Since

$$\operatorname{dist}(\overline{B_k}, \partial G) \ge \operatorname{dist}(z_k, \partial G) - r_k \ge 3r_k,$$

we get the first part of (iii).

Noting that

$$|x_0 - z_k| \le l(\gamma(x_0, z_k)) = 4br_k,$$

we see that $x_0 \in B(z_k, 5br_k)$, which is the first part of (iv).

For each $y \in B(z_k, r_k)$ we have

$$\delta(y) \le \delta(z_k) + r_k \le \delta(x_0) + 4br_k + r_k \le (8b+1)r_k.$$

Thus we get the first part of (vi).

Noting $r_k = l(\gamma(z_0, z_k))/(4b) \le (a/(8b))|x_0 - y_0|$, we have the first part of (vii).

We next consider $(-\gamma)(y_0, z')$ instead of $\gamma(x_0, z')$ and can construct B'_k $(k = 0, 1, \ldots, n)$ by the same method. $\square$

3. Maximal functions.

In this section we introduce two kinds of maximal functions, which will play important roles in later sections. Without loss of generality we may assume that $r_1 \ge 3R$ in the condition (b).

Since ∂D is a β-set, there exists a positive number s_2 such that

$$\int_{\{\delta(y) < \varepsilon\} \cap B(z, r) \cap D} dy \le s_2 r^\beta \varepsilon^{d-\beta} \tag{3.1}$$

for all positive real numbers r, ε satisfying $0 < \varepsilon \le r \le 3R$ (cf. [**W1**, Lemma 2.1]).

Moreover, if D satisfies the condition (b), then

$$s_1 r^\beta \varepsilon^{d-\beta} \le \int_{\{\delta(y)<\varepsilon\}\cap B(z,r)\cap D} dy \tag{3.2}$$

for all positive real numbers r, ε satisfying $0 < \varepsilon \le r \le 3R$ (cf. [**W4**, Lemma 2.1]).

We fix positive real numbers s_1, s_2 satisfying (3.2), (3.1), respectively and define, for $t > 0$,

$$A_t(D) \equiv A_t = \left\{ y \in D; \frac{t}{2}\left(\frac{s_1}{s_2}\right)^{1/(d-\beta)} \le \delta(y) < t \right\}. \tag{3.3}$$

Let $t > 0$, $0 < \lambda < d - \beta$. We define a Borel measure $\nu_{\lambda,t}$ on A_t by

$$\nu_{\lambda,t}(E) = \int_{E\cap A_t} \delta(y)^{-\lambda}\, dy$$

for each Borel set E. We also define a Borel measure ν_λ^+ on D by

$$\nu_\lambda^+(E) = \int_{E\cap D} \delta(y)^{-\lambda}\, dy.$$

These two measures have the following properties.

LEMMA 3.1. *Assume that D satisfies the condition* (b). *Let $0 < \lambda < d - \beta$, $0 < t \le 1$, $x \in A_t(D)$ and $2\delta(x) \le r \le R$. Then*

$$\nu_{\lambda,t}(B(x,r)) \le c_1 r^{d-\lambda} \le c_2 \nu_\lambda^+(B(x,r)) \le c_3 r^{d-\lambda}, \tag{3.4}$$

where c_1, c_2 and c_3 are constants independent of x, t and r.

PROOF. Choose $x' \in \partial D$ satisfying $|x - x'| = \mathrm{dist}(x, \partial D)$. Since for every $y \in B(x,r)$,

$$|x' - y| \le |x' - x| + |x - y| \le \delta(x) + r < 2r,$$

we have, by (3.1),

$$\begin{aligned} \nu_{\lambda,t}(B(x,r)) &\le c_1 t^{-\lambda} |B(x',2r) \cap \{y \in D; \delta(y) < t\}| \\ &\le c_2 t^{-\lambda}(2r)^\beta t^{d-\beta} = c_3 r^\beta t^{d-\beta-\lambda} \\ &\le c_4 r^\beta r^{d-\beta-\lambda} = c_4 r^{d-\lambda}, \end{aligned}$$

which shows the first inequality of (3.4).

We next show the second inequality. Noting that $r \ge 2\delta(x)$, we have, by (3.2),

$$\begin{aligned} \nu_\lambda^+(B(x,r)) &\ge \int_{B(x',r/2)\cap D} \delta(y)^{-\lambda}\, dy \\ &= \int_{(r/2)^{-\lambda}}^\infty \left| B\left(x', \frac{r}{2}\right) \cap \{y \in D; \delta(y) < s^{-1/\lambda}\} \right| ds \\ &\ge s_1 \left(\frac{r}{2}\right)^\beta \int_{(r/2)^{-\lambda}}^\infty s^{(\beta-d)/\lambda}\, ds \ge c_5 r^{d-\lambda}. \end{aligned}$$

The third inequality of (3.4) is obtained by the inequality

$$
\begin{aligned}
v_\lambda^+(B(x,r)) &\le \int_{B(x',2r)\cap D} \delta(y)^{-\lambda}\,dy \\
&\le \int_{(2r)^{-\lambda}}^{\infty} |B(x',2r)\cap\{y\in D;\delta(y)<s^{-1/\lambda}\}|\,ds \\
&\le s_2(2r)^\beta \int_{(2r)^{-\lambda}}^{\infty} s^{(\beta-d)/\lambda}\,ds \le c_6 r^\beta t^{d-\beta-\lambda} \le c_6 r^{d-\lambda}. \qquad \square
\end{aligned}
$$

Fix $t>0$ and define, for $f\in L^1(v_\lambda^+)$, a maximal function $\mathscr{M}(v_{\lambda,t},v_\lambda^+)(f)$ of f by

$$
\mathscr{M}(v_{\lambda,t},v_\lambda^+)(f)(x)=\sup\left\{\frac{1}{r^{d-\lambda}}\int_{B(x,r)}|f(y)|\,dv_\lambda^+(y); 2\delta(x)\le r\le R\right\}
$$

for each $x\in A_t(D)$. Then the maximal function has the following properties as usual.

Lemma 3.2. *Assume that D satisfies the condition* (b). *Let $0<\lambda<d-\beta$.*
(i) *Let $f\in L^1(v_\lambda^+)$, $s>0$ and set*

$$
E_s=\{x\in A_t;\mathscr{M}(v_{\lambda,t},v_\lambda^+)(f)(x)>s\}.
$$

Then

$$
v_{\lambda,t}(E_s)\le\frac{c}{t}\int|f|\,dv_\lambda^+,
$$

where c is a constant independent of f, s.
(ii) *Let $1<p<\infty$ and $f\in L^p(v_\lambda^+)$. Then*

$$
\int \mathscr{M}(v_{\lambda,t},v_\lambda^+)(f)^p\,dv_{\lambda,t}\le c\int|f|^p\,dv_\lambda^+,
$$

where c is a constant independent of f.

Proof. (i) For each $x\in E_s$ there is a positive real number r_x such that $r_x\le R$ and

$$
\frac{1}{r_x^{d-\lambda}}\int_{B(x,r_x)}|f(y)|\,dv_\lambda^+(y)>s.
$$

Using the Besicovitch covering theorem, we can find a subcovering $\{B(x_j,r_j)\}$ of $\{B(x,r_x)\}_{x\in E_s}$ such that

$$
E_s\subset\bigcup_j B(x_j,r_j)
$$

and each point $x\in E_s$ is contained in at most N numbers of $B(x_j,r_j)$. Lemma 3.1 yields

$$\begin{aligned} v_{\lambda,t}(E_s) &\le \sum_j v_{\lambda,t}(B(x_j, r_j)) \le c_1 \sum_j r_j^{d-\lambda} \\ &\le c_1 \frac{1}{s} \sum_j \int_{B(x_j, r_j)} |f(y)|\, dv_\lambda^+(y) \le \frac{c_1 N}{s} \int_D |f(y)|\, dv_\lambda^+(y). \end{aligned}$$

This shows (i).

(ii) The assertion (ii) is deduced from (i) by the usual method. □

We next consider another maximal function. To do so, let $0 < \lambda < d - \beta$ and define, for a Borel set E,

$$v_\lambda^-(E) = \int_{(B(0,2R)\setminus \bar{D}) \cap E} \delta(y)^{-\lambda}\, dy.$$

We can easily show the following lemma as the proof of Lemma 3.1.

Lemma 3.3. *Assume that $\boldsymbol{R}^d \setminus \bar{D}$ satisfies the condition* (b). *Let $0 < \lambda < d - \beta$. Further let $x \in D$ and $(4/3)\delta(x) \le r \le 3R$. Then*

$$v_\lambda^+(B(x,r)) \le c_1 r^{d-\lambda} \le c_2 v_\lambda^-(B(x,r)) \le c_3 r^{d-\lambda}, \tag{3.5}$$

where c_1, c_2 and c_3 are constants independent of x and r.

Let $u \in L^1(v_\lambda^-)$. We define a maximal function $\mathscr{M}(v_\lambda^+, v_\lambda^-)(u)$ of u by

$$\begin{aligned} &\mathscr{M}(v_\lambda^+, v_\lambda^-)(u)(x) \\ &\quad = \sup\left\{ \frac{\int_{B(x,r)\cap(B(0,2R)\setminus\bar{D})} |u|\, dv_\lambda^-}{v_\lambda^-(B(x,r))} ; \frac{4}{3}\delta(x) \le r \le 3R \right\} \end{aligned} \tag{3.6}$$

for $x \in D$.

Using Lemma 3.3, we can prove the following lemma by the same method as in the proof of Lemma 3.2.

Lemma 3.4. *Assume that $\boldsymbol{R}^d \setminus \bar{D}$ satisfies the condition* (b). *Let $0 < \lambda < d - \beta$.*

(i) *Let $u \in L^1(v_\lambda^-)$, $s > 0$ and set*

$$F_s = \{x \in D; \mathscr{M}(v_\lambda^+, v_\lambda^-)(u)(x) > s\}.$$

Then

$$v_\lambda^+(F_s) \le \frac{c}{s} \int |u|\, dv_\lambda^-,$$

where c is a constant independent of u, s.

(ii) *Let $1 < p < \infty$ and $u \in L^p(v_\lambda^-)$. Then*

$$\int \mathscr{M}(v_\lambda^+, v_\lambda^-)(u)^p\, dv_\lambda^+ \le c \int |u|^p\, dv_\lambda^-,$$

where c is a constant independent of u.

REMARK 3.1. For $t > 0$ and $0 < \lambda < d - \beta$ we define

$$v'_{\lambda,t}(E) = \int_{E \cap A_t(\boldsymbol{R}^d \setminus \bar{D})} \delta(y)^{-\lambda}\, dy$$

for each Borel set E. Lemma 3.1 remains valid if we replace D, $v_{\lambda,t}$, v_λ^+ with $\boldsymbol{R}^d \setminus \bar{D}$, $v'_{\lambda,t}$, v_λ^-, respectively. We also see that Lemma 3.3 is valid even if we exchange D and v_λ^+ for $\boldsymbol{R}^d \setminus \bar{D}$ and v_λ^-, respectively. Therefore the maximal functions $\mathscr{M}(v'_{\lambda,t}, v_\lambda^-)(f)$ and $\mathscr{M}(v_\lambda^-, v_\lambda^+)(u)$ are defined and assertions corresponding to Lemmas 3.2 and 3.4 hold.

4. Proof of Theorem 1.

In this section we assume that D is a bounded domain such that ∂D is a β-set. We prepare some lemmas.

The estimate for the Besov norm in the product of two balls is as follows.

LEMMA 4.1. *Let* $1 \leq p < \infty$, $p - p\alpha - d + \beta > 0$ *and* $x_0 \in \boldsymbol{R}^d$. *Further let* $r > 0$ *and* $u \in C^1(\overline{B(x_0, r)})$. *Then*

$$\int_{B(x_0,r)} dy \int_{B(x_0,r)} \frac{|u(x) - u(y)|^p}{|x - y|^{d+p\alpha+d-\beta}} dx \leq c r^{p-p\alpha-d+\beta} \int_{B(x_0,r)} |\nabla u(y)|^p\, dy,$$

where c *is a constant independent of* u, x_0 *and* r.

PROOF. Let $x, y \in B(x_0, r)$. From

$$\begin{aligned} u(x) - u(y) &= \int_0^1 \frac{d}{dt} u(y + t(x - y))\, dt \\ &= (x - y) \cdot \int_0^1 \nabla u(y + t(x - y))\, dt, \end{aligned}$$

we deduce

$$\frac{|u(x) - u(y)|^p}{|x - y|^{d+p\alpha+d-\beta}} \leq |x - y|^{p-d-p\alpha-d+\beta} \int_0^1 |\nabla u(y + t(y - x))|^p\, dt.$$

Let $0 < s < 2r$. Then

$$\begin{aligned} I &\equiv \int_{\partial B(y,s) \cap B(x_0,r)} \frac{|u(x) - u(y)|^p}{|x - y|^{d+p\alpha+d-\beta}} d\mathscr{H}^{d-1}(x) \\ &\leq s^{p-d-p\alpha-d+\beta} \int_0^1 dt \int_{\partial B(y,s) \cap B(x_0,r)} |\nabla u(y + t(x - y))|^p\, d\mathscr{H}^{d-1}(x) \\ &\leq s^{p-d-p\alpha-d+\beta} \int_0^1 \frac{dt}{t^{d-1}} \int_{\partial B(y,st) \cap B(x_0,r)} |\nabla u(w)|^p\, d\mathscr{H}^{d-1}(w) \\ &\leq s^{p-d-p\alpha+\beta-2} \int_0^s \frac{dt'}{t'^{d-1}} \int_{\partial B(y,t') \cap B(x_0,r)} |\nabla u(w)|^p\, d\mathscr{H}^{d-1}(w). \end{aligned}$$

Since $p-d-p\alpha+\beta>0$, we choose a positive number $\varepsilon<1$ such that $p-d-p\alpha+\beta-1+\varepsilon>0$. Then we have

$$\begin{aligned}
I &\le s^{p-d-p\alpha+\beta-2+\varepsilon}\int_0^s \frac{1}{t'^{d-1+\varepsilon}}\,dt'\int_{\partial B(y,t')\cap B(x_0,r)} |\nabla u(w)|^p\,d\mathscr{H}^{d-1}(w) \\
&\le s^{p-d-p\alpha+\beta-2+\varepsilon}\int_0^{2r} dt'\int_{\partial B(y,t')\cap B(x_0,r)} \frac{|\nabla u(w)|^p}{|y-w|^{d-1+\varepsilon}}\,d\mathscr{H}^{d-1}(w) \\
&= s^{p-d-p\alpha+\beta-2+\varepsilon}\int_{B(x_0,r)} \frac{|\nabla u(w)|^p}{|y-w|^{d-1+\varepsilon}}\,dw.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{B(x_0,r)} \frac{|u(x)-u(y)|^p}{|x-y|^{d+p\alpha+d-\beta}}\,dx \\
&\qquad = \int_0^{2r} I\,ds \le c_1 r^{p-d-p\alpha+\beta-1+\varepsilon}\int_{B(x_0,r)} \frac{|\nabla u(w)|^p}{|y-w|^{d-1+\varepsilon}}\,dw.
\end{aligned}$$

Noting that $d-1+\varepsilon<d$ and integrating over $B(x_0,r)$ with respect to y, we have the conclusion. □

In [**W6**, Theorem 2] we gave the following theorem.

Lemma A. *Suppose that G is a domain in $\boldsymbol{R}^d$ such that ∂G is compact and a β-set $(d-1\le\beta<d)$ and satisfies the condition* (b). *Let $1\le p<\infty$ and $\alpha+(d-\beta)/p<\lambda<1$. If f is λ-Hölder continuous on $\overline{G}$, then*

$$\begin{aligned}
&\int_{\partial G}\int_{\partial G} \frac{|f(x)-f(y)|^p}{|x-y|^{\beta+p\alpha}}\,d\mu(x)d\mu(y) \\
&\qquad \le c\liminf_{t\to 0}\int_{A_t(G)}\int_{A_t(G)} \frac{|f(x)-f(y)|^p}{|x-y|^{d+p\alpha+d-\beta}}\,dxdy,
\end{aligned}$$

where c is a constant independent of f and $A_t(G)$ is the set defined by (3.3).

Note that we assumed in [**W6**, Theorem 2] that G is bounded. But the theorem is valid under our assumption with a small change in the proof.

Lemma 4.2. *Let $p-p\alpha-d+\beta>0$ and $u\in C^1(D)$. Set*

$$\eta_{x,r}=\frac{\int_{B(x,r)} u\,dy}{|B(x,r)|} \quad \textit{and} \quad r_x=\frac{\delta(x)}{4b}$$

for $x\in D$ and $r>0$ satisfying $B(x,r)\subset D$ and b is the constant in (1.8). *If $0<c_1\le 1$ and $c_2\ge 1$, then*

$$\sum_{j=-1}^{\infty}\int_{A_t}dx\int_{c_1 2^j\delta(x)\le|x-y|<c_2 2^{j+1}\delta(x)}\frac{|u(x)-\eta_{x,r_x}|^p}{|x-y|^{d+p\alpha+d-\beta}}dy$$
$$\le c\int_D|\nabla u(y)|^p\delta(y)^{p-p\alpha-d+\beta}\,dy,$$

where c is a constant independent of u and t.

Proof. Since $A_t\subset\bigcup_{x\in A_t}B(x,\delta(x)/(20b))$, we can find $\{x_k\}\subset A_t$ such that

$$A_t\subset\bigcup_{k=1}^{\infty}B\left(x_k,\frac{\delta(x_k)}{4b}\right)\tag{4.1}$$

and $\{B(x_k,\delta(x_k)/(20b))\}$ is mutually disjoint.

Fix a natural number k and let $x\in B(x_k,\delta(x_k)/(4b))\cap A_t$. Then

$$|u(x)-\eta_{x,r_x}|^p\le\frac{1}{r_x^{dp}}\left(\int_{B(x,r_x)}|u(x)-u(z)|\,dz\right)^p\le c_3\frac{1}{r_x^d}\int_{B(x,r_x)}|u(x)-u(z)|^p\,dz.$$

By the same method as in the proof of Lemma 4.1 we have

$$\frac{1}{r_x^d}\int_{B(x,r_x)}|u(x)-u(z)|^p\,dz\le c_4\frac{1}{r_x^d}\int_{B(x,r_x)}\frac{|\nabla u(w)|^p}{|x-w|^{d-\varepsilon}}dw\int_0^{2r_x}s^{p+d-1-\varepsilon}\,ds$$
$$\le c_5\delta(x)^{p-\varepsilon}\int_{B(x,r_x)}\frac{|\nabla u(w)|^p}{|x-w|^{d-\varepsilon}}dw,$$

where ε is a positive number satisfying $p-p\alpha-d+\beta-\varepsilon>0$. Hence

$$\int_{c_1 2^j\delta(x)\le|x-y|<c_2 2^{j+1}\delta(x)}\frac{|u(x)-\eta_{x,r_x}|^p}{|x-y|^{d+p\alpha+d-\beta}}dy$$
$$\le c_6(2^j)^{-p\alpha-d+\beta}\delta(x)^{p-p\alpha-d+\beta-\varepsilon}\int_{B(x,r_x)}\frac{|\nabla u(w)|^p}{|x-w|^{d-\varepsilon}}dw.$$

Take $z_k\in\partial D$ satisfying $\delta(x_k)=|x_k-z_k|$. Since

$$|z_k-x|\le|z_k-x_k|+|x_k-x|<\left(1+\frac{1}{4b}\right)\delta(x_k),$$

we see that $\delta(x)<(1+1/(4b))\delta(x_k)$. Similarly $\delta(x)\ge(1-1/(4b))\delta(x_k)$.

From these we deduce, for every $w\in B(x,r_x)$,

$$|x-w|<\frac{1}{4b}\delta(x)\le\frac{1}{4b}\left(1+\frac{1}{4b}\right)\delta(x_k)$$

and

$$\delta(w)\ge\delta(x)-r_x=\left(1-\frac{1}{4b}\right)\delta(x)\ge\left(1-\frac{1}{4b}\right)^2\delta(x_k).$$

Setting $r_k=\delta(x_k)/(4b)$, we have

$$\int_{B(x_k,r_k)} dx \int_{c_1 2^j\delta(x)\le|x-y|<c_2 2^{j+1}\delta(x)} \frac{|u(x)-\eta_{x,r}|^p}{|x-y|^{d+p\alpha+d-\beta}}\,dy$$
$$\le c_7(2^j)^{-p\alpha-d+\beta}\int_{B(x_k,4r_k)\cap D} |\nabla u(w)|^p\delta(w)^{p-p\alpha-d+\beta-\varepsilon}\,dw \int_{|x-w|\le c_8\delta(w)} \frac{dx}{|w-x|^{d-\varepsilon}}$$
$$\le c_9(2^j)^{-p\alpha-d+\beta}\int_{B(x_k,4r_k)\cap D} |\nabla u(w)|^p\delta(w)^{p-p\alpha-d+\beta}\,dw.$$

Since $\{B(x_k,r_k/5)\}$ are mutually disjoint and $(s_1/s_2)^{1/(d-\beta)}t/(8b)\le r_k<t/(4b)$, we have

$$\int_{A_t} dx \int_{c_1 2^j\delta(x)\le|x-y|<c_2 2^{j+1}\delta(x)} \frac{|u(x)-\eta_{x,r}|^p}{|x-y|^{d+p\alpha+d-\beta}}\,dy$$
$$\le c_{10}(2^j)^{-p\alpha-d+\beta}\int_D |\nabla u(w)|^p\delta(w)^{p-p\alpha-d+\beta}\,dw,$$

whence

$$\sum_{j=-1}^{\infty}\int_{A_t} dx \int_{c_1 2^j\delta(x)\le|x-y|<c_2 2^{j+1}\delta(x)} \frac{|u(x)-\eta_{x,r}|^p}{|x-y|^{d+p\alpha+d-\beta}}\,dy$$
$$\le c_{11}\int_D |\nabla u(w)|^p\delta(w)^{p-p\alpha-d+\beta}\,dw.$$

Thus we have the conclusion. □

Lemma 4.3. *Suppose that for each $t\le t_0$ each pair of points $x_0,y_0\in F_t(D)$ is joined by a rectifiable arc γ satisfying* (1.8) *in D. Let $1<p<\infty$, $1-(d-\beta)<\alpha<1-(d-\beta)/p$ and $u\in C^1(D)$. Then*

$$\liminf_{t\to 0}\int_{A_t}\int_{A_t}\frac{|u(x)-u(y)|^p}{|x-y|^{d+p\alpha+d-\beta}}\,dxdy\le c\int_D |\nabla u(y)|^p\delta(y)^{p-p\alpha-d+\beta}\,dy,$$

where c is a constant independent of u.

Proof. We may assume that

$$\int_D |\nabla u(y)|^p\delta(y)^{p-p\alpha-d+\beta}\,dy<\infty.$$

Let $x\in A_t$ and write

$$\int_{A_t}\frac{|u(x)-u(y)|^p}{|x-y|^{d+p\alpha+d-\beta}}\,dy$$
$$=\int_{A_t\cap\{|x-y|<\delta(x)/2\}}\frac{|u(x)-u(y)|^p}{|x-y|^{d+p\alpha+d-\beta}}\,dy+\int_{A_t\cap\{|x-y|\ge\delta(x)/2\}}\frac{|u(x)-u(y)|^p}{|x-y|^{d+p\alpha+d-\beta}}\,dy$$
$$\equiv I_1+I_2.$$

Take the covering $\{B(x_k,\delta(x_k)/(4b))\}$ of (4.1) and assume that $x\in B(x_k,\delta(x_k)/(4b))$ and $|x-y|<\delta(x)/2$. We note that $b\ge 1$. From

$$|x_k - y| \le |x_k - x| + |x - y| \le \frac{\delta(x_k)}{4} + \frac{\delta(x)}{2}$$
$$\le \frac{\delta(x_k)}{4} + \frac{5\delta(x_k)}{8} = \frac{7\delta(x_k)}{8}$$

we deduce that $y \in B(x_k, 7\delta(x_k)/8)$. Hence, by Lemma 4.1,

$$\begin{aligned}
&\int_{A_t \cap B(x_k, \delta(x_k)/(4b))} I_1\, dx \\
&\quad \le \int_{A_t \cap B(x_k, \delta(x_k)/(4b))} dx \int_{A_t \cap B(x_k, 7\delta(x_k)/8)} \frac{|u(x) - u(y)|^p}{|x - y|^{d+p\alpha+d-\beta}}\, dy \\
&\quad \le \int_{B(x_k, 7\delta(x_k)/8)} dx \int_{B(x_k, 7\delta(x_k)/8)} \frac{|u(x) - u(y)|^p}{|x - y|^{d+p\alpha+d-\beta}}\, dy \\
&\quad \le c_1 \delta(x_k)^{p-p\alpha-d+\beta} \int_{B(x_k, 7\delta(x_k)/8)} |\nabla u(y)|^p\, dy.
\end{aligned}$$

On the other hand, if $y \in B(x_k, 7\delta(x_k)/8)$, then

$$\delta(x_k) \le \delta(y) + |x_k - y| \le \delta(y) + \frac{7\delta(x_k)}{8}$$

and hence $\delta(x_k)/8 \le \delta(y)$. Consequently

$$\int_{A_t \cap B(x_k, \delta(x_k)/(4b))} I_1\, dx \le c_2 \int_{B(x_k, 7\delta(x_k)/8)} |\nabla u(y)|^p \delta(y)^{p-p\alpha-d+\beta}\, dy.$$

Noting that $B(x_k, \delta(x_k)/(20b))$ are mutually disjoint, we have

$$\begin{aligned}
\int_{A_t} I_1\, dx &\le \sum_k \int_{B(x_k, 7\delta(x_k)/8)} |\nabla u(y)|^p \delta(y)^{p-p\alpha-d+\beta}\, dy \\
&\le c_3 \int_D |\nabla u(y)|^p \delta(y)^{p-p\alpha-d+\beta}\, dy.
\end{aligned} \tag{4.2}$$

We next estimate I_2. Let y be a point in A_t such that $2^j\delta(x) \le |x - y| < 2^{j+1}\delta(x)$. For $x_0 = x$ and $y_0 = y$ we choose families $\{B_k\}$ ($B_k = B(z_k, r_k)$) and $\{B'_k\}$ ($B'_k = B(z'_k, r'_k)$) satisfying (i)–(vii) of Lemma 2.2. Noting that $B(z_m, r_m) = B(z'_n, r'_n)$, we write

$$\begin{aligned}
|u(x) - u(y)| &\le |u(x) - \eta_{x, r_0}| + \sum_{k=0}^{m-1} |\eta_{z_k, r_k} - \eta_{z_{k+1}, r_{k+1}}| \\
&\quad + |u(y) - \eta_{y, r'_0}| + \sum_{k=0}^{n-1} |\eta_{z'_k, r'_k} - \eta_{z'_{k+1}, r'_{k+1}}| \\
&\equiv J(x) + \sum_{k=0}^{m-1} J_k + J'(y) + \sum_{k=0}^{n-1} J'_k.
\end{aligned}$$

Lemma 4.2 yields

$$\sum_{j=-1}^{\infty}\int_{A_t}dx\int_{\{2^j\delta(x)\le|x-y|<2^{j+1}\delta(x)\}\cap A_t}\frac{J(x)^p}{|x-y|^{d+p\alpha+d-\beta}}dy \tag{4.3}$$

$$\le c_4\int_D|\nabla u(y)|^p\delta(y)^{p-p\alpha-d+\beta}\,dy$$

and

$$\sum_{j=-1}^{\infty}\int_{A_t}dx\int_{\{2^j\delta(x)\le|x-y|<2^{j+1}\delta(x)\}\cap A_t}\frac{J'(y)^p}{|x-y|^{d+p\alpha+d-\beta}}dy \tag{4.4}$$

$$\le \sum_{j=-1}^{\infty}\int_{A_t}dy\int_{\{d_12^j\delta(y)\le|x-y|<d_22^{j+1}\delta(y)\}\cap A_t}\frac{J'(x)^p}{|x-y|^{d+p\alpha+d-\beta}}dx$$

$$\le c_5\int_D|\nabla u(y)|^p\delta(y)^{p-p\alpha-d+\beta}\,dy.$$

We next consider $\sum_{k=0}^{m-1}J_k$. Noting that

$$B(y_k,r_k/2)\subset B(z_k,r_k)\cap B(z_{k+1},r_{k+1})$$

for some y_k, we have, by Poincaré's inequality and Lemma 2.2, (iii),

$$\int_{B(y_k,r_k/2)}J_k\,dy$$

$$\le\int_{B(z_k,r_k)}|u(y)-\eta_{z_k,r_k}|\,dy+\int_{B(z_{k+1},r_{k+1})}|u(y)-\eta_{z_{k+1},r_{k+1}}|\,dy$$

$$\le c_6\left(r_k\int_{B(z_k,r_k)}|\nabla u(y)|\,dy+r_{k+1}\int_{B(z_{k+1},r_{k+1})}|\nabla u(y)|\,dy\right)$$

$$\le c_7\left(\int_{B(z_k,r_k)}|\nabla u(y)|\delta(y)\,dy+\int_{B(z_{k+1},r_{k+1})}|\nabla u(y)|\delta(y)\,dy\right).$$

Hence

$$J_k\le c_8\frac{1}{(r_k)^d}\int_{B(z_k,r_k)}|\nabla u(y)|\delta(y)\,dy$$

$$+c_8\frac{1}{(r_{k+1})^d}\int_{B(z_{k+1},r_{k+1})}|\nabla u(y)|\delta(y)\,dy.$$

Set

$$I_k=\frac{1}{(r_k)^d}\int_{B(z_k,r_k)}|\nabla u(y)|\delta(y)\,dy.$$

Choose $\varepsilon > 0$ satisfying $0 < d - \beta - p\varepsilon$ and set $\lambda = d - \beta - p\varepsilon$. With the aid of Lemma 2.2, (iii), (vi), (vii) we have

$$\begin{aligned} I_k &\le c_9 \frac{r_k^{\alpha+\varepsilon} \int_{B(x,10br_k)\cap D} |\nabla u(y)| \delta(y)^{1-\alpha-\varepsilon} \delta(y)^{-\lambda}\, dy}{r_k^{d-\lambda}} \\ &\le c_{10} |x-y|^{\alpha+\varepsilon} \mathcal{M}(v_{\lambda,t}, v_\lambda^+)(f)(x), \end{aligned}$$

where $f(y) = |\nabla u(y)| \delta(y)^{1-\alpha-\varepsilon}$. Here we note that

$$10br_k \ge 10br_1 = \frac{5}{2}\delta(x).$$

Therefore we have, by Lemma 2.2, (ii),

$$\sum_{k=0}^{m-1} J_k \le c_{11} |x-y|^{\alpha+\varepsilon} j \mathcal{M}(v_{\lambda,t}, v_\lambda^+)(f)(x),$$

whence

$$\begin{aligned} &\int_{2^j\delta(x) \le |x-y| < 2^{j+1}\delta(x)} \frac{(\sum_{k=0}^{m-1} J_k)^p}{|x-y|^{d+p\alpha+d-\beta}}\, dy \\ &\quad \le c_{12} \int_{2^j\delta(x) \le |x-y| < 2^{j+1}\delta(x)} \frac{j^p \mathcal{M}(v_{\lambda,t}, v_\lambda^+)(f)(x)^p}{|x-y|^{d+d-\beta-p\varepsilon}}\, dy \\ &\quad \le c_{13} (2^j)^{-d+\beta+p\varepsilon} j^p \delta(x)^{-d+\beta+p\varepsilon} \mathcal{M}(v_{\lambda,t}, v_\lambda^+)(f)(x)^p. \end{aligned}$$

Using Lemma 2.2 and Lemma 3.2, we have

$$\begin{aligned} (4.5) \quad &\int_{A_t} dx \sum_{j=-1}^{\infty} \int_{2^j\delta(x) \le |x-y| < 2^{j+1}\delta(x)} \frac{(\sum_{k=0}^{m-1} J_k)^p}{|x-y|^{d+p\alpha+d-\beta}}\, dy \\ &\le c_{14} \int_{A_t} \mathcal{M}(v_{\lambda,t}, v_\lambda^+)(f)(x)^p \delta(x)^{-d+\beta+p\varepsilon}\, dx = c_{14} \int_{A_t} \mathcal{M}(v_{\lambda,t}, v_\lambda^+)(f)(x)^p\, dv_{\lambda,t}(x) \\ &\le c_{15} \int_D |\nabla u(y)|^p \delta(y)^{p-p\alpha-p\varepsilon}\, dv_\lambda^+(y) = c_{15} \int_D |\nabla u(y)|^p \delta(y)^{p-p\alpha-d+\beta}\, dy. \end{aligned}$$

On the other hand, if $y \in A_t$ and $2^j\delta(x) \le |x-y| < 2^{j+1}\delta(x)$, then

$$2^{j-1} \left(\frac{s_1}{s_2}\right)^{1/(d-\beta)} \delta(y) \le |x-y| < 2^{j+2} \left(\frac{s_2}{s_1}\right)^{1/(d-\beta)} \delta(y).$$

So, for $\sum_{k=0}^{n-1} J'_k$, we also obtain the same estimate as $\sum_{k=0}^{m-1} J_k$.

Combining the above facts with (4.2), (4.3), (4.4), (4.5) we have the conclusion. □

Let us prove Theorem 1.

PROOF OF THEOREM 1. Let u be λ-Hölder continuous on $\bar{D}$ and a C^1-function in D. Lemmas A and 4.3 yield

$$\begin{aligned}\iint \frac{|u(x)-u(y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y) &\le c_1 \liminf_{t\to 0} \int_{A_t}\int_{A_t} \frac{|u(x)-u(y)|^p}{|x-y|^{d+p\alpha+d-\beta}} dxdy \\ &\le c_2 \int_D |\nabla u(y)|^p \delta(y)^{p-p\alpha-d+\beta}\, dy.\end{aligned}$$

This completes the proof. □

REMARK 4.1. We note that Lemmas 4.2 and 4.3 are valid if we replace D with $\boldsymbol{R}^d\backslash\bar{D}$.

5. Extention operator $\mathscr{E}$.

In this section we consider the relation of the L^p-norm of $|\nabla\mathscr{E}(f)|\delta(x)^{\lambda}$ and the norm $\|f\|_{\alpha,p}$. To do so, we first mention the property of $\mathscr{E}$ (cf. [**JW**], [**W3**], [**W4**]). Recall that $\mathscr{V}(\boldsymbol{R}^d\backslash\partial D)$ stands for a Whitney decomposition of $\boldsymbol{R}^d\backslash\partial D$.

PROPOSITION B. *Assume that $\bar{D}\subset B(0,R/2)$. Then there exists a linear operator $\mathscr{E}$ from $L^p(\mu)$ to $L^p(\boldsymbol{R}^d)$ having the properties* (i)–(vi):

(i) *$\mathscr{E}(f)$ is a C^∞-function in $\boldsymbol{R}^d\backslash\partial D$,*
(ii) *$\mathscr{E}(f)=f$ on ∂D,*
(iii) $\operatorname{supp}\mathscr{E}(f)\subset B(0,2R)$,
(iv) *$\mathscr{E}(1)=1$ on $\overline{B(0,R)}$,*
(v)

$$\int |\mathscr{E}(f)|^p\, dy \le c\int |f|^p\, d\mu,$$

where c is a constant independent f,

(vi) *Let $Q\in\mathscr{V}(\boldsymbol{R}^d\backslash\partial D)$ be a cube with common side-length l. Then, for each $y\in Q$,*

$$|\nabla\mathscr{E}(f)(y)| \le cl^{-\beta-1}\int_{B(a,b''l)} |f(z)|\, d\mu(z),$$

where a is a boundary point satisfying $\operatorname{dist}(\partial D,Q)=\operatorname{dist}(a,Q)$ *and $b''=6\sqrt{d}$, and c is a constant independent of l, y and f.*

Set

$$H=\{y;\delta(y)\le R\}$$

and denote by ν_0 the positive measure on $B(0,2R)\backslash\partial D$ defined by

$$\nu_0(E)=\int_{E\cap(B(0,2R)\backslash\partial D)\cap H}\delta(y)^{2\beta-d}\, dy$$

for a Borel set E.

We begin with estimates for two measures $\mu \times \mu$ and ν_0.

Lemma 5.1.

$$\nu_0(B(y,r)) \le c_1 r^{2\beta} \le c_2 \int_{B(y,r)\cap\partial D} \int_{B(y,r)\cap\partial D} d\mu(x)d\mu(z)$$

for every $y \in H$ *and every* r *satisfying* $(11/10)\delta(y) \le r \le (3/2)R$.

Proof. Let $y \in H$ and $(11/10)\delta(y) \le r \le (3/2)R$. Since $d \ge 2$ and $d-1 \le \beta < d$, $2\beta - d \ge 0$,

$$\int_{B(y,r)\cap H} \delta(x)^{2\beta-d}\, dx \le c_1 \left(\frac{10}{11} r\right)^{2\beta-d} r^d \le c_2 r^{2\beta},$$

which is the first inequality.

Denote by y' a boundary point such that $\delta(y) = |y - y'|$. Since $B(y', (1/11)r) \subset B(y,r)$ and ∂D is a β-set, we also get the second inequality. □

We introduce the following maximal function on $\partial D \times \partial D$. We define, for $h \in L^p(\mu\times\mu)$ and $y \in H$,

$$\begin{aligned}&\mathscr{M}(\nu_0, \mu\times\mu)(h)(y)\\ &\quad = \sup\left\{ \frac{1}{\mu(B(y,r)\cap\partial D)^2} \int_{B(y,r)\cap\partial D}\int_{B(y,r)\cap\partial D} |h(x,z)|\, d\mu(x)d\mu(z);\right.\\ &\qquad\qquad \left. \frac{11}{10}\delta(y) \le r \le \frac{R}{4}\right\}.\end{aligned}$$

Using Vitali's covering lemma and Lemma 5.1, we can prove the following lemma by the same method as the proof of Lemma 3.2.

Lemma 5.2. (i) *Let* $t > 0$, $h \in L^1(\mu\times\mu)$ *and set*

$$E_t = \{y \in H; \mathscr{M}(\nu_0, \mu\times\mu)(h)(y) > t\}.$$

Then

$$\nu_0(E_t) \le c t^{-1} \iint |h(x,z)|\, d\mu(x)d\mu(z),$$

where c *is a constant independent of* f *and* t.

(ii) *Let* $p > 1$ *and* $h \in L^p(\mu\times\mu)$. *Then*

$$\int \mathscr{M}(\nu_0, \mu\times\mu)(h)(y)^p\, d\nu_0(y) \le c \iint |h(x,z)|^p\, d\mu(x)d\mu(z).$$

Proof of Theorem 2. Let $\{Q_j\}$ be a Whitney decomposition of $\boldsymbol{R}^d \setminus \partial D$ (cf. [**S**]). Denote by l_j and a_j the common side-length of Q_j and a boundary point satisfying $\operatorname{dist}(\partial D, Q_j) = \operatorname{dist}(a_j, Q_j)$, respectively. Put

$$b_j = \frac{1}{\mu(B(a_j, \eta l_j))} \int_{B(a_j, \eta l_j)} f(w)\, d\mu(w),$$

where η is a fixed positive real number satisfying $0 < \eta < 1/4$ and used in the definition $\mathscr{E}(f)$ in [**W3**].

With the aid of Proposition B we have, for each $y \in Q_j$

$$\begin{aligned}
&|\nabla \mathscr{E}(f - b_j)(y)| \\
&\quad \le c_1 \frac{1}{l_j^{\beta+1} l_j^{\beta}} \int_{B(a_j, b'' l_j)} d\mu(z) \int_{B(a_j, \eta l_j)} |f(z) - f(w)|\, d\mu(w) \\
&\quad \le c_2 l_j^{\beta/p + \alpha - 2\beta - 1} \int_{B(a_j, b'' l_j)} d\mu(z) \int_{B(a_j, \eta l_j)} \frac{|f(z) - f(w)|}{|z - w|^{\beta/p + \alpha}} d\mu(w).
\end{aligned}$$

Further, let $y \in Q_j$ and x_j be a point in Q_j satisfying $|a_j - x_j| = \operatorname{dist}(a_j, Q_j)$. If $z \in B(a_j, b'' l_j) \cap \partial D$, then

$$\begin{aligned}
|y - z| &\le |y - x_j| + |x_j - a_j| + |a_j - z| \\
&\le \sqrt{d} l_j + 4\sqrt{d} l_j + b'' l_j = 11\sqrt{d} l_j.
\end{aligned}$$

Putting $s' = 11\sqrt{d}$, we have

$$\begin{aligned}
(5.1)\qquad &|\nabla \mathscr{E}(f - b_j)(y)| \delta(y)^{1-\alpha-\beta/p} \\
&\quad \le c_3 \frac{1}{l_j^{2\beta}} \int_{B(y, s' l_j) \cap \partial D} d\mu(z) \int_{B(y, s' l_j) \cap \partial D} |h(z, w)|\, d\mu(w),
\end{aligned}$$

where $h(z, w) = |f(z) - f(w)| / |z - w|^{\beta/p + \alpha}$.

Put $s'' = R/(s' 5\sqrt{d})$. First, let $l_j \le s''$ and $y \in Q_j$. Then

$$s' l_j \le s' \frac{R}{s' 5\sqrt{d}} < \frac{R}{5}$$

and

$$s' l_j \ge s' \frac{\delta(y)}{5\sqrt{d}} = \frac{11\delta(y)}{5}.$$

Noting that

$$\mu(B(y, s' l_j) \cap \partial D) \le \mu(B(a_j, 2s' l_j) \cap \partial D) \le c_4 l_j^{\beta},$$

we have, by (5.1),

$$|\nabla \mathscr{E}(f - b_j)(y)| \delta(y)^{1-\alpha-\beta/p} \le c_5 \mathscr{M}(v_0, \mu \times \mu)(h)(y).$$

Since the interiors of Q_j are mutually disjoint, we obtain, by Lemma 5.2,

$$(5.2)\qquad \sum_{l_j \le s''} \int_{Q_j} |\nabla \mathscr{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta}\, dy$$

$$\le \sum_{l_j \le s''} \int_{Q_j} |\nabla \mathscr{E}(f-b_j)(y)|^p \delta(y)^{p-p\alpha-\beta} \delta(y)^{2\beta-d}\, dy$$

$$\le c_6 \sum_{l_j \le s''} \int_{Q_j} \mathscr{M}(v_0, \mu \times \mu)(h)^p\, dv_0 \le c_7 \iint h(z,w)^p\, d\mu(z)\mu(w).$$

We next assume that $l_j \ge s''$. Then, by $y \in Q_j$, (vi) in Proposition B implies

$$|\nabla \mathscr{E}(f)(y)| \le c_8 l_j^{-\beta-1} \int_{B(a_j, b'' l_j)} |f(z)|\, d\mu(z) \le c_9 (s'')^{-\beta/p-1} \|f\|_p.$$

Noting that $\operatorname{supp} \mathscr{E}(f) \subset B(0,2R)$, we have

$$(5.3)\qquad \sum_{l_j \ge s''} \int_{Q_j} |\nabla \mathscr{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta}\, dy$$

$$\le c_{10} (s'')^{-\beta-p} \|f\|_p^p \int_{B(0,2R)} (2R)^{p-p\alpha-d+\beta}\, dy \le c_{11} \|f\|_p^p.$$

Thus we have, by (5.2) and (5.3),

$$\int_{\boldsymbol{R}^d} |\nabla \mathscr{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta}\, dy \le c_{12} \|f\|_{\alpha,p}^p,$$

which completes the proof. □

6. Boundedness of operators.

In this section we shall prove the boundedness of operators K_1. To do so, we first prove Theorem 3.

Proof of Theorem 3. Denote by $\mathscr{V}_n(D)$ the set of all n-cubes in D. If $x \in Q \in \mathscr{V}_n(D)$, then

$$\sqrt{d}\, 2^{-n} \le \delta(x) \le 5\sqrt{d}\, 2^{-n}.$$

Consider $Q \in \mathscr{V}_n(D)$ and denote by x_0 the center of Q. For each $x \in Q$ we write

$$I \equiv \left| \frac{\partial \Phi f}{\partial x_i}(x) \right| = \left| \frac{\partial}{\partial x_i} (\Phi f - \Phi f(x_0))(x) \right|$$

$$= \left| \int_{\boldsymbol{R}^d \setminus \bar{D}} \langle \nabla \mathscr{E}(f)(y), \nabla_y \frac{\partial}{\partial x_i} (N(x-y) - N(x_0 - y)) \rangle\, dy \right|$$

$$\le c_1 |x - x_0| \int_{B(0,2R) \setminus \bar{D}} |\nabla \mathscr{E}(f)(y)| (|x-y|^{-d-1} + |y - x_0|^{-d-1})\, dy.$$

Since

$$|x_0 - y| \geq |x - y| - |x_0 - x| \geq |x - y| - \frac{\sqrt{d}}{2} 2^{-n}$$
$$\geq |x - y| - \frac{\delta(x)}{2} \geq \frac{1}{2}|x - y|$$

for $y \in B(0, 2R) \backslash D$, we write

$$I \leq c_2 \delta(x) \int_{(B(0,2R)\backslash \bar{D}) \cap \{\delta(y) \leq R/2\}} |\nabla \mathscr{E}(f)(y)| \, |x - y|^{-d-1} \, dy$$
$$+ c_2 \delta(x) \int_{(B(0,2R)\backslash \bar{D}) \cap \{\delta(y) > R/2\}} |\nabla \mathscr{E}(f)(y)| \, |x - y|^{-d-1} \, dy \equiv I_1 + I_2.$$

Then

$$I_1 \leq c_2 \delta(x) \sum_{k=1}^{m_x} \int_{(B(0,2R)\backslash \bar{D}) \cap \{2^{k-1}\delta(x) < |x-y| \leq 2^k \delta(x)\}} |\nabla \mathscr{E}(f)(y)| \, |x - y|^{-d-1} \, dy$$
$$\leq c_3 \delta(x) \sum_{k=1}^{m_x} (2^{k-1}\delta(x))^{-d-1} \int_{(B(0,2R)\backslash \bar{D}) \cap \{2^{k-1}\delta(x) < |x-y| \leq 2^k \delta(x)\}} |\nabla \mathscr{E}(f)(y)| \, dy,$$

where m_x stands for the natural number satisfying $2^{m_x - 1}\delta(x) < 3R/2 \leq 2^{m_x}\delta(x)$.

Set $q = p/(p-1)$ and $\lambda = q(1 - \alpha - (d - \beta)/p) > 0$. Then $d - \beta - \lambda = q(d - \beta + \alpha - 1) > 0$. Hence, by Lemma 3.3,

$$I_1 \delta(x)^{\lambda} \leq c_4 \sum_{k=1}^{m_x} (2^k)^{-1-\lambda} \frac{\int_{(B(0,2R)\backslash \bar{D}) \cap \{|x-y| \leq 2^k \delta(x)\}} |\nabla \mathscr{E}(f)(y)| \delta(y)^{\lambda} \, d\nu_{\lambda}^{-}(y)}{(2^k \delta(x))^{d-\lambda}}$$
$$\leq c_5 \sum_{k=1}^{\infty} (2^k)^{-1-\lambda} \mathscr{M}(\nu_{\lambda}^{+}, \nu_{\lambda}^{-})((|\nabla \mathscr{E}(f)| \delta(y)^{\lambda}))(x)$$
$$\leq c_6 \mathscr{M}(\nu_{\lambda}^{+}, \nu_{\lambda}^{-})((|\nabla \mathscr{E}(f)| \delta(y)^{\lambda}))(x).$$

Lemma 3.4 yields

$$\int_D I_1^p \delta(x)^{p\lambda} \delta(x)^{-\lambda} \, dx \leq c_7 \int_D \mathscr{M}(\nu_{\lambda}^{+}, \nu_{\lambda}^{-})((|\nabla \mathscr{E}(f)| \delta(y)^{\lambda}))(x)^p \delta(x)^{-\lambda} \, dy$$
$$\leq c_8 \int_{B(0,2R)\backslash \bar{D}} |\nabla \mathscr{E}(f)(y)|^p \delta(y)^{p\lambda} \delta(y)^{-\lambda} \, dy.$$

Since $p\lambda - \lambda = p - p\alpha - d + \beta$, we have

$$\int_D I_1^p \delta(x)^{p - p\alpha - d + \beta} \, dx \leq c_9 \int_{B(0,2R)\backslash \bar{D}} |\nabla \mathscr{E}(f)(y)|^p \delta(y)^{p - p\alpha - d + \beta} \, dy. \tag{6.1}$$

We next estimate I_2. Since

$$I_2 \le c_2 \int_{(B(0,2R)\setminus\bar{D})\cap\{\delta(y)>R/2\}} |x-y|\,|\nabla\mathscr{E}(f)(y)|\,|x-y|^{-d-1}\,dy$$

$$\le c_{10}\left(\frac{R}{2}\right)^{-d} \int_{(B(0,2R)\setminus\bar{D})\cap\{\delta(y)>R/2\}} |\nabla\mathscr{E}(f)(y)|\,dy$$

$$\le c_{11} \int_{(B(0,2R)\setminus\bar{D})\cap\{\delta(y)>R/2\}} |\nabla\mathscr{E}(f)(y)|\,dy,$$

we have

$$\int_D I_2^p\,dx \le c_{12}\int_D dx \int_{B(0,2R)\setminus\bar{D}} |\nabla\mathscr{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta}\,dy \tag{6.2}$$

$$\le c_{13}\int_{B(0,2R)\setminus\bar{D}} |\nabla\mathscr{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta}\,dy.$$

The inequalities (6.1) and (6.2) lead to the conclusion. □

We next prove Theorem 4.

Proof of Theorem 4. We show that K_1 is bounded from $\Lambda_\alpha^p(\partial D)$ to $\Lambda_\alpha^p(\partial D)$. In [**W3**, Proof of Theorem] we saw that K_1 is bounded from $\Lambda_\alpha^p(\partial D)$ to $L^p(\mu)$. So it suffices to prove that

$$\iint \frac{|K_1 f(x) - K_1 f(z)|^p}{|x-z|^{\beta+p\alpha}}\,d\mu(x)d\mu(z) \le c_1 \|f\|_{\alpha,p}^p, \tag{6.3}$$

for every $f \in \Lambda_\alpha^p(\partial D)$.

For this purpose let f be a Lipschitz function on ∂D. Define

$$u(x) = \int_{\boldsymbol{R}^d\setminus\bar{D}} \langle \nabla\mathscr{E}(f)(y), \nabla_y N(x-y)\rangle\,dy$$

for $x \in \bar{D}$. Then $u(x) = \Phi f(x)$ for $x \in D$ and $u(x) = K_1 f(x)$ for $x \in \partial D$.

We choose a real number λ satisfying $\alpha + (d-\beta)/p < \lambda < 1$. Then we see by [**W2**, Lemma 3.2] that u is λ-Hölder continuous on $\bar{D}$ and a C^1-function in D. Theorem 1, Theorem 3 and Theorem 2 yield

$$\iint \frac{|K_1 f(x) - K_1 f(z)|^p}{|x-z|^{\beta+p\alpha}}\,d\mu(x)d\mu(z) \le c_6 \int_D |\nabla\Phi f(x)|^p \delta(x)^{p-p\alpha-d+\beta}\,dx \tag{6.4}$$

$$\le c_7 \int_{\boldsymbol{R}^d\setminus\bar{D}} |\nabla\mathscr{E}(f)(y)|\delta(y)^{p-p\alpha-d+\beta}\,dy \le c_8 \|f\|_{\alpha,p}^p$$

for every Lipschitz function f on ∂D.

We next consider $f \in \Lambda_\alpha^p(\partial D)$. We have known that the set of all Lipschitz functions on ∂D is dense in $\Lambda_\alpha^p(\partial D)$ (cf. [**W5**, Lemma 3.1]). So we take a sequence $\{f_n\}$ of Lipschitz functions on ∂D such that $\|f_n - f\|_{\alpha,p} \to 0$. By (6.4) we have

$$\iint \frac{|K_1 f_n(x) - K_1 f_n(z)|^p}{|x-z|^{\beta+p\alpha}} d\mu(x) d\mu(z) \le c_9 \|f_n\|_{\alpha,p}^p.$$

Noting that $\|K_1 f_n - K_1 f\|_p \to 0$ and $\|f_n - f\|_p \to 0$, we can find a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that $f_{n_j}(z) \to f(z)$ μ-*a.e.* $z \in \partial D$ and $K_1 f_{n_j}(z) \to K_1 f(z)$ μ-*a.e.* $z \in \partial D$. We also use $\{g_j\}$ instead of $\{f_{n_j}\}$. With the aid of Fatou's lemma we obtain

$$\begin{aligned}
&\iint \frac{|K_1 f(x) - K_1 f(z)|^p}{|x-z|^{\beta+p\alpha}} d\mu(x) d\mu(z) \\
&\quad \le \liminf_{j\to\infty} \iint \frac{|K_1 g_j(x) - K_1 g_j(z)|^p}{|x-z|^{\beta+p\alpha}} d\mu(x) d\mu(z) \\
&\quad \le c_{10} \liminf_{j\to\infty} \|f_n\|_{\alpha,p}^p = c_{10} \|f\|_{\alpha,p}^p.
\end{aligned}$$

Thus we have (6.3) and we see that K_1 is bounded from $\Lambda_\alpha^p(\partial D)$ from $\Lambda_\alpha^p(\partial D)$. □

PROOF OF THEOREM 5. By Lemma 2.1 the domain $\boldsymbol{R}^d \setminus \bar{D}$ satisfies the condition (b). Hence, by Theorem 4, the operator K_1 is bounded on $\Lambda_\alpha^p(\partial D)$. Noting Remarks 3.1 and 4.1, we can prove by the same method as in the proof of Theorem 4 that K_2 is also bounded. Since

$$K = \frac{K_1 + K_2}{2},$$

we also see that K is bounded on $\Lambda_\alpha^p(\partial D)$. □

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