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Perturbation of non-exponentially-bounded α-times integrated C-semigroups

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Abstract. Let $T(\cdot)$ be a (not necessarily exponentially bounded, not necessarily nondegenerate) α-times integrated C-semigroup and let $-B$ be the generator of a (C_0)-group $S(\cdot)$ commuting with $T(\cdot)$ and C. Under suitable conditions on $T(\cdot)$ and $S(\cdot)$ we prove the existence of an α-times integrated C-semigroup $V(\cdot)$, which has generator $\overline{A+B}$ provided that $T(\cdot)$ is nondegenerate and has generator A. Explicit expressions of $V(\cdot)$ in terms of $T(\cdot)$ and $S(\cdot)$ are obtained. In particular, when B is bounded, $V(\cdot)$ can be constructed by means of a series in terms of $T(\cdot)$ and powers of B.

0. Introduction.

This paper is concerned with the perturbation of α-times integrated C-semigroups which may be degenerate and may be not exponentially bounded.

We first recall some related definitions. Let X be a complex Banach space and let $B(X)$ be the Banach algebra of all bounded (linear) operators on X. For $r \in [-1, \infty)$, let $j_r : [0, \infty) \to R$ be defined as $j_{-1} :=$ the Dirac measure at 0; $j_0 \equiv 1$; $j_r(t) := t^r/\Gamma(r+1)$, $t > 0$, and $j_r(0) = 0$ for $r > -1$ with $r \neq 0$, where $\Gamma(\cdot)$ is the Gamma function.

For $\alpha > 0$, a family of operators $\{T(t); t \geq 0\} \subset B(X)$ is called an *α-times integrated C-semigroup* on X (cf. [**9**]–[**14**], [**22**]) if

(a) $T(\cdot)x : [0, \infty) \to X$ is continuous for each $x \in X$;

(b) $T(0) = 0$, $CT(\cdot) = T(\cdot)C$, and

$$T(t)T(s)x = \frac{1}{\Gamma(\alpha)}\left[\int_0^{s+t} - \int_0^{s} - \int_0^{t}\right](t+s-r)^{\alpha-1}T(r)Cx\,dr$$

for $x \in X$ and $t, s \geq 0$.

$T(\cdot)$ is called a (0-times integrated) C-semigroup (cf. [**2**]–[**4**], [**20**], [**21**]) if $T(0) = C$ and $T(t)T(s) = T(t+s)C$ for all $t, s \geq 0$.

When $C = I$, an α-times integrated C-semigroup reduces to an α-times integrated semigroup (cf. [**1**], [**3**], [**7**], [**15**], [**16**]), and a C-semigroup becomes a classical (C_0)-semigroup (cf. [**5**], [**8**]).

$T(\cdot)$ is called exponentially bounded if there exist $M > 0$, $w \geq 0$ such that $\|T(t)\| \leq Me^{wt}$ for all $t \geq 0$. If $C = I$ and each $T(t)$ is a hermitian operator, then $T(\cdot)$ has to be exponentially bounded [**11**]. But, unlike (C_0)-semigroups, in general, an α-times integrated C-semigroup may be not exponentially bounded (cf. [**9**]).

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For convenience, we use the notation $(j_r * T)(t)$ for the operator defined by

$$(j_r * T)(t)x := \int_0^t j_r(t-s)T(s)x\,ds \quad \text{for all } x \in X.$$

$T(\cdot)$ is said to be nondegenerate if $T(t)x = 0$ for all $t > 0$ implies $x = 0$. If $T(\cdot)$ is nondegenerate, then C is injective and one can define a subgenerator as a closed operator A_1 which satisfies $CD(A_1) \subset D(A_1)$, $CA_1x = A_1Cx$ for $x \in D(A_1)$, $R((1 * T)(t)) \subset D(A_1)$ and

$$(1 * T)(t)A_1 \subset A_1(1 * T)(t) = T(t) - j_\alpha(t)C, \quad t \geq 0.$$

It is known (cf. [**10**], [**12**]) that $A := C^{-1}A_1C$ is also a subgenerator and it is an extension of all subgenerators, that is, A is the maximal subgenerator. We call this A the generator of $T(\cdot)$. It follows that $C^{-1}AC = A$ and we have

$$x \in D(A) \text{ and } Ax = y \Leftrightarrow T(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx = \int_0^t T(s)y\,ds \quad \text{for all } t \geq 0.$$

A is well-defined as a closed linear operator. In general A is not densely defined and the resolvent set $\rho(A)$ is not necessarily nonempty.

To our knowledge, all known perturbation theorems for integrated semigroups are obtained under the assumption of exponential boundedness. For instance, Xiao and Liang [**23**, Theorem 1.3.5] proved that if A is the generator of an exponentially bounded α-times integrated semigroup $T(\cdot)$ and $B \in B(X)$ commutes with A, then $A + B$ is also a generator of an α-times integrated semigroup. See also [**17**] and [**18**] for the case of $\alpha \in N$. Perturbation theorems for nondegenerate C-semigroups can be found in [**19**]. Based on Propositions 1.1 and 1.2 to be given in Section 1, we attempt to prove in this paper some perturbation theorems for α-times integrated C-semigroups without the assumptions of exponential boundedness and nondegeneracy.

As a motivation we first consider a C-semigroup $T(\cdot)$ and a (C_0)-group $S(\cdot)$ (with generator $-B$ satisfying $T(t)S(s) = S(s)T(t)$, $t \geq 0$, $s \in R$. Clearly, the family $\{V(t) := S(-t)T(t); t \geq 0\}$ is also a C-semigroup, and $V(\cdot)$ is nondegenerate if and only if $T(\cdot)$ is. It is known [**23**, Theorem 1.3.6] that if $T(\cdot)$ has generator A and if $B \in B(X)$, then $A + B$ is the generator of $V(\cdot)$. When B is unbounded, $A + B$ may be not closed, and so not a generator (cf. [**6**, p. 39]). Is $A + B$ closable? and, if yes, is $\overline{A + B}$ the generator of $V(\cdot)$? The answers are affirmative; we shall see that $\overline{A + B}$ is the generator of $V(\cdot)$. We further observe that $V(\cdot)$ satisfies $S(t)V(s) = V(s)S(t)$ and $1 * [S(1 * V)] = (1S) * (1 * T)$, i.e.,

$$\begin{aligned}
&\int_0^t S(u)(1 * V)(u)\,du \\
&\quad = \int_0^t S(u)\int_0^u S(-s)T(s)\,dsdu = \int_0^t\int_s^t S(u-s)T(s)\,duds \\
&\quad = \int_0^t\int_0^{t-s} S(u)T(s)\,duds = \int_0^t S(u)\int_0^{t-u} T(s)\,dsdu \\
&\quad = \int_0^t S(u)(1 * T)(t-u)\,du.
\end{aligned}$$

As will be seen, actually this condition is also sufficient for a function $V(\cdot)$ to be a C-semigroup.

In Section 2, these facts will be generalized to the case that $T(\cdot)$ is an α-times integrated C-semigroup. It is proved in Theorem 2.1 that if there is a strongly continuous function $V : [0, \infty) \to B(X)$ such that $V(0) = \delta_{0,\alpha}C$, $CV(t) = V(t)C$, $S(t)V(s) = V(s)S(t)$, and

$$(*) \qquad (j_\alpha * [S(1 * V)])(t) = [(j_\alpha S) * (1 * T)](t) \quad \text{for all } t \geq 0,$$

then $V(\cdot)$ is an α-times integrated C-semigroup. Moreover, if A is the generator of $T(\cdot)$, then $A + B$ is closable and $\overline{A + B}$ is the generator of $V(\cdot)$.

When is there a $V(\cdot)$ satisfying $V(0) = \delta_{0,\alpha}C$, $CV(t) = V(t)C$, $S(t)V(s) = V(s)S(t)$, and $(*)$? and how to construct it? Respective sufficient conditions on $S(\cdot)$ and on $T(\cdot)$ for the existence of $V(\cdot)$ will be given in Section 3 and Section 4; in Section 3 we prove that the generator $-B$ of $S(\cdot)$ being bounded is sufficient, and in Section 4, a sufficient condition is given on $T(\cdot)$ for the case that $\alpha = 1$. In both cases, explicit formulas ((3.13), (3.14), (4.1), (4.2)) for the expression of $V(\cdot)$ in terms of $T(\cdot)$ and $S(\cdot)$ are obtained. For use in Sections 2 and 3, we collect some characterization results and two combinatorial lemmas in Section 1.

1. Preliminaries.

We prepare some propositions and lemmas in this section for use in the latter sections. The following proposition gives a characterization of an α-times integrated C-semigroup (see also [**10**, Proposition 2.3]).

Proposition 1.1. *$T(\cdot)$ is an α-times integrated C-semigroup if and only if $T(\cdot)$ commutes with C and satisfies $T(0) = \delta_{0,\alpha}C$ and*

$$(1.1) \qquad [T(t) - j_\alpha(t)C](1 * T)(s) = (1 * T)(t)[T(s) - j_\alpha(s)C] \quad \textit{for all } s, t \geq 0.$$

Proof. Let $U(t)x := (1 * T)(t)$. Suppose $T(\cdot)$ is an α-times integrated C-semigroup on X. We can write the equation in (b) as

$$T(s)T(t)x = \int_0^s [j_{\alpha-1}(r)CT(s + t - r) - j_{\alpha-1}(s + t - r)CT(r)]x\,dr.$$

Integrating it with respect to t and using integration by parts, we obtain:

$$(1.2) \qquad T(s)U(t)x = \int_0^s [j_{\alpha-1}(r)CU(s + t - r) - j_\alpha(s + t - r)CT(r)]x\,dr$$
$$= \left(\int_t^{s+t} - \int_0^s\right) j_{\alpha-1}(s + t - r)CU(r)x\,dr - j_\alpha(t)CU(s)x$$

and (after interchanging s and t)

$$(1.3)\qquad T(t)U(s)x = \int_0^t [j_{\alpha-1}(r)CU(s+t-r) - j_\alpha(s+t-r)CT(r)]x\,dr$$
$$= \left(\int_s^{s+t} - \int_0^t\right) j_{\alpha-1}(s+t-r)CU(r)x\,dr - j_\alpha(s)CU(t)x$$

for $x \in X$ and $s, t \geq 0$. Comparing (1.2) and (1.3), we obtain

$$T(s)U(t)x + j_\alpha(t)CU(s)x = \left(\int_0^{s+t} - \int_0^t - \int_0^s\right) j_{\alpha-1}(s+t-r)CU(r)x\,dr$$
$$= T(t)U(s)x + j_\alpha(s)CU(t)x.$$

Since $U(t)$ commutes with C and $T(s)$, we obtain (1.1).

Conversely, we suppose that $T(\cdot)$ satisfies (1.1). We show that $U(\cdot)$ is an $(\alpha+1)$-times integrated C-semigroup. Then $T(\cdot)$ is an α-times integrated C-semigroup. First, we replace s by $s+t-r$ and t by r in (1.1). Then we have for $x \in X$

$$T(r)U(s+t-r)x - U(r)T(s+t-r)x = j_\alpha(r)CU(s+t-r)x - U(r)j_\alpha(s+t-r)Cx.$$

By integrating the right-hand-side with respect to r from 0 to t, we obtain from $CT(\cdot) = T(\cdot)C$ that

$$\int_0^t j_\alpha(r)CU(s+t-r)x\,dr - \int_0^t U(r)j_\alpha(s+t-r)Cx\,dr$$
$$= \int_s^{s+t} j_\alpha(s+t-r)CU(r)x\,dr - \int_0^t j_\alpha(s+t-r)CU(r)x\,dr$$
$$= \left(\int_0^{s+t} - \int_0^s - \int_0^t\right) j_\alpha(s+t-r)CU(r)x\,dr.$$

On the other hand, from the left-hand-side we have

$$\int_0^t T(r)U(s+t-r)x\,dr - \int_0^t U(r)T(s+t-r)x\,dr$$
$$= U(r)U(s+t-r)x|_0^t + \int_0^t U(r)T(s+t-r)x\,dr - \int_0^t U(r)T(s+t-r)x\,dr$$
$$= U(t)U(s) - U(0)U(s+t) = U(t)U(s)$$

for $t, s \geq 0$. Therefore $U(\cdot)$ is an $(\alpha+1)$-times integrated C-semigroup. This completes the proof. □

We also need the following characterization theorem, which is proved in [**12**] for $\alpha = n \in \boldsymbol{N}$ and in [**10**] for general real $\alpha \geq 0$.

Proposition 1.2. *$T(\cdot)$ is an α-times integrated C-semigroup with generator A if and only if it commutes with C and A is a closed operator satisfying $C^{-1}AC = A$, $R((1 * T)(t)) \subset D(A)$ and*

$$(1.4)\qquad (1 * T)(t)A \subset A(1 * T)(t) = T(t) - j_\alpha(t)C \quad \text{for all } t \geq 0.$$

We shall also need the following lemmas.

Lemma 1.3 (cf. [**10**, Lemma 2.1]). *Let* $r, s \geq -1$.
(a) *If* $r + s > -2$, *then* $j_r * j_s = j_{r+s+1}$.
(b) *Let* $f : [0, b] \to X$ *be Bochner integrable. If* $j_r * f \equiv 0$ *on* $[0, b]$, *then* $f = 0$ *almost everywhere*.

Lemma 1.4. *Let* A *be a closed linear operator on* X, *let* $r \geq 0$ *and* $f, g \in C([0, a], X)$. *Then* $(j_r * f)[0, a] \subset D(A)$ *and* $A(j_r * f) \equiv (j_r * g)$ *if and only if* $f[0, a] \subset D(A)$ *and* $Af \equiv g$.

Proof. The sufficiency follows from the closedness of A and the existence of $j_r * f$ and $j_r * g$. To show the converse, take an $s > 0$ such that $n := r + s + 1$ is a positive integer. Since A is closed, we have for every $t \in [0, a]$

$$\begin{aligned}(j_n * g)(t) &= j_s * (j_r * g)(t) = j_s * (A(j_r * f))(t) \\ &= A(j_s * (j_r * f))(t) = A(j_n * f)(t).\end{aligned}$$

Taking differentiation $n + 1$ times, we obtain, again by the closedness of A, that $f[0, a] \subset D(A)$ and $Af \equiv g$. □

As usual, we use the notations $\binom{r}{n} = r(r-1)\cdots(r-n+1)/n!$ and $\binom{r}{0} = 1$ for any real number r. Let $\{a_{-1}, a_0, a_1, \ldots\}$ be real numbers defined by $a_{-1} := \binom{-\alpha}{0} = 1$ and

$$a_n := (-1)^{n+1}\binom{n+\alpha}{n+1} = \binom{-\alpha}{n+1}, \quad n = 0, 1, 2, \ldots.$$

Lemma 1.5. *Let* $\{a_{-1}, a_0, a_1, \ldots\}$ *be as defined above. Then for* $n = 0, 1, 2, \ldots$

$$\sum_{k=0}^{n+1} a_{n-k}\binom{n+\alpha}{k} = 0. \tag{1.5}$$

Proof. We have for $n = 0, 1, 2, \ldots$

$$\begin{aligned}\sum_{k=0}^{n+1} a_{n-k}\binom{n+\alpha}{k} &= \sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n-k+\alpha}{n+1-k}\binom{n+\alpha}{k} \\ &= \sum_{k=0}^{n+1}(-1)^{n+1-k}\frac{(n-k+\alpha)(n-k-1+\alpha)\cdots(1+\alpha)\alpha}{(n+1-k)!} \\ &\quad \cdot\frac{(n+\alpha)(n-1+\alpha)\cdots(n-k+1+\alpha)}{k!} \\ &= \frac{(n+\alpha)(n-1+\alpha)\cdots\alpha}{(n+1)!}\sum_{k=0}^{n+1}(-1)^{n+1-k}\frac{(n+1)!}{(n+1-k)!k!} \\ &= \binom{n+\alpha}{n+1}\cdot(1-1)^{n+1} = 0.\end{aligned}$$

□

Lemma 1.6. *For every $n = 0, 1, 2, \ldots$ and real number x,*

$$\sum_{k=0}^{n} \binom{x}{k+1} \binom{n}{k} = \binom{n+x}{n+1}. \tag{1.6}$$

Proof. First, we suppose $x = m$ is a positive integer. Define a function

$$f(t) := \sum_{n=0}^{\infty} \binom{n+m}{n+1} j_n(t), \quad t \geq 0.$$

Then we have for every $t \geq 0$

$$(j_{m-1} * f)(t) = \sum_{n=0}^{\infty} \binom{n+m}{n+1} j_{n+m}(t) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!(m-1)!} t^{n+m}$$
$$= \sum_{n=0}^{\infty} j_{n+1}(t) j_{m-1}(t) = (e^t - 1) j_{m-1}(t).$$

Differentiating the left hand side m times, we obtain

$$f(t) = \sum_{k=1}^{m} \binom{m}{k} \left[\frac{d^k}{dt^k}(e^t - 1) \right] \left[\frac{d^{m-k}}{dt^{m-k}} j_{m-1}(t) \right] = \sum_{k=1}^{m} \binom{m}{k} e^t j_{k-1}(t)$$
$$= \sum_{k=1}^{\infty} \binom{m}{k} \left[\sum_{n=0}^{\infty} j_n(t) \right] j_{k-1}(t) \quad \left(\text{since } \binom{m}{k} = 0 \text{ for } k \geq m+1 \right)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{m}{k+1} \binom{n+k}{k} j_{n+k}(t) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{m}{k+1} \binom{n}{k} j_n(t)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{m}{k+1} \binom{n}{k} j_n(t).$$

Comparing the last expression and the definition of f, we obtain from the uniqueness of coefficients of a power series that $\binom{n+m}{n+1} = \sum_{k=0}^{n} \binom{m}{k+1} \binom{n}{k}$ for all $n = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$ Therefore (1.6) holds for every $n = 0, 1, 2, \ldots$ and for all positive integers x. Since both $\binom{n+x}{n+1}$ and $\sum_{k=0}^{n} \binom{x}{k+1} \binom{n}{k}$ are polynomials with the same degree $n+1$, it follows from the fundamental theorem of algebra that they are identical. This proves (1.6). □

2. A general perturbation theorem.

The next theorem is the main result in this section.

Theorem 2.1. *Let $T(\cdot)$ be an α-times integrated C-semigroup on a Banach space X and let $S(\cdot)$ be a (C_0)-group with generator $-B$. Suppose $S(t)T(s) = T(s)S(t)$ and $S(t)C = CS(t)$ for all $s, t \geq 0$. There is at most one strongly continuous function $V : [0, \infty) \to B(X)$ such that $V(0) = \delta_{0,\alpha} C$, $CV(t) = V(t)C$, $S(t)V(s) = V(s)S(t)$, and*

$$\int_0^t j_\alpha(t-u)S(u)(1*V)(u)\,du = \int_0^t j_\alpha(u)S(u)(1*T)(t-u)\,du \tag{2.1}$$

for all $s, t \geq 0$. If $V(\cdot)$ is such a function, then

(a) *$V(\cdot)$ is an α-times integrated C-semigroup.*

(b) *$T(\cdot)$ is nondegenerate if and only if $V(\cdot)$ is nondegenerate.*

(c) *If A is the generator of $T(\cdot)$, then $A+B$ is closable and $\overline{A+B}$ is the generator of $V(\cdot)$. If $A+B$ is closed, then actually $A+B$ is the generator. This is true in particular when $B \in B(X)$.*

To prove this theorem, we need the following proposition.

Proposition 2.2. *Let $T(\cdot)$ and $S(\cdot)$ be two commuting α-times and β-times integrated C-semigroups on X with generators A and B, respectively. Then the following hold.*

(i) *$A+B$ is closable and satisfies:*

$$(A+B) \subset C^{-1}(A+B)C \quad \text{and} \quad \overline{A+B} \subset C^{-1}\overline{A+B}C.$$

(ii) *If either one of $T(\cdot)$ and $S(\cdot)$ is a (C_0)-semigroup, then*

$$C^{-1}\overline{A+B}C = \overline{A+B}.$$

Proof. (i) First, we show that $A+B$ is closable. Let $\{x_n\}$ be a null sequence in $D(A+B)$ such that $(A+B)x_n$ converges to a vector $y \in X$. We need to show $y = 0$. Observe that $S(t)T(s) = T(s)S(t)$ implies that $S(t)Ax = AS(t)x$ for $x \in D(A)$. Hence we have

$$\begin{aligned}(1*T)(t)(1*S)(s)y &= \lim_{n\to\infty}(1*T)(t)(1*S)(s)(A+B)x_n \\ &= \lim_{n\to\infty}\{[T(t)-j_\alpha(t)C](1*S)(s)x_n + (1*T)(t)[S(s)-j_\beta(t)C]x_n\} \\ &= [T(t)-j_\alpha(t)C](1*S)(s)0 + (1*T)(t)[S(s)-j_\beta(s)C]0 = 0\end{aligned}$$

and then $T(t)S(s)y = 0$ for all $s, t \in (0, \infty)$, by differentiation. Then the nondegeneracy of $T(\cdot)$ and $S(\cdot)$ imply $y = 0$. Therefore $A+B$ is closable.

Let $x \in D(A+B) = D(A) \cap D(B)$. Since A and B are generators, by Proposition 1.2, we have $C^{-1}ACx = Ax$ and $C^{-1}BCx = Bx$, so that $Cx \in D(A) \cap D(B) = D(A+B)$ and $ACx = CAx$ and $BCx = CBx$. Hence $(A+B)Cx = ACx + BCx = C(A+B)x$ and so $x \in D(C^{-1}(A+B)C)$ and $(A+B)x = C^{-1}(A+B)Cx$. Hence $(A+B) \subset C^{-1}(A+B)C$. Next, we show $\overline{A+B} \subset C^{-1}\overline{A+B}C$. If $x \in D(\overline{A+B})$, then there is a sequence $\{x_n\}$ in $D(A+B)$ such that $(x_n, (A+B)x_n) \to (x, \overline{A+B}x)$. As above, we have $(A+B)Cx_n = C(A+B)x_n \to C\overline{A+B}x$. This with the fact that $Cx_n \to Cx$ implies that $Cx \in D(\overline{A+B})$ and $\overline{A+B}Cx = C\overline{A+B}x$, or $\overline{A+B}x = C^{-1}\overline{A+B}Cx$. Therefore $\overline{A+B} \subset C^{-1}\overline{A+B}C$.

(ii) Assume $S(\cdot)$ is a (C_0)-semigroup. It remains to show the inclusion: $C^{-1}\overline{A+B}C \subset \overline{A+B}$. Let $x \in D(C^{-1}\overline{A+B}C)$ and $y := C^{-1}\overline{A+B}Cx$. Then $Cy = \overline{A+B}Cx$. So, there is a sequence $\{z_n\}$ in $D(A+B)$ such that $(z_n, (A+B)z_n) \to (Cx, Cy)$ strongly as $n \to \infty$. Therefore we have for every $s, t \in [0, \infty)$

$$(1 * T)(s)(1 * S)(t)Cy = \lim_{n\to\infty}(1 * T)(s)(1 * S)(t)(A + B)z_n$$
$$= \lim_{n\to\infty}[(1 * S)(t)[T(s) - j_\alpha(s)C]z_n + (1 * T)(s)(S(t) - I)z_n]$$
$$= (1 * S)(t)[T(s) - j_\alpha(s)C]Cx + (1 * T)(s)(S(t) - I)Cx.$$

Since $T(\cdot)$, $S(\cdot)$, and C commute, it follows from the injectivity of C that

$$(1 * T)(s)[(1 * S)(t)y - (S(t) - I)x] = [T(s) - j_\alpha(s)C](1 * S)(t)x$$

for every $s, t \in [0, \infty)$. By the definition of generator, this implies that $(1 * S)(t)x \in D(A)$ and

$$A(1 * S)(t)x = (1 * S)(t)y - (S(t) - I)x = (1 * S)(t)y - B(1 * S)(t)x$$

for all $t \geq 0$. Hence we have for every $t \geq 0$

$$(1 * S)(t)y = (A + B)(1 * S)(t)x = \overline{A + B}(1 * S)(t)x.$$

By differentiation, we have $S(t)x \in D(\overline{A + B})$ and $S(t)y = \overline{A + B}S(t)x$. Since $S(0) = I$, this implies that $x \in D(\overline{A + B})$ and $y = \overline{A + B}x$. Therefore $C^{-1}\overline{A + B}C \subset \overline{A + B}$. This completes the proof. □

Proof of Theorem 2.1. Suppose $V_1(\cdot)$ and $V_2(\cdot)$ are two functions with the desired properties. Then it follows from (2.1) that the function $V(\cdot) := V_1(\cdot) - V_2(\cdot)$ satisfies $\int_0^t j_\alpha(t - u)S(u)(1 * V)(u)\,du = 0$ for all $t \geq 0$. By Lemma 1.3, we have $S(t)(1 * V)(t) = 0$ for all $t \geq 0$. Since $S(t)$ is injective, we must have $V(\cdot) \equiv 0$.

(a) Differentiating (2.1) we obtain

$$\int_0^t j_{\alpha-1}(t - u)S(u)(1 * V)(u)\,du = \int_0^t j_\alpha(u)S(u)T(t - u)\,du \tag{2.2}$$

for all $t \geq 0$. Since $1 * V$ commutes with $S(\cdot)$, it commutes with the generator $-B$, i.e., $(1 * V)(u)x \in D(B)$ and $B(1 * V)(u)x = (1 * V)(u)Bx$ for $x \in D(B)$. Thus

$$S'(u)(1 * V)(u)x = -BS(u)(1 * V)(u)x = -S(u)B(1 * V)(u)x$$
$$= -S(u)(1 * V)(u)Bx$$

for all $u \geq 0$. Using integration by parts, the closedness of B, and (2.1), we obtain for $x \in D(B)$

$$\int_0^t j_{\alpha-1}(t - u)S(u)(1 * V)(u)x\,du$$
$$= -\int_0^t j_\alpha(t - u)S(u)(1 * V)(u)Bx\,du + \int_0^t j_\alpha(t - u)S(u)V(u)x\,du$$
$$= -\int_0^t j_\alpha(t - u)BS(u)(1 * V)(u)x\,du + \int_0^t j_\alpha(t - u)S(u)V(u)x\,du$$
$$= -B\int_0^t j_\alpha(u)S(u)(1 * T)(t - u)x\,du + \int_0^t j_\alpha(t - u)S(u)V(u)x\,du.$$

Combining this and (2.2), and by the closedness of B again, we obtain that

$$\begin{aligned}(2.3)\qquad &\int_0^t j_\alpha(u)S(u)(1*T)(t-u)\,duB \\ &\subset B\int_0^t j_\alpha(u)S(u)(1*T)(t-u)\,du \\ &= -\int_0^t j_\alpha(u)S(u)T(t-u)\,du + \int_0^t j_\alpha(t-u)S(u)V(u)\,du\end{aligned}$$

for every $t \geq 0$. Since $(j_\alpha S) * j_\alpha(t)C = j_\alpha * (j_\alpha S)(t)C$ for all $t \geq 0$ and $T(\cdot)$ is an α-times integrated C-semigroup, using (2.1), (2.3), the commutativity of $S(\cdot)$ and $T(\cdot)$, and (1.1), we have for all $s, t \geq 0$

$$\begin{aligned}&\int_0^t\int_0^s j_\alpha(t-u)S(u)j_\alpha(s-v)S(v)\{(1*V)(u)[V(v)-j_\alpha(v)C] \\ &\quad -[V(u)-j_\alpha(u)C](1*V)(v)\}\,dvdu \\ &= \int_0^t j_\alpha(t-u)S(u)(1*V)(u)\,du\cdot\int_0^s j_\alpha(s-v)S(v)[V(v)-j_\alpha(v)C]\,dv \\ &\quad -\int_0^t j_\alpha(t-u)S(u)[V(u)-j_\alpha(u)C]\,du\cdot\int_0^s j_\alpha(s-v)S(v)(1*V)(v)\,dv \\ &= \int_0^t j_\alpha(u)S(u)(1*T)(t-u)\,du\cdot\left[\int_0^s j_\alpha(v)S(v)T(s-v)\,dv\right. \\ &\quad \left.+B\int_0^s j_\alpha(v)S(v)(1*T)(s-v)\,dv - \int_0^s j_\alpha(v)S(v)j_\alpha(s-v)C\,dv\right] \\ &\quad -\left[\int_0^t j_\alpha(u)S(u)T(t-u)\,du + B\int_0^t j_\alpha(u)S(u)(1*T)(t-u)\,du\right. \\ &\quad \left.-\int_0^t j_\alpha(u)S(u)j_\alpha(t-u)C\,du\right]\cdot\int_0^s j_\alpha(v)S(v)(1*T)(s-v)\,dv \\ &= \int_0^t\int_0^s j_\alpha(u)S(u)j_\alpha(v)S(v)\{(1*T)(t-u)[T(s-v)-j_\alpha(s-v)C] \\ &\quad -[T(t-u)-j_\alpha(t-u)C](1*T)(s-v)\}\,dvdu \\ &\quad +\left[\int_0^t j_\alpha(u)S(u)(1*T)(t-u)\,duB - B\int_0^t j_\alpha(u)S(u)(1*T)(t-u)\,du\right] \\ &\quad \cdot\int_0^s j_\alpha(v)S(v)(1*T)(s-v)\,dv \\ &= \int_0^t\int_0^s j_\alpha(u)S(u)j_\alpha(v)S(v)0\,dvdu + 0 = 0.\end{aligned}$$

Therefore, by Lemma 1.3 and the invertibility of $S(u)$ we have

$$(1 * V)(u)[V(v) - j_\alpha(v)C] = [V(u) - j_\alpha(u)C](1 * V)(v) \quad \text{for all } u, v \geq 0.$$

Since $V(\cdot)$ is assumed to satisfy $V(0) = \delta_{0,\alpha}C$ and $V(t)C = CV(t)$ for all $t \geq 0$, it follows from Proposition 1.1 that $V(\cdot)$ is an α-times integrated C-semigroup.

(b) If $T(\cdot)$ is nondegenerate and $V(t)x = 0$ for $t > 0$ and some $x \in X$, then (2.1) implies $(1 * ((j_\alpha S) * T))(t)x = (j_\alpha S) * ((1 * T))(t)x = 0$, so that, by Lemma 1.3(b), $S(t)\int_0^t j_\alpha(t-s)(S(-s)T(s))x\,ds = ((j_\alpha S) * T)(t)x = 0$ for all $t > 0$. Since $S(t)$ is injective, $\int_0^t (j_\alpha(t-s)(S(-s)T(s))x\,ds = 0$ and hence $S(-t)T(t)x = 0$ for all $t > 0$. Then the injectivity of $S(-t)$ and the nondegeneracy of $T(\cdot)$ imply $x = 0$. Conversely, if $V(\cdot)$ is nondegenerate and $T(t)x = 0$ for all $t \geq 0$ and some $x \in X$, then (2.1) implies $S(t)(1 * V)(t)x = 0$ for all $t \geq 0$, by Lemma 1.3. Therefore the injectivity of $S(t)$ for all $t \geq 0$ together with the nondegeneracy of $V(\cdot)$ implies $x = 0$.

(c) The closability of $A + B$ follows from Proposition 2.2(i). Since $(1 * T)(t)A \subset A(1 * T)(t) = T(t) - j_\alpha(t)C$ for $t \geq 0$, and since A is closed and $S(t)Ay = AS(t)y$ for $y \in D(A)$ we have $R(\int_0^t j_\alpha(u)S(u)(1 * T)(t-u)x\,du) \subset D(A)$ and

$$\begin{aligned}
&\int_0^t j_\alpha(u)S(u)(1 * T)(t-u)\,du A \\
&\quad \subset A\int_0^t j_\alpha(u)S(u)(1 * T)(t-u)\,du = \int_0^t j_\alpha(u)S(u)A(1 * T)(t-u)\,du \\
&\quad = \int_0^t j_\alpha(u)S(u)[T(t-u) - j_\alpha(t-u)C]\,du \\
&\quad = \frac{d}{dt}\int_0^t j_\alpha(u)S(u)(1 * T)(t-u)\,du - \int_0^t j_\alpha(t-u)S(u)j_\alpha(u)C\,du \\
&\quad = \frac{d}{dt}\int_0^t j_\alpha(t-u)S(u)(1 * V)(u)\,du - \int_0^t j_\alpha(t-u)S(u)j_\alpha(u)C\,du \\
&\quad = \int_0^t j_{\alpha-1}(t-u)S(u)(1 * V)(u)\,du - \int_0^t j_\alpha(t-u)S(u)j_\alpha(u)C\,du
\end{aligned}$$

for all $t \geq 0$. This and (2.1) imply that

$$\begin{aligned}
&\int_0^t j_\alpha(t-u)S(u)(1 * V)(u)A\,du \\
&\quad \subset A\int_0^t j_\alpha(t-u)S(u)(1 * V)(u)\,du \\
&\quad = \int_0^t j_{\alpha-1}(t-u)S(u)(1 * V)(u)\,du - \int_0^t j_\alpha(t-u)S(u)j_\alpha(u)C\,du.
\end{aligned}$$

Then by Lemma 1.4 we have

$$\begin{aligned}
(2.4) \qquad \int_0^t S(u)(1 * V)(u)A\,du &\subset A\int_0^t S(u)(1 * V)(u)\,du \\
&= S(t)(1 * V)(t) - \int_0^t S(u)j_\alpha(u)C\,du.
\end{aligned}$$

On the other hand, we obtain from (2.1), (2.2) and (2.3) that

$$\begin{aligned}\int_0^t & j_\alpha(t-u)S(u)(1*V)(u)B\,du \\ &\subset B\int_0^t j_\alpha(t-u)S(u)(1*V)(u)\,du \\ &= \int_0^t j_\alpha(t-u)S(u)V(u)\,du - \int_0^t j_{\alpha-1}(t-u)S(u)(1*V)(u)\,du.\end{aligned}$$

Since B is closed, application of Lemma 1.4 yields

$$\begin{aligned}\int_0^t S(u)(1*V)(u)B\,du &\subset B\int_0^t S(u)(1*V)(u)\,du \\ &= \int_0^t S(u)V(u)\,du - S(t)(1*V)(t).\end{aligned} \tag{2.5}$$

Hence, from (2.4) and (2.5) we have for every $t \geq 0$

$$\begin{aligned}\int_0^t S(u)(1*V)(u)(A+B)\,du &\subset (A+B)\int_0^t S(u)(1*V)(u)\,du \\ &= \int_0^t S(u)[V(u) - j_\alpha(u)C]\,du.\end{aligned} \tag{2.6}$$

Since $A+B$ is closable, by Lemma 1.4, we have $R(S(t)(1*V)(t)) \subset D(\overline{A+B})$ and

$$S(t)(1*V)(t)\overline{A+B} \subset \overline{A+B}S(t)(1*V)(t) = S(t)[V(t) - j_\alpha(t)C]. \tag{2.7}$$

Since $S(t)$ is injective, $(1*V)(t)\overline{A+B} \subset V(t) - j_\alpha(t)C$. On the other hand, since $S(\cdot)$ commutes with $V(\cdot)$, we have $R((1*V)(t)S(t)) \subset D(\overline{A+B})$ and $\overline{A+B}(1*V)(t)S(t) = [V(t) - j_\alpha(t)C]S(t)$. Then, by the surjectivity of $S(t)$, we obtain that $R((1*V)(t)) \subset D(\overline{A+B})$ and

$$\overline{A+B}(1*V)(t) = [V(t) - j_\alpha(t)C].$$

Hence $\overline{A+B}$ is a subgenerator of $V(\cdot)$. By Proposition 2.2(ii), we have $C^{-1}\overline{A+B}C = \overline{A+B}$. It follows from Proposition 1.2 that $\overline{A+B}$ is the generator of $V(\cdot)$. If $B \in B(X)$, it is clear that $A+B$ is closed and hence $A+B$ is the generator of $V(\cdot)$. This is also the result of Section 3. □

3. Perturbation by bounded operators.

In order to construct the desired $V(\cdot)$ for the case that $B \in B(X)$, we first define the bounded operators $Q_{m,n}$, $m = 0, 1, 2, \ldots$, $n = -1, 0, 1, \ldots$, on X by

$$Q_{m,n}(t) := j_m(t)(j_n * T)(t), \quad t \geq 0,\ m,n \geq 0, \tag{3.1}$$

$$Q_{m,-1}(t) := j_m(t)T(t), \quad t \geq 0,\ m \geq 0. \tag{3.2}$$

Define the strongly continuous families $G_n(\cdot)$ by

$$G_n(t) := \sum_{k=0}^{n+1} a_{n-k} Q_{k,n-k}(t), \quad t \geq 0, \ n = -1, 0, 1, 2, \dots. \tag{3.3}$$

We first prove two lemmas about the operators $Q_{m,n}$ and G_n.

LEMMA 3.1. (i) *For* $t \geq 0$ *and* $n = -1, 0, 1, \dots,$ *we have*

$$(1 * Q_{0,n})(t) = Q_{0,n+1}(t) \tag{3.4}$$

and for $m \geq 1$

$$(1 * Q_{m,n})(t) = Q_{m,n+1}(t) - (1 * Q_{m-1,n+1})(t). \tag{3.5}$$

(ii) *If* A *is the generator of* $T(\cdot)$, *then for* $t \geq 0$ *and* $n = -1, 0, 1, \dots,$ *we have*

$$A(1 * Q_{0,n})(t) = Q_{0,n}(t) - j_{n+\alpha+1}(t)C \tag{3.6}$$

and

$$A(1 * Q_{m,n})(t) = Q_{m,n}(t) - (1 * Q_{m-1,n})(t) - \binom{m+n+\alpha}{m} j_{m+n+1+\alpha}(t)C. \tag{3.7}$$

PROOF. (i) If $m = 0$, then we have for $n = -1, 0, 1, \dots$ and $t \geq 0$

$$(1 * Q_{0,n})(t) = (1 * (j_n * T))(t) = ((1 * j_n) * T)(t) = (j_{n+1} * T)(t).$$

If $m \geq 1$, integrating (3.1) and using integration by parts, we have for $n = -1, 0, 1, \dots$ and $t \geq 0$

$$\begin{aligned}(1 * Q_{m,n})(t) &= \int_0^t j_m(s)(j_n * T)(s)\,ds \\ &= j_m(t)(j_{n+1} * T)(t) - \int_0^t j_{m-1}(s)(j_{n+1} * T)(s)\,ds \\ &= Q_{m,n+1}(t) - (1 * Q_{m-1,n+1})(t).\end{aligned}$$

(ii) Integrating (1.4) n-times, we obtain from the closedness of A that

$$A(j_n * T)(t) = (j_{n-1} * T)(t) - j_{n+\alpha}(t)C, \quad t \geq 0. \tag{3.8}$$

By (3.8) and (3.1), we have

$$\begin{aligned}A(1 * Q_{0,n})(t) &= (j_n * A(1 * T))(t) = (j_n * (T - j_\alpha)C)(t) \\ &= (j_n * T)(t) - j_{n+1+\alpha}(t)C = Q_{0,n}(t) - j_{n+1+\alpha}(t)C,\end{aligned}$$

$$\begin{aligned}AQ_{m,n+1}(t) &= j_m(t)(j_n * [T - j_\alpha C])(t) \\ &= Q_{m,n}(t) - j_m(t)(j_n * j_\alpha C)(t) \\ &= Q_{m,n}(t) - \binom{m+n+1+\alpha}{m} j_{m+n+1+\alpha}(t)C\end{aligned} \tag{3.9}$$

and

$$A(1 * Q_{m-1,n+1})(t) \tag{3.10}$$
$$= \int_0^t j_{m-1}(s) A(j_{n+1} * T)(s)\, ds$$
$$= \int_0^t j_{m-1}(s)(j_n * [T - j_\alpha C])(s)\, ds$$
$$= \int_0^t [Q_{m-1,n}(s) - j_{m-1}(s) j_{n+1+\alpha}(s) C]\, ds$$
$$= (1 * Q_{m-1,n})(t) - \int_0^t \binom{m+n+\alpha}{m-1} j_{m+n+\alpha}(s) C\, ds$$
$$= (1 * Q_{m-1,n})(t) - \binom{m+n+\alpha}{m-1} j_{m+n+1+\alpha}(t) C.$$

Since $\binom{r+1}{m} = \binom{r}{m-1} + \binom{r}{m}$ for all $r \in \boldsymbol{R}$ and $m \in \boldsymbol{N}$, (3.7) follows from (3.5), (3.9) and (3.10). □

Lemma 3.2. *For* $t \geq 0$,
(a) $A(1 * G_{-1})(t) = G_{-1}(t) - j_\alpha(t) C$ *and*
(b) $A(1 * G_n)(t) = G_n(t) - (1 * G_{n-1})(t)$ *for* $n = 0, 1, 2, \ldots$.

Proof. Since $G_{-1}(t) = a_{-1} Q_{0,-1}(t) = T(t)$, (a) is (1.4). We show (b). Using Lemma 3.1 and (1.5), we have for every $t \geq 0$ and $n = 0, 1, 2, \ldots$

$$\begin{aligned}
A(1 * G_n)(t) &= \sum_{k=0}^{n+1} a_{n-k} A(1 * Q_{k,n-k})(t) \\
&= a_n A(1 * Q_{0,n})(t) + \sum_{k=1}^{n+1} a_{n-k} A(1 * Q_{k,n-k})(t) \\
&= a_n [Q_{0,n}(t) - j_{n+1+\alpha}(t) C] \\
&\quad + \sum_{k=1}^{n+1} a_{n-k} \left[Q_{k,n-k}(t) - (1 * Q_{k-1,n-k})(t) - \binom{n+\alpha}{k} j_{n+1+\alpha}(t) C \right] \\
&= \sum_{k=0}^{n+1} a_{n-k} Q_{k,n-k}(t) - \left(1 * \sum_{k=0}^{n} a_{n-1-k} Q_{k,n-1-k} \right)(t) \\
&\quad - \left(a_n + \sum_{k=1}^{n+1} a_{n-k} \binom{n+\alpha}{k} \right) j_{n+1+\alpha}(t) C \\
&= G_n(t) - (1 * G_{n-1})(t).
\end{aligned}$$

This completes the proof. □

Proposition 3.3. *For a given* $B \in B(X)$, *let* $V_n(t) := \sum_{k=-1}^{n} B^{k+1} G_k(t)$ *for* $t \geq 0$

and $n = 0, 1, 2, \ldots$. *Then* $\{V_n(\cdot)\}$ *converges in operator norm to a strongly continuous function* $V(\cdot)$, *uniformly for* t *in compact subsets of* $[0, \infty)$.

Proof. Let $\ell := [\alpha]$, the largest integer less than or equal to α, and let $\beta_t := \sup_{0 \le s \le t} \|T(s)\|$ for $t \ge 0$. Then

$$|a_n| = \left|\binom{n+\alpha}{n+1}\right| \le \binom{n+\ell+1}{n+1} = \binom{n+\ell+1}{\ell} \le (n+1+\ell)^{\ell}.$$

Thus we have $|a_n| \le (n+1+\ell)^{\ell}$. Then for $t \ge 0$ and every $n = 0, 1, 2, \ldots$

$$\begin{aligned} \|G_n(t)\| &\le \sum_{k=0}^{n+1} \|a_{n-k} Q_{k,n-k}(t)\| \\ &\le \sum_{k=0}^{n+1} (n-k+1+\ell)^{\ell} j_k(t) \|(j_{n-k} * T)(t)\| \\ &\le \sum_{k=0}^{n+1} (n-k+1+\ell)^{\ell} j_k(t) j_{n-k+1}(t) \beta_t \\ &\le \frac{\beta_t (n+1+\ell)^{\ell}}{(n+1)!} \sum_{k=0}^{n+1} \binom{n+1}{k} t^{n+1} \\ &= \beta_t \frac{(2t)^{n+1} (n+1+\ell)^{\ell}}{(n+1)!}. \end{aligned} \tag{3.11}$$

Therefore we have for all $0 \le s \le t$

$$\begin{aligned} \sum_{n=-1}^{\infty} \|B^{n+1} G_n(s)\| &\le \sum_{n=-1}^{\infty} \sum_{k=0}^{n+1} \|B^{n+1}\| \, \|a_{n-k} Q_{k,n-k}(t)\| \\ &\le \sum_{n=-1}^{\infty} \frac{(2t)^{n+1} (n+1+\ell)^{\ell}}{(n+1)!} \|B\|^{n+1} \beta_t < \infty. \end{aligned} \tag{3.12}$$

It follows from the M-test that $V(t) := \sum_{n=-1}^{\infty} B^{n+1} G_n(t)$ converges absolutely and uniformly on compact subsets of $[0, \infty)$. Hence $V(\cdot)$ is strongly continuous on $[0, \infty)$. □

Theorem 3.4. (i) *The function* $V(\cdot)$ *in Proposition* 3.3 *has the following two expressions*:

$$\begin{aligned} V(t) &= \sum_{n=-1}^{\infty} B^{n+1} G_n(t) = e^{tB} \sum_{n=0}^{\infty} a_{n-1} B^n (j_{n-1} * T)(t) \\ &= e^{tB} \sum_{n=0}^{\infty} \binom{-\alpha}{n} B^n (j_{n-1} * T)(t) \\ &= e^{tB} \sum_{n=0}^{\infty} \binom{n-1+\alpha}{n} (-B)^n (j_{n-1} * T)(t), \end{aligned} \tag{3.13}$$

$$V(t) = \sum_{n=0}^{\infty} \binom{\alpha}{n} (-B)^n (j_{n-1} * e^{\cdot B} T)(t), \tag{3.14}$$

the series being absolutely convergent in operator norm and uniformly for t in any compact subset of $[0, \infty)$.

(ii) $V(\cdot)$ *satisfies the equations*:

$$(1 * V)(t) = e^{tB} \sum_{n=0}^{\infty} \binom{n+\alpha}{n} (-B)^n (j_n * T)(t), \tag{3.15}$$

$$\int_0^t j_\alpha(t-s) e^{-sB} (1 * V)(s)\, ds = \int_0^t j_\alpha(s) e^{-sB} (1 * T)(t-s)\, ds \tag{3.16}$$

for all $t \geq 0$.

Proof. (i) Since $e^{tB} = \sum_{k=0}^{\infty} B^k j_k(t)$ for $t \geq 0$, it follows from (3.5) and (3.12) that

$$\begin{aligned} V(t) &= \sum_{n=-1}^{\infty} B^{n+1} G_n(t) = \sum_{n=0}^{\infty} B^n G_{n-1}(t) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n-1-k} B^n Q_{k,n-1-k}(t) \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{n-1-k} B^n Q_{k,n-1-k}(t) \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{n-1} B^{n+k} Q_{k,n-1}(t) \\ &= \sum_{n=0}^{\infty} a_{n-1} B^n \left[\sum_{k=0}^{\infty} B^k j_k(t) \right] (j_{n-1} * T)(t) \\ &= e^{tB} \sum_{n=0}^{\infty} a_{n-1} B^n (j_{n-1} * T)(t) \end{aligned}$$

for every $t \geq 0$. This proves (3.13). Moreover, the series converges in operator norm and absolutely and uniformly on any compact subset of $[0, \infty)$.

Since $\binom{-\alpha}{n} = \binom{n-1+\alpha}{n}(-1)^n$ for all $n \geq 0$, we obtain from (3.13) and Lemma 1.6 that for every $t \geq 0$

$$\begin{aligned} V(t) &= e^{tB} \sum_{n=0}^{\infty} \binom{-\alpha}{n} B^n (j_{n-1} * T)(t) \\ &= e^{tB} \sum_{n=0}^{\infty} \binom{n-1+\alpha}{n} (-B)^n (j_{n-1} * T)(t) \\ &= e^{tB} \left[\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{\alpha}{k+1} \binom{n-1}{k} (-B)^n (j_{n-1} * T)(t) + T(t) \right] \end{aligned}$$

$$= e^{tB}\left[\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{\alpha}{k+1}\binom{n}{k}(-B)^{n+1}(j_n * T)(t) + T(t)\right]$$

$$= e^{tB}\left[\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\binom{\alpha}{k+1}\binom{n}{k}(-B)^{n+1}(j_n * T)(t) + T(t)\right]$$

$$= e^{tB}\left[\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{\alpha}{k+1}\binom{n+k}{k}(-B)^{n+k+1}(j_{n+k} * T)(t) + T(t)\right]$$

$$= e^{tB}\left[\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{\alpha}{k+1}(-B)^{n+k+1}((j_n j_k) * T)(t) + T(t)\right]$$

$$= e^{tB}\left[\sum_{k=0}^{\infty}\binom{\alpha}{k+1}(-B)^{k+1}\left(\left(\left[\sum_{n=0}^{\infty}(-B)^n j_n\right] j_k\right) * T\right)(t) + T(t)\right]$$

$$= e^{tB}\left[\sum_{k=1}^{\infty}\binom{\alpha}{k}(-B)^{k}\int_0^t j_{k-1}(t-s)e^{-(t-s)B}T(s)\,ds + T(t)\right]$$

$$= \sum_{k=0}^{\infty}\binom{\alpha}{k}(-B)^{k}\int_0^t j_{k-1}(t-s)e^{sB}T(s)\,ds.$$

This completes the proof of (3.14).

(ii) To see (3.15), by (1.5) and (3.5) we have for every $t \geq 0$

$$(1 * Q_{m,n})(t) = \sum_{k=0}^{m}(-1)^k Q_{m-k,n+1+k}(t) \quad \text{for all } m = 0,1,2,\ldots,\ n = -1,0,1,\ldots.$$

Since $\sum_{n=0}^{k}\binom{n-1+\alpha}{n} = \binom{k+\alpha}{k}$ for $k = 0,1,2,\ldots$, it follows from the above proof of (3.13) that

$$(1 * V)(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{n-1+\alpha}{n}(-1)B^{n+k}(1 * Q_{k,n-1})(t)$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{k}\binom{n-1+\alpha}{n}(-1)^n B^{n+k}(-1)^i Q_{k-i,n+i}(t)$$

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\binom{n-1+\alpha}{n}(-1)^{n+i} B^{n+k} Q_{k-i,n+i}(t)$$

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\binom{n-1+\alpha}{n}(-1)^{n+i} B^{n+k+i} Q_{k,n+i}(t)$$

$$= e^{tB}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\binom{n-1+\alpha}{n}(-1)^{n+i} B^{n+i}(j_{n+i} * T)(t)$$

$$= e^{tB}\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\binom{n-1+\alpha}{n}\right)(-1)^k B^k(j_k * T)(t)$$

$$= e^{tB}\sum_{k=0}^{\infty}\binom{k+\alpha}{k}(-1)^k B^k(j_k * T)(t).$$

To prove (3.16), put $W(t) := e^{-tB}(1 * V)(t)$ for $t \geq 0$. Then we have for every $t \geq 0$

$$W(t) = \sum_{k=0}^{\infty}\binom{k+\alpha}{k}(-1)^k B^k(j_k * T)(t).$$

Since

$$j_{\alpha-1} * \left(\sum_{k=0}^{\infty}\binom{k+\alpha}{k}(-1)^k B^k j_k\right)(t) = \sum_{k=0}^{\infty}\binom{k+\alpha}{k}(-1)^k B^k j_{k+\alpha}(t)$$

$$= \sum_{k=0}^{\infty}(-B)^k j_k(t) j_\alpha(t) = j_\alpha(t)e^{-tB},$$

we have

$$\int_0^t j_\alpha(t-s)e^{-sB}(1 * V)(s)\,ds = (j_\alpha * W)(t) = \int_0^t j_\alpha(s)e^{-sB}(1 * T)(t-s)\,ds.$$

The proof is complete. □

Finally, by applying Theorems 1.3 and 3.4, we obtain the following bounded perturbation theorem.

THEOREM 3.5. *Let $T(\cdot)$ be an α-times integrated C-semigroup on X and let $B \in B(X)$ be commuting with $T(\cdot)$ and C. Then the function $V(\cdot)$, given by* (3.13) *and* (3.14), *is an α-times integrated C-semigroup. Moreover, if $T(\cdot)$ is nondegenerate and has generator A, then $V(\cdot)$ is nondegenerate and has generator $A + B$.*

PROOF. Clearly, $V(0) = \delta_{0,\alpha}C$. The assumption that B commutes with C and $T(\cdot)$ implies that $CV(t) = V(t)C$ and $S(t)V(s) = V(s)S(t)$ for $s, t \geq 0$. Hence the theorem follows from Theorem 3.4(ii) and Theorem 2.1.

Next, we give a direct proof for the case that $T(\cdot)$ is nondegenerate. By Lemma 3.2 and Proposition 3.3, we obtain from the closedness of A that

$$(A+B)(1 * V)(t)$$

$$= e^{tB}\left\{\sum_{n=1}^{\infty}\binom{n+\alpha}{n}[(-B)^n A(j_n * T)(t) - (-B)^{n+1}(j_n * T)(t)] + (A+B)(1 * T)(t)\right\}$$

$$= e^{tB}\Bigg\{\sum_{n=1}^{\infty}\left[\binom{n+1+\alpha}{n+1}-\binom{n+\alpha}{n}\right](-B)^{n+1}(j_n * T)(t)$$

$$+\binom{1+\alpha}{1}(-B)(1*T)(t)-\sum_{n=1}^{\infty}\binom{n+\alpha}{n}(-B)^n j_{n-1}*j_\alpha(t)C$$

$$+(A+B)(1*T)(t)\Bigg\}$$

$$= e^{tB}\Bigg\{\left[\sum_{n=0}^{\infty}\binom{n+\alpha}{n+1}(-B)^{n+1}(j_n * T)(t)+T(t)\right]$$

$$-\sum_{n=0}^{\infty}\binom{n+\alpha}{n}(-B)^n(j_{n-1}*j_\alpha C)(t)\Bigg\}$$

$$= e^{tB}\sum_{n=0}^{\infty}a_{n-1}B^n(j_{n-1}*T)(t)-e^{tB}\sum_{n=0}^{\infty}\binom{n+\alpha}{n}(-B)^n j_{n+\alpha}(t)C$$

$$= V(t)-j_\alpha(t)C$$

for $t \geq 0$. Note that $BC = CB$ and $BT(\cdot) = T(\cdot)B$ imply that $A + B$ commutes with $V(t)$ and C. Therefore $A + B$ is a subgenerator of $V(\cdot)$. Then the same argument in the proof of (c) of Theorem 2.1 shows that $A + B$ is actually the generator of $V(\cdot)$. □

Remarks. (i) Theorem 3.5 generalizes and extends a result in [**23**, Theorem 1.3.5], therein Xiao and Liang proved that formula (3.14) defines an α-times integrated semigroup with generator $A + B$ if A generates an exponentially bounded α-times integrated semigroup $T(\cdot)$ and $B \in B(X)$ commutes with A.

(ii) If $T(\cdot)$ is an n-times integrated C-semigroup, then the expression (3.14) of $V(\cdot)$ reduces to the finite series:

$$V(t) = \sum_{k=0}^{n}\binom{n}{k}(-B)^k(j_{k-1} * e^{\cdot B}T)(t), \quad t \geq 0.$$

In particular, if $n = 0$ and $C = I$, then we have $V(t) = e^{tB}T(t) = e^{t(A+B)}$ as the perturbation (C_0)-semigroup. This also follows from the classical bounded perturbation theorem for (C_0)-semigroups (cf. [**5**, Theorem III.1.3]).

4. Perturbation of nondegenerate once integrated C-semigroups.

The next theorem presents sufficient conditions on a once integrated C-semigroup $T(\cdot)$ so that a once integrated C-semigroup $V(\cdot)$ as described in Theorem 2.1 exists.

Theorem 4.1. *Let $T(\cdot)$ be a once integrated C-semigroup on X and let $S(\cdot)$ be a (C_0)-group on X with generator $-B$ such that $S(t)T(s) = T(s)S(t)$ and $S(t)C = CS(t)$ for all $s, t \geq 0$. Suppose there is a nonempty bounded subset E of X^* such that the following conditions hold*:

(i) $\|x\| \le \sup_{x^* \in E} |\langle x, x^* \rangle|$ *for all* $x \in X$;

(ii) *For every* $x^* \in E$ *and* $x \in X$, $\langle T(\cdot)x, x^* \rangle$ *is continuously differentiable on* $(0, \infty)$;

(iii) $F(t; x, x^*) := (d/dt)\langle T(t)x, x^* \rangle$, $t > 0$, *is linear on* x *and for every* $t > 0$ *there is a number* $M_t > 0$ *such that* $\sup_{0 < s \le t} |F(s, x, x^*)| \le M_t \|x\| \cdot \|x^*\|$ *for all* $t > 0$, $x \in X$, *and* $x^* \in E$.

Then for every $t \ge 0$ *the linear operator* $V(t) : D(B) \to X$ *defined by*

$$V(t)x := T(t)\bar{S}(t)x - \int_0^t T(u)\bar{S}(u)Bx\,du \quad \text{for } x \in D(B) \tag{4.1}$$

can be extended to the whole space X *and the extended operator function, still denoted by* $V(\cdot)$, *is a once integrated C-semigroup. Moreover, if* $T(\cdot)$ *is nondegenerate and has generator* A, *then* $V(\cdot)$ *is nondegenerate and has generator* $\overline{A + B}$.

Proof. It is clear that $V(\cdot)x$ is strongly continuous for every $x \in D(B)$. Let $x^* \in E$ and $x \in D(B)$. Since

$$\begin{aligned}\frac{d}{du}\langle T(u)\bar{S}(u)x, x^* \rangle &= \lim_{h \to 0} h^{-1}\langle T(u+h)(\bar{S}(u+h) - \bar{S}(u))x, x^* \rangle \\ &\quad + \lim_{h \to 0} h^{-1}\langle (T(u+h) - T(u))\bar{S}(u)x, x^* \rangle \\ &= \langle T(u)\bar{S}(u)Bx, x^* \rangle + F(u, \bar{S}(u)x, x^*),\end{aligned}$$

we have for every $t \ge 0$

$$\begin{aligned}\langle V(t)x, x^* \rangle &= \langle T(t)\bar{S}(t)x, x^* \rangle - \int_0^t \langle T(u)\bar{S}(u)Bx, x^* \rangle\,du \\ &= \langle T(t)\bar{S}(t)x, x^* \rangle - \langle T(u)\bar{S}(u)x, x^* \rangle|_0^t + \int_0^t F(u, \bar{S}(u)x, x^*)\,du \\ &= \int_0^t F(u, \bar{S}(u)x, x^*)\,du.\end{aligned}$$

Thus we obtain from conditions (i) and (iii) that

$$\begin{aligned}\|V(t)x\| &\le \sup_{x^* \in E} |\langle V(t)x, x^* \rangle| \\ &= \sup_{x^* \in E} \left| \int_0^t F(u, \bar{S}(u)x, x^*)\,du \right| \\ &\le tM_t \sup_{-t \le u \le 0} \|S(u)\| \cdot \|x\| \cdot \sup_{x^* \in E} \|x^*\|.\end{aligned}$$

Since E is bounded, this implies that $V(\cdot)|_{D(B)}$ is uniformly bounded on compact subset of $[0, \infty)$. Since $D(B)$ is dense in X, each $V(t)$ can be extended to a bounded linear operator on X. We still denote it as $V(\cdot)$. Since $V(\cdot)|_{D(B)}$ is strongly continuous on $[0, \infty)$, its extension $V(\cdot)$ is also strongly continuous on $[0, \infty)$.

By the definition of $V(\cdot)$, it is easy to see that $V(\cdot)$ commutes with $S(\cdot)$, $T(\cdot)$, and C. On the other hand, we have for every $x \in D(B)$

$$S(t)(1 * V)(t)x = S(t)\int_0^t \bar{S}(u)T(u)x\,du - S(t)\int_0^t j_1(t-u)\bar{S}(u)T(u)Bx\,du$$
$$= (S * T)(t)x - [(j_1 S) * T](t)Bx.$$

Since $1 * (j_1 S)(t)Bx = -uS(u)x|_0^t + \int_0^t S(u)x\,du = -j_1(t)S(t)x + (1 * S)(t)x$, we have for every $x \in D(B)$

$$[j_1 * (S(1 * V))](t)x = j_1 * S * T(t)x - j_1 * T * (j_1 S)(t)Bx$$
$$= j_1 * S * T(t)x - 1 * T * [-j_1 S + 1 * S](t)x$$
$$= (j_1 S) * (1 * T)(t)x.$$

It follows from the denseness of $D(B)$ that $[j_1 * (S(1 * V))](t) = (j_1 S) * (1 * T)(t)$ for all $t \geq 0$. That is, (2.1) holds for $\alpha = 1$. Hence the conclusion follows from Theorem 2.1. □

Example 1. Let $X := C([0, \infty), Y)$ with Y a Banach space, and let $T_0(\cdot)$ be the translation semigroup on X. Define $[T(t)x](s) := \int_0^t [T_0(u)x](s)\,du$ for $s, t \geq 0$ and $x \in X$. It is known that $T(\cdot)$ is a nondegenerate once integrated semigroup. So, it has the generator A. Let us take $E := \{\delta_s \in X^*; \delta_s x = x(s) \text{ for } x \in X, s \geq 0\}$. It is clear that E satisfies the three conditions of Theorem 4.1. Suppose $S(\cdot)$ is a (C_0)-group on X with generator $-B$ and suppose $S(\cdot)$ commutes with $T(\cdot)$. Then Theorem 4.1 asserts that $\overline{A + B}$ is the generator of a once integrated semigroup $V(\cdot)$, which is the extension of the operator defined by (4.1).

Lemma 4.2. *For the once integrated semigroup* $T(\cdot)$ *in Example* 1, *the function* $V(\cdot)$ *determined by* (4.1) *has the expression*:

$$[V(t)x](s) = \int_0^t [T_0(u)\bar{S}(u)x](s)\,du \tag{4.2}$$
$$= \int_0^t [\bar{S}(u)x](s+u)\,du, \quad s, t \geq 0, \ x \in X.$$

Proof. Indeed, denoting by $V(\cdot)$ the function defined by the last integral and using integration by parts, we have for $x^* := \delta_s \in E$ and $x \in D(B)$

$$\langle V(t)x, x^* \rangle = \int_0^t \langle T_0(r)[x + (1 * \bar{S})(r)Bx], x^* \rangle\,dr$$
$$= \langle T(t)x, x^* \rangle + \langle T(r)(1 * \bar{S}(r)Bx, x^* \rangle|_0^t - \int_0^t \langle T(r)\bar{S}(r)Bx, x^* \rangle\,dr$$
$$= \langle T(t)x, x^* \rangle + \langle T(t)(1 * \bar{S})(t)Bx, x^* \rangle - \langle [1 * (T\bar{S})](t)Bx, x^* \rangle$$
$$= \langle T(t)\bar{S}(t)x - [1 * (T\bar{S})](t)Bx, x^* \rangle.$$

Therefore we get (4.1) for all $x \in D(B)$ and $t \geq 0$. □

Example 2. Let $T(\cdot)$ be a once integrated C-semigroup on X. Suppose $T(t)x := \int_0^t T_0(u)x\,du$, $x \in X$, $t \geq 0$ for some operator function $T_0(\cdot)$ which is locally bounded and strongly continuous on $(0, \infty)$. Then the conditions (i)–(iii) of Theorem 4.1 are satisfied when E is the closed unit ball of X^*. So, if $S(\cdot)$ is a (C_0)-group on X with generator $-B$, and if $S(\cdot)$ commutes with $T(\cdot)$ and C, then it follows from Theorem 4.1 that the function $V(\cdot)$ given by (4.1) is a once integrated C-semigroup, which has generator $\overline{A+B}$ if $T(\cdot)$ has generator A. A similar calculation as in Lemma 4.2 yields the expression: $V(t) = \int_0^t T_0(u)\overline{S}(u)\,du$, $t \geq 0$.

In particular, if $T(\cdot)$ is a hermitian once integrated C-semigroup on X with generator A. (For instance, $X = C(\Omega)$ and $T(t) = \int_0^t qe^{sp}\,ds$ for $t \geq 0$, where Ω is a compact Hausdorff space, $q \in X$ is real-valued, and p a real-valued measurable function defined on Ω such that $T(\cdot)$ is strongly continuous.) Then $T(\cdot)$ is norm infinitely differentiable on $(0, \infty)$ and there is an operator-value function $T_0 : [0, \infty) \to B(X)$ such that $T_0(0) = C$, $T_0(\cdot)$ is locally bounded on $[0, \infty)$, and $T(t)x = \int_0^t T_0(u)x\,du$ for all $t \geq 0$ and $x \in X$ (see [**11**, Theorem 2.3(d)] and [**13**]). Hence hermitian once integrated C-semigroups are particular case of this example.

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