J. Math. Soc. Japan
Vol. 55, No. 4, 2003

Extendible and stably extendible vector bundles over real projective spaces

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(Received May 11, 2001)
(Revised May 23, 2002)

Abstract. The purpose of this paper is to study extendibility and stable extendibility of vector bundles over real projective spaces. We determine a necessary and sufficient condition that a vector bundle ζ over the real projective n-space RP^n is extendible (or stably extendible) to RP^m for every $m > n$ in the case where ζ is the complexification of the tangent bundle of RP^n and in the case where ζ is the normal bundle associated to an immersion of RP^n in the Euclidean $(n+k)$-space R^{n+k} or its complexification, and give examples of the normal bundle which is extendible to RP^N but is not stably extendible to RP^{N+1}.

1. Introduction and results.

Let F stand for any one of the real number field $\boldsymbol{R}$, the complex number field $\boldsymbol{C}$ or the quaternion number field $\boldsymbol{H}$. Let X be a space and A its subspace. An F-vector bundle ζ of dimension t over A is said to be *extendible* (respectively *stably extendible*) to X, if there is a t-dimensional F-vector bundle over X whose restriction to A is equivalent (respectively stably equivalent) to ζ as F-vector bundles, that is, if ζ is equivalent (respectively stably equivalent) to the induced bundle $i^*\alpha$ of a t-dimensional F-vector bundle α over X under the inclusion map $i : A \to X$ (cf. [**15**, p. 20], [**16**, p. 191] and [**3**, p. 273]).

An example of an $\boldsymbol{R}$-vector bundle that is stably extendible but is not extendible is given by the tangent bundle $\tau(S^n)$ of the n-sphere S^n in the $(n+1)$-sphere S^{n+1} for $n \neq 1, 3, 7$. In fact, $\tau(S^n) \oplus 1$ is the $(n+1)$-dimensional trivial bundle over S^n and so $\tau(S^n) \oplus 1 = (i^*n) \oplus 1$, where $i : S^n \to S^{n+1}$ is the inclusion map, 1 denotes the trivial $\boldsymbol{R}$-line bundle over S^n, n denotes the trivial $\boldsymbol{R}$-vector bundle over S^{n+1} of dimension n and $\oplus$ denotes the Whitney sum. Hence $\tau(S^n)$ is stably extendible to S^{n+1}. But $\tau(S^n)$ is not extendible to S^{n+1} for $n \neq 1, 3, 7$ (cf. [**10**, Proof of Theorem 2.2]).

It is important in topology and in algebraic geometry to determine whether an F-vector bundle ζ is stably equivalent to a sum of F-line bundles. Let $F = \boldsymbol{R}$ or $\boldsymbol{C}$ and let ζ be an F-vector bundle of dimension t over the projective n-space RP^n. Then ζ is stably equivalent to a sum of t F-line bundles if and only if ζ is stably extendible to RP^m for every $m > n$ (Theorem 3.2). So in this paper we firstly study the problem: Determine the condition that an F-vector bundle over RP^n is extendible (or stably extendible) to RP^m for every $m > n$. As for the problem, several results have been obtained (cf. [**5**]–[**10**], [**12**] and [**15**]).

2000 *Mathematics Subject Classification.* Primary 55R50; Secondary 57R25.
Key Words and Phrases. Vector bundle, extendible, stably extendible, tangent bundle, span, immersion, normal bundle, K-theory, KO-theory, real projective space.

Let $\tau(RP^n)$ denote the tangent bundle of RP^n. We have proved in [**6**, Theorem 6.6] and [**8**, Theorem 4.2] that *the following three conditions are equivalent*:

(1) $\tau(RP^n)$ *is extendible to* RP^m *for every* $m > n$.

(2) $\tau(RP^n)$ *is stably extendible to* RP^m *for every* $m > n$.

(3) $n = 1, 3$ *or* 7.

For an $\boldsymbol{R}$-vector bundle ζ, denote by $c\zeta$ the complexification of ζ. Then we obtain

THEOREM 1. *The following three conditions are equivalent*:

(1) $c\tau(RP^n)$ *is extendible to* RP^m *for every* $m > n$.

(2) $c\tau(RP^n)$ *is stably extendible to* RP^m *for every* $m > n$.

(3) $1 \leqq n \leqq 5$ *or* $n = 7$.

Let ξ_n be the canonical $\boldsymbol{R}$-line bundle over RP^n. Then we have

THEOREM 2. *Let* $\nu(f)$ *be the normal bundle associated to an immersion* f *of* RP^n *in the Euclidean* $(n+k)$*-space* $\boldsymbol{R}^{n+k}$, *where* $k > 0$. *Then the following hold.*

(i) $\nu(f)$ *is stably extendible to* RP^m *for every* $m > n$ *if and only if* $\nu(f)$ *is stably equivalent to* $s\xi_n$ *for some integer* s *with* $0 \leqq s \leqq k$.

(ii) $c\nu(f)$ *is stably extendible to* RP^m *for every* $m > n$ *if and only if* $c\nu(f)$ *is stably equivalent to* $sc\xi_n$ *for some integer* s *with* $0 \leqq s \leqq k$.

Secondly, we study the problem: Determine the dimension m for which an F-vector bundle over RP^n is extendible to RP^m. The answer for $c\tau(RP^n)$ is obtained in [**5**, Theorem 3] (and [**12**, Theorem 4.1]) as follows:

If $n = 6$ *or* $n > 7$, $c\tau(RP^n)$ *is extendible to* RP^{2n+1}, *but is not stably extendible to* RP^{2n+2}.

For an $\boldsymbol{R}$-vector bundle α, denote by $\operatorname{span}\alpha$ the maximum of the number of cross-sections of α which are nowhere linearly dependent. For a differentiable manifold M, let $\operatorname{span} M$ stand for $\operatorname{span}\tau(M)$, where $\tau(M)$ is the tangent bundle of M. Let $\phi(n)$ denote the number of integers s such that $0 < s \leqq n$ and $s \equiv 0, 1, 2,$ or $4 \mod 8$, and let $N(n) = 2^{\phi(n)} - n - 2$. Then we have the following table for $1 \leqq n \leqq 11$.

n	1	2	3	4	5	6	7	8	9	10	11
$\phi(n)$	1	2	2	3	3	3	3	4	5	6	6
$N(n)$	-1	0	-1	2	1	0	-1	6	21	52	51

We prove

THEOREM 3. *Let* $\nu(f)$ *be the normal bundle associated to an immersion* f *of* RP^n *in* $\boldsymbol{R}^{n+k}$, *and let* $N(= N(n)) = 2^{\phi(n)} - n - 2$. *Suppose* $0 < k < N + 1 \leqq \operatorname{span} RP^N + k + 1$. *Then* $\nu(f)$ *is extendible to* RP^N, *but is not stably extendible to* RP^{N+1}.

This paper is arranged as follows. We prove Theorems 1, 2 and 3 in Sections 2, 3 and 4 respectively. In Section 5 we give some examples of Theorem 3 and give a proof of Theorem 4.4 that is stated in Section 4 and is used for the proof of Theorem 3.

The authors would like to thank the referee for his valuable comments and suggestions.

2. Proof of Theorem 1.

The following is clear by definition.

LEMMA 2.1. *Let A be a subspace of a space X, and let ζ and η be stably equivalent F-vector bundles of same dimension over A, where $F = \boldsymbol{R}, \boldsymbol{C}$, or $\boldsymbol{H}$. Then ζ is stably extendible to X if and only if η is stably extendible to X.*

In the following, we use the same letter for a vector bundle and its equivalence class. Let d denote $\dim_{\boldsymbol{R}} F$, where $F = \boldsymbol{R}, \boldsymbol{C}$, or $\boldsymbol{H}$. Then the following fact is known.

THEOREM 2.2 (cf. [**2**, Theorem 1.5, p. 100]). *If α and β are two t-dimensional F-vector bundles over an n-dimensional CW-complex X such that $\langle (n+2)/d - 1 \rangle \leqq t$ and $\alpha \oplus k = \beta \oplus k$ for some k-dimensional trivial F-bundle k over X, then $\alpha = \beta$, where $\langle x \rangle$ denotes the smallest integer m with $x \leqq m$.*

Let ξ_n denote the canonical $\boldsymbol{R}$-line bundle over RP^n and $c\xi_n$ its complexification.

PROOF OF THEOREM 1 (cf. [**12**, Theorem 4.1]). It is clear that (1) implies (2). That (2) implies (3) is proved in [**5**, Theorem 3]. Hence it suffices to prove that (3) implies (1).

Note that the condition $1 \leqq n \leqq 5$ or $n = 7$ is equivalent to the condition $2^{[n/2]} \leqq n+1$, where $[x]$ denotes the integral part of a real number x.

Complexifying the equality $\tau(RP^n) \oplus 1 = (n+1)\xi_n$, and using the fact that $c\xi_n - 1$ is of order $2^{[n/2]}$ (cf. [**1**, Theorem 7.3]), we have

$$c\tau(RP^n) = (n+1)c\xi_n - 1 = (n+1-2^{[n/2]})c\xi_n + 2^{[n/2]} - 1$$

in $K(RP^n)$. Here $n+1-2^{[n/2]} \geqq 0$ and $2^{[n/2]} - 1 \geqq 0$. As $\langle (n+2)/2 - 1 \rangle \leqq \dim c\tau(RP^n)$, we have, by Theorem 2.2,

$$c\tau(RP^n) = (n+1-2^{[n/2]})c\xi_n \oplus (2^{[n/2]} - 1).$$

Since $c\xi_n$ and the trivial bundle are both extendible to RP^m for any $m > n$, so is $c\tau(RP^n)$. □

3. Proof of Theorem 2.

The following "stably extendible version" of Theorem 6.5 in [**6**] is obtained from (2.3) in [**9**] which is the "stably extendible version" of Theorem 6.2 in [**6**]. For completeness we give a proof.

THEOREM 3.1. *Let ζ be a t-dimensional $\boldsymbol{R}$-vector bundle over RP^n. Then the following hold.*

(i) *For $n \neq 1, 3, 7$, ζ is stably equivalent to a sum of t $\boldsymbol{R}$-line bundles if ζ is stably extendible to RP^N, where $N = 2^{\phi(n)} - 1$.*

(ii) *For $n = 1, 3$, or 7, ζ is stably equivalent to a sum of t $\boldsymbol{R}$-line bundles.*

PROOF. There is an integer ℓ such that

$$\zeta - t = (t + \ell)(\xi_n - 1) \in \widetilde{KO}(RP^n).$$

Since $\xi_n - 1$ is of order $2^{\phi(n)}$ (cf. [**1**, Theorem 7.4]), we have $0 \leqq t + \ell < 2^{\phi(n)}$. If $\ell > 0$, $n < t + \ell$ and ζ is not stably extendible to $RP^{t+\ell}$ by (2.3) in [**9**]. If $n \neq 1, 3, 7$, the latter contradicts the assumption of (i), and if $n = 1, 3$, or 7, the former contradicts. We therefore have $\ell \leqq 0$. Hence we obtain (i) and (ii). □

Using Theorem 3.1, we have

Theorem 3.2. *Let $F = \boldsymbol{R}$ or $\boldsymbol{C}$, and let ζ be a t-dimensional F-vector bundle over RP^n. Then ζ is stably extendible to RP^m for every $m > n$ if and only if ζ is stably equivalent to a sum of t F-line bundles.*

Proof. For $F = \boldsymbol{R}$, the "only if" part follows from Theorem 3.1 (or from the "stably extendible version" of Corollary to Theorem 3 in [**15**]). For a positive integer n and a group G, let $K(G, n)$ denote the Eilenberg-MacLane space of type (G, n), let $BO(n)$ and $BU(n)$ denote the classifying spaces for the orthogonal group $O(n)$ and the unitary group $U(n)$ respectively, and $[X, Y]$ denote the set of all homotopy classes of continuous maps (not necessarily base point preserving) from X to Y. Then we have

$$[RP^n, BO(1)] = [RP^n, K(\boldsymbol{Z}/2, 1)] = H^1(RP^n; \boldsymbol{Z}/2) = \boldsymbol{Z}/2.$$

Hence $\boldsymbol{R}$-line bundles over RP^n are ξ_n and the trivial $\boldsymbol{R}$-line bundle. Since they are extendible to RP^m for every $m > n$, a t-dimensional $\boldsymbol{R}$-vector bundle which is stably equivalent to a sum of t $\boldsymbol{R}$-line bundles is stably extendible to RP^m for every $m > n$, by Lemma 2.1. This proves the "if" part for $F = \boldsymbol{R}$.

For $F = \boldsymbol{C}$, the "only if" part is Theorem A in [**8**]. For $n \geqq 2$, we have

$$[RP^n, BU(1)] = [RP^n, K(\boldsymbol{Z}, 2)] = H^2(RP^n; \boldsymbol{Z}) = \boldsymbol{Z}/2.$$

Hence $\boldsymbol{C}$-line bundles over RP^n for $n \geqq 2$ are $c\xi_n$ and the trivial $\boldsymbol{C}$-line bundle. Clearly any $\boldsymbol{C}$-line bundle over RP^1 $(= S^1)$ is trivial. Since $c\xi_n$ and the trivial $\boldsymbol{C}$-line bundle are extendible to RP^m for every $m > n$, a t-dimensional $\boldsymbol{C}$-vector bundle which is stably equivalent to a sum of t $\boldsymbol{C}$-line bundles is stably extendible to RP^m for every $m > n$, by Lemma 2.1. This proves the "if" part for $F = \boldsymbol{C}$. □

Proof of Theorem 2. (i) If $\nu(f)$ is stably equivalent to $s\xi_n$ for some integer s with $0 \leqq s \leqq k$, $\nu(f)$ is stably equivalent to $s\xi_n \oplus (k - s)$ and $\dim \nu(f) = k = \dim(s\xi_n \oplus (k - s))$. Hence the "if" part follows from Lemma 2.1. The "only if" part follows from that of Theorem 3.2 for $F = \boldsymbol{R}$.

(ii) The proof is similar to that of (i). □

4. Proof of Theorem 3.

Let $N = N(n) = 2^{\phi(n)} - n - 2$. The following is Theorem 3.5 in [**10**] (cf. also [**10**, (3.3)]).

Theorem 4.1. *Let $\nu(f)$ be the normal bundle associated to an immersion f of RP^n in $\boldsymbol{R}^{n+k}$. Suppose $0 < k < N + 1$. Then $n < N + 1$ (that is, $n \geqq 9$) and $\nu(f)$ is not stably extendible to RP^m for any $m \geqq \min\{N + 1, n + k + 1\}$.*

We have

THEOREM 4.2. *Let f be an immersion of RP^n in $\boldsymbol{R}^{n+k}$ and m an integer with $m \geqq n$. Provided $0 < k < N+1$ and $\operatorname{span}(N+1)\xi_m \geqq N-k+1$, the normal bundle $\nu(f)$ associated to f is stably extendible to RP^m. If $n < k$, in addition, $\nu(f)$ is extendible to RP^m.*

PROOF. Since $\tau(RP^n) \oplus \nu(f) = n+k$ and $\tau(RP^n) \oplus 1 = (n+1)\xi_n$, $(n+1)\xi_n \oplus \nu(f) = n+k+1$. Using the fact that $\xi_n - 1$ is of order $2^{\phi(n)}$ (cf. [**1**, Theorem 7.4]), we have

$$\nu(f) + (N-k+1) = (N+1)\xi_n$$

in $KO(RP^n)$. Let $i : RP^n \to RP^m$ be the standard inclusion. Then

$$(N+1)\xi_n = (N+1)i^*\xi_m = i^*((N+1)\xi_m).$$

By the assumption, there is a k-dimensional $\boldsymbol{R}$-vector bundle α over RP^m such that $(N+1)\xi_m = \alpha \oplus (N-k+1)$. Hence

$$\nu(f) + (N-k+1) = (i^*\alpha) \oplus (N-k+1),$$

and $\nu(f)$ is stably equivalent to $i^*\alpha$. So $\nu(f)$ is stably extendible to RP^m, since $\dim \nu(f) = k = \dim \alpha$.

The latter part follows from [**10**, Theorem 2.2]. □

The following is proved in [**14**, Theorem 2.4] (cf. also [**11**]).

THEOREM 4.3. $\operatorname{span}(n+1)\xi_n = \operatorname{span} RP^n + 1$.

The proof of the following theorem will be given in the next section.

THEOREM 4.4. *If $n \geqq 9$, $\operatorname{span} RP^N < N - n$, where $N = 2^{\phi(n)} - n - 2$.*

PROOF OF THEOREM 3. The latter part follows from Theorem 4.1.

By Theorem 4.1, it follows from the inequalities $0 < k < N+1$ that $n \leqq N$, namely $n \geqq 9$. Furthermore, it follows from the inequality $N+1 \leqq \operatorname{span} RP^N + k + 1$ that $N+1 \leqq \operatorname{span}(N+1)\xi_N + k$ by Theorem 4.3, and that, for $n \geqq 9$, $N+1 < N-n+k+1$, namely $n < k$, by Theorem 4.4. Hence we have the latter part of Theorem 3, by setting $m = N$ in Theorem 4.2. □

5. Examples.

We give some examples of Theorem 3.

The following is well-known (cf. [**1**], [**13**] and [**4**]).

THEOREM 5.1. *Write $n+1 = (2a+1)2^{c+4d}$, where a, c and d are non-negative integers and $0 \leqq c \leqq 3$. Then $\operatorname{span} RP^n = \operatorname{span} S^n = 2^c + 8d - 1$.*

We have

PROPOSITION 5.2. *Let $\nu(f)$ be the normal bundle associated to an immersion f of RP^n in $\boldsymbol{R}^{n+k}$, where $k = 20$ or 21 if $n = 9$, $k = 52$ if $n = 10$, and $k = 48, 49, 50$ or 51 if $n = 11$. Then $\nu(f)$ is extendible to RP^N but not stably extendible to RP^{N+1}, where $N = 21$ if $n = 9$, $N = 52$ if $n = 10$ and $N = 51$ if $n = 11$.*

Proof. Let us consider the case $n = 9, 10$ or 11. Since $N = 2^{\phi(n)} - n - 2$, we have $N = 21$ if $n = 9$, $N = 52$ if $n = 10$ and $N = 51$ if $n = 11$, and conclude $\operatorname{span} RP^N = 1$ if $n = 9$, $\operatorname{span} RP^N = 0$ if $n = 10$ and $\operatorname{span} RP^N = 3$ if $n = 11$, from Theorem 5.1. Therefore, the assumption

$$0 < k < N + 1 \leqq \operatorname{span} RP^N + k + 1$$

in Theorem 3 is equivalent to

$$k < 22 \leqq k + 2 \text{ if } n = 9, \quad k < 53 \leqq k + 1 \text{ if } n = 10, \quad k < 52 \leqq k + 4 \text{ if } n = 11,$$

namely

$$k = 20, 21 \text{ if } n = 9, \quad k = 52 \text{ if } n = 10, \quad k = 48, 49, 50, 51 \text{ if } n = 11.$$

Hence, the proposition follows from Theorem 3. □

Finally, we give a proof of Theorem 4.4 in the previous section. We prepare two lemmas for the proof.

Lemma 5.3. *$\operatorname{span} RP^n (= \operatorname{span} S^n) < n/2$ if and only if $n \neq 1, 3, 7, 15$.*

Proof. Write $n + 1 = (2a + 1)2^{c+4d}$, where a, c and d are non-negative integers and $0 \leqq c \leqq 3$. Then, by Theorem 5.1, $\operatorname{span} RP^n = 2^c + 8d - 1$. Hence we have $\operatorname{span} RP^n < n/2$ if and only if $2^{c+1} + 16d - 1 < (2a + 1)2^{c+4d}$. We see easily that the inequality above holds if and only if $(a, c, d) \neq (0, 0, 1), (0, 1, 0), (0, 2, 0), (0, 3, 0)$, that is, $n \neq 15, 1, 3, 7$. □

Lemma 5.4. *If $n \geqq 9$, $3n + 2 < 2^{\phi(n)}$.*

Proof. Let $n = 8k + r$, where k is a positive integer and r is an integer with $0 \leqq r \leqq 7$. The inequality $3n + 2 < 2^{\phi(n)}$ holds for every n with $9 \leqq n \leqq 16$ clearly. Assume that the inequality $3n + 2 < 2^{\phi(n)}$ holds for some n with $n \geqq 9$. Then

$$\begin{aligned} 2^{\phi(n+8)} - (3(n+8) + 2) &= 2^{\phi(n)+4} - (3n + 26) = 16 \cdot 2^{\phi(n)} - (3n + 26) \\ &> 16(3n + 2) - (3n + 26) = 45n + 6 > 0. \end{aligned}$$

Hence the result follows by induction on n. □

Proof of Theorem 4.4. Assume $n \geqq 9$. Then $3n + 2 < 2^{\phi(n)}$ by Lemma 5.4. Hence $N = 2^{\phi(n)} - n - 2 > 2n \geqq 18$, and so $\operatorname{span} RP^N < N/2$ by Lemma 5.3. On the other hand, $N/2 < 2^{\phi(n)} - 2n - 2$ for $n \geqq 9$. Therefore we have $\operatorname{span} RP^N < 2^{\phi(n)} - 2n - 2$ if $n \geqq 9$, as desired. □

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