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Convergence of the Feynman path integral in the weighted Sobolev spaces and the representation of correlation functions

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Abstract. There are many ways to give a rigorous meaning to the Feynman path integral. In the present paper especially the method of the time-slicing approximation determined through broken line paths is studied. It was proved that these time-slicing approximate integrals of the Feynman path integral in configuration space and also in phase space converge in L^2 space as the discretization parameter tends to zero. In the present paper it is shown that these time-slicing approximate integrals converge in some weighted Sobolev spaces as well. Next as an application of this convergence result in the weighted Sobolev spaces, the path integral representation of correlation functions is studied of the position and the momentum operators. We note that their path integral representation is given in phase space. It is shown that the approximate integrals of correlation functions converge or diverge as the discretization parameter tends to zero. We note that the divergence of the approximate integrals reflects the uncertainty principle in quantum mechanics.

1. Introduction.

It was an interested and important problem to give the description of quantization, i.e. of passing from classical physical systems to the corresponding quantum ones, from the moment that quantum mechanics came into existence. In the end Heisenberg and Schrödinger succeeded in giving the description based on the notion of operators. On the other hand in 1948 Feynman proposed an essentially new description in [**6**] based on the notion of the so-called Feynman path integrals. His description is that the probability amplitudes can be constructed from the classical systems in a direct way with the physical meaning. In 1951 Feynman himself gave the description reformulated by means of the path integrals in phase space in [**7**]. Now we know that his description is very useful and applied to wide areas in physics (cf. [**14**], [**28**]).

Since Feynman published his papers, many ways have been proposed and much work has been done to give a rigorous meaning to the Feynman path integral: the method of analytic continuation from Wiener integrals, the formulation by means of the product formula of Kato and Trotter, the formulation by means of an improper integral in the Hilbert manifold of paths, the formulation by means of pseudomeasures, the method of the time-slicing approximation determined through piecewise classical paths and so on. See [**1**], [**3**], [**4**], [**5**], [**9**], [**10**], [**11**], [**12**], [**13**], [**20**], [**21**], [**23**], [**29**], [**32**] and their references.

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In the present paper we study especially the method of the time-slicing approximation determined through broken line paths of the Feynman path integral. This method is very simple and very familiar in physics (cf. [**8**], [**14**], [**28**], [**29**]). Let $L^2 = L^2(\boldsymbol{R}^n)$ be the space of all square integrable functions in $\boldsymbol{R}^n$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Truman in [**30**] first studied this method, using an improper integral in the Hilbert manifold of paths and proved that the time-slicing approximate integrals converge in L^2 as the discretization parameter tends to zero. His result was generalized by the author in [**16**], [**17**] by means of the theory of the oscillatory integral operators and the pseudo-differential operators. In addition, the author defined in [**18**], [**19**] the time-slicing approximate integrals in phase space that are proved to be equal to the time-slicing approximate integrals in configuration space. Thus we know that the time-slicing approximate integrals, determined through broken line paths in configuration space and also in phase space, converge in L^2 space as the discretization parameter tends to zero.

One of our aims in the present paper is to show that the time-slicing approximate integrals stated above, i.e. determined through broken line paths in configuration space and also in phase space, converge in the weighted Sobolev spaces $B^a := \{f \in L^2; \|f\|_{B^a} := \|f\| + \sum_{|\alpha|=a}(\|x^\alpha f\| + \|\partial_x^\alpha f\|) < \infty\}$ $(a = 1, 2, \dots)$ as well (Theorem 1 in the present paper). Here for an $x = (x_1, \dots, x_n) \in \boldsymbol{R}^n$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ we write $|\alpha| = \sum_{j=1}^n \alpha_j$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. We write $B^0 = L^2$. Next as an application of our Theorem 1 we study the approximate integrals of correlation functions of the position and the momentum operators. We note that these approximate integrals are defined in phase space. As is well known, correlation functions are some of the most important quantities in quantum mechanics and quantum field theory (cf. [**24**], [**28**]). We prove the convergence or the divergence of the approximate integrals of correlation functions as the discretization parameter tending to zero (Theorem 2 in the present paper). The second aim in the present paper is to show this. We note that the divergence of the approximate integrals of correlation functions reflects the uncertainty principle in quantum mechanics.

The outline of the proof of our main theorems is as follows. The approximate integral of the Feynman path integral is determined correspondingly to each subdivision of the time interval. We consider the family of all approximate integrals. We first show the uniform boundedness of the family of approximate integrals in B^a $(a = 0, 1, \dots)$ (Theorem 3.5 in the present paper). This result is essential in our proof. By means of this result of the boundedness we show the equicontinuity of the family of approximate integrals in B^a on the finite time interval. Then, by applying the abstract Ascoli-Arzelà theorem we can prove our Theorem 1, i.e. the convergence of the approximate integrals of the Feynman path integral in B^a. We note that our method of proving convergence is direct, compared to that in [**16**]–[**19**], in the sense that we don't use the result of the corresponding Schrödinger equation. Our Theorem 2, i.e. the convergence or the divergence of the approximate integrals of correlation functions is proved by means of Theorem 1 and studying further the oscillatory integral operators in phase space. We note that we use the results in the preceding papers by the author, especially in [**15**], [**17**] for the proof.

The plan of the present paper is as follows. In §2 we state the main results and some remarks. §3 and §4 are devoted to the proofs of the uniform boundedness and the equicontinuity of the approximate integrals of the Feynman path integral in B^a, respectively. In §5 we prove Theorem 1. Theorem 2 is proved in §6.

2. Main Theorems.

We consider some charged non-relativistic particles in an electromagnetic field. For the sake of simplicity we suppose the charge and the mass of every particle to be one and $m > 0$, respectively. We consider $x \in \boldsymbol{R}^n$ and $t \in [0, T]$. Let $E(t,x) = (E_1, \dots, E_n) \in \boldsymbol{R}^n$ and $(B_{jk}(t,x))_{1 \le j<k \le n} \in \boldsymbol{R}^{n(n-1)/2}$ denote electric strength and magnetic strength tensor, respectively and $(V(t,x), A(t,x)) = (V, A_1, \dots, A_n) \in \boldsymbol{R}^{n+1}$ an electromagnetic potential, i.e.

$$E = -\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x},$$

$$d\left(\sum_{j=1}^{n} A_j\, dx_j\right) = \sum_{1 \le j<k \le n} B_{jk}\, dx_j \wedge dx_k \quad \text{on } \boldsymbol{R}^n, \tag{2.1}$$

where $\partial V/\partial x = (\partial V/\partial x_1, \dots, \partial V/\partial x_n)$. Then the Lagrangian function $\mathscr{L}(t,x,\dot{x})$ $(\dot{x} \in \boldsymbol{R}^n)$ is given by

$$\mathscr{L}(t,x,\dot{x}) = \frac{m}{2}|\dot{x}|^2 + \dot{x} \cdot A - V. \tag{2.2}$$

The Hamiltonian function $\mathscr{H}(t,x,p)$ $(p \in \boldsymbol{R}^n)$ is defined through the Legendre transformation of $\mathscr{L}$ by

$$\mathscr{H}(t,x,p) = \frac{1}{2m}|p - A|^2 + V. \tag{2.3}$$

Let $T^*\boldsymbol{R}^n = \boldsymbol{R}^n_x \times \boldsymbol{R}^n_p$ denote the phase space, and $(\boldsymbol{R}^n)^{[s,t]}$ and $(T^*\boldsymbol{R}^n)^{[s,t]}$ the spaces of all paths $q : [s,t] \ni \theta \to q(\theta) \in \boldsymbol{R}^n$ and $(q,p) : [s,t] \ni \theta \to (q(\theta), p(\theta)) \in T^*\boldsymbol{R}^n$, respectively. The classical actions $S_c(t,s;q)$ for $q \in (\boldsymbol{R}^n)^{[s,t]}$ in configuration space and $S(t,s;q,p)$ for $(q,p) \in (T^*\boldsymbol{R}^n)^{[s,t]}$ in phase space are given by

$$S_c(t,s;q) = \int_s^t \mathscr{L}(\theta, q(\theta), \dot{q}(\theta))\, d\theta, \quad \dot{q}(\theta) = \frac{dq}{d\theta}(\theta) \tag{2.4}$$

and

$$S(t,s;q,p) = \int_s^t p(\theta) \cdot \dot{q}(\theta) - \mathscr{H}(\theta, q(\theta), p(\theta))\, d\theta, \tag{2.5}$$

respectively (cf. [**2**]).

Let $\Delta : 0 = \tau_0 < \tau_1 < \cdots < \tau_\nu = T$ be a subdivision of the interval $[0,T]$. We set $|\Delta| = \max_{1 \le j \le \nu}(\tau_j - \tau_{j-1})$. Let $0 \le s \le t \le T$ and $f \in C_0^\infty(\boldsymbol{R}^n)$, where $C_0^\infty(\boldsymbol{R}^n)$ is the space of all infinitely differentiable functions in $\boldsymbol{R}^n$ with compact support. For Δ above we define the time-slicing approximate integrals $\mathscr{C}_\Delta(t,s)f$ and $G_\Delta(t,s)f$ of the Feynman path integrals in configuration space and in phase space respectively as follows.

At first we define $\mathscr{C}_\Delta(t,s)f$. We set $\mathscr{C}_\Delta(s,s)f = f$. Let $0 \le s < t \le T$. We take $1 \le \mu' \le \mu \le \nu$ such that $\tau_{\mu'-1} \le s < \tau_{\mu'}$ and $\tau_{\mu-1} < t \le \tau_\mu$. For $y, x^{(j)}$ $(j = \mu', \mu'+1, \dots, \mu-1)$ and x in $\boldsymbol{R}^n$ let's define $q_\Delta(\theta; y, x^{(\mu')}, \dots, x^{(\mu-1)}, x) \in (\boldsymbol{R}^n)^{[s,t]}$ by the broken line path joining points y at s, $x^{(j)}$ at τ_j $(j = \mu', \mu'+1, \dots, \mu-1)$ and x at t in order. We define $\mathscr{C}_\Delta(t,s)f$ by

$$(\mathscr{C}_\Delta(t,s)f)(x) = \sqrt{\frac{m}{2\pi i\hbar(t-\tau_{\mu-1})}}^{\,n}\left(\prod_{j=\mu'+1}^{\mu-1}\sqrt{\frac{m}{2\pi i\hbar(\tau_j-\tau_{j-1})}}^{\,n}\right)\sqrt{\frac{m}{2\pi i\hbar(\tau_{\mu'}-s)}}^{\,n}$$
$$\times \operatorname{os}-\int\cdots\int(\exp i\hbar^{-1}S_c(t,s;q_\Delta))f(y)\,dy\,dx^{(\mu')}\cdots dx^{(\mu-1)}. \tag{2.6}$$

Here $\operatorname{os}-\int\cdots\int g(y, x^{(\mu')}, \dots, x^{(\mu-1)})\,dy\,dx^{(\mu')}\cdots dx^{(\mu-1)}$ means the oscillatory integral (cf. [**22**]).

We define $G_\Delta(t,s)$. For the sake of simplicity we set $s = 0$. The general case can be defined in the same way that $\mathscr{C}_\Delta(t,s)$ was done. We set $G_\Delta(0,0)f = f$. For $0 < t \le T$ take a $1 \le \mu \le \nu$ such that $\tau_{\mu-1} < t \le \tau_\mu$. For $v^{(j)} \in \boldsymbol{R}^n$ $(j = 0, 1, \dots, \mu-1)$ in velocity space we define $v_\Delta(\theta; v^{(0)}, \dots, v^{(\mu-1)}) \in (\boldsymbol{R}^n)^{[0,t]}$ in velocity space by the piecewise constant path taking $v^{(0)}$ at $\theta = 0$, $v^{(j)}$ for $\tau_j < \theta \le \tau_{j+1}$ $(j = 0, 1, \dots, \mu-2)$ and $v^{(\mu-1)}$ for $\tau_{\mu-1} < \theta \le t$. Let $q_\Delta(\theta; x^{(0)}, \dots, x^{(\mu-1)}, x) \in (\boldsymbol{R}^n)^{[0,t]}$ $(x^{(0)} = y)$ be the path in configuration space defined above. Then we determine the path $p_\Delta(\theta; x^{(0)}, \dots, x^{(\mu-1)}, x, v^{(0)}, \dots, v^{(\mu-1)}) \in (\boldsymbol{R}^n)^{[0,t]}$ in momentum space by

$$p_\Delta(\theta) := \frac{\partial\mathscr{L}}{\partial\dot{x}}(\theta, q_\Delta(\theta), v_\Delta(\theta)) = mv_\Delta(\theta) + A(\theta, q_\Delta(\theta)). \tag{2.7}$$

We define $G_\Delta(t,0)f$ by

$$(G_\Delta(t,0)f)(x) = (2\pi\hbar/m)^{-n\mu}\operatorname{os}-\int\cdots\int(\exp i\hbar^{-1}S(t,0;q_\Delta,p_\Delta))$$
$$\times f(x^{(0)})\,dv^{(0)}dx^{(0)}dv^{(1)}dx^{(1)}\cdots dv^{(\mu-1)}dx^{(\mu-1)}. \tag{2.8}$$

In [**16**]–[**18**] we proved the following.

Theorem A. *Let $\partial_x^\alpha E_j(t,x)$ $(j = 1, 2, \dots, n)$, $\partial_x^\alpha B_{jk}(t,x)$ and $\partial_t B_{jk}(t,x)$ $(1 \le j < k \le n)$ be continuous in $[0,T] \times \boldsymbol{R}^n$ for all α. We suppose*

$$|\partial_x^\alpha E_j(t,x)| \le C_\alpha, \quad |\alpha| \ge 1, \qquad |\partial_x^\alpha B_{jk}(t,x)| \le C_\alpha\langle x\rangle^{-(1+\delta)}, \quad |\alpha| \ge 1 \tag{2.9}$$

in $[0,T] \times \boldsymbol{R}^n$ for some constants $\delta > 0$ and C_α, where $\langle x\rangle = \sqrt{1+|x|^2}$ and δ is independent of α. Then there exists a constant $\rho^* > 0$ such that we have for an arbitrary potential (V, A) with continuous $V, \partial V/\partial x_j, \partial A_j/\partial t$ and $\partial A_j/\partial x_k$ $(j, k = 1, 2, \dots, n)$ in $[0,T] \times \boldsymbol{R}^n$: (1) Let $|\Delta| \le \rho^*$. Then both of $\mathscr{C}_\Delta(t,s)$ and $G_\Delta(t,s)$ on C_0^∞ are well-defined and can be extended to bounded operators on L^2. They are equal to one another. (2) Let $|\Delta| \le \rho^*$. Then there exists a constant $K \ge 0$ independent of Δ such that

$$\|\mathscr{C}_\Delta(t,s)f\| \le e^{K(t-s)}\|f\|, \quad 0 \le s \le t \le T \tag{2.10}$$

for all $f \in L^2$. (3) *As* $|\Delta| \to 0$, $\mathscr{C}_\Delta(t,s)f$ *for* $f \in L^2$ *converges in* L^2 *uniformly in* $0 \le s \le t \le T$ *and this limit satisfies the Schrödinger equation*

$$i\hbar\frac{\partial}{\partial t}u(t) = H(t)u(t), \quad u(s) = f, \tag{2.11}$$

where

$$H(t) = \frac{1}{2m}\sum_{j=1}^{n}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - A_j\right)^2 + V. \tag{2.12}$$

We note that in fact the constant ρ^* in Theorem A is defined from Proposition 3.1 stated later in the present paper. This constant $\rho^* > 0$ is fixed throughout the present paper. We write $\int(\exp i\hbar^{-1}S_c(t,s;q))f(q(s))\mathscr{D}q$ and $\iint(\exp i\hbar^{-1}S(t,s;q,p))f(q(s))\mathscr{D}p\mathscr{D}q$ for the limit of $\mathscr{C}_\Delta(t,s)f$ and $G_\Delta(t,s)f$ as $|\Delta| \to 0$, respectively.

Remark 2.1. In (2.8) we make the change of variables: $\boldsymbol{R}^{n\mu} \ni (v^{(0)},\dots,v^{(\mu-1)}) \to (p^{(0)},\dots,p^{(\mu-1)}) \in \boldsymbol{R}^{n\mu}$, setting $p^{(j)} = mv^{(j)} + A(\tau_j, x^{(j)})$. Then $G_\Delta(t,0)f$ is written

$$(G_\Delta(t,0)f)(x) = (2\pi\hbar)^{-n\mu}\text{os}-\int\cdots\int(\exp i\hbar^{-1}S(t,0;q_\Delta,p_\Delta)) \times f(x^{(0)})\,dp^{(0)}dx^{(0)}dp^{(1)}dx^{(1)}\cdots dp^{(\mu-1)}dx^{(\mu-1)}$$

in the form of an integral on the phase space as a product space.

Remark 2.2. In Theorem A only smooth electromagnetic fields are considered. We can apply Theorem A as follows to the case that electromagnetic fields have singularities. For example consider atomic Hamiltonians

$$H = -\frac{\hbar^2}{2m}\sum_{j=1}^{n}\Delta_j - \sum_{j=1}^{n}\frac{n}{|x^{(j)}|} + \sum_{1\le j<k\le n}\frac{1}{|x^{(j)}-x^{(k)}|},$$

where $x^{(j)} \in \boldsymbol{R}^3$ and Δ_j denotes the Laplacian in $x^{(j)}$. Let χ_l $(l = 1,2,\dots)$ be real valued infinitely differentiable functions in $\boldsymbol{R}^3$ such that $\sup_{x\in\boldsymbol{R}^3}|\partial_x^\alpha\chi_l(x)| < \infty$ for $|\alpha| \ge 2$ and

$$\lim_{l\to\infty}\chi_l(x) = -\frac{1}{|x|} \quad \text{in } L^2(\boldsymbol{R}^3) + L^\infty(\boldsymbol{R}^3).$$

We set

$$H_l = -\frac{\hbar^2}{2m}\sum_{j=1}^{n}\Delta_j + \sum_{j=1}^{n}n\chi_l(x^{(j)}) - \sum_{1\le j<k\le n}\chi_l(x^{(j)}-x^{(k)}).$$

We know that $e^{-i\hbar^{-1}(t-s)H_l}$ converges to $e^{-i\hbar^{-1}(t-s)H}$ strongly in L^2 as $l \to \infty$. See Example 2 of §X.2 in [**26**] and Theorems VIII.21, VIII.25 in [**25**], and also see [**31**]. It follows from Theorem A in the present paper that $e^{-i\hbar^{-1}(t-s)H_l}f$ for $f \in L^2$ can be written in the form of our path integrals. So we see that $e^{-i\hbar^{-1}(t-s)H}f$ can be written in the form of the limit of our path integrals. The same argument can be applied to the general case of electromagnetic fields having singularities.

The following is the first main theorem in the present paper.

Theorem 1. *Besides the assumptions of Theorem A we suppose*

$$|\partial_x^\alpha A_j| \le C_\alpha, \quad |\alpha| \ge 1, \qquad |\partial_x^\alpha V| \le C_\alpha \langle x \rangle, \quad |\alpha| \ge 1 \tag{2.13}$$

for $j = 1, 2, \ldots, n$ *in* $[0, T] \times \boldsymbol{R}^n$. *Let* $a = 0, 1, \ldots$ *and* $|\Delta| \le \rho^*$. *Then we have*: (1) *There exists a constant* $K_a \ge 0$ *independent of* Δ *such that*

$$\|\mathscr{C}_\Delta(t,s)f\|_{B^a} \le e^{K_a(t-s)}\|f\|_{B^a}, \quad 0 \le s \le t \le T \tag{2.14}$$

for all $f \in B^a$. *In addition,* $\mathscr{C}_\Delta(t,s)f$ *for* $f \in B^a$ *is continuous as a* B^a*-valued function in* $0 \le s \le t \le T$. (2) *As* $|\Delta| \to 0$, $\mathscr{C}_\Delta(t,s)f$ *for* $f \in B^a$ *converges to the solution of* (2.11) *in* B^a *uniformly in* $0 \le s \le t \le T$.

Remark 2.3. Suppose that E_j $(j = 1, 2, \ldots, n)$ and B_{jk} $(1 \le j < k \le n)$ satisfy the assumptions of Theorem A. Then we can find a potential (V', A') satisfying (2.13), which is proved in Lemma 6.1 of [**17**]. We define $\mathscr{C}_\Delta(t,s)'$ by (2.6) for these (V', A'). Let (V, A) be an arbitrary potential such that $V, \partial V/\partial x_j, \partial A_j/\partial t$ and $\partial A_j/\partial x_k$ $(j, k = 1, 2, \ldots)$ are continuous in $[0, T] \times \boldsymbol{R}^n$. Then we have a continuously differentiable function $\psi(t, x)$ in $[0, T] \times \boldsymbol{R}^n$ such that

$$-V'\,dt + \sum_{j=1}^n A_j'\,dx_j = -V\,dt + \sum_{j=1}^n A_j\,dx_j + d\psi.$$

So we have from (2.6)

$$\mathscr{C}_\Delta(t,s)f = e^{-i\hbar^{-1}\psi(t,\cdot)}\mathscr{C}_\Delta(t,s)'(e^{i\hbar^{-1}\psi(s,\cdot)}f), \quad |\Delta| \le \rho^*, \ f \in L^2.$$

See the proof of Theorem in [**17**] for details. Hence Theorem A follows from Theorem 1. That is, Theorem 1 is a generalization of Theorem A.

Remark 2.4. Let $\mathscr{E}^0_{t,s}([0,T]; B^{a+2}) \cap \mathscr{E}^1_{t,s}([0,T]; B^a)$ denote the space of all B^{a+2}-valued continuous and B^a-valued continuously differentiable functions in $0 \le s \le t \le T$. Suppose (2.13) and consider the Schrödinger equation (2.11) for $f \in \bigcup_{a=0}^\infty B^a$. Then the uniqueness of the solutions in $\bigcup_{a=-\infty}^\infty (\mathscr{E}^0_{t,s}([0,T]; B^{a+2}) \cap \mathscr{E}^1_{t,s}([0,T]; B^a))$ has been proved in [**15**], where B^{-a} $(a = 1, 2, \ldots)$ denote the dual space of B^a. So we write the solution of (2.11) as $U(t,s)f$ hereafter.

Let $\Delta : 0 = \tau_0 < \tau_1 < \cdots < \tau_\nu = T$ be a subdivision and $(q_\Delta(\theta; x^{(0)}, \ldots, x^{(\nu-1)}, x), p_\Delta(\theta; x^{(0)}, \ldots, x^{(\nu-1)}, x, v^{(0)}, \ldots, v^{(\nu-1)})) \in (T^*\boldsymbol{R}^n)^{[0,T]}$ the path determined before. Let $0 \le t_1 \le t_2 \le \cdots \le t_k \le T$. For $z = q$ or p we write

$$\iint (\exp i\hbar^{-1} S(T,0;q_\Delta,p_\Delta))(z_\Delta)_{j_k}(t_k)\cdots(z_\Delta)_{j_1}(t_1)f(q_\Delta(0))\mathscr{D}p_\Delta\mathscr{D}q_\Delta$$
$$:= \mathrm{os}-\int\cdots\int (\exp i\hbar^{-1} S(T,0;q_\Delta,p_\Delta))(z_\Delta)_{j_k}(t_k)\cdots(z_\Delta)_{j_1}(t_1)$$
$$\times f(x^{(0)})(2\pi\hbar/m)^{-n\nu}\, dv^{(0)}dx^{(0)}dv^{(1)}dx^{(1)}\cdots dv^{(\nu-1)}dx^{(\nu-1)} \tag{2.15}$$

and

$$\int (\exp i\hbar^{-1} S_c(T,0;q_\Delta))(q_\Delta)_{j_k}(t_k)\cdots(q_\Delta)_{j_1}(t_1)f(q_\Delta(0))\mathscr{D}q_\Delta$$
$$:= \mathrm{os}-\int\cdots\int (\exp i\hbar^{-1} S_c(T,0;q_\Delta))(q_\Delta)_{j_k}(t_k)\cdots(q_\Delta)_{j_1}(t_1)f(x^{(0)})$$
$$\times \prod_{j=1}^{\nu}\sqrt{\frac{m}{2\pi i\hbar(\tau_j-\tau_{j-1})}}^{\,n}\, dx^{(0)}dx^{(1)}\cdots dx^{(\nu-1)}, \tag{2.16}$$

where $(z_\Delta)_j$ is the j-th component of $z_\Delta \in (\boldsymbol{R}^n)^{[0,T]}$. We state the second main theorem.

Theorem 2. *Let $0 \le t_1 \le t_2 \le \cdots \le t_k \le T$, $a = 0,1,\ldots$ and $|\Delta| \le \rho^*$. Then under the assumptions of Theorem* 1 *we have*: (1) *The operator* (2.15) *on C_0^∞ is well-defined and can be extended to a bounded operator from B^{a+k} into B^a. In more detail, we have*

$$\Big\| \iint (\exp i\hbar^{-1} S(T,0;q_\Delta,p_\Delta))(z_\Delta)_{j_k}(t_k)\cdots$$
$$\times (z_\Delta)_{j_1}(t_1)f(q_\Delta(0))\mathscr{D}p_\Delta\mathscr{D}q_\Delta\Big\|_{B^a} \le C_a\|f\|_{B^{a+k}}, \tag{2.17}$$

where C_a is a constant independent of $\Delta, t_1,\ldots,t_{k-1}$ and t_k. (2) *We assume $t_i \ne t_j$ $(i\ne j)$. Then as $|\Delta|\to 0$,* (2.15) *for $f\in B^{a+k}$ converges in B^a, which we write $\iint(\exp i\hbar^{-1}S(T,0;q,p))z_{j_k}(t_k)\cdots z_{j_1}(t_1)f(q(0))\mathscr{D}p\mathscr{D}q$. This limit is equal to $U(T,t_k)\hat{z}_{j_k}\cdot U(t_k,t_{k-1})\cdots\hat{z}_{j_1}U(t_1,0)f$, where $\hat{z}_j$ denotes the multiplication operator x_j when $z=q$ and denotes $i^{-1}\hbar\partial_{x_j}$ when $z=p$.* (3) *Let $t\in[0,T]$ and $f\in B^{a+2}$. We take a μ for each Δ so that $\tau_{\mu-1}<t\le\tau_\mu$. When $t=0$, we take $\mu=1$. Then we have*

$$\lim_{|\Delta|\to 0}\iint (\exp i\hbar^{-1} S(T,0;q_\Delta,p_\Delta))(q_\Delta)_j(t)(p_\Delta)_k(t)f(q_\Delta(0))\mathscr{D}p_\Delta\mathscr{D}q_\Delta$$
$$= U(T,t)\hat{q}_j\hat{p}_kU(t,0)f + \frac{\hbar}{i}\delta_{jk}\lim_{|\Delta|\to 0}\left(\frac{\tau_\mu - t}{\tau_\mu-\tau_{\mu-1}}\right)U(T,0)f \tag{2.18}$$

in B^a, where δ_{jk} is the Kronecker delta. We note that the right-hand side above is divergent if $j=k$ and $0<t<T$. (4) *Here we don't assume $t_i\ne t_j$ $(i\ne j)$. Then the operator* (2.16) *on C_0^∞ is well-defined and is equal to* (2.15) *where $z=q$. In addition, in this case, i.e. in the case of all $z=q$ the operator* (2.15) *for $f\in B^{a+k}$ converges in B^a as $|\Delta|\to 0$ and this limit is equal to $U(T,t_k)\hat{q}_{j_k}U(t_k,t_{k-1})\cdots\hat{q}_{j_1}U(t_1,0)f$.*

See Remark 6.1 for the other results related to Theorem 2.

We write $\int(\exp i\hbar^{-1}S_c(T,0;q))q_{j_k}(t_k)\cdots q_{j_1}(t_1)f(q(0))\mathscr{D}q$ for the limit of (2.16) as $|\Delta|\to 0$. Let's use the notations of the Heisenberg picture of quantum mechanics: $\hat{z}_j(t)=U(t,0)^{-1}\hat{z}_jU(t,0)$, $|f,t\rangle=U(t,0)^{-1}f$ and $\langle f,t|=|f,t\rangle^*$, where g^* is the complex conjugate of g.

Corollary. *Under the assumptions of Theorem* 1 *we have*: (1) *Let* $0\le t_1<t_2<\cdots<t_k\le T$, $g\in L^2$ *and* $f\in B^k$. *Then we obtain the path integral representation of correlation functions*

$$\langle g,T|\hat{z}_{j_k}(t_k)\cdots\hat{z}_{j_1}(t_1)|f,0\rangle\ (:=(|g,T\rangle,\hat{z}_{j_k}(t_k)\cdots\hat{z}_{j_1}(t_1)|f,0\rangle))$$
$$=\left(g,\iint(\exp i\hbar^{-1}S(T,0;q,p))z_{j_k}(t_k)\cdots z_{j_1}(t_1)f(q(0))\mathscr{D}p\mathscr{D}q\right). \tag{2.19}$$

We also have for $0\le t_1\le t_2\le\cdots\le t_k\le T$

$$\langle g,T|\hat{q}_{j_k}(t_k)\cdots\hat{q}_{j_1}(t_1)|f,0\rangle$$
$$=\left(g,\int(\exp i\hbar^{-1}S_c(T,0;q))q_{j_k}(t_k)\cdots q_{j_1}(t_1)f(q(0))\mathscr{D}q\right). \tag{2.20}$$

(2) *Let* $0\le t'<t\le T$ *and* $f\in B^2$. *Then we have for* $j,k=1,2,\dots,n$

$$\lim_{t'\to t}\iint(\exp i\hbar^{-1}S(T,0;q,p))(p_j(t)q_k(t')-q_k(t)p_j(t'))f(q(0))\mathscr{D}p\mathscr{D}q$$
$$=\frac{\hbar}{i}\delta_{jk}\iint(\exp i\hbar^{-1}S(T,0;q,p))f(q(0))\mathscr{D}p\mathscr{D}q \tag{2.21}$$

in L^2.

Proof. Since

$$U(T,t_k)\hat{z}_{j_k}U(t_k,t_{k-1})\cdots\hat{z}_{j_1}U(t_1,0)f=U(T,0)\hat{z}_{j_k}(t_k)\cdots\hat{z}_{j_1}(t_1)f, \tag{2.22}$$

we can easily prove (2.19) and (2.20) from the assertions (2) and (4) of Theorem 2. It follows from the assertion (2) of Theorem 2 that the left-hand side of (2.21) is equal to

$$\lim_{t'\to t}(U(T,t)\hat{p}_jU(t,t')\hat{q}_kU(t',0)f-U(T,t)\hat{q}_kU(t,t')\hat{p}_jU(t',0)f).$$

Here let's use the fact that $\|U(t,s)g\|_{B^a}\le e^{K_a(t-s)}\|g\|_{B^a}$ and $U(t,s)g$ for $g\in B^a$ is continuous as a B^a-valued function in $0\le s\le t\le T$, which follows from Theorem 1. Then

$$\|U(t,t')\hat{q}_kU(t',0)f-\hat{q}_kU(t,0)f\|_{B^1}$$
$$\le e^{K_1(t-t')}\|\hat{q}_k(U(t',0)-U(t,0))f\|_{B^1}+\|U(t,t')\hat{q}_kU(t,0)f-\hat{q}_kU(t,0)f\|_{B^1}$$

and so $\lim_{t'\to t}U(t,t')\hat{q}_kU(t',0)f=\hat{q}_kU(t,0)f$ in B^1. Consequently we have

$$\lim_{t'\to t}U(T,t)\hat{p}_jU(t,t')\hat{q}_kU(t',0)f=U(T,t)\hat{p}_j\hat{q}_kU(t,0)f$$

in L^2. In the same way we can prove (2.21). □

REMARK 2.5. (i) The path integral representation (2.20) of correlation functions of the position operators has been well known in physics, though it has not been rigorous (cf. [**24**], [**28**]). We note that our result (2.19) gives a more general representation of correlation functions including the momentum operators. (ii) It follows from Theorem 2 and (2.22) that the equation (2.21) is equivalent to

$$\lim_{t'\nearrow t}(\hat{p}_j(t)\hat{q}_k(t')f - \hat{q}_k(t)\hat{p}_j(t')f) = \frac{\hbar}{i}\delta_{jk}f, \tag{2.23}$$

i.e. the canonical commutation relations.

EXAMPLE 2.1. Let (V, A) be an electromagnetic potential such that

$$|\partial_x^\alpha V| + \langle x\rangle^{1+\delta}\sum_{j=1}^n |\partial_x^\alpha A_j| \le C_\alpha, \quad |\alpha| \ge 2, \qquad \sum_{j=1}^n |\partial_x^\alpha \partial_t A_j| \le C_\alpha, \quad |\alpha| \ge 1$$

in $[0, T] \times \boldsymbol{R}^n$ for some constant $\delta > 0$. Then since we have $E_j = -\partial A_j/\partial t - \partial V/\partial x_j$ and $B_{jk} = \partial A_k/\partial x_j - \partial A_j/\partial x_k$ from (2.1), we can see that the assumptions of Theorems 1 and 2 are satisfied.

3. Uniform boundedness of the family of the approximate integrals.

Hereafter we set $\hbar = 1$ for the sake of simplicity. We set

$$q_{x,y}^{t,s}(\theta) = y + \frac{\theta - s}{t - s}(x - y) = x - \frac{t - \theta}{t - s}(x - y), \quad s \le \theta \le t. \tag{3.1}$$

Let $p(x, w)$ be an infinitely differentiable function in $\boldsymbol{R}^{2n}$. We define the operator $P(t, s)$ on C_0^∞ by

$$P(t,s)f = \begin{cases} \sqrt{m/(2\pi i(t-s))}^{\,n}\displaystyle\int(\exp iS_c(t,s;q_{x,y}^{t,s})) \\ \qquad \times p(x,(x-y)/\sqrt{t-s})f(y)\,dy, \quad s < t, \\ \sqrt{m/(2\pi i)}^{\,n}\,\mathrm{os}-\displaystyle\int(\exp im|w|^2/2) \\ \qquad \times p(x,w)\,dwf(x), \qquad\qquad s = t \end{cases} \tag{3.2}$$

as in [**16**]–[**19**]. In particular, when $p(x, w)$ is identically one, we write $P(t, s)$ as $\mathscr{C}(t, s)$.

Let's define $B^a := \{f \in L^2; \|f\|_{B^a} = \|\langle\cdot\rangle^a f\| + \|\langle\cdot\rangle^a \hat{f}\| < \infty\}$ for positive numbers $a \ne 1, 2, \ldots$, where $\hat{f}$ is the Fourier transform $\int e^{-ix\cdot\xi}f(x)\,dx$. We state the results obtained in [**17**], which will be used in the present paper. See Lemma 2.1, Theorems 3.3 and 4.4 of [**17**] for their proofs.

PROPOSITION 3.1. *Let $M \ge 0$ and suppose*

$$|\partial_w^\alpha \partial_x^\beta p(x, w)| \le C_{\alpha,\beta}\langle x; w\rangle^M \tag{3.3}$$

in $\boldsymbol{R}^{2n}$ for all α and β, where $\langle x; w\rangle = \sqrt{1 + |x|^2 + |w|^2}$. Then we have under the assumptions of Theorem 1: (1) *Let $f \in C_0^\infty$. Then $\partial_x^\alpha(P(t, s)f)$ are continuous in $0 \le$*

$s \le t \le T$ and $x \in \boldsymbol{R}^n$ for all α. (2) *There exists a constant $\rho^* > 0$ independent of M and $p(x,w)$ such that if $0 \le t - s \le \rho^*$, $P(t,s)$ on C_0^∞ can be extended to a bounded operator from B^{M+a} $(a = 0, 1, \ldots)$ into B^a. In more detail, we have for constants C_a independent of t and s*

$$\|P(t,s)f\|_{B^a} \le C_a \|f\|_{B^{M+a}}, \quad 0 \le t - s \le \rho^*. \tag{3.4}$$

(3) *There exists a constant $K \ge 0$ such that*

$$\|\mathscr{C}(t,s)f\| \le e^{K(t-s)}\|f\|, \quad 0 \le t - s \le \rho^* \tag{3.5}$$

for $f \in L^2$. (4) *Let $|\Delta| \le \rho^*$. Then $\mathscr{C}_\Delta(t,s)$ on C_0^∞ given by (2.6) is well-defined and can be extended to a bounded operator on L^2. In addition, for $\tau_{\mu-1} < t \le \tau_\mu$ and $\tau_{\mu'-1} \le s < \tau_{\mu'}$ we have*

$$\mathscr{C}_\Delta(t,s)f = \mathscr{C}(t,\tau_{\mu-1})\mathscr{C}(\tau_{\mu-1},\tau_{\mu-2})\cdots\mathscr{C}(\tau_{\mu'+1},\tau_{\mu'})\mathscr{C}(\tau_{\mu'},s)f, \quad f \in L^2. \tag{3.6}$$

We can write from (2.2), (2.4) and (3.1)

$$\begin{aligned} S_c(t,s;q^{t,s}_{x,y}) &= \frac{m|x-y|^2}{2(t-s)} + (x-y)\cdot\int_0^1 A(t-\theta(t-s), x-\theta(x-y))\,d\theta \\ &\quad - \int_s^t V\left(\theta, x - \frac{t-\theta}{t-s}(x-y)\right)d\theta \\ &= \frac{m|x-y|^2}{2(t-s)} + (x-y)\cdot\int_0^1 A(t-\theta(t-s), x-\theta(x-y))\,d\theta \\ &\quad - (t-s)\int_0^1 V(t-\theta(t-s), x-\theta(x-y))\,d\theta. \end{aligned} \tag{3.7}$$

Making the change of variables: $\boldsymbol{R}^n \ni y \to w = (x-y)/\sqrt{t-s} \in \boldsymbol{R}^n$ in (3.2), we have

$$P(t,s)f = \sqrt{\frac{m}{2\pi i}}^{\,n} \operatorname{os} - \int e^{i\phi(t,s;x,w)} p(x,w) f(x - \sqrt{\rho}w)\,dw, \quad s \le t, \tag{3.8}$$

where $\rho = t - s$ and

$$\begin{aligned} \phi(t,s;x,w) &= \frac{m}{2}|w|^2 + \psi(t,s;x,\sqrt{\rho}w) \\ &:= \frac{m}{2}|w|^2 + \sqrt{\rho}w\cdot\int_0^1 A(t-\theta\rho, x-\theta\sqrt{\rho}w)\,d\theta \\ &\quad - \rho\int_0^1 V(t-\theta\rho, x-\theta\sqrt{\rho}w)\,d\theta. \end{aligned} \tag{3.9}$$

The following lemma is fundamental.

Lemma 3.2. *Assume* (2.13) *and let $\kappa = (\kappa_1, \ldots, \kappa_n)$ be an arbitrary multi-index. Let $0 \le s \le t \le T$. Then both of $\partial_x^\kappa(\mathscr{C}(t,s)f) - \mathscr{C}(t,s)(\partial_x^\kappa f)$ and $x^\kappa(\mathscr{C}(t,s)f) - \mathscr{C}(t,s)\cdot(x^\kappa f)$ for $f \in C_0^\infty$ are written in the form*

$$(t-s)\sum_{|\gamma|\le|\kappa|}\tilde P_\gamma(t,s)(\partial_x^\gamma f) := (t-s)\sum_{|\gamma|\le|\kappa|}\sqrt{\frac{m}{2\pi i}}^{\,n}\,\mathrm{os}-\int e^{i\phi(t,s;x,w)}$$

$$\times\, p_\gamma(t,s;x,\sqrt{\rho}w)(\partial_x^\gamma f)(x-\sqrt{\rho}w)\,dw, \tag{3.10}$$

where γ are multi-indices and $p_\gamma(t,s;x,\zeta)$ satisfy

$$|\partial_\zeta^\alpha\partial_x^\beta p_\gamma(t,s;x,\zeta)| \le C_{\alpha,\beta}\langle x;\zeta\rangle^{|\kappa|-|\gamma|} \tag{3.11}$$

for all α and β.

Proof. We note from (2.13) and (3.9)

$$|\partial_\zeta^\alpha\partial_x^\beta\psi(t,s;x,\zeta)| \le C_{\alpha,\beta}\langle x;\zeta\rangle,\quad |\alpha+\beta|\ge 1. \tag{3.12}$$

We first consider $\partial_x^\kappa(\mathscr{C}(t,s)f)-\mathscr{C}(t,s)(\partial_x^\kappa f)$. We prove the assertion by induction with respect to $|\kappa|$. Let $|\kappa|=0$. Then the assertion is clear. Suppose that the assertion for $|\kappa|=l$ is true. Let λ be an arbitrary multi-index such that $|\lambda|=l$. Then $\partial_x^\lambda(\mathscr{C}(t,s)f)-\mathscr{C}(t,s)(\partial_x^\lambda f)$ has the form (3.10). Consequently it follows from (3.12) that $\partial_{x_j}(\partial_x^\lambda(\mathscr{C}(t,s)f)-\mathscr{C}(t,s)(\partial_x^\lambda f))$ is written in the form (3.10) where $|\kappa|=l+1$. Hence to complete the induction we will prove that $\partial_{x_j}(\mathscr{C}(t,s)\partial_x^\lambda f)-\mathscr{C}(t,s)(\partial_{x_j}\partial_x^\lambda f)$ can be written in the form (3.10) where $|\kappa|=l+1$. We have from (3.8) with $p(x,w)=1$ and (3.9)

$$\begin{aligned}
&\partial_{x_j}(\mathscr{C}(t,s)\partial_x^\lambda f)-\mathscr{C}(t,s)(\partial_{x_j}\partial_x^\lambda f)\\
&\quad= i\sqrt{\rho}\sqrt{\frac{m}{2\pi i}}^{\,n}\,\mathrm{os}-\int e^{i\phi}w\cdot\int_0^1\frac{\partial A}{\partial x_j}(t-\theta\rho,x-\theta\sqrt{\rho}w)\,d\theta(\partial_x^\lambda f)(x-\sqrt{\rho}w)\,dw\\
&\quad\quad- i\rho\sqrt{\frac{m}{2\pi i}}^{\,n}\,\mathrm{os}-\int e^{i\phi}\int_0^1\frac{\partial V}{\partial x_j}(t-\theta\rho,x-\theta\sqrt{\rho}w)\,d\theta(\partial_x^\lambda f)(x-\sqrt{\rho}w)\,dw.
\end{aligned}$$

We see from (2.13) that the second term above is of the form (3.10). Making integration by parts in w, we can write the first term above as

$$\begin{aligned}
-\frac{\sqrt{\rho}}{m}\sqrt{\frac{m}{2\pi i}}^{\,n}\sum_{k=1}^n\mathrm{os}-\int e^{im|w|^2/2}\frac{\partial}{\partial w_k}\Bigg(&e^{i\psi(t,s;x,\sqrt{\rho}w)}\\
&\times\int_0^1\frac{\partial A_k}{\partial x_j}(t-\theta\rho,x-\theta\sqrt{\rho}w)\,d\theta(\partial_x^\lambda f)(x-\sqrt{\rho}w)\Bigg)\,dw,
\end{aligned}$$

which proves itself from (2.13) and (3.12) to be of the form (3.10). Thus we have completed the induction.

We consider $x^\kappa(\mathscr{C}(t,s)f)-\mathscr{C}(t,s)(x^\kappa f)$. The assertion for $|\kappa|=0$ is clear. Suppose that the assertion for $|\kappa|=l$ is true. Let λ be an arbitrary multi-index such that $|\lambda|=l$. Then since $x^\lambda(\mathscr{C}(t,s)f)-\mathscr{C}(t,s)(x^\lambda f)$ has the form (3.10), $x_j(x^\lambda(\mathscr{C}(t,s)f)-\mathscr{C}(t,s)(x^\lambda f))$ has the form (3.10) where $|\kappa|=l+1$. Hence to complete the induction we will show that $x_j\mathscr{C}(t,s)(x^\lambda f)-\mathscr{C}(t,s)(x_jx^\lambda f)$ is of the form (3.10) where $|\kappa|=l+1$. We have from (3.8) and (3.9)

$$x_j\mathscr{C}(t,s)(x^\lambda f) - \mathscr{C}(t,s)(x_j x^\lambda f)$$
$$= \sqrt{\rho}\sqrt{\frac{m}{2\pi i}}^n \text{os} - \int e^{i\phi} w_j (x - \sqrt{\rho}w)^\lambda f(x - \sqrt{\rho}w)\, dw$$
$$= \frac{i\sqrt{\rho}}{m}\sqrt{\frac{m}{2\pi i}}^n \text{os} - \int e^{im|w|^2/2} \frac{\partial}{\partial w_j}(e^{i\psi(t,s;x,\sqrt{\rho}w)}$$
$$\times (x - \sqrt{\rho}w)^\lambda f(x - \sqrt{\rho}w))\, dw,$$

which proves itself from (3.12) to be of the form (3.10). Thus we have completed the induction. □

Example 3.1. Let $V = 0$ and $A = 0$. Then

$$\mathscr{C}(t,s) = \sqrt{\frac{m}{2\pi i}}^n \text{os} - \int e^{im|w|^2/2} f(x - \sqrt{\rho}w)\, dw$$

for $0 \le s \le t \le T$. So, we have

$$\partial_x^\kappa(\mathscr{C}(t,s)f) = \mathscr{C}(t,s)(\partial_x^\kappa f)$$

and as in the proof of Lemma 3.2

$$x_j(\mathscr{C}(t,s)f) = \mathscr{C}(t,s)(x_j f) - \frac{i\rho}{m}\mathscr{C}(t,s)(\partial_{x_j} f)$$

for $j = 1, 2, \dots, n$.

Lemma 3.3. *Let $a \ge 0$ and $\Gamma_a = \langle x\rangle^a + \langle D_x\rangle^a$ denote the pseudo-differential operator with symbol $\langle x\rangle^a + \langle \eta\rangle^a$ (cf. [**22**]). Then we have: (1) There exist a constant $d_a \ge 0$ and $z_a(x,\eta)$ such that*

$$|\partial_\eta^\alpha \partial_x^\beta z_a(x,\eta)| \le C_{\alpha,\beta}(\langle x\rangle^a + \langle\eta\rangle^a)^{-1}$$

for all α and β and $Z_a(x,D_x)(d_a + \Gamma_a)f = (d_a + \Gamma_a)Z_a(x,D_x)f = f$ for $f \in C_0^\infty$, where $Z_a(x,D_x)$ is the pseudo-differential operator with symbol $z_a(x,\eta)$. (2) The norm $\|\cdot\|_{B^a}$ is equivalent to $\|(d_a + \Gamma_a)\cdot\|$. (3) Let $l \ge 0$. Then the operators $(d_a + \Gamma_a)$ and Z_a are bounded from B^{l+a} into B^l and from B^l into B^{l+a}, respectively.

Proof. The assertions (1) and (2) follow from Lemmas 2.3 and 2.4 of [**15**] respectively, where we take $s = 1$ and $a = b$. We see from the assertions (1) and (2) that $\|(d_a + \Gamma_a)f\|_{B^l}$ is equivalent to $\|(d_l + \Gamma_l)(d_a + \Gamma_a)f\| = \|(d_l + \Gamma_l)(d_a + \Gamma_a)Z_{l+a}\cdot(d_{l+a} + \Gamma_{l+a})f\|$. It follows from Lemmas 2.1 and 2.5 of [**15**], where we take $a = b = 1$, that $(d_l + \Gamma_l)(d_a + \Gamma_a)Z_{l+a}$ is bounded on L^2. So we have $\|(d_a + \Gamma_a)f\|_{B^l} \le \text{Const.}\|(d_{l+a} + \Gamma_{l+a})f\| \le \text{Const.}\|f\|_{B^{l+a}}$. In the same way we can prove the boundedness of Z_a. □

Proposition 3.4. *Under the assumptions of Theorem* 1 *there exist constants $K_a \ge 0$ for $a = 0, 1, \dots$ such that*

$$\|\mathscr{C}(t,s)f\|_{B^a} \le e^{K_a(t-s)}\|f\|_{B^a}, \quad 0 \le t - s \le \rho^*. \tag{3.13}$$

Proof. Let $|\kappa| = a$ and $0 \le t - s \le \rho^*$. We have from Lemma 3.2

$$\|\partial_x^\kappa(\mathscr{C}(t,s)f)\| \le \|\mathscr{C}(t,s)(\partial_x^\kappa f)\| + (t-s)\sum_{|\gamma|\le a}\|\tilde{P}_\gamma(t,s)(\partial_x^\gamma f)\|.$$

Noting (3.11), we apply the assertions (2) and (3) in Proposition 3.1 to the right-hand side above. Then

$$\|\partial_x^\kappa(\mathscr{C}(t,s)f)\| \le e^{K(t-s)}\|\partial_x^\kappa f\| + C_a(t-s)\sum_{|\gamma|\le a}\|\partial_x^\gamma f\|_{B^{a-|\gamma|}}$$

$$\le e^{K(t-s)}\|\partial_x^\kappa f\| + C_a'(t-s)\|f\|_{B^a}.$$

Here we used

$$\|\partial_x^\gamma f\|_{B^{a-|\gamma|}} \le \text{Const.}\|(d_{a-|\gamma|} + \Gamma_{a-|\gamma|})\partial_x^\gamma f\| \le \text{Const.}\|f\|_{B^a}, \tag{3.14}$$

which can be proved as in the proof of (3) in Lemma 3.3. In the same way we have

$$\|x^\kappa(\mathscr{C}(t,s)f)\| \le e^{K(t-s)}\|x^\kappa f\| + C_a''(t-s)\|f\|_{B^a}.$$

Since $\|\mathscr{C}(t,s)f\|_{B^a} = \|\mathscr{C}(t,s)f\| + \sum_{|\kappa|=a}(\|x^\kappa(\mathscr{C}(t,s)f)\| + \|\partial_x^\kappa(\mathscr{C}(t,s)f)\|)$, we have for a constant $K_a \ge 0$

$$\|\mathscr{C}(t,s)f\|_{B^a} \le e^{K(t-s)}\|f\|_{B^a} + C_a'''(t-s)\|f\|_{B^a} \le e^{K_a(t-s)}\|f\|_{B^a},$$

which proves Proposition 3.4. □

Theorem 3.5. *Let $|\Delta| \le \rho^*$. Then under the assumptions of Theorem* 1 *we have* (2.14).

Proof. We can easily prove (2.14) by applying Proposition 3.4 to (3.6). □

4. Equicontinuity of the family of the approximate integrals.

Let $H(t)$ be the Hamiltonian operator defined by (2.12).

Lemma 4.1. *Suppose that there exists a constant $M' \ge 0$ satisfying*

$$|\partial_x^\alpha V| + \sum_{j=1}^n |\partial_x^\alpha A_j| + \sum_{j=1}^n |\partial_x^\alpha \partial_t A_j| \le C_\alpha \langle x\rangle^{M'}$$

in $[0,T]\times \boldsymbol{R}^n$ for all α. Then there exist continuous functions $r(t,s;x,w)$ and $r'(t,s;x,w)$ in $0 \le s \le t \le T$ and $x, w \in \boldsymbol{R}^n$ satisfying (3.3) *for an $M \ge 0$ such that*

$$i\frac{\partial}{\partial t}\mathscr{C}(t,s)f - H(t)\mathscr{C}(t,s)f$$

$$= \sqrt{t-s}R(t,s)f$$

$$:= \sqrt{t-s}\sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}\int(\exp iS_c(t,s;q_{x,y}^{t,s}))r\left(t,s;x,\frac{x-y}{\sqrt{t-s}}\right)f(y)\,dy \tag{4.1}$$

and

$$i\frac{\partial}{\partial s}\mathscr{C}(t,s)f + \mathscr{C}(t,s)H(s)f = \sqrt{t-s}R'(t,s)f \tag{4.2}$$

for $f \in C_0^\infty$ *and* $s < t$.

Remark 4.1. In [**16**], [**17**], supposing $|\partial_x^\alpha \partial_t V| \le C_\alpha \langle x \rangle^{M'}$ for all α besides the assumptions of Lemma 4.1, we proved (4.1).

Proof. In this proof we write $S_c(t,s;q_{x,y}^{t,s})$ as $S(q_{x,y}^{t,s})$ for simplicity. Direct calculations show from (2.12) and (3.2)

$$i\frac{\partial}{\partial t}\mathscr{C}(t,s)f - H(t)\mathscr{C}(t,s)f = -\sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}\int e^{iS(q_{x,y}^{t,s})} \times \left(r_1(t,s;x,y) + \frac{i}{2m}r_2(t,s;x,y)\right)f(y)\,dy, \tag{4.3}$$

$$r_1 = \partial_t S(q_{x,y}^{t,s}) + \frac{1}{2m}\sum_{j=1}^n (\partial_{x_j} S(q_{x,y}^{t,s}) - A_j(t,x))^2 + V(t,x), \tag{4.4}$$

$$r_2 = \frac{nm}{t-s} - \Delta_x S(q_{x,y}^{t,s}) + \sum_{j=1}^n (\partial_{x_j} A_j)(t,x). \tag{4.5}$$

Set $\rho = t - s$. Under the assumptions we can easily prove from the first equation of (3.7)

$$\partial_t S(q_{x,y}^{t,s}) = -\frac{m|x-y|^2}{2(t-s)^2} - V(t,x) + \sqrt{\rho}p_1\left(t,s;x,\frac{x-y}{\sqrt{t-s}}\right),$$

where $p_1(t,s;x,w)$ satisfies (3.3) for an M. It has been proved in (2.23) of [**16**] that

$$\frac{1}{2m}\sum_{j=1}^n (\partial_{x_j} S(q_{x,y}^{t,s}) - A_j(t,x))^2 = \frac{m|x-y|^2}{2(t-s)^2} + \sqrt{\rho}p_2\left(t,s;x,\frac{x-y}{\sqrt{t-s}}\right).$$

Hence we have $r_1 = \sqrt{\rho}p_3(t,s;x,(x-y)/\sqrt{t-s})$ from (4.4). It has also been proved in (2.25) of [**16**] that

$$\Delta_x S(q_{x,y}^{t,s}) = \frac{dm}{t-s} + \sum_{j=1}^n \partial_{x_j} A_j(t,x) + \sqrt{\rho}p_4\left(t,s;x,\frac{x-y}{\sqrt{t-s}}\right).$$

Hence $r_2 = \sqrt{\rho}p_5(t,s;x,(x-y)/\sqrt{t-s})$ follows from (4.5). Hence we can prove (4.1) from (4.3).

We consider (4.2). We also have by direct calculations

$$i\frac{\partial}{\partial s}\mathscr{C}(t,s)f + \mathscr{C}(t,s)H(s)f = -\sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}\int e^{iS(q^{t,s}_{x,y})}$$
$$\times\left(r'_1(t,s;x,y) + \frac{i}{2m}r'_2(t,s;x,y)\right)f(y)\,dy, \qquad (4.6)$$

$$r'_1 = \partial_s S(q^{t,s}_{x,y}) - \frac{1}{2m}\sum_{j=1}^n(\partial_{y_j}S(q^{t,s}_{x,y}) + A_j(s,y))^2 - V(s,y), \qquad (4.7)$$

$$r'_2 = -\frac{nm}{t-s} + \Delta_y S(q^{t,s}_{x,y}) + \sum_{j=1}^n(\partial_{x_j}A_j)(s,y). \qquad (4.8)$$

From (4.6)–(4.8) we can prove (4.2) as in the proof of (4.1). □

Theorem 4.2. *Under the assumptions of Theorem* 1 *we can find an integer* $M \ge 2$ *such that*

$$\|\mathscr{C}_\Delta(t,s)f - \mathscr{C}_\Delta(t',s')f\|_{B^a} \le C_a(|t-t'| + |s-s'|)\|f\|_{B^{a+M}} \qquad (4.9)$$

for $0 \le s \le t \le T,\ 0 \le s' \le t' \le T,\ |\Delta| \le \rho^*$ *and* $a = 0, 1, \dots.$

Proof. We have $\partial A/\partial t = E - \partial V/\partial x$ from (2.1) and so

$$\sum_{j=1}^n |\partial_x^\alpha \partial_t A_j| \le C_\alpha\langle x\rangle \qquad (4.10)$$

for all α from the assumptions. Consequently Lemma 4.1 holds. Let's take an integer $M \ge 2$ in Lemma 4.1 and fix it. It is clear from the assumption (2.13) that we have

$$\|H(t)f\|_{B^a} \le \text{Const.}\|f\|_{B^{a+2}} \le \text{Const.}\|f\|_{B^{a+M}}, \quad t \in [0,T] \qquad (4.11)$$

by using Lemma 3.3 and its proof. Let $f \in B^{a+M}$ and $t, t' > s$. Then we have from (2) of Proposition 3.1 and Lemma 4.1

$$i(\mathscr{C}(t,s)f - \mathscr{C}(t',s)f) = \int_{t'}^t (H(\theta)\mathscr{C}(\theta,s)f + \sqrt{\theta - s}R(\theta,s)f)\,d\theta \qquad (4.1)'$$

in B^a. It follows from (1) of Proposition 3.1 that $(4.1)'$ is valid for $t, t' \ge s$. In the same way we have

$$i(\mathscr{C}(t,s)f - \mathscr{C}(t,s')f) = -\int_{s'}^s (\mathscr{C}(t,\theta)H(\theta)f - \sqrt{t-\theta}R'(t,\theta)f)\,d\theta \qquad (4.2)'$$

in B^a for $t, t' \ge s$.

We first consider (4.9) in the case of $s' = s$. We may assume $t' \le t$. Let $\tau_j < t \le \tau_{j+1}$ and $\tau_k < t' \le \tau_{k+1}$ for some j and k. Suppose $j = k$ and $s \le \tau_j$. Then we have from (3.6) and $(4.1)'$

$$
\begin{aligned}
&i(\mathscr{C}_\Delta(t,s)-\mathscr{C}_\Delta(t',s))\\
&\quad = i(\mathscr{C}(t,\tau_j)-\mathscr{C}(t',\tau_j))\mathscr{C}_\Delta(\tau_j,s)\\
&\quad = \int_{t'}^{t} H(\theta)\mathscr{C}_\Delta(\theta,s)\,d\theta + \int_{t'}^{t}\sqrt{\theta-\tau_j}R(\theta,\tau_j)\,d\theta\mathscr{C}_\Delta(\tau_j,s).
\end{aligned}
\tag{4.12}
$$

Suppose $j > k$ and $s \le \tau_k$. Then

$$
\begin{aligned}
&\mathscr{C}_\Delta(t,s)-\mathscr{C}_\Delta(t',s)\\
&\quad = \mathscr{C}_\Delta(t,s)-\mathscr{C}_\Delta(\tau_j,s)+\sum_{l=1}^{j-k-1}(\mathscr{C}_\Delta(\tau_{j-l+1},s)-\mathscr{C}_\Delta(\tau_{j-l},s))\\
&\qquad + \mathscr{C}_\Delta(\tau_{k+1},s)-\mathscr{C}_\Delta(t',s).
\end{aligned}
$$

The same proof as of (4.12) shows

$$
\begin{aligned}
&i(\mathscr{C}_\Delta(t,s)-\mathscr{C}_\Delta(t',s))\\
&\quad = \int_{t'}^{t} H(\theta)\mathscr{C}_\Delta(\theta,s)\,d\theta + \int_{\tau_j}^{t}\sqrt{\theta-\tau_j}R(\theta,\tau_j)\,d\theta\mathscr{C}_\Delta(\tau_j,s)\\
&\qquad + \sum_{l=1}^{j-k-1}\int_{\tau_{j-l}}^{\tau_{j-l+1}}\sqrt{\theta-\tau_{j-l}}R(\theta,\tau_{j-l})\,d\theta\mathscr{C}_\Delta(\tau_{j-l},s)\\
&\qquad + \int_{t'}^{\tau_{k+1}}\sqrt{\theta-\tau_k}R(\theta,\tau_k)\,d\theta\mathscr{C}_\Delta(\tau_k,s).
\end{aligned}
\tag{4.13}
$$

Apply (2) in Proposition 3.1, Theorem 3.5 and (4.11) to the right-hand side of (4.13). Then

$$
\begin{aligned}
&i\|\mathscr{C}_\Delta(t,s)f-\mathscr{C}_\Delta(t',s)f\|_{B^a}\\
&\quad \le \text{Const.}\Bigg(\int_{t'}^{t}\|\mathscr{C}_\Delta(\theta,s)f\|_{B^{a+M}}\,d\theta + \sqrt{|\Delta|}\int_{\tau_j}^{t}d\theta\|\mathscr{C}_\Delta(\tau_j,s)f\|_{B^{a+M}}\\
&\qquad + \sum_{l=1}^{j-k-1}\sqrt{|\Delta|}\int_{\tau_{j-l}}^{\tau_{j-l+1}}d\theta\|\mathscr{C}_\Delta(\tau_{j-l},s)f\|_{B^{a+M}} + \sqrt{|\Delta|}\int_{t'}^{\tau_{k+1}}d\theta\|\mathscr{C}_\Delta(\tau_k,s)f\|_{B^{a+M}}\Bigg)\\
&\quad \le \text{Const.}\,e^{K_{a+M}T}(1+\sqrt{\rho^*})|t-t'|\,\|f\|_{B^{a+M}},
\end{aligned}
$$

which shows (4.9). In the same way we can prove (4.9) generally in the case of $s' = s$.

We consider the case of $t' = t$. We may assume $s' \le s$. Let $\tau_j \le s < \tau_{j+1}$ and $\tau_k \le s' < \tau_{k+1}$. Suppose $j = k$ and $t \ge \tau_{j+1}$. Then we have from (3.6) and (4.2)$'$

$$
\begin{aligned}
&i(\mathscr{C}_\Delta(t,s)-\mathscr{C}_\Delta(t,s'))\\
&\quad = -\int_{s'}^{s}\mathscr{C}_\Delta(t,\theta)H(\theta)\,d\theta + \mathscr{C}_\Delta(t,\tau_{j+1})\int_{s'}^{s}\sqrt{\tau_{j+1}-\theta}R'(\tau_{j+1},\theta)\,d\theta
\end{aligned}
\tag{4.14}
$$

as in the proof of (4.12). Supposing $j > k$ and $t \ge \tau_{j+1}$, we have

$$\begin{aligned} & i(\mathscr{C}_\Delta(t,s) - \mathscr{C}_\Delta(t,s')) \\ &= i(\mathscr{C}_\Delta(t,s) - \mathscr{C}_\Delta(t,\tau_j)) + i\sum_{l=1}^{j-k-1}(\mathscr{C}_\Delta(t,\tau_{j-l+1}) - \mathscr{C}_\Delta(t,\tau_{j-l})) \\ &\quad + i(\mathscr{C}_\Delta(t,\tau_{k+1}) - \mathscr{C}_\Delta(t,s')) \\ &= -\int_{s'}^{s}\mathscr{C}_\Delta(t,\theta)H(\theta)\,d\theta + \mathscr{C}_\Delta(t,\tau_{j+1})\int_{\tau_j}^{s}\sqrt{\tau_{j+1}-\theta}R'(\tau_{j+1},\theta)\,d\theta \\ &\quad + \sum_{l=1}^{j-k-1}\mathscr{C}_\Delta(t,\tau_{j-l+1})\int_{\tau_{j-l}}^{\tau_{j-l+1}}\sqrt{\tau_{j-l+1}-\theta}R'(\tau_{j-l+1},\theta)\,d\theta \\ &\quad + \mathscr{C}_\Delta(t,\tau_{k+1})\int_{s'}^{\tau_{k+1}}\sqrt{\tau_{k+1}-\theta}R'(\tau_{k+1},\theta)\,d\theta \end{aligned} \tag{4.15}$$

as in the proof of (4.14). Hence we can prove (4.9) from (4.14) and (4.15) as in the proof of the case of $s' = s$. In the same way we can prove (4.9) generally in the case of $t' = t$. We can easily complete the proof of Theorem 4.2 from the results above. □

5. Proof of Theorem 1.

Let $M \ge 2$ be the integer in Theorem 4.2. We fix M through this section.

LEMMA 5.1. *Let $\{\Delta_j\}_{j=1}^{\infty}$ be a family of subdivisions of $[0,T]$ such that $|\Delta_j| \le \rho^*$ and $\lim_{j\to\infty}|\Delta_j| = 0$. Let $a = 0,1,\dots$ and $f \in B^{a+2M}$. Then under the assumptions of Theorem 1 we can find a subsequence $\{\Delta_{j_k}\}_{k=1}^{\infty}$, which may depend on f, such that: (1) As $k \to \infty$, $\mathscr{C}_{\Delta_{j_k}}(t,s)f$ converges in B^{a+M} uniformly in $0 \le s \le t \le T$. (2) The limit of $\mathscr{C}_{\Delta_{j_k}}(t,s)f$ belongs to $\mathscr{E}_{t,s}^0([0,T];B^{a+M}) \cap \mathscr{E}_{t,s}^1([0,T];B^a)$ and satisfies the Schrödinger equation* (2.11).

PROOF. It is proved from the Rellich criterion (cf. Theorem XIII. 65 in [**27**]) that the embedding map from B^M into L^2 is compact. Consequently we can easily see from Lemma 3.3 that the embedding map from B^{a+2M} into B^{a+M} is compact as well. Since $f \in B^{a+2M}$, it follows from Theorem 3.5 that $\{\mathscr{C}_{\Delta_j}(t,s)f\}_{j=1}^{\infty}$ is uniformly bounded as a family of B^{a+2M}-valued functions in $0 \le s \le t \le T$. Consequently $\{\mathscr{C}_{\Delta_j}(t,s)f\}_{j=1}^{\infty}$ for each t and s makes a relatively compact set in B^{a+M}. We also know from Theorem 4.2 that $\{\mathscr{C}_{\Delta_j}(t,s)f\}_{j=1}^{\infty}$ makes a equicontinuous family of B^{a+M}-valued functions in $0 \le s \le t \le T$. Hence applying the Ascoli-Arzelà theorem, we can find a subsequence $\{\Delta_{j_k}\}_{k=1}^{\infty}$ such that $\mathscr{C}_{\Delta_{j_k}}(t,s)f$ converges in B^{a+M} uniformly in $0 \le s \le t \le T$ as $k \to \infty$.

We set $W(t,s)f := \lim_{k\to\infty}\mathscr{C}_{\Delta_{j_k}}(t,s)f$. Let's apply Proposition 3.1 and Theorem 3.5 to (4.13) where $t' = s$. Then we have from (4.11)

$$i(W(t,s)f - f) = \int_s^t H(\theta)W(\theta,s)f\,d\theta \tag{5.1}$$

in B^a, which shows that $W(t,s)f$ belongs to $\mathscr{E}^1_{t,s}([0,T];B^a)$ and satisfies (2.11). Thus we could complete the proof. □

Remark 5.1. Let $f \in B^{a+2M}$ and $W(t,s)f$ the function defined in the proof of Lemma 5.1, i.e. $W(t,s)f = \lim_{k\to\infty} \mathscr{C}_{\Delta_{j_k}}(t,s)f$. Then, using (4.15) where $s = t$, we can prove

$$i(f - W(t,s)f) = -\int_s^t W(t,\theta)H(\theta)f\,d\theta$$

in B^a. Consequently we have

$$i\frac{\partial}{\partial s}W(t,s)f = -W(t,s)H(s)f. \tag{5.2}$$

Proposition 5.2. *Let $a = 0,1,\dots$ and $f \in B^{a+2M}$. Then under the assumptions of Theorem 1 we see that as $|\Delta| \to 0$, $\mathscr{C}_\Delta(t,s)f$ converges in B^{a+M} uniformly in $0 \le s \le t \le T$. In addition, this limit belongs to $\mathscr{E}^0_{t,s}([0,T];B^{a+M}) \cap \mathscr{E}^1_{t,s}([0,T];B^a)$ and satisfies* (2.11).

Proof. We can easily prove by means of the energy method that the solutions of (2.11) in $\mathscr{E}^0_{t,s}([0,T];B^{a+M}) \cap \mathscr{E}^1_{t,s}([0,T];B^a)$ are unique. Proposition 5.2 can be shown from this uniqueness result and Lemma 5.1. □

Proof of Theorem 1. Let $a = 0,1,\dots$. The inequality (2.14) has been already proved in Theorem 3.5. Let $|\Delta| \le \rho^*$ and $f \in B^a$. Then for any $\varepsilon > 0$ we can find a $g \in B^{a+M}$ such that $\|g - f\|_{B^a} < \varepsilon$. We have from (2.14)

$$\|\mathscr{C}_\Delta(t,s)f - \mathscr{C}_\Delta(t',s')f\|_{B^a} < \|\mathscr{C}_\Delta(t,s)g - \mathscr{C}_\Delta(t',s')g\|_{B^a} + 2e^{K_aT}\varepsilon.$$

Hence we have from Theorem 4.2

$$\overline{\lim}_{t'\to t,\,s'\to s}\|\mathscr{C}_\Delta(t,s)f - \mathscr{C}_\Delta(t',s')f\|_{B^a} \le 2e^{K_aT}\varepsilon,$$

which shows $\mathscr{C}_\Delta(t,s)f \in \mathscr{E}^0_{t,s}([0,T];B^a)$.

We show the assertion (2). Let $f \in B^a$. For any $\varepsilon > 0$ we take a $g \in B^{a+2M}$ such that $\|g - f\|_{B^a} < \varepsilon$. Let Δ and Δ' be subdivisions such that $|\Delta|, |\Delta'| \le \rho^*$. Then we have from (2.14)

$$\begin{aligned}\|\mathscr{C}_\Delta(t,s)f - \mathscr{C}_{\Delta'}(t,s)f\|_{B^a} &< \|\mathscr{C}_\Delta(t,s)g - \mathscr{C}_{\Delta'}(t,s)g\|_{B^a} + 2e^{K_aT}\varepsilon \\ &\le \|\mathscr{C}_\Delta(t,s)g - \mathscr{C}_{\Delta'}(t,s)g\|_{B^{a+M}} + 2e^{K_aT}\varepsilon.\end{aligned}$$

Proposition 5.2 gives

$$\overline{\lim}_{|\Delta|,|\Delta'|\to 0}\max_{0\le s\le t\le T}\|\mathscr{C}_\Delta(t,s)f - \mathscr{C}_{\Delta'}(t,s)f\|_{B^a} \le 2e^{K_aT}\varepsilon,$$

which shows that as $|\Delta| \to 0$, $\mathscr{C}_\Delta(t,s)f$ converges in B^a uniformly in $0 \le s \le t \le T$. In the same way we can easily complete the proof of Theorem 1.

6. Proof of Theorem 2.

Let x, y and v be in $\boldsymbol{R}^n$. Using the path $q_{x,y}^{t,s} \in (\boldsymbol{R}^n)^{[s,t]}$ defined by (3.1), we set

$$\zeta_{x,y,v}^{t,s}(\theta) = \left(q_{x,y}^{t,s}(\theta), \frac{\partial \mathscr{L}}{\partial \dot{x}}(\theta, q_{x,y}^{t,s}(\theta), v)\right) \in (T^*\boldsymbol{R}^n)^{[s,t]}. \tag{6.1}$$

Then we have from (2.5)

$$\begin{aligned} S(t,s;\zeta_{x,y,v}^{t,s}) &= -\frac{m}{2}(t-s)\left|v - \frac{x-y}{t-s}\right|^2 + S_c(t,s;q_{x,y}^{t,s}) \\ &= -\frac{m}{2}(t-s)|v|^2 + (x-y)\cdot mv + (x-y)\cdot\int_0^1 A(t-\theta\rho, x-\theta(x-y))\,d\theta \\ &\quad - (t-s)\int_0^1 V(t-\theta\rho, x-\theta(x-y))\,d\theta, \quad \rho = t-s \end{aligned} \tag{6.2}$$

(cf. Corollary 3.4 in [**18**]). Let $\varepsilon > 0$ and set for $f \in C_0^\infty$

$$G_\varepsilon(t,s)f = \begin{cases} (2\pi/m)^{-n}\displaystyle\iint(\exp iS(t,s;\zeta_{x,y,v}^{t,s}))\chi(\varepsilon v)f(y)\,dvdy, & s < t, \\ f, \quad s = t, \end{cases} \tag{6.3}$$

where $\chi \in C_0^\infty(\boldsymbol{R}^n)$ such that $\chi(0) = 1$.

Lemma 6.1. *Let $0 < t-s \le \rho^*$ and $s \le t' \le t$. We consider an infinitely differentiable function $b(x)$ such that*

$$|\partial_x^\alpha b(x)| \le C_\alpha \langle x\rangle^{M-1}, \quad |\alpha| \ge 1,$$

where $M \ge 0$ is a constant independent of α. Then we have under the assumptions of Theorem 1: (1) Let γ be an arbitrary multi-index. Then we have for $f \in C_0^\infty(\boldsymbol{R}^n)$

$$\begin{aligned} &(2\pi/m)^{-n}\iint(\exp iS(t,s;\zeta_{x,y,v}^{t,s}))\chi(\varepsilon v)(mv)^\gamma b(q_{x,y}^{t,s}(t'))f(y)\,dvdy \\ &= \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}\int(\exp iS_c(t,s;q_{x,y}^{t,s}))\sum_{0\le\gamma'\le\gamma}\left(p_{\gamma'}\left(t,t',s;x,\frac{x-y}{\sqrt{t-s}}\right)\right. \\ &\quad \left. + p_{\gamma',\varepsilon}\left(t,t',s;x,\frac{x-y}{\sqrt{t-s}}\right)\right)\partial_y^{\gamma'}f(y)\,dy, \end{aligned} \tag{6.4}$$

$$|\partial_w^\alpha\partial_x^\beta p_{\gamma'}(t,t',s;x,w)| \le C_{\alpha,\beta}\langle x;w\rangle^{M+|\gamma|-|\gamma'|}, \tag{6.5}$$

$$|\partial_w^\alpha\partial_x^\beta p_{\gamma',\varepsilon}(t,t',s;x,w)| \le C'_{\alpha,\beta}\langle x;w\rangle^{M+|\gamma|-|\gamma'|}, \tag{6.6}$$

$$\lim_{\varepsilon\to 0}\partial_w^\alpha\partial_x^\beta p_{\gamma',\varepsilon}(t,t',s;x,w) = 0, \ \textit{pointwisely} \tag{6.7}$$

for all α and β, where $C_{\alpha,\beta}$ are independent of t, t' and s and $C'_{\alpha,\beta}$ are independent of

$0 < \varepsilon \le 1$, *but dependent on* $t-s$. *The notation* $\gamma' \le \gamma$ *means* $\gamma'_j \le \gamma_j$ $(j = 1, 2, \dots, n)$. (2) *We study the case of* $\gamma = 0$ *in* (1) *in more detail. We have for* $f \in C_0^\infty$

$$(2\pi/m)^{-n} \iint (\exp iS(t,s;\zeta^{t,s}_{x,y,v}))\chi(\varepsilon v) b(q^{t,s}_{x,y}(t')) f(y)\, dvdy$$

$$= \mathscr{C}(t,s)(bf) + \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n} \int (\exp iS_c(t,s;q^{t,s}_{x,y}))\biggl(\sqrt{t-s}$$

$$\times p'_0\left(t,t',s;x,\frac{x-y}{\sqrt{t-s}}\right) + p_{0,\varepsilon}\left(t,t',s;x,\frac{x-y}{\sqrt{t-s}}\right)\biggr) f(y)\, dy, \tag{6.8}$$

where $p'_0(t,t',s;x,w)$ *satisfies the same estimates as of* (6.5) *with* $\gamma = \gamma' = 0$. (3) *We study the case of* $|\gamma| = 1$ *in* (1) *in more detail. We have for* $f \in C_0^\infty$

$$(2\pi/m)^{-n} \iint (\exp iS(t,s;\zeta^{t,s}_{x,y,v}))\chi(\varepsilon v) m v_j b(q^{t,s}_{x,y}(t')) f(y)\, dvdy$$

$$= \mathscr{C}(t,s)\left(\frac{b}{i}\frac{\partial}{\partial x_j} - A_j(s) b + \frac{t-t'}{i(t-s)}\frac{\partial b}{\partial x_j}\right) f + \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}$$

$$\times \int (\exp iS_c(t,s;q^{t,s}_{x,y}))\Biggl(\sqrt{t-s}\; p''_0\left(t,t',s;x,\frac{x-y}{\sqrt{t-s}}\right)$$

$$+ \sqrt{t-s}\; p''_{e_j}\left(t,t',s;x,\frac{x-y}{\sqrt{t-s}}\right)\partial_{y_j}$$

$$+ \sum_{0 \le \gamma' \le e_j} p_{\gamma',\varepsilon}\left(t,t',s;x,\frac{x-y}{\sqrt{t-s}}\right)\partial_y^{\gamma'}\Biggr) f(y)\, dy, \tag{6.9}$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ *and* $p''_0(t,t',s;x,w)$ *and* $p''_{e_j}(t,t',s;x,w)$ *satisfy the same estimates as of* (6.5) *with* $|\gamma| = 1$, $\gamma' = 0$ *and with* $|\gamma| = |\gamma'| = 1$, *respectively*.

Proof. (1) We first consider the case of $\gamma = 0$. Then we can prove from (3.1) and (6.2) that the left-hand side of (6.4), i.e. (6.8) is equal to

$$(2\pi/m)^{-n} \int (\exp iS_c(t,s;q^{t,s}_{x,y})) b\left(y + \frac{t'-s}{t-s}(x-y)\right) f(y)\, dy$$

$$\times \int \left(\exp -i\frac{m}{2}(t-s)\left|v - \frac{x-y}{t-s}\right|^2\right)\chi(\varepsilon v)\, dv$$

$$= \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n} \int (\exp iS_c(t,s;q^{t,s}_{x,y})) b\left(y + \frac{t'-s}{t-s}(x-y)\right) f(y)\, dy$$

$$\times \sqrt{i/2\pi}^{\,n} \int (\exp -i|z|^2/2)\chi\left(\varepsilon\left(\sqrt{\frac{1}{m(t-s)}}z + \frac{x-y}{t-s}\right)\right) dz$$

$$= \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n} \int (\exp iS_c(t,s;q^{t,s}_{x,y})) b\left(y + \frac{t'-s}{t-s}(x-y)\right) f(y)\, dy$$

$$+\sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}\int(\exp iS_c(t,s;q^{t,s}_{x,y}))b\Big(y+\frac{t'-s}{t-s}(x-y)\Big)f(y)\,dy$$

$$\times\sqrt{i/2\pi}^{\,n}\int(\exp-i|z|^2/2)\Big\{\chi\Big(\varepsilon\Big(\sqrt{\frac{1}{m(t-s)}}z+\frac{x-y}{t-s}\Big)\Big)-1\Big\}\,dz. \tag{6.10}$$

Writing $y+(t'-s)(x-y)/(t-s)=x-(t-t')(x-y)/(t-s)$, we get (6.4)–(6.7). Next we consider the case of $|\gamma|=1$, i.e. $v^\gamma=v_j$. Then we can prove from (6.2) that the left-hand side of (6.4), i.e. (6.9) is equal to

$$(2\pi/m)^{-n}\iint\Big(\exp-i\frac{m}{2}(t-s)|v|^2+i(x-y)\cdot mv\Big)\chi(\varepsilon v)$$

$$\times\frac{1}{i}\frac{\partial}{\partial y_j}\Big\{\Big(\exp i(x-y)\cdot\int_0^1 A(t-\theta\rho,x-\theta(x-y))\,d\theta$$

$$-i(t-s)\int_0^1 V(t-\theta\rho,x-\theta(x-y))\,d\theta\Big)b\Big(y+\frac{t'-s}{t-s}(x-y)\Big)f(y)\Big\}\,dvdy. \tag{6.11}$$

Consequently, noting the assumption (2.13), we can easily prove (6.4)–(6.7) as in the proof of the case of $\gamma=0$. In the same way we can prove (1) generally.

(2) We can write

$$\sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}\int(\exp iS_c(t,s;q^{t,s}_{x,y}))b\Big(y+\frac{t'-s}{t-s}(x-y)\Big)f(y)\,dy$$

$$=\mathscr{C}(t,s)(bf)+\sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}\int(\exp iS_c(t,s;q^{t,s}_{x,y}))$$

$$\times\Big(b\Big(y+\frac{t'-s}{t-s}(x-y)\Big)-b(y)\Big)f(y)\,dy.$$

Consequently we can prove (6.8) from (6.10). In the same way (6.9) can be proved from (6.11). □

Lemma 6.2. *Let* $0\le t-s\le\rho^*$, $0\le M$ *and* $\{p_\varepsilon(x,w)\}_{0<\varepsilon\le1}$ *a family satisfying*

$$\sup_{0<\varepsilon\le1}|\partial_w^\alpha\partial_x^\beta p_\varepsilon(x,w)|\le C_{\alpha,\beta}\langle x;w\rangle^M, \tag{6.12}$$

$$\lim_{\varepsilon\to0}\partial_w^\alpha\partial_x^\beta p_\varepsilon(x,w)=0,\ \textit{pointwisely} \tag{6.13}$$

for all α *and* β. *We define the operator* $P_\varepsilon(t,s)$ *by* (3.2) *for* $p_\varepsilon(x,w)$. *Let* $a=0,1,\dots$. *Then under the assumptions of Theorem* 1 *we have*

$$\lim_{\varepsilon\to0}\|P_\varepsilon(t,s)f\|_{B^a}=0$$

for all $f\in B^{M+a}$.

Proof. Reviewing the proof of (3.4), i.e. the proof of Theorem 4.4 of [**17**], we prove this lemma. At first we see from Lemmas 4.1 and 4.2 of [**17**] that $P_\varepsilon(t,s)^*P_\varepsilon(t,s)$

can be written in the form of the pseudo-differential operator $Q_\varepsilon(t,s;x,D_x)$ with properties

$$\sup_{0<\varepsilon\le 1} |\partial_\eta^\alpha \partial_x^\beta q_\varepsilon(t,s;x,\eta)| \le C'_{\alpha,\beta}\langle x;\eta\rangle^{2M},$$

$$\lim_{\varepsilon\to 0} \partial_\eta^\alpha \partial_x^\beta q_\varepsilon(t,s;x,\eta) = 0, \text{ pointwisely}$$

for all α and β. Let $d_M + \Gamma_M$ and Z_M be the pseudo-differential operators in Lemma 3.3. Then we can easily see $\lim_{\varepsilon\to 0}\|Z_M Q_\varepsilon(t,s)f\| = 0$ for $f \in C_0^\infty$ (cf. Lemma 2.2 of [**15**]) and $\|Z_M Q_\varepsilon f\| = \|(Z_M Q_\varepsilon Z_M)(d_M + \Gamma_M)f\| \le C\|f\|_{B^M}$ with a constant $C \ge 0$ independent of ε as in the proof of Lemma 3.3, which show $\lim_{\varepsilon\to 0}\|Z_M Q_\varepsilon(t,s)f\| = 0$ for $f \in B^M$. Since $(P_\varepsilon(t,s)f, P_\varepsilon(t,s)f) = (Z_M Q_\varepsilon(t,s)f, (d_M + \Gamma_M)f)$, we have

$$\lim_{\varepsilon\to 0} \|P_\varepsilon(t,s)f\| = 0 \tag{6.14}$$

for $f \in B^M$.

Let $|\alpha| = a$ and $f \in B^{M+a}$. We can prove from (6.14)

$$\lim_{\varepsilon\to 0} \|x^\alpha P_\varepsilon(t,s)f\| = 0. \tag{6.15}$$

As was shown in the proof of Theorem 4.4 of [**17**], we have from (3.8)

$$\partial_x^\alpha(P_\varepsilon(t,s)f) = \sum_{\beta\le\alpha} P_{\beta,\varepsilon}(t,s)(\partial_x^{\alpha-\beta} f),$$

where $\{p_{\beta,\varepsilon}(x,w)\}_{0<\varepsilon\le 1}$ satisfies (6.12) having M replaced with $M + |\beta|$ and (6.13). Since $\partial_x^{\alpha-\beta} f \in B^{M+|\beta|}$ follows from (3.14), we have from (6.14)

$$\lim_{\varepsilon\to 0} \|P_{\beta,\varepsilon}(t,s)(\partial_x^{\alpha-\beta} f)\| = 0$$

and so

$$\lim_{\varepsilon\to 0} \|\partial_x^\alpha(P_\varepsilon(t,s)f)\| = 0. \tag{6.16}$$

Hence we can complete the proof together with (6.14) and (6.15). □

Lemma 6.3. *Let* $0 \le t - s \le \rho^*$ *and* $a = 0, 1, \ldots.$ *Then under the assumptions of Theorem* 1 *we have*

$$\lim_{\varepsilon\to 0} G_\varepsilon(t,s)f = \mathscr{C}(t,s)f \ \text{ in } B^a, \quad \sup_{0<\varepsilon\le 1} \|G_\varepsilon(t,s)f - \mathscr{C}(t,s)f\|_{B^a} \le C_a\|f\|_{B^a}$$

for $f \in B^a$, *where the constant* C_a *may depend on* $t - s$.

Proof. We have from (6.3), (6.10) and (1) of Lemma 6.1

$$G_\varepsilon(t,s)f = \mathscr{C}(t,s)f + \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n} \int (\exp iS_c(t,s;q_{x,y}^{t,s})) \times p_{0,\varepsilon}\left(t,s;x,\frac{x-y}{\sqrt{t-s}}\right) f(y)\,dy.$$

Here $p_{0,\varepsilon}(t,s;x,w)$ satisfies (6.12) with $M=0$ and (6.13), where the constants $C_{\alpha,\beta}$ in (6.12) may depend on $t-s$. Consequently the assertion follows from Proposition 3.1 and Lemma 6.2. □

Proof of Theorem 2. For the sake of simplicity we prove (1) and (2) of Theorem 2 in the case of $0<t_1\le t_2\le T$, $z_{j_1}=q_j$ and $z_{j_2}=p_k$. The general case can be proved in the same way. We write for $s\le t'\le t$

$$\tilde{G}_{1,\varepsilon}(t,t',s)f := (2\pi/m)^{-n}\iint(\exp iS(t,s;\zeta^{t,s}_{x,y,v}))\chi(\varepsilon v)(q^{t,s}_{x,y})_j(t')f(y)\,dvdy, \tag{6.17}$$

$$\tilde{G}_{2,\varepsilon}(t,t',s)f := (2\pi/m)^{-n}\iint(\exp iS(t,s;\zeta^{t,s}_{x,y,v}))\chi(\varepsilon v) \cdot (mv_k + A_k(t',q^{t,s}_{x,y}(t')))f(y)\,dvdy, \tag{6.18}$$

$$\tilde{\mathscr{C}}_1(t,s)f := \mathscr{C}(t,s)(\hat{q}_jf) = \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}\int(\exp iS_c(t,s;q^{t,s}_{x,y}))y_jf(y)\,dy, \tag{6.19}$$

$$\tilde{\mathscr{C}}_2(t,s)f := \mathscr{C}(t,s)(\hat{p}_kf) = \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}\int(\exp iS_c(t,s;q^{t,s}_{x,y}))\frac{1}{i}\frac{\partial f}{\partial x_k}(y)\,dy. \tag{6.20}$$

We note (4.10). Then we have the following from (6.8), (6.9) and Lemma 6.2. There exist $p_N(t,t',s;x,w)$ $(N=1,2)$ satisfying (6.5) with $M+|\gamma|=1$ and $\gamma'=0$ such that

$$\lim_{\varepsilon\to 0}\|\tilde{G}_{N,\varepsilon}(t,t',s)f - \tilde{\mathscr{C}}_N(t,s)f - \sqrt{t-s}P_N(t,t',s)f\|_{B^a} = 0 \tag{6.21}$$

for $f\in B^{a+1}$.

Let Δ be a subdivision such that $|\Delta|\le\rho^*$ and fix such a Δ for a while. We take μ and μ' for Δ so that $\tau_{\mu-1}<t_1\le\tau_\mu$ and $\tau_{\mu'-1}<t_2\le\tau_{\mu'}$. Suppose $\mu<\mu'$. Then we can write from (2.7), (2.15) and (6.3)

$$\begin{aligned}
&\iint(\exp iS(T,0;q_\Delta,p_\Delta))(p_\Delta)_k(t_2)(q_\Delta)_j(t_1)f(q_\Delta(0))\mathscr{D}p_\Delta\mathscr{D}q_\Delta\\
&\quad=\lim_{\varepsilon\to 0}G_\varepsilon(T,\tau_{\nu-1})\chi(\varepsilon\cdot)G_\varepsilon(\tau_{\nu-1},\tau_{\nu-2})\cdots\chi(\varepsilon\cdot)G_\varepsilon(\tau_{\mu'+1},\tau_{\mu'})\\
&\qquad\cdot\chi(\varepsilon\cdot)\tilde{G}_{2,\varepsilon}(\tau_{\mu'},t_2,\tau_{\mu'-1})\chi(\varepsilon\cdot)G_\varepsilon(\tau_{\mu'-1},\tau_{\mu'-2})\cdots\chi(\varepsilon\cdot)G_\varepsilon(\tau_{\mu+1},\tau_\mu)\\
&\qquad\cdot\chi(\varepsilon\cdot)\tilde{G}_{1,\varepsilon}(\tau_\mu,t_1,\tau_{\mu-1})\chi(\varepsilon\cdot)G_\varepsilon(\tau_{\mu-1},\tau_{\mu-2})\cdots\chi(\varepsilon\cdot)G_\varepsilon(\tau_1,0)f.
\end{aligned} \tag{6.22}$$

Here we note

$$\sup_{0<\varepsilon\le 1}(\|\tilde{G}_{1,\varepsilon}(\tau_\mu,t_1,\tau_{\mu-1})f\|_{B^a} + \|\tilde{G}_{2,\varepsilon}(\tau_{\mu'},t_2,\tau_{\mu'-1})f\|_{B^a}) \le C_\Delta\|f\|_{B^{a+1}},$$

where the constant C_Δ may depend on Δ. This inequality can be proved as in the proof of Lemma 6.3. Then applying the equality

$$K_\nu\chi(\varepsilon\cdot)K_{\nu-1}\chi(\varepsilon\cdot)\cdots\chi(\varepsilon\cdot)K_1 f - K'_\nu K'_{\nu-1}\cdots K'_1 f$$
$$= \sum_{j=1}^{\nu} K_\nu\chi(\varepsilon\cdot)\cdots\chi(\varepsilon\cdot)K_{j+1}\chi(\varepsilon\cdot)(K_j - K'_j)K'_{j-1}\cdots K'_1 f$$
$$+ \sum_{j=1}^{\nu-1} K_\nu\chi(\varepsilon\cdot)\cdots\chi(\varepsilon\cdot)K_{j+1}(\chi(\varepsilon\cdot)-1)K'_j\cdots K'_1 f \tag{6.23}$$

to (6.22), we can prove from (6.21) together with (3.6) and Lemma 6.3

$$\iint (\exp iS(T,0;q_\Delta,p_\Delta))(p_\Delta)_k(t_2)(q_\Delta)_j(t_1)f(q_\Delta(0))\mathscr{D}p_\Delta\mathscr{D}q_\Delta$$
$$= \mathscr{C}_\Delta(T,\tau_{\mu'})(\tilde{\mathscr{C}}_2(\tau_{\mu'},\tau_{\mu'-1}) + \sqrt{\tau_{\mu'}-\tau_{\mu'-1}}P_2(\tau_{\mu'},t_2,\tau_{\mu'-1}))\mathscr{C}_\Delta(\tau_{\mu'-1},\tau_\mu)$$
$$\cdot(\tilde{\mathscr{C}}_1(\tau_\mu,\tau_{\mu-1}) + \sqrt{\tau_\mu-\tau_{\mu-1}}P_1(\tau_\mu,t_1,\tau_{\mu-1}))\mathscr{C}_\Delta(\tau_{\mu-1},0)f \tag{6.24}$$

in B^a for $f \in B^{a+2}$. Hence we can prove the assertion (1) from (2.14) of Theorem 1 and (3.4) of Proposition 3.1. In the same way we can prove (1) in the case of $\mu' = \mu$ as well. See also the proof of the assertion (3) below.

We consider the assertion (2). Let $t_1 < t_2$. We may assume $\mu < \mu'$. So we have (6.24). It is proved from (2.14) and (6.19) that

$$\|\mathscr{C}_\Delta(\tau_{\mu'-1},\tau_\mu)\tilde{\mathscr{C}}_1(\tau_\mu,\tau_{\mu-1})\mathscr{C}_\Delta(\tau_{\mu-1},0)f - U(t_2,t_1)\hat{q}_jU(t_1,0)f\|_{B^{a+1}}$$
$$\le \|\mathscr{C}_\Delta(\tau_{\mu'-1},\tau_{\mu-1})\hat{q}_j(\mathscr{C}_\Delta(\tau_{\mu-1},0) - U(\tau_{\mu-1},0))f\|_{B^{a+1}}$$
$$+ \|\mathscr{C}_\Delta(\tau_{\mu'-1},\tau_{\mu-1})\hat{q}_j(U(\tau_{\mu-1},0) - U(t_1,0))f\|_{B^{a+1}}$$
$$+ \|(\mathscr{C}_\Delta(\tau_{\mu'-1},\tau_{\mu-1}) - U(\tau_{\mu'-1},\tau_{\mu-1}))\hat{q}_jU(t_1,0)f\|_{B^{a+1}}$$
$$+ \|(U(\tau_{\mu'-1},\tau_{\mu-1}) - U(t_2,t_1))\hat{q}_jU(t_1,0)f\|_{B^{a+1}}$$
$$\le e^{K_{a+1}T}\|\hat{q}_j(\mathscr{C}_\Delta(\tau_{\mu-1},0) - U(\tau_{\mu-1},0))f\|_{B^{a+1}} + e^{K_{a+1}T}\|\hat{q}_j(U(\tau_{\mu-1},0)$$
$$- U(t_1,0))f\|_{B^{a+1}} + \|(\mathscr{C}_\Delta(\tau_{\mu'-1},\tau_{\mu-1}) - U(\tau_{\mu'-1},\tau_{\mu-1}))\hat{q}_jU(t_1,0)f\|_{B^{a+1}}$$
$$+ \|(U(\tau_{\mu'-1},\tau_{\mu-1}) - U(t_2,t_1))\hat{q}_jU(t_1,0)f\|_{B^{a+1}}.$$

Hence it follows from the uniform convergence of $\mathscr{C}_\Delta(t,s)f$ to $U(t,s)f$ as $|\Delta| \to 0$ and the continuity of $U(t,s)f$ in $0 \le s \le t \le T$, proved in Theorem 1, that

$$\lim_{|\Delta|\to 0}\|\mathscr{C}_\Delta(\tau_{\mu'-1},\tau_\mu)\tilde{\mathscr{C}}_1(\tau_\mu,\tau_{\mu-1})\mathscr{C}_\Delta(\tau_{\mu-1},0)f - U(t_2,t_1)\hat{q}_jU(t_1,0)f\|_{B^{a+1}} = 0 \tag{6.25}$$

for $f \in B^{a+2}$. In the same way we can prove the assertion (2) from (6.24).

The proof of the assertion (3) is almost the same as of the assertion (2) above. Let's use

$$\tilde{G}_\varepsilon(t,t',s)f := (2\pi/m)^{-n}\iint(\exp iS(t,s;\zeta^{t,s}_{x,y,v}))\chi(\varepsilon v)(q^{t,s}_{x,y})_j(t')$$
$$\cdot(mv_k + A_k(t',q^{t,s}_{x,y}(t'))f(y)\,dvdy$$

in place of $\tilde{G}_{N,\varepsilon}(t,t',s)f$ $(N=1,2)$. Then we have as in the proof of (6.24)

$$\Big\| \iint (\exp iS(T,0;q_\varDelta,p_\varDelta))(q_\varDelta)_j(t)(p_\varDelta)_k(t) f(q_\varDelta(0))\mathscr{D}p_\varDelta\mathscr{D}q_\varDelta - \mathscr{C}_\varDelta(T,\tau_\mu) \cdot \mathscr{C}(\tau_\mu,\tau_{\mu-1})\left(\hat{q}_j\hat{p}_k + \frac{1}{i}\delta_{jk}\frac{\tau_\mu - t}{\tau_\mu-\tau_{\mu-1}}\right)\mathscr{C}_\varDelta(\tau_{\mu-1},0)f\Big\|_{B^a} \le C'\sqrt{|\varDelta|}\,\|f\|_{B^{a+2}},$$

which completes the proof of (3).

We study the assertion (4). Let $b(x)$ be a function in Lemma 6.1 and $s \le t' \le t$. Then it follows from (6.10) and Lemma 6.2 that

$$\lim_{\varepsilon\to 0}(2\pi/m)^{-n}\int(\exp iS(t,s;\zeta^{t,s}_{x,y,v}))\chi(\varepsilon v)b(q^{t,s}_{x,y}(t'))f(y)\,dvdy = \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n}\int(\exp iS_c(t,s;q^{t,s}_{x,y}))b(q^{t,s}_{x,y}(t'))f(y)\,dy \tag{6.26}$$

in B^a for $f\in B^{a+M}$. Consider (2.15) where all z are q. Then we can prove from (6.26) as in the proof of (6.24) that (2.15) is equal to (2.16). In addition, the convergence of (2.15) as $|\varDelta|\to 0$ can be proved as in the proof of the assertions (2) and (3). Thus we could complete the proof of Theorem 2.

REMARK 6.1. We state the other results related to Theorem 2 under the assumptions of Theorem 1. Let $0\le t_1\le t_2\le\cdots\le t_k\le T$, $j_1,j_2,\dots,j_k = 1,2,\dots,n$ and $f\in B^{a+k}$. We define

$$\iint(\exp iS(T,0;q_\varDelta,p_\varDelta))(p_\varDelta(t_k)-A(t_k,q_\varDelta(t_k)))_{j_k}\cdots \times (p_\varDelta(t_1)-A(t_1,q_\varDelta(t_1)))_{j_1}f(q_\varDelta(0))\mathscr{D}p_\varDelta\mathscr{D}q_\varDelta \tag{6.27}$$

in the similar way to (2.15). We note $p_\varDelta(t)-A(t,q_\varDelta(t)) = mv_\varDelta(t)$ from (2.7). Then we can prove as in the proof of (2) and (4) of Theorem 2 that (6.27) converges in B^a as $|\varDelta|\to 0$.

Next we consider for $0\le t\le T$

$$\iint(\exp iS(T,0;q_\varDelta,p_\varDelta))(p_\varDelta)_k(t)(p_\varDelta)_j(t)f(q_\varDelta(0))\mathscr{D}p_\varDelta\mathscr{D}q_\varDelta, \tag{6.28}$$

which is equal to

$$\iint(\exp iS(T,0;q_\varDelta,p_\varDelta))\{(mv_\varDelta)_k(t)(mv_\varDelta)_j(t) + A_k(t,q_\varDelta(t))A_j(t,q_\varDelta(t))\}f(q_\varDelta(0))\mathscr{D}p_\varDelta\mathscr{D}q_\varDelta$$
$$+\iint(\exp iS(T,0;q_\varDelta,p_\varDelta))\{(mv_\varDelta)_k(t)A_j(t,q_\varDelta(t)) + (mv_\varDelta)_j(t)A_k(t,q_\varDelta(t))\}f(q_\varDelta(0))\mathscr{D}p_\varDelta\mathscr{D}q_\varDelta.$$

Let $f \in B^{a+2}$. Then it is proved from the result above and the similar result to (4) of Theorem 2 that the first term above converges in B^a as $|\Delta| \to 0$. On the other hand we can prove from (6.9) as in the proof of (3) of Theorem 2 that the limit of the second term in B^a as $|\Delta| \to 0$ is equal to

$$U(T,t)(A_j(t)(\hat{p}_k - A_k(t)) + A_k(t)(\hat{p}_j - A_j(t)))U(t,0)f$$
$$+ \frac{1}{i} \lim_{|\Delta| \to 0} \frac{\tau_\mu - t}{\tau_\mu - \tau_{\mu-1}} U(T,t)\left(\frac{\partial A_j}{\partial x_k}(t) + \frac{\partial A_k}{\partial x_j}(t)\right)U(t,0)f, \tag{6.29}$$

where we take a μ for t and Δ as in (3) of Theorem 2. These imply that (6.28) does not converge in general as $|\Delta| \to 0$.

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