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Equivariant classification of Gorenstein open log del Pezzo surfaces with finite group actions

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Abstract. We classify equivariantly Gorenstein log del Pezzo surfaces with boundaries at infinity and with finite group actions such that the quotient surface modulo the finite group has Picard number one. We also determine the corresponding finite groups.

0. Introduction.

Let $\bar{X}$ be a normal projective rational surface with at worst rational double singularities and let $\bar{D}$ be a reduced divisor on $\bar{X}$. We, further, assume that there is a finite group G acting faithfully on $\bar{X}$ so that $\bar{D}$ is G-stable. We assume that $(\bar{X}, \bar{D})$ has log terminal singularities (cf. [**7**], [**8**], [**12**]) and $\bar{\kappa}(\bar{X} \backslash \bar{D}) = -\infty$. Let $f : X \to \bar{X}$ be the minimal resolution. Let D be the proper transform of $\bar{D}$. We can write $f^*(\bar{D}) = D + \Delta$, where Δ is a positive $\boldsymbol{Q}$-divisor such that $\operatorname{Supp}(\Delta)$ is the exceptional locus of f arising only from the singular points lying on $\bar{D}$ (cf. *ibid.*). It is also known (cf. *ibid.*) that the exceptional graph of $f^{-1}(P)$ with $P \in \bar{D} \cap \operatorname{Sing}(\bar{X})$ is a linear (-2)-chain and that one of the end components of the chain meets transversally D in one point. The G-action on $\bar{X}$ lifts up to a G-action on X such that $D + \Delta$ is G-stable.

Our objective is to describe a pair $(\bar{X}, \bar{D})$ with finite group action of G. In the present article we shall determine the geometric structure of the minimal resolution $(X, D + \Delta_{\mathrm{red}} + A)$ (see Remark (2) below) of $(\bar{X}, \bar{D})$ as well as the group action G on $\bar{X}$. We assume the following:

HYPOTHESIS (H). *$\bar{X}$ has at worst rational double singularities, $(\bar{X}, \bar{D})$ is log terminal, $\bar{D} \neq 0$, $\kappa(\bar{X} \backslash \bar{D}) = -\infty$ and $\rho(\bar{X} /\!\!/ G) = 1$.*

THEOREM A. *Assume the Hypothesis (H). Then either $K_{\bar{X}}^2 \geq 8$ and one of the cases $(1a)$–$(1f)$ in Lemma 4 occurs, or $2 \leq K_{\bar{X}}^2 \leq 6$ and $f^{-1}\bar{D} + f^{-1}(\operatorname{Sing} \bar{X}) = D + \Delta_{\mathrm{red}} + A$* (see Remark (2) below) *is given in Figure m for some* $1 \leq m \leq 43$ (see Section 2).

REMARK. (1) In [**21**], the equivariant classification of the pair (X, G) where X is smooth, is treated. In [**13**], the authors dealt with the pair $(\bar{X}, G)$ where $\rho(\bar{X} /\!\!/ G) \geq 2$, but $\bar{X}$ is assumed to be only log terminal. In [**22**], a finiteness criterion for $|\operatorname{Aut}(X)|$ is given, where X is Gorenstein del Pezzo of Picard number one.

(2) We can write $f^{-1}(\bar{D}) = D + \Delta_{\mathrm{red}}$ and $f^{-1}(\operatorname{Sing} \bar{X}) = \Delta_{\mathrm{red}} + A$, where A is contractible to singular points on $\bar{X}$ but not on $\bar{D}$. We let $A = \sum_i A_i$ be the irreducible decomposition.

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(3) Note that either D is irreducible or $D = D_1 + D_2$ is a linear chain of two smooth rational curves (see Lemma 3). Figure m contains the graph of $D + \varDelta_{\mathrm{red}} + A +$ (some (-1)-curves like E, E_i, F_j), where each (-1)-curve (resp. each other curve) is represented by a broken (resp. solid line) with self intersection typed next to it.

(4) Also shown in each Figure m is a $\boldsymbol{P}^1$-fibration $\Phi : X \to B$ $(\cong \boldsymbol{P}^1)$ with all its singular fibres drawn vertically; thus one can read off from each Figure m, the Picard number $\rho(X)$ and K_X^2 by blowing down X to a Hirzebruch surface. See Lemma 14.

(5) We use the notation $\varDelta = \sum_{i=1}^{t} \varDelta_i$, $(\varDelta_i)_{\mathrm{red}} = \sum_{j=1}^{s_i} \varDelta_i(j)$ in Lemma 2. In each Figure m, if $\varDelta$ or $\varDelta_i$ is irreducible we use the same letter to denote its support (which is a reduced irreducible curve). If $t = 1$, we set $\varDelta(j) := \varDelta_1(j)$.

THEOREM B. *Concerning the group G acting faithfully on $\overline{X}$ (or X), the following assertions hold*:

(1) *With the notations and assumptions in Theorem A, each G is determined in Section 2.*

(2) *Conversely, given Figure m of $D + \varDelta_{\mathrm{red}} + A$ on a surface X for some $1 \le m \le 43$ in Section 2, we let $f : X \to \overline{X}$ be the contraction of $\varDelta + A$ and $\overline{D} = f_* D$. Then we can find a finite group G specified in §2 acting on $\overline{X}$ faithfully such that the Hypothesis (H) is satisfied.*

THEOREM C. *With the notations and assumptions in Theorem A, assume further that $K_X^2 \le 4$. Then either G is soluble or $|G : H| \le 2$ for some group H in $\{g = (a_{ij}) \in PGL_2(\boldsymbol{C}) \mid a_{ij} \ne 0$ only when $i = j$ or $(i, j) = (3, 1)\}$. We have $G = H$ except the case of Figure 25.*

Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of the form above can act on some $\overline{X}$ faithfully so that the Hypothesis (H) is satisfied.

We assume throughout the article that the ground field k is an algebraically closed field of characteristic zero.

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1. Geometric structure of the surface X.

Let us begin with the following result. We assume Hypothesis (H) in §1.

LEMMA 1. *The following conditions are equivalent*:

(1) *The Picard number $\rho(\overline{X} /\!\!/ G) = 1$.*

(2) $(\operatorname{Pic} \overline{X})^G \cong \boldsymbol{Z}$.

(3) $(\operatorname{Pic} \overline{X})^G \otimes \boldsymbol{Q} = ((\operatorname{Pic} \overline{X}) \otimes \boldsymbol{Q})^G \cong \boldsymbol{Q}$.

PROOF. Since the pull back of the quotient map $\overline{X} \to \overline{X} /\!\!/ G$ induces an isomorphism between $(\operatorname{Pic}(\overline{X} /\!\!/ G)) \otimes \boldsymbol{Q}$ and $(\operatorname{Pic} \overline{X})^G \otimes \boldsymbol{Q}$, (1) and (3) are equivalent. Since $\overline{X}$

has at worst quotient singularities, the resolution f induces an isomorphism $\pi_1(X) \to \pi_1(\overline{X})$ by Theorem 7.8 in [**9**]. So $\pi_1(\overline{X}) = 1$ and hence $\operatorname{Pic} \overline{X}$ and $(\operatorname{Pic} \overline{X})^G$ have no torsion elements; thus (2) and (3) are equivalent. This proves the lemma. $\square$

From now on, we assume one of the equivalent conditions of Lemma 1. We call such a pair $(\overline{X}, \overline{D})$ with a finite group action of G a *Gorenstein open log del Pezzo surface* provided $\overline{D} \neq 0$.

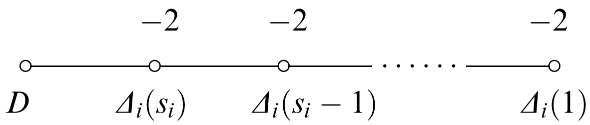
Lemma 2. *The following assertions hold.*

(1) $\overline{D}, -K_{\overline{X}}$ *and* $-(K_{\overline{X}} + \overline{D})$ *are all* $\boldsymbol{Q}$*-ample divisors. Each of them generates* $(\operatorname{Pic} \overline{X}) \otimes \boldsymbol{Q}$. *The divisors* $D + \Delta, -K_X$ *and* $-(K_X + D + \Delta)$ *are all nef and big and* $\boldsymbol{Q}$*-proportional to one another.*

(2) $f : X \to \overline{X}$ *is nothing but the contraction of all* (-2)*-curves on* X. *If* C *is a curve on* X *with* $C^2 < 0$, *then* C *is either a* (-1)*-curve or a* (-2)*-curve. If* $X \to \Sigma_d$ *is a birational morphism to a Hirzebruch surface of degree* d, *then* $d = 0, 1, 2$.

(3) *If* $\Phi : X \to \boldsymbol{P}^1$ *is a* $\boldsymbol{P}^1$*-fibration and* Γ_1 *a singular fibre, then either* (*type* I_n) $\Gamma_1 = E_1 + A_1 + \cdots + A_n + E_2$ *is an ordered linear chain where* $n \geq 0$, *or* (*type* II_n) $\Gamma_1 = 2(E + A_1 + \cdots + A_n) + A_{n+1} + A_{n+2}$ *where* $n \geq 1$ *and both* $E + A_1 + \cdots + A_n$ *and* $A_{n+1} + A_n + A_{n+2}$ *are ordered linear chains, or* (*type* II_0) $\Gamma_1 = A_1 + 2E + A_2$ *where* $A_1 + E + A_2$ *is an ordered linear chain*; *here the* E, E_i *are* (-1)*-curves and the* A_ℓ *are* (-2)*-curves.*

(4) *Let* $\Delta = \sum_{i=1}^{t} \Delta_i$ *be the decomposition into the connected components and let* $(\Delta_i)_{\mathrm{red}} = \sum_{j=1}^{s_i} \Delta_i(j)$ *be the irreducible decomposition with the dual graph below. We set* $\Delta(j) := \Delta_1(j)$ *when* $t = 1$.



Then we have

$$\Delta_i = \sum_{j=1}^{s_i} \frac{j}{s_i + 1} \Delta_i(j).$$

Proof. (1) Note that both $K_{\overline{X}}$ and $\overline{D}$ are in $(\operatorname{Pic} \overline{X})^G \otimes \boldsymbol{Q}$. We shall show that $-(K_{\overline{X}} + \overline{D})$ is $\boldsymbol{Q}$-ample. Note that $\kappa(X, K_X + D + \Delta_{\mathrm{red}}) = -\infty$. Suppose either $K_{\overline{X}} + \overline{D} \equiv 0$ or $K_{\overline{X}} + \overline{D}$ is $\boldsymbol{Q}$-ample. Consider the pull-back $K_X + D + \Delta = f^*(K_{\overline{X}} + \overline{D})$. Then either $n(K_X + D + \Delta) \sim 0$ because X is rational or $n(K_X + D + \Delta) > 0$ for a positive integer n. Then we have

$$-\infty = \kappa(X, K_X + D + \Delta_{\mathrm{red}}) \geq \kappa(X, K_X + D + \Delta) \geq 0,$$

which is absurd. So, $-(K_{\overline{X}} + \overline{D})$ is a $\boldsymbol{Q}$-ample divisor. Clearly, $\overline{D}$ is $\boldsymbol{Q}$-ample. Since $-K_{\overline{X}} = -(K_{\overline{X}} + \overline{D}) + \overline{D}$, the divisor $-K_{\overline{X}}$ is ample.

(2) follows from the ampleness of $-K_{\overline{X}}$ and that $-K_X = f^*(-K_{\overline{X}})$.

(3) is a consequence of (2) (see also Lemma 1.3 in [**20**]).

(4) follows from the fact that $-(K_X + D + \Delta).\Delta_i(j) = 0$. $\square$

The following result describes roughly the shape of the divisor D.

Lemma 3. *We have the following assertions.*

(1) *Either* $D \cong \boldsymbol{P}^1$, *or* $D = D_1 + D_2$, *where* $D_i \cong \boldsymbol{P}^1$ *and* $D_1.D_2 = 1$. *In both cases,* $D.(D + K_X) = -2$.

(2) $D.\Delta < 2$, *and* $D_i.\Delta < 1$ *if* $D = D_1 + D_2$.

(3) $0 < (K_X + D + \Delta)^2 = (K_X + D).K_X + \Delta.D - 2 < (K_X + D).K_X$.

(4) *Suppose* X *contains a* (-1)*-curve* E *which is not a component of* D. *Then* $E \cap D = \emptyset$ *and* $E.\Delta < 1$. *Furthermore,* $\Delta \neq 0$ *and* $E.\Delta > 0$. See also Lemma 8.

Proof. Let D_1 be an irreducible component of D. Then we have

$$\begin{aligned} 0 &< -(K_X + D + \Delta).D_1 \\ &= 2 - 2p_a(D_1) - (D - D_1).D_1 - \Delta.D_1 \le 2 - D_1.(D - D_1). \end{aligned}$$

This implies that $D_1.(D - D_1) \le 1$ and $D_1 \cong \boldsymbol{P}^1$. Since $D + \Delta = f^*(\bar{D})$ is nef and big, it is 1-connected, whence connected; see the proof of Lemma 1 in [**1**]. We note that if D is not connected then $D + \Delta$ is not connected either. This is because each connected component of Δ meets exactly one irreducible component of D. The assertions (1) and (2) are thus proved. To verify the assertion (3), note that $\Delta.(K_X + D + \Delta) = 0$ and $K_X.\Delta = 0$. In view of the assertions (1) and (2), the computation is made as follows:

$$\begin{aligned} 0 &< (K_X + D + \Delta)^2 = (K_X + D + \Delta).(K_X + D) \\ &= (K_X + D)^2 + \Delta.D = -2 + (K_X + D).K_X + \Delta.D < (K_X + D).K_X. \end{aligned}$$

Let E now be a (-1)-curve as in the assertion (4). Then we have

$$0 < -(K_X + D + \Delta).E = 1 - D.E - \Delta.E,$$

where $\Delta.E \ge 0$. This implies that $D.E = 0$ and $E.\Delta < 1$. Suppose $E.\Delta = 0$. Then $E \cap (D + \Delta) = \emptyset$ and the image on $\bar{X}$ of E is disjoint from $\bar{D}$. This contradicts the ampleness of $\bar{D}$. $\square$

In case $\bar{X}$ has no singular points on $\bar{D}$, we can determine the surface X.

Lemma 4. *We have the following assertions.*

(1) *If* $K_X^2 \ge 8$ *then one of the following cases occurs, where* Σ_2 *is the Hirzebruch surface with a minimal section* M *such that* $M^2 = -2$.

(1a) $X = \boldsymbol{P}^2$ *and* $\deg D = 1, 2$,

(1b) $X = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ *and* $\operatorname{bideg} D = (1, 1)$; *there is an element* g *in* G *which interchanges the two different rulings on* X,

(1c) $\bar{X} = \bar{\Sigma}_2$ (*the quadric cone in* $\boldsymbol{P}^3$) *and* $\bar{D}$ *is a hyperplane not passing through the vertex of the cone,*

(1d) $X = \Sigma_2$, $\Delta_{\mathrm{red}} = M$ *and* D *is a fibre,*

(1e) $X = \Sigma_2$, $\Delta_{\mathrm{red}} = M$ *and* D *is a section with self-intersection* 4,

(1f) $X = \Sigma_2$, $\Delta_{\mathrm{red}} = M$ *and* D *is the sum of a fibre and a section* (*disjoint from* M) *with self-intersection* 2.

(2) *If* $\Delta = 0$ *then* $K_X^2 \ge 8$ *and hence case* (1a), (1b), *or* (1c) *occurs.*

Proof. For the assertion (1), note that $X = \boldsymbol{P}^2$ or X is a Hirzebruch surface Σ_d of degree $d \leq 2$ (see Lemma 2). So if $\Delta \neq 0$, then $X = \Sigma_2$ and $\Delta = (1/2)M$. Now (1) follows from the fact that both $D + \Delta$ and $-(K_X + D + \Delta)$ are nef and big. The last part in (1b) follows from the fact that $\rho(\overline{X}/\!\!/G) = 1$.

Let $\Delta = 0$. Suppose the contrary that $K_X^2 \leq 7$. Then X contains a (-1)-curve E. By Lemma 3, E is contained in D. Since D is nef and big we have $D = D_1 + D_2$ with $D_1 = E$ and $D_2^2 \geq 0$. Then both D_i are G-stable. Hence $D_1 = \alpha D_2$ with $\alpha > 0$. Indeed, D_i is the total transform of its image on $\overline{X}/\!\!/G$, and the images of the D_i on $\overline{X}/\!\!/G$ differ by a constant multiple which is a rational number $\alpha > 0$. Then we have

$$-1 = D_1^2 = \alpha D_1.D_2 = \alpha > 0,$$

which is a contradiction. Now (2) follows from (1). $\square$

From now on, we assume that $\Delta \neq 0$.

Lemma 5. *If $D = D_1 + D_2$, we may assume that $D_2^2 \leq 0$.*

Proof. Suppose the contrary that $D_i^2 \geq 1$ for both $i = 1, 2$. We have

$$h^0(X, D_i) = D_i^2 + 2,$$

which follows from an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D_i) \to \mathcal{O}_{\boldsymbol{P}^1}(D_i^2) \to 0.$$

Then one can find a member $\tilde{D}_1 \in |D_1|$ such that $\#(\tilde{D}_1 \cap D_2) \geq D_1^2 + 1 \geq 2$. Then D_2 is a component of $\tilde{D}_1$ since $D_1.D_2 = 1$. Similarly, there exists a member $\tilde{D}_2 \in |D_2|$ such that D_1 is a component of $\tilde{D}_2$. This implies that $D_1 \sim D_2$ and $D_1^2 = 1$. Since $\Delta \neq 0$, there is an irreducible component $\Delta_i(s)$ of Δ such that $D.\Delta_i(s) = 1$. This is absurd for $D = D_1 + D_2 \sim 2D_1$. So it is wrong to assume that $D_i^2 \geq 1$ for both $i = 1, 2$. Hence we may assume that $D_2^2 \leq 0$. $\square$

Lemma 6. *Suppose that both D_1 and D_2 are G-stable and $K_X^2 \leq 7$. Then Δ has two connected components Δ_i, where Δ_1 is irreducible and Δ_2 has length 2. $f^{-1}\overline{D} + f^{-1}(\mathrm{Sing}\,\overline{X}) = D + \Delta_{\mathrm{red}} + A$ is given in Figure 1 in Section 2.*

Proof. We use the notation $\Delta = \sum_{i=1}^{t} \Delta_i$ and $\Delta_i = \sum_{j=1}^{s_i} \Delta_i(j)$ of Lemma 2. We may assume that Δ_i meets D_1 (resp. D_2) for $1 \leq i \leq t_1$ (resp. for $t_1 + 1 \leq i \leq t = t_1 + t_2$). By Lemma 3, $1 > D_1.\Delta = \sum_{i=1}^{t_1} s_i/(s_i + 1) \geq t_1/2$. Hence $t_1 = 0, 1$. Similarly, $t_2 = 0, 1$. Thus $t = t_1 + t_2 = 1, 2$. So we may assume that D_i meets the connected component Δ_i of Δ of length s_i. We put $s_i = 0$ if $t_i = 0$.

Since $\overline{D}_1 = \alpha \overline{D}_2$ on $\overline{X}$ with $\alpha > 0$, we have

$$D_1 + \Delta_1 = \alpha(D_2 + \Delta_2). \tag{1}$$

Taking the intersection of (1) with D_1, we have

$$D_1^2 + \frac{s_1}{s_1 + 1} = \alpha. \tag{2}$$

Taking the intersection of (1) with D_2, we obtain

$$1 = \alpha\left(D_2^2 + \frac{s_2}{s_2+1}\right). \tag{3}$$

Since D_2^2 is an integer, the equation (3) implies that $D_2^2 \geq 0$. Then $D_2^2 = 0$ by Lemma 5 and hence $\alpha = (s_2+1)/s_2$. Plugging the value of α in the equation (2), we have

$$D_1^2 = \alpha - \frac{s_1}{s_1+1} = \frac{s_2+1}{s_2} - \frac{s_1}{s_1+1} = \frac{1}{s_2} + \frac{1}{s_1+1}.$$

Since D_1^2 is an integer, we have the following possibilities:

$$(s_1, s_2; \alpha; D_1^2) = (0, 1; 2; 2),\ (1, 2; 3/2; 1).$$

We shall show the assertion that the first case (resp. the second case) implies that $X = \Sigma_2$ and $K_X^2 = 8$ (resp. implies the result of the lemma). Indeed, in the first (resp. second) case, if Γ_2 is a singular fibre of the $\boldsymbol{P}^1$-fibration φ induced by $|D_2|$ containing no components of Δ (resp. containing $\Delta_2(1)$ but no Δ_1), then it is of type I_n in Lemma 2 and the cross-section D_1 must meet a (-1)-curve E_1 of Γ_2, contradicting Lemma 3 (4). In the second case, if Γ_2 is the fibre of φ containing both $\Delta_2(1)$ and Δ_1 then it is of type II_0. So the assertion is true and the lemma proved. $\square$

Now consider the case where $D = D_1 + D_2$ and $g(D_1) = D_2$ for some g in G. Then $D_1^2 = D_2^2$. By the proof of Lemma 6, Δ has two connected components Δ_i and we may assume that D_i meets Δ_i so that $g(\Delta_1) = \Delta_2$ and hence Δ_1 and Δ_2 are (-2)-linear chains of the same length $s \geq 1$. Since $\bar{D}$ is not contractible to a point, it follows that $D_1^2 = D_2^2 \geq -1$. By Lemma 5, we have $D_1^2 = -1$ or 0. Write $(\Delta_i)_{\mathrm{red}} = \sum_{j=1}^{s_i} \Delta_i(j)$ as in Lemma 2.

Lemma 7. *Suppose that $g(D_1) = D_2$ for some g in G. Then $D_1^2 = -1$. Furthermore, there is a $\boldsymbol{P}^1$-fibration $\Phi : X \to B$ ($B \cong \boldsymbol{P}^1$) such that $D_1 + D_2$ is a fibre. According to the possible types of the singular fibres of Φ, we have five different cases*; *see Figures* 2–6 *in Section* 2, *each of which also contains the graph $f^{-1}\bar{D} + f^{-1}(\mathrm{Sing}\,\bar{X}) = D + \Delta_{\mathrm{red}} + A$.*

$G = \langle g \rangle \cong \boldsymbol{Z}/(2)$ is realizable; *indeed either X or its blow-down of the G-stable curve E* (*for Figures* 2 *and* 3) *is obtained from the Hirzebruch surface Σ_2* (*or one point Q_i blow-up of Σ_2 for Figures* 2, 5, 6) *by taking a double cover ramifying along a smooth* (*or singular at Q_i*) *irreducible member of $|-K_{\Sigma_2}|$ with G equal to the Galois group* $\mathrm{Gal}(X/\Sigma_2)$.

Proof. Suppose $D_1^2 \neq -1$. Then $D_1^2 = D_2^2 = 0$ as shown above. Consider the $\boldsymbol{P}^1$-fibration $\Phi_{|D_2|}$ for which D_1 and the component $\Delta_2(s)$ are cross-sections. Since $\Delta_1 \neq 0$, the map $\Phi_{|D_2|}$ has a singular fibre Γ_1 comprising $(\Delta_1)_{\mathrm{red}}$ and $(\Delta_2)_{\mathrm{red}} - \Delta_2(s)$. Note that there are no other singular fibres because no (-1)-curves lying outside of D meet the cross-section D_1 by Lemma 3. The singular fibre Γ_1 consists of (-2)-curves and (-1)-curves. Since Δ_1 and Δ_2 have the same length s, only possibility for Γ_1 is that $s = 1$ and the dual graph of $\Gamma = E_1 + \Delta_1(1) + E_2$ is of type I_1 in Lemma 2 such that E_1 meets the cross-section $\Delta_2(1)$. Then $E_1.\Delta = 1$, which is a contradiction to Lemma 3.

Now assume that $D_1^2 = D_2^2 = -1$. Then $|D_1 + D_2|$ defines a $\boldsymbol{P}^1$-fibration $\Phi : X \to B$, where $B \cong \boldsymbol{P}^1$. The $\Delta_i(s)$ are the cross-sections of Φ. Suppose that Δ_1 has length $s \geq 2$. Let Γ_1 be the singular fibre of Φ containing $(\Delta_1)_{\mathrm{red}} - \Delta_1(s) = \Delta_1(s-1) + \Delta_1(s-2) + \cdots + \Delta_1(1)$. If Γ_1 contains no components of Δ_2, then $\Gamma_1 = E_1 + \Delta_1(s-1) + \cdots + \Delta_1(1) + E_2$ is of type I_{s-1} in Lemma 2 so that E_i meets the cross-section $\Delta_2(s)$ for $i = 1$ or 2. Then we have by Lemma 2,

$$\Delta.E_i \geq \frac{1}{s+1} + \frac{s}{s+1} = 1,$$

which is impossible by Lemma 3. So, the fibre Γ_1 contains also $(\Delta_2)_{\mathrm{red}} - \Delta_2(s)$. This implies that $s = 2$ and $\Gamma_1 = \Delta_1(1) + 2E + \Delta_2(1)$ is of type II_0 in Lemma 2, where E is a (-1)-curve. In particular, the length of the (-2) linear chain Δ_i is less than or equal to 2. The possible cases of all singular fibres of Φ are exhausted by the following four (see Fact 12 in the proof of Lemma 11); this proves the lemma (the realization part is easy to check).

1. $s = 2$; $K_X^2 = 3$; $\Gamma_0 := D_1 + D_2$, $\Gamma_1 := \Delta_1(1) + 2E + \Delta_2(1)$ and $\Gamma_2 := E_1 + A + E_2$ are of types I_0, II_0, I_1 in Lemma 2. See Figure 2.
2. $s = 2$; $K_X^2 = 3$; $\Gamma_0 := D_1 + D_2$, $\Gamma_1 := \Delta_1(1) + 2E + \Delta_2(1)$ and $\Gamma_{i+1} := E_i + F_i$ $(i = 1, 2)$ are of types I_0, II_0, I_0, I_0. See Figure 3.
3. $s = 1$; $K_X^2 = 4$; $\Gamma_0 := D_1 + D_2$ and $\Gamma_i := E_i + F_i$ $(i = 1, 2, 3)$ are all of types I_0. See Figure 4.
4. $s = 1$; $K_X^2 = 4$; $\Gamma_0 := D_1 + D_2$ and $\Gamma_1 = E_1 + A_1 + A_2 + E_2$ are of types I_0, I_2. See Figure 5.
5. $s = 1$; $K_X^2 = 4$; $\Gamma_0 := D_1 + D_2$, $\Gamma_1 = E_1 + F_1$ and $\Gamma_2 = E_2 + A + F_2$ are of types I_0, I_0, I_1. See Figure 6. □

Now we switch to the case D is irreducible. Note that $\Delta \neq 0$ is assumed.

LEMMA 8. *Suppose D is irreducible. Then the following assertions hold.*

(1) $h^0(-K_X - D) \geq K_X.(K_X + D) > 0$. *Hence* $|-(K_X + D)| \neq \emptyset$.

(2) $|-(K_X + D + \Delta_{\mathrm{red}})| \neq \emptyset$.

(3) *Let* E $(\neq D)$ *be a* (-1)*-curve. Then* $E.\Delta_{\mathrm{red}} = 1, 2$. *If* $E.\Delta_{\mathrm{red}} = 2$, *then* $D.\Delta_{\mathrm{red}} = 2$ (*i.e.*, $t = 2$ *in notation of Lemma* 2) *and* $E + D + \Delta_{\mathrm{red}}$ *is a simple loop and linearly equivalent to* $-K_X$.

(4) *We have* $D^2 \geq -1$. *The number* t *of connected components of* Δ *is at most* 3. *If* $t = 3$, *then, in notation of Lemma* 2, $(s_1 + 1, s_2 + 1, s_3 + 1) = (2, 2, n)$ $(n \geq 2)$ *or* $(2, 3, n)$ $(n = 3, 4, 5)$ (those triplets are called the Platonic numbers).

PROOF. By the Riemann-Roch theorem, we have

$$\begin{aligned} h^0(-K_X - D) - h^1(-K_X - D) + h^0(2K_X + D) &= \frac{(-K_X - D).(-2K_X - D)}{2} + 1 \\ &= K_X^2 + \frac{D^2}{2} + \frac{3K_X.D}{2} + 1 \\ &= K_X^2 + K_X.D > 0; \end{aligned}$$

here we used that $K_X.D + D^2 = -2$ and $K_X^2 + K_X.D > 0$ in Lemma 3. Note that

$h^0(2K_X + D) \le h^0(2(K_X + D + \Delta)) = 0$ because $-(K_X + D + \Delta)$ is nef and big. Now the assertion (1) follows.

Let $(\Delta_1)_{\mathrm{red}} = A_s + A_{s-1} + \cdots + A_1$ be a connected component of Δ_{red} such that $D.A_s = 1$. Since $-(K_X + D).A_s = -D.A_s < 0$, we have $|-(K_X + D + A_s)| \neq \varnothing$. Suppose $|-(K_X + D + A_s + \cdots + A_i)| \neq \varnothing$. Since $-(K_X + D + A_s + \cdots + A_i).A_{i-1} = -A_i.A_{i-1} < 0$, it follows that $|-(K_X + D + A_s + \cdots + A_{i-1})| \neq \varnothing$. So $|-(K_X + D + (\Delta_1)_{\mathrm{red}})| \neq \varnothing$. Likewise, $|-(K_X + D + \Delta_{\mathrm{red}})| \neq \varnothing$.

Let E be a (-1)-curve not in D. Then $E.\Delta_{\mathrm{red}} > 0$ by Lemma 3. Let $p : X \to Y$ be the blow-down of E. If $E.\Delta_{\mathrm{red}} \ge 2$ then $|K_Y + p_*(D + \Delta_{\mathrm{red}})| \neq \varnothing$ by the Riemann-Roch theorem or Lemma 2.1.3 in [**11**], page 7. Since $|-(K_Y + p_*(D + \Delta_{\mathrm{red}}))| \neq \varnothing$ by the assertion (2), it follows that $K_Y + p_*(D + \Delta_{\mathrm{red}}) \sim 0$. So $t = 2$ and E meets the end component of each Δ_i which is located on the opposite side of D. Namely, $D + \Delta_{\mathrm{red}} + E$ is a simple loop.

Since $D.\Delta < 2$ by Lemma 3, in notation of Lemma 2, we have

$$\sum_{i=1}^{t} \left(1 - \frac{1}{s_i + 1}\right) < 2.$$

It then follows that $t \le 3$. Moreover, if $t = 3$ then $\{s_1 + 1, s_2 + 1, s_3 + 1\}$ is one of the Platonic triplets upto permutations. So always $D^2 \ge -1$, for otherwise $D + \Delta$ is contractible, contradicting the nef and bigness of $D + \Delta$ (see Satz 2.11 in [**2**]). $\square$

Consider the case Δ has three connected components Δ_i (see Lemma 2). Let C_i be the component of Δ_i meeting D, i.e., $C_i = \Delta_i(s_i)$ in notation of Lemma 2.

Lemma 9. *Suppose that D is irreducible and Δ has three connected components. Then $D^2 = -1$. Also Δ_{red} consists of three disjoint irreducible curves $C_i = (\Delta_i)_{\mathrm{red}}$ $(i = 1, 2, 3)$ and $-K_X = 2D + C_1 + C_2 + C_3$. Furthermore, $K_X^2 = 2$ and there is a birational morphism $q : X \to \boldsymbol{P}^2$ such that $q_*(D + C_1 + C_2 + C_3)$ is a union of a line and a conic touching each other in one point. $|2D + C_1 + C_2|$ defines a $\boldsymbol{P}^1$-fibration $\Phi : X \to B$, and according to the different types of the singular fibres of Φ, there are seven possible cases; see Figures* 7–13, *each of which also contains the graph $f^{-1}\bar{D} + f^{-1}(\mathrm{Sing}\,\bar{X}) = D + \Delta_{\mathrm{red}} + A$.*

Proof. By Lemma 8, $D^2 \ge -1$. Consider the case $D^2 = -1$. Let $p : X \to Y$ be the blow-down of D and let $\bar{C}_i = p(C_i)$. Then the $\bar{C}_i$ share one point in common. So $|K_Y + \bar{C}_1 + \bar{C}_2 + \bar{C}_3| \neq \varnothing$ by the Riemann-Roch theorem or Lemma 2.1.3 in [**11**], page 7. Since $|-(K_Y + \bar{\Delta}_{\mathrm{red}})| \neq \varnothing$ by Lemma 8, we have $-(\bar{\Delta}_{\mathrm{red}} - \bar{C}_1 - \bar{C}_2 - \bar{C}_2) \ge 0$, where $\bar{\Delta} = p_*(\Delta)$. Hence it follows that $\bar{\Delta}_{\mathrm{red}} = \bar{C}_1 + \bar{C}_2 + \bar{C}_3$ and $K_Y + \bar{C}_1 + \bar{C}_2 + \bar{C}_3 = 0$. Thence follows the assertion on the expression of $-K_X$. In order to obtain the morphism q, we let $q_1 : X \to X_1$ be the blow-down of $D + C_3$ and continue blowing down further to reach a relatively minimal model Σ_d with $d = 0, 1, 2$ (see Lemma 2). Then one can bypass the blow-down steps to reach $\boldsymbol{P}^2$. By making use of the property that $-K_Z$ for a surface Z appearing in the blow-down step is the sum of the images of C_1 and C_2, any (-1)-curve on Z meets transversally exactly one of the images of C_1 and C_2 in one point. If we set $B_1 := q(C_1)$ and $B_2 := q(C_2)$, then $B_1 + B_2$ is a cubic curve and $B_1 \cap B_2$ consists of a single point. Hence we may assume that B_1 is a line and B_2 is a conic. Let $\Gamma_0 := 2D + (\Delta_1)_{\mathrm{red}} + (\Delta_2)_{\mathrm{red}}$ and let Φ be the $\boldsymbol{P}^1$-fibration with Γ_0 as a fibre. Since $-K_X = 2D + \sum_i (\Delta_i)_{\mathrm{red}}$ supports a fibre and a 2-section $(\Delta_3)_{\mathrm{red}}$, every (-2)-curve, i.e., every component of $f^{-1}(\mathrm{Sing}\,\overline{X})$ other than $(\Delta_3)_{\mathrm{red}}$ is contained in a fibre. So $f^{-1}(\overline{D}) + f^{-1}(\mathrm{Sing}\,\overline{X}) = D + \Delta_{\mathrm{red}} + A$ is given in one of Figures 7–13 in Section 2. See Lemma 2 for possible types of singular fibres; see also Fact 12 in the proof of Lemma 11. To be precise, the following cases are not included but reduced to other cases, and Figures 7–7′ appear on the same X with two different fibrations.

Case 9.1. $\Gamma_0, \Gamma_1 = E_1 + A_1 + A_2 + A_3 + E_2$ which is of type I_3 in Lemma 2, are the only singular fibres of Φ. Also the 2-section $(\Delta_3)_{\mathrm{red}}$ meets each E_i. By going to a Hirzebruch surface Σ_d $(d \le 2)$, we see that there is a (-1)-curve E on X such that $E.A_2 = E.(\Delta_i)_{\mathrm{red}} = 1$ for $i = 1$ or 2 say for $i = 1$. Then $\Gamma_0' := 2D + (\Delta_2)_{\mathrm{red}} + (\Delta_3)_{\mathrm{red}}$ is the singular fibre of a new $\boldsymbol{P}^1$-fibration Φ', and $\Gamma_1' := 2(E + A_2) + A_1 + A_3$ is also a singular fibre of Φ'. So Φ' fits Figure 8 after relabeling Δ_i.

Case 9.2. Γ_0, and $\Gamma_i = E_i + A_i + F_i$ $(i = 1, 2)$ each of which is of type I_1 in Lemma 2, are the only singular fibres of Φ. Also the 2-section $(\Delta_3)_{\mathrm{red}}$ meets each of E_i, F_j. We can find a (-1)-curve E on X such that $E.A_1 = E.A_2 = E.\Delta_i = 1$ for $i = 1$ or 2 say for $i = 1$. Then $\Gamma_0' := 2D + (\Delta_2)_{\mathrm{red}} + (\Delta_3)_{\mathrm{red}}$ is the singular fibre of a new $\boldsymbol{P}^1$-fibration Φ', and $\Gamma_1' := 2E + A_1 + A_2$ is also a singular fibre of Φ'. So Φ' fits Figure 10 after relabeling Δ_i.

Consider the case $D^2 = 0$. Then $|D|$ defines a $\boldsymbol{P}^1$-fibration $\Phi : X \to B$ for which the curves C_1, C_2, C_3 are cross-sections. Suppose $\{s_1 + 1, s_2 + 1, s_3 + 1\} = \{2, 2, n\}$ with $n \ge 3$. Write $\Delta_3 = C_3 + A_m + \cdots + A_1$ with $m = n - 2 \ge 1$. Then there exists a singular fibre Γ_1 of Φ such that $\Gamma_1 = E_1 + A_m + \cdots + A_1 + E_2$ is an ordered linear chain and of type I_m in Lemma 2 so that E_i meets the cross-section C_2 for $i = 1$ or 2. Then $E_i + D + \Delta_{\mathrm{red}}$ contains a loop and $(\Delta_1)_{\mathrm{red}}$, contradicting Lemma 8. In the case $\{s_1 + 1, s_2 + 1, s_3 + 1\} = \{2, 2, 2\}$, we note that Φ is not a relatively minimal $\boldsymbol{P}^1$-fibration. Hence Φ has a singular fibre Γ_1 of type I_k and Γ_1 contains a (-1)-curve E_1 meeting two of C_1, C_2, C_3, say meeting C_1, C_2. Then $E + D + \Delta_{\mathrm{red}}$ contains a loop and C_3, contradicting Lemma 8. The case $\{s_1 + 1, s_2 + 1, s_3 + 1\} = \{2, 3, n\}$ with $n = 3, 4, 5$ also leads to a contradiction.

Consider the case $D^2 = 1$. Let $p : X \to \boldsymbol{P}^2$ be the birational morphism defined by $|D|$. Since $D^2 = 1$ and hence D is linearly equivalent to the pull-back of a line, the morphism p is a composite of the blow-downs of (-1)-curves which are disjoint from D and its images. This implies that $B_i := p(C_i)$ $(i = 1, 2, 3)$ is a curve. Since

$$-K_{\boldsymbol{P}^2} = -p_*(K_X) \ge p_*(D + C_1 + C_2 + C_3),$$

it follows that $\deg(-K_{\boldsymbol{P}^2}) \ge 4$, which is a contradiction.

Consider the case $D^2 \ge 2$. Then we have by Lemma 3

$$K_X^2 \ge 1 - D.K_X = 3 + D^2 \ge 5.$$

Hence we have

$$5 \ge 10 - K_X^2 = \rho(X) \ge \rho(\overline{X}) + \#\Delta \ge 1 + \#\Delta,$$

where $\#\Delta$ signifies the number of the irreducible components of Δ. So, $s_1 + s_2 + s_3 =$

$\#\Delta \le 4$. Thus the possible cases of $\{s_1+1, s_2+1, s_3+1\}$ are $\{2,2,2\}$ and $\{2,2,3\}$ up to permutations. If $\{s_1+1, s_2+1, s_3+1\} = \{2,2,3\}$, then $\rho(\overline{X}) = 1$, $\operatorname{Sing}\overline{X} = 2A_1 + A_2$ and $K_X^2 = 5$. But this case cannot occur by the classifications of the distributions of singular points (cf. Lemma 3 in part I of [**14**]). If $\{s_1+1, s_2+1, s_3+1\} = \{2,2,2\}$, then either $\rho(X) = 4$ or $\rho(X) = 5$. In the first case, we have $\rho(\overline{X}) = 1$ and $\operatorname{Sing}\overline{X} = 3A_1$, which is also impossible [**14**]. In the second case, we have either $\rho(\overline{X}) = 1$ and $\operatorname{Sing}\overline{X} = 4A_1$, or $\rho(\overline{X}) = 2$ and $\operatorname{Sing}\overline{X} = 3A_1$. The case $\rho(\overline{X}) = 1$ is ruled out by [**14**] and the case $\rho(\overline{X}) = 2$ by [**19**]. □

Next we consider the case where Δ is connected. Write $\Delta_{\mathrm{red}} = \Delta(s) + \cdots + \Delta(1)$ as an ordered linear chain so that $D.\Delta(s) = 1$.

LEMMA 10. *Suppose that D is irreducible, Δ is connected of length s and $K_X^2 \le 7$. Then the following assertions hold, where each of Figures* 14–23 *contains the graph $f^{-1}\overline{D} + f^{-1}(\operatorname{Sing}\overline{X}) = D + \Delta_{\mathrm{red}} + A$.*

(1) *For any (-1)-curve E on X, it holds that $E.\Delta_{\mathrm{red}} = 1$ and $E \cap D = \emptyset$.*

(2) *$D^2 = 0, 1, 2$.*

(3) *If $D^2 = 0$, then $s = 2$ (resp. 4) and $K_X^2 = 6$ (resp. $K_X^2 = 5$). There are three possible cases (see Figures* 14–16). *In Figure* 16, *there is an element g in G such that $g(E_1) = E_2$.*

(4) *If $D^2 = 2$, then $s = 2$, $K_X^2 = 6$ and there are two possible cases (see Figures* 17 *and* 18). *In Figure* 17 *(resp.* 18), *E (resp. E_1, E_2) are the only (-1)-curve(s) on X. In Figure* 18, *we have $g(E_1) = E_2$ for some g in G.*

(5) *If $D^2 = 1$, then either $s = 4$ and $K_X^2 = 5$ (see Figure* 19), *or $s = 1$ and $K_X^2 = 6$ (see Figures* 20–21). *In Figures* 19–20, *the E is the only (-1)-curve on X.*

PROOF. (1) It follows from Lemmas 3 and 8 (see also (2)).

(2) By Lemma 3, we have

$$7 \ge K_X^2 > 2 - \Delta.D - K_X.D = 4 + D^2 - \Delta.D = 3 + D^2 + \frac{1}{s+1}.$$

Since $D + \Delta$ is nef and big, we have $D^2 \ge 0$, for otherwise $D + \Delta_{\mathrm{red}}$ is contractible to a point, a contradiction. Hence $7 \ge K_X^2 \ge 4 + D^2 \ge 4$. So, $D^2 = 0, 1, 2, 3$.

Suppose $D^2 = 3$. Then $K_X^2 = 7$ and

$$3 = \rho(X) \ge \rho(\overline{X}) + s \ge 1 + s \ge 2,$$

whence $s = 1, 2$. If $s = 2$, then $\rho(\overline{X}) = 1$ and $\operatorname{Sing}\overline{X} = A_2$, and this case does not occur (cf. [**14**]). So, $s = 1$. If $\rho(\overline{X}) = 1$, then $\operatorname{Sing}\overline{X} = A_1 + A_1$, and this case does not occur either (cf. *ibid.*). Thus $\rho(\overline{X}) = 2$, $K_X^2 = 7$ and $\operatorname{Sing}\overline{X} = A_1$.

Take a (-1)-curve E_1 on X. Then $E_1.\Delta_{\mathrm{red}} = 1$ by the assertion (1). The blow-down of E_1 brings X to the Hirzebruch surface Σ_1, as the image of Δ_{red} becomes a (-1)-curve. Let P be the image of E_1 on Σ_1. Then the proper transform E_2 of a fibre of the ruling on Σ_1 passing through P is a (-1)-curve meeting D. This contradicts the assertion (1). Hence $D^2 = 0, 1, 2$.

(3) Let $D^2 = 0$. Let $\Phi : X \to B$ be the $\boldsymbol{P}^1$-fibration with D as a fibre. Since D is G-stable, G permutes fibres of Φ.

Suppose $D^2 = 0$ and $s = 1$. Then there is a singular fibre $\Gamma = E_1 + A_1 + \cdots + A_n + E_2$ of type I_n, where we may assume that the (-1)-curve E_1 meets the cross-section Δ_{red}. Then $E_2 \cap \Delta = \emptyset$, contradicting Lemma 3.

Suppose $D^2 = 0$ and $s \geq 2$. Let Γ_1 be the singular fibre containing $\Delta(s-1) + \cdots + \Delta(1)$. Then Γ_1 is G-stable. If Φ contains a second singular fibre Γ_2, then we can reach the same contradiction as above. Hence Γ_1 is the only singular fibre of Φ. If $s \geq 3$ and $\Gamma_1 = E_1 + \Delta(1) + \cdots + \Delta(s-1) + E_2$ is an ordered linear chain and a singular fiber of type I_{s-1}, then the image on $\overline{X}$ of E_1 is G-stable and contractible, contradicting $\rho(\overline{X}/\!\!/G) = 1$. By the arguments above, all possible types of Γ_1 are given in Figures 14–16. In Figure 16, since Γ_1 is G-stable, each element in G either stabilizes E_i or interchanges E_1, E_2. If E_i is G-stable then the image on $\overline{X}$ of E_i is G-stable and contractible, contradicting $\rho(\overline{X}/\!\!/G) = 1$.

(4) Suppose $D^2 = 2$. Since $7 \geq K_X^2 \geq 4 + D^2 = 6$, we have $K_X^2 = 6$ or 7. Accordingly, $\rho(X) = 4$ or 3. Since $2 \leq 1 + s \leq \rho(\overline{X}) + s \leq \rho(X)$, we have $s = 3, 2, 1$.

Suppose $s = 3$. Then $\rho(X) = 4$, $\rho(\overline{X}) = 1$ and $\operatorname{Sing} \overline{X} = A_3$. This case does not occur by [**14**].

Suppose $s = 2$. If $\rho(X) = 3$ then $\rho(\overline{X}) = 1$ and $\operatorname{Sing} \overline{X} = A_2$, and this case does not occur either [**14**]. If $\rho(X) = 4$, either $\rho(\overline{X}) = 1$, $\operatorname{Sing} \overline{X} = A_1 + A_2$ and there is only one (-1)-curve on X, or $\rho(\overline{X}) = 2$, $\operatorname{Sing} \overline{X} = A_2$ and there are only two (-1)-curves on X; see Figures 5 and 6 in [**19**].

Take a (-1)-curve E_1 on X. Note that $E_1 . \Delta_{\mathrm{red}} = 1$ by the assertion (1). Let $X \to \boldsymbol{P}^2$ be the blow-down of $E_1 + \Delta(1) + \Delta(2)$ and $\tilde{D}$ the image of D. If $E_1 . \Delta(1) = 1$, then $\tilde{D}^2 = 3$, which is impossible on $\boldsymbol{P}^2$. Hence $E_1 . \Delta(2) = 1$ and $\tilde{D}^2 = 4$. Let P be the point on $\tilde{D}$ which is the fundamental point of the blow-down $X \to \boldsymbol{P}^2$. Let ℓ be a line which is tangent to $\tilde{D}$ at P. Reverse the above blow-down. Let L be the proper transform of ℓ. There are two possibilities according as $E_1 \cap L \neq \emptyset$ or $E_1 \cap L = \emptyset$. In the former case, L is a (-2)-curve (see Figure 17 where $A := L$ and $\rho(\overline{X}) = 1$) and, in the latter case, L is a (-1)-curve (see Figure 18 where $E_2 := L$ and $\rho(\overline{X}) = 2$). Note that E (resp. E_1 and E_2) is/are the only (-1)-curve(s) on X (see Figures 5 and 6 in [**19**]). In Figure 18, we have $g(E_1) = E_2$ for some g in G (see the argument for Figure 16).

Suppose $s = 1$. Take a (-1)-curve E_1. Then $E_1 . \Delta(1) = 1$. Let $p : X \to Y$ be the blow-down of E_1 and $\Delta(1)$. Since $K_X^2 \geq 6$, we have $K_Y^2 \geq 8$ and $\tilde{D}^2 = 3$. Since there are no curves $\tilde{D}$ on $\boldsymbol{P}^2$ with $\tilde{D}^2 = 3$, $K_Y^2 = 8$. Hence $Y \cong \Sigma_d$ with $d = 0, 1, 2$ (see Lemma 2). But any curve on Σ_d with $d = 0, 2$ has self-intersection number divisible by 2. So, $Y \cong \Sigma_1$. Let M and ℓ be respectively the minimal section and a fibre on Σ_1. Then $\tilde{D} \sim M + 2\ell$. Let P be the fundamental point of p. If $P = \tilde{D} \cap M$, then $\Delta(1) + p'(M)$ is a (-2)-chain, for otherwise there appears a (-3)-curve on X; since $s = 1$, this is a contradiction. If $P \neq \tilde{D} \cap M$, then $p'(M)$ is a (-1)-curve meeting D, contradicting Lemma 3.

(5) Now assume that $D^2 = 1$. Note that $7 \geq K_X^2 \geq 4 + D^2 = 5$. On the other hand, since

$$2 \leq s + 1 \leq s + \rho(\overline{X}) \leq \rho(X) = 10 - K_X^2 \leq 6 - D^2 = 5,$$

we have $1 \leq s \leq 4$. We consider all possible cases according to the value of s. We note that $E . \Delta_{\mathrm{red}} = 1$ for any (-1)-curve E on X by the assertion (1).

CASE $s = 4$. Then $K_X^2 = 5$, $\rho(\overline{X}) = 1$ and $\mathrm{Sing}\,\overline{X} = A_4$. Take a (-1)-curve E on X. If $E.\Delta(i) = 1$ for $i = 1$ or 4, then one can blow down $E + \Delta_{\mathrm{red}}$ and the resulting surface Y has $K_Y^2 = 10$. This is a contradiction. Suppose $E.\Delta(2) = 1$. Then D is a component of a fibre of the $\boldsymbol{P}^1$-fibration defined by $|2(E + \Delta(2)) + \Delta(1) + \Delta(3)|$. Since $D^2 = 1$, this is a contradiction. Consequently, $E.\Delta(3) = 1$. We thus obtain Figure 19, where E is the only (-1)-curve on X (see Figure 5 in [**19**]).

CASE $s = 3$. We claim that this case does not take place. Note that $\rho(X) = 4$ or 5. If $\rho(X) = 4$ then $\rho(\overline{X}) = 1$ and $\mathrm{Sing}\,\overline{X} = A_3$, which is not the case by [**14**]. So $\rho(X) = 5$. If $\rho(\overline{X}) = 1$ then $\mathrm{Sing}\,\overline{X} = A_1 + A_3$, which is not the case either by [**14**]. So $\rho(\overline{X}) = 2$, $\mathrm{Sing}\,\overline{X} = A_3$ and $K_X^2 = 5$. Then there are only two (-1)-curves E_1, E_2 on X with $E_i.\Delta_{\mathrm{red}} = 1$ $(i = 1, 2)$ and $E_1.\Delta(2) = E_2.(\Delta(1) + \Delta(3)) = 1$ (see Figure 6 in [**19**]). So both E_i are G-stable, and the image on $\overline{X}$ of E_2 is G-stable and contractible, contradicting $\rho(\overline{X}/\!\!/G) = 1$.

CASE $s = 2$. We shall show that this case does not take place. Let E be a (-1)-curve. Then $E.\Delta_{\mathrm{red}} = 1$. Let $p : X \to Y$ be the blow-down of $E + \Delta(1) + \Delta(2)$. Since $K_X^2 \geq 5$, we have $K_Y^2 \geq 8$. If $E.\Delta(2) = 1$, then $p_*(D)^2 = 3$. As in the proof of the assertion (4) for the case $D^2 = 2$ and $s = 1$, Y must be the Hirzebruch surface Σ_1. Let M be the minimal section of Σ_1. Since $s = 2$, the fundamental point of p is different from the point $M \cap p_*(D)$. Then $E_1 := p'(M)$ is a (-1)-curve such that $E_1.D = 1$, contradicting Lemma 3. Hence $E.\Delta(1) = 1$. Then $p_*(D)^2 = 2$ and $K_Y^2 \geq 8$. Hence Y is the Hirzebruch surface Σ_d. Since $p_*(D)^2 = 2$, one can readily show that $d = 0, 2$ (see Lemma 2).

Suppose first that $Y \cong \Sigma_2$. Let M be the minimal section. Then $M \cap p_*(D) = \emptyset$. Reversing the above blow-down and noting that the length of Δ_{red} is 2, we can show that $E_1 := p'(\ell)$ is a (-1)-curve meeting $\Delta(2)$, where ℓ is the fibre passing through the fundamental point of p. So, we are lead to a contradiction by the above case.

Suppose $Y \cong \Sigma_0$. Let ℓ be one of the fibres (of the two different $\boldsymbol{P}^1$-fibrations) passing through the fundamental point of p. Then it follows that $E_1 := p'(\ell)$ is a (-1)-curve meeting $\Delta(2)$. Again, we are lead to a contradiction. Consequently, the case $s = 2$ does not occur.

CASE $s = 1$. Let E be a (-1)-curve. Then $E.\Delta(1) = 1$. Let $p : X \to Y$ be the blow-down of $E + \Delta(1)$. Since $K_X^2 \geq 5$, we have $K_Y^2 \geq 7$.

Suppose $K_X^2 = 5$. Then Y has a $\boldsymbol{P}^1$-fibration π, which is not relatively minimal but contains a singular fibre consisting of two (-1)-curves $E_1 + E_2$. Since $p_*(D)^2 = 2$, $p_*(D)$ is not contained in a fibre of π. This implies that $p_*(D) \cap E_i \neq \emptyset$ for $i = 1$ or $i = 2$, say for $i = 1$. Then the fundamental point P of the morphism p is not contained in $p_*(D) \cap E_1$, for otherwise $s \geq 2$ or $p'(E_1)^2 \leq -3$, a contradiction. Hence $p'(E_1)$ remains as a (-1)-curve on X which meets D. This contradicts Lemma 3. We have therefore $K_X^2 = 6$ and $Y \cong \Sigma_d$ with $d = 0, 2$ because $p_*(D)^2 = 2$ (see Lemma 2).

Suppose $Y \cong \Sigma_2$. Let M be the minimal section and let ℓ be the fibre passing through the fundamental point P of p. Consider the inverse of the morphism p. After blowing up the point P, there are two possibilities of taking the centre Q of the next blow-up. Namely, Q lies (resp. does not lie) on the proper transform of ℓ. The first case gives rise to Figure 20, where $\rho(\overline{X}) = 1$, $\operatorname{Sing}\overline{X} = A_1 + A_2$ and E is the only (-1)-curve on X (see [**19**], Figure 5). In the second case, $E + \Delta(1) + \ell' + M'$ has the dual graph

$$(-1) - (-2) - (-1) - (-2)$$

where ℓ', M' are the proper transforms of ℓ, M. If $\rho(\overline{X}) = 1$, then $\operatorname{Sing}\overline{X} = A_1 + A_2$ by [**14**], but X has two (-1)-curves, contradicting Figure 5 in [**19**]. If $\rho(\overline{X}) = 2$, then $\operatorname{Sing}\overline{X} = 2A_1$, and E, ℓ' are the only (-1)-curves on X by Figure 6 in [**19**]. So both (-1)-curves are G-stable, hence the image on $\overline{X}$ of E is G-stable and contractible, contradicting $\rho(\overline{X}/\!\!/G) = 1$.

Suppose $Y \cong \Sigma_0$. We also consider the inverse of the morphism p. Let ℓ_i be the fibres of the two different $\boldsymbol{P}^1$-fibrations which pass through the fundamental point P. After blowing up the point P, there are two choices of taking the center Q of the next blow-up. Namely, if Q is on the proper transform of ℓ_i with $i = 1$ say, then $\ell_2' + \Delta(1) + E + \ell_1'$ has the dual graph in the above paragraph and we will reach the same contradiction; if Q does not lie on the proper transforms of ℓ_i, then we obtain Figure 21 ($E := E_1$). We have $\rho(\overline{X}) = 3$ and $\operatorname{Sing}\overline{X} = A_1$ (see Figure 5 and 6 in [**19**]). We have thus verified all the assertions of Lemma 10. □

We finally consider the case where Δ has two connected components Δ_1 and Δ_2. As in Lemma 2, write

$$(\Delta_1)_{\mathrm{red}} = \Delta_1(s) + \cdots + \Delta_1(1)$$
$$(\Delta_2)_{\mathrm{red}} = \Delta_2(t) + \cdots + \Delta_2(1),$$

where $D.\Delta_1(s) = D.\Delta_2(t) = 1$.

Lemma 11. *Suppose that D is irreducible, that Δ_{red} has two connected components $(\Delta_i)_{\mathrm{red}}$ $(i = 1, 2)$ of lengths s, t and that $K_X^2 \le 7$. Then the following assertions hold, where each of Figures* 22–43 *contains the graph of $f^{-1}\overline{D} + f^{-1}(\operatorname{Sing}\overline{X}) = D + \Delta_{\mathrm{red}} + A$.*

(0) *If $s \ne t$ then both Δ_i are G-stable.*
(1) *$K_X^2 \ge 3 + D^2$, and $K_X^2 \ge 4 + D^2$ provided $s = t = 1$.*
(2) *$2 \le s + t \le 6 - D^2$, and $s + t \le 5 - D^2$ provided $s = t = 1$.*
(3) *$-1 \le D^2 \le 4$, and $0 \le D^2 \le 3$ provided $s = t = 1$.*
(4) *For any (-1)-curve E $(\ne D)$ on X, we have either $E.\Delta_{\mathrm{red}} = 1$, or $E.\Delta_{\mathrm{red}} = 2$, $E.\Delta_i(1) = 1$ $(i = 1, 2)$ and $-K_X = E + D + \Delta_{\mathrm{red}}$. One can say simply that $D + \Delta_{\mathrm{red}} + E$ is a simple loop in the latter case.*
(5) *In case $D^2 = -1$, there are ten possibilities for $D + \Delta$ (see Figures* 22–31). *In Figure* 25 *(resp.* 27*), there is an element g in G such that $g(E_1) = E_2$ (resp. $g(\Delta_1) = \Delta_2$). In Figures* 28 *and* 31, *no E_i or F_j is G-stable.*
(6) *In case $D^2 = 0$, there are nine possibilities (see Figures* 32–40). *In Figures* 34, 36 and 40, *no E_i or F_j is G-stable. In Figures* 33, 35 *and* 39, *there is an element g in G such that $g(\Delta_1) = \Delta_2$.*
(7) *The case $D^2 = 4$ is impossible. In case $D^2 = 3$, there is one possibility; see Figure* 41 *where E is the only (-1)-curve on X.*
(8) *The case $D^2 = 2$ is impossible.*

(9) *In case $D^2 = 1$, there are two possibilities* (*see Figures* 42 *and* 43). *In Figure* 42, *the E_1 and E_2 are the only (-1)-curves on X. In Figure* 43, *E, E_1, E_2 are the only (-1)-curves on X and there is an element g in G such that $g(E_1) = E_2$.*

Proof. The last part of assertion (5) or (6) follows from the arguments for Figure 16 in Lemma 10 and Figure 31 below. Indeed, in Figures 27, 33 and 39, the argument of Figure 16 shows that $g(E_2) = F_2$ for some g in G; in Figure 35, the argument of Figure 31 shows that $g(E_1) = F_1$ for some g in G.

(0) is clear for $D + \varDelta$ is G-stable.

(1) Since

$$K_X^2 > 2 - \varDelta.D - D.K_X = 4 + D^2 - \varDelta.D = 2 + D^2 + \frac{1}{s+1} + \frac{1}{t+1}$$

by Lemma 3, it follows that $K_X^2 \geq 3 + D^2$, and that $K_X^2 \geq 4 + D^2$ if $s = t = 1$.

(2) Since

$$1 + s + t \leq s + t + \rho(\overline{X}) \leq \rho(X) = 10 - K_X^2 < 8 - D^2 - \left(\frac{1}{s+1} + \frac{1}{t+1}\right),$$

we conclude that $2 \leq s + t \leq 6 - D^2$, and that $s + t \leq 5 - D^2$ if $s = t = 1$.

(3) By Lemma 8, $D^2 \geq -1$. Since $K_X^2 \leq 7$, we have $D^2 \leq 4$. If $s = t = 1$ and $D^2 = -1$, then $D + \varDelta_{\mathrm{red}}$ is negative semi-definite. Hence $D^2 \geq 0$ if $s = t = 1$. Furthermore, $D^2 \leq 3$ if $s = t = 1$ (cf. the assertion (1)).

(4) It follows from Lemmas 3 and 8.

(5) Let $D^2 = -1$. Then $s + t \geq 3$ by the assertion (3). We may assume that $s \leq t$. Set $\varGamma_0 := \varDelta_1(s) + 2D + \varDelta_2(t)$. Then $\varPhi := \varPhi_{|\varGamma_0|} : X \to B$ $(B \cong \boldsymbol{P}^1)$ is a $\boldsymbol{P}^1$-fibration with a singular fibre $\varGamma_0$ and cross-sections $\varDelta_1(s-1)$ (if $s \geq 2$) and $\varDelta_2(t-1)$ (if $t \geq 2$). Since $\varGamma_0$ is G-stable, G permutes fibres of $\varPhi$. We will often use the following:

Fact 12. *Suppose $s \geq 2$. There is then a composite of blow downs $X \to \Sigma_2$ of all (-1)-curves in fibres not meeting the cross-section $\varDelta_1(s-1)$ so that $\varDelta_1(s-1)$ becomes the minimal section M and $\varDelta_2(t-1)$ becomes a section disjoint from M and with self-intersection* 2.

Suppose $s = 1$ and $t = 2$. If $\varGamma_1$ $(\neq \varGamma_0)$ is a singular fibre, then it is of type I_n with two (-1)-curves E_1, E_2 as in Lemma 2, and we may assume that the cross-section $\varDelta_2(1)$ meets E_2; then $E_1 \cap \varDelta = \varnothing$, contradicting Lemma 3. So $\varPhi$ has only one singular fibre $\varGamma_0$. Hence $K_X^2 = 6$. See Figure 22.

Suppose $s = 1$ and $t \geq 3$. Let $\varGamma_1$ be the singular fibre of $\varPhi$ containing $\varDelta_2(t-2) + \cdots + \varDelta_2(1)$. By the argument above, $\varGamma_1$ is the only singular fibre besides $\varGamma_0$. If $\varGamma_1$ is of type I_n $(n = t-2)$ then no (-1)-curve in $\varGamma_1$ is G-stable and hence $t = 3$; see the argument for Figure 16. Now according to the different types of $\varGamma_1$ in Lemma 2, we have three cases: Figure 23 with $t = 3$ and $K_X^2 = 4$, Figure 24 with $t = 5$ and $K_X^2 = 3$ and Figure 25 with $t = 3$ and $K_X^2 = 4$.

Suppose $s = t = 2$. According to the number of singular fibres and using Fact 12 and assertion (4), we have three cases: Figures 26, 27, 28 all with $K_X^2 = 3$.

Suppose $s=2$ and $t \geq 3$. Let Γ_1 be the fibre of Φ containing $\Delta_2(1)+\cdots+\Delta_2(t-2)$. Then $\Gamma_1 = E_1 + \Delta_2(1)+\cdots+\Delta_2(t-2)+E_2$ is an ordered linear chain and a singular fiber of type I_{t-2} in Lemma 2, and we may assume that E_1 intersects the cross-section $\Delta_1(1)$. Note that Γ_1, E_i are all G-stable (see the assertion (0)); so if $t=3$ then the image on $\overline{X}$ of E_2 is G-stable and contractible, contradicting $\rho(\overline{X}/\!\!/G)=1$. Thus $t \geq 4$ and $t=4,5$. If Φ has exactly one type I_0 singular fibre $\Gamma_2 = F_1+F_2$, then Γ_2 is G-stable; we may assume F_1 (resp. F_2) intersects the cross-section $\Delta_1(1)$ (resp. $\Delta_2(t-1)$) and hence both F_i are G-stable, but then the image on $\overline{X}$ of F_1 is G-stable and contractible, contradicting $\rho(\overline{X}/\!\!/G)=1$. By the above arguments and by Fact 12 and assertion (4), we see that $t=5$, $K_X^2=2$ and Γ_0, Γ_1 are the only singular fibres of Φ. See Figure 29.

Suppose $s=t=3$. We also consider the possible singular fibres of Φ. Then by Fact 12 and assertion (4), two cases given in Figures 30 and 31 survive. In Figure 31, if one of E_i, F_j say E_1 is G-stable, then $2(E_1+\Delta_1(2))+\Delta_1(3)+\Delta_1(1)$ is a G-stable fibre of another $\boldsymbol{P}^1$-fibration Ψ where all exceptional divisors of $f: X \to \overline{X}$ are contained in fibres; this Ψ induces a $\boldsymbol{P}^1$-fibration on $\overline{X}/\!\!/G$, whence $\rho(\overline{X}/\!\!/G) \geq 2$, a contradiction.

The remaining case is $s=3$ and $t=4$. In this case we can show by the argument for the case $s=t=3$ that there is no possibility.

(6) Let $D^2=0$. Then $K_X^2 \geq 3+D^2=3$ and $s+t \leq 6-D^2=6$. We again assume $s \leq t$. Let $\Phi: X \to B$ $(B \cong \boldsymbol{P}^1)$ be the $\boldsymbol{P}^1$-fibration defined by $|D|$, for which $\Delta_1(s)$ and $\Delta_2(t)$ are cross-sections. Since D is G-stable, G permutes fibres of Φ.

Suppose $s=t=1$. Again, we consider all possibilities of the singular fibres of Φ listed up in Lemma 2. By Fact 12 and the assertion (4), there are five possibilities: Figures 32–36.

Suppose $s=1$ and $t \geq 2$. Let Γ_1 be the singular fibre of Φ containing $(\Delta_2)_{\mathrm{red}} - \Delta_2(t)$. Then $\Gamma_1 = E_1+\Delta_2(1)+\cdots+\Delta_2(t-1)+E_2$ is an ordered linear chain as in Lemma 2 and we may assume that E_1 meets the cross-section $(\Delta_1)_{\mathrm{red}}$. Note that Γ_1, E_i are all G-stable. If $t=2$, then the image on $\overline{X}$ of E_2 is G-stable and contractible, contradicting $\rho(\overline{X}/\!\!/G)=1$; so $t \geq 3$. If $t=3$, then we reach a contradiction as in Figure 31 by considering another $\boldsymbol{P}^1$-fibration Φ_1 defined by $|2(E_2+\Delta_2(2))+\Delta_2(3)+\Delta_2(1)|$. Thus $t \geq 4$ and hence $t=4,5$. By the argument for the case $D^2=-1$, $s=2$, $t \geq 4$, it is impossible that Φ contains a unique type I_0 singular fibre. By the arguments above and by Fact 12 and assertion (4), we see that $t=5$, $K_X^2=3$ and Γ_1 is the only singular fibre of Φ. See Figure 37.

Suppose $s \geq 2$ and $t \geq 2$. Then $(s,t)=(2,2),(2,3),(2,4)$ or $(3,3)$. If $s+t=6$, then

$$7 \leq s+t+\rho(\overline{X}) \leq \rho(X) = 10-K_X^2 \leq 7.$$

So $\rho(\overline{X})=1$ and $\operatorname{Sing}\overline{X} = A_s+A_t$ with $(s,t)=(2,4),(3,3)$. But these cases are impossible by [**14**]. Suppose $(s,t)=(2,3)$. Then we have

$$6 \leq s+t+\rho(\overline{X}) \leq \rho(X) \leq 7.$$

Hence either $\rho(X)=6$ or $\rho(X)=7$. If $\rho(X)=6$, then $\rho(\overline{X})=1$ and $\operatorname{Sing}\overline{X}=A_2+A_3$ which is impossible by [**14**]. Suppose $\rho(X)=7$. If $\rho(\overline{X})=1$ then $\operatorname{Sing}\overline{X}=A_1+A_2+$

A_3, which is impossible by [**14**]. If $\rho(\overline{X}) = 2$ then $\operatorname{Sing} \overline{X} = A_2 + A_3$, which is impossible by [**19**]. So the remaining case is $(s, t) = (2, 2)$.

Let Γ_i be the singular fibre of Φ containing $\Delta_i(1)$. If Γ_1 is of type I_1 in Lemma 2, then so is Γ_2, and we can write $\Gamma_i = E_i + \Delta_i(1) + F_i$ such that F_1 (resp. E_2) meets the cross-section $\Delta_2(2)$ (resp. $\Delta_1(2)$); since $D + \Delta$ is G-stable, the image on $\overline{X}$ of $E_1 + F_2$ is G-stable and also contractible, contradicting $\rho(\overline{X}/\!\!/G) = 1$. The above argument, Fact 12 and the assertion (4) imply that there are only three possibilities. See Figures 38–40, where $K_X^2 = 3$ in all three cases.

(7) Consider the case $D^2 = 4$. Then $K_X^2 = 7$ and $s + t \le 6 - D^2 = 2$. Hence $(s, t) = (1, 1)$, while then $s + t \le 5 - D^2 = 1$, which is absurd. So the case $D^2 = 4$ does not occur.

Consider the case $D^2 = 3$. Then $K_X^2 \ge 3 + D^2 = 6$, whence $K_X^2 = 6$ or 7. Meanwhile, $s + t \le 6 - D^2 = 3$. So, $(s, t) = (1, 1), (1, 2)$.

Suppose $(s, t) = (1, 1)$. Then $K_X^2 \ge 4 + D^2 = 7$. Hence $K_X^2 = 7$ by the assumption. Since we have

$$3 = \rho(X) \ge \rho(\overline{X}) + s + t \ge 1 + 2,$$

we have $\rho(\overline{X}) = 1$ and $\operatorname{Sing} \overline{X} = 2A_1$, which is impossible by [**14**].

Suppose $(s, t) = (1, 2)$. Then we have

$$4 \ge 10 - K_X^2 = \rho(X) \ge \rho(\overline{X}) + s + t \ge 1 + 3,$$

whence follows that $K_X^2 = 6$, $\rho(\overline{X}) = 1$ and $\operatorname{Sing} \overline{X} = A_1 + A_2$. Note that there is only one (-1)-curve E on X and $E.(\Delta_1)_{\mathrm{red}} = E.\Delta_2(1) = 1$ by the assertion (4) and by Figure 5 in [**19**]. See Figure 41.

(8) We shall show that $D^2 = 2$ is impossible. In fact, since $2 \le s + t \le 6 - D^2 = 4$, we have $(s, t) = (1, 1), (1, 2), (1, 3)$ or $(2, 2)$, where we assume $s \le t$. Note that $K_X^2 \ge 3 + D^2 = 5$. If $s + t = 4$, we have $\rho(\overline{X}) = 1$ and $\operatorname{Sing} \overline{X} = A_s + A_t$ with $(s, t) = (1, 3), (2, 2)$. But we cannot find these cases in the table in [**14**].

Suppose $(s, t) = (1, 1)$. Then $K_X^2 \ge 4 + D^2 = 6$, whence $K_X^2 = 6, 7$. We utilize the inequality

$$3 = 1 + s + t \le \rho(\overline{X}) + s + t \le \rho(X) = 10 - K_X^2.$$

If $K_X^2 = 7$, we have $\rho(\overline{X}) = 1$ and $\operatorname{Sing} \overline{X} = 2A_1$, which is impossible by [**14**]. If $K_X^2 = 6$, then either $\rho(\overline{X}) = 1$ and $\operatorname{Sing} \overline{X} = 3A_1$, or $\rho(\overline{X}) = 2$ and $\operatorname{Sing} \overline{X} = 2A_1$. The former case is impossible by [**14**]. In the latter case, take a (-1)-curve E. If $E.\Delta_{\mathrm{red}} = 1$, say $E.(\Delta_1)_{\mathrm{red}} = 1$, the blowing-down of $E + (\Delta_1)_{\mathrm{red}}$ brings X to Σ_2, while the image $\tilde{D}$ of D satisfies $\tilde{D}^2 = 3$, which is impossible on Σ_2. If $E.\Delta_{\mathrm{red}} = 2$, then $E.\Delta = 1$, contradicting Lemma 3.

Suppose $(s, t) = (1, 2)$. By an argument similar to the above using the inequalities

$$4 = 1 + s + t \le \rho(\overline{X}) + s + t \le \rho(X) = 10 - K_X^2 \le 5,$$

we see that $\operatorname{Sing} \overline{X} = A_1 + A_2$ and either $K_X^2 = 6$ and $\rho(\overline{X}) = 1$, or $K_X^2 = 5$ and $\rho(\overline{X}) = 2$. In the first case, there is only one (-1)-curve on X and $E.\Delta_i(1) = 1$ $(i = 1, 2)$ by the assertion (4) and Figure 5 in [**19**]. Let $p : X \to Y$ be the blow-down of $E, \Delta_2(1), \Delta_2(2)$.

Since $K_Y^2 = K_X^2 + 3 = 9$, we have $Y \cong \boldsymbol{P}^2$. However, $p_*(D)^2 = 3$, which is impossible on $\boldsymbol{P}^2$. In the second case, there are only three (-1)-curves on X one of which is disjoint from Δ (see Figure 6 in [**19**]); this contradicts Lemma 3.

(9) Now we treat the case $D^2 = 1$. Note that $K_X^2 \geq 3 + D^2 = 4$ and $s + t \leq 6 - D^2 = 5$. We shall prove the following claim.

Claim 13. *There exists a* (-1)*-curve, say* E*, on* X *such that* $E.\Delta_{\mathrm{red}} = 2$.

Proof. Consider the morphism $q : X \to \boldsymbol{P}^2$ defined by $|D|$. Then D is the pull-back of a line by q. Since D is not touched, $\Delta_1(s)$ and $\Delta_2(t)$ are mapped to lines ℓ_1 and ℓ_2, respectively. Let $P := \ell_1 \cap \ell_2$. Then P is one of the fundamental point of the morphism. We consider to reverse the morphism. Let E_1 be the (-1)-curve appearing by the blowing-up of P. If E_1 stays as a (-1)-curve on X, then it is a (-1)-curve we require for. Otherwise, one of the intersection points P_1, P_2 of E_1 with the proper transforms of ℓ_1 and ℓ_2 is blown up, but both points are not; if both points are blown up, there will appear a $(-n)$-curve with $n \geq 3$, a contradiction. Then the proper transform of E_1 on X is contained in Δ_{red}. Now blow up one of the points P_1, P_2 and apply the same argument as above to the (-1)-curve E_2 appearing from the blow-up. We have just only to continue this argument. $\square$

In case $s + t = 5$, we have $6 \leq s + t + \rho(\overline{X}) \leq \rho(X) = 10 - K_X^2 \leq 6$. Hence $\rho(\overline{X}) = 1$ and $\operatorname{Sing} \overline{X} = A_s + A_t$ with $(s,t) = (1,4), (2,3)$. But these cases are impossible by [**14**].

Suppose $s + t = 4$. Then we have

$$5 \leq s + t + \rho(\overline{X}) \leq \rho(X) = 10 - K_X^2 \leq 6.$$

If $K_X^2 = 5$, then $\rho(\overline{X}) = 1$ and $\operatorname{Sing} \overline{X} = A_s + A_t$ with $(s,t) = (1,3), (2,2)$. These cases do not exist by [**14**]. If $K_X^2 = 4$, by [**14**], [**19**], we have $(s,t) = (1,3)$, and either $\rho(\overline{X}) = 1$ and $\operatorname{Sing} \overline{X} = 2A_1 + A_3$ or $\rho(\overline{X}) = 2$ and $\operatorname{Sing} \overline{X} = A_1 + A_3$. See Figures 42 and 43, where E_1 or E is as in Claim 13. The part about the uniqueness of the (-1)-curves in the assertion (9) follows from Figures 5 and 6 in [**19**]. Since G acts on the set of (-1)-curves on X, in Figure 43, E is G-stable and each element of G either stabilizes or switches E_1 and E_2; so the existence of g in G with $g(E_1) = E_2$ follows from $\rho(\overline{X}/\!/G) = 1$ (see the argument for Figure 16).

Suppose $(s,t) = (1,1)$. Then a (-1)-curve E as in Claim 13 has $E.\Delta = 1$, contradicting Lemma 3.

Suppose $(s,t) = (1,2)$. Consider the $\boldsymbol{P}^1$-fibration $\Phi : X \to B$ defined by $|\Gamma_0|$ where $\Gamma_0 := \Delta_1(1) + 2E + \Delta_2(1)$. Then we can make the Hirzebruch surface Σ_2 out of X with the image of $\Delta_2(2)$ as the minimal section. The blow-down of $E, \Delta_1(1)$ increases D^2 by 1. Since $D.\Delta_2(2) = 1$, the blow-down of $E, \Delta_1(1)$ is not enough to bring X to Σ_2. Hence there exists a singular fibre Γ_1 of type I_n which then contains a (-1)-curve E_1 meeting the cross-section D. This is a contradiction by Lemma 3. This ends the proof of Lemma 11.

2. Determination of the group G action on X.

In this section, we shall consider all 43 triplets $(\overline{X}, \overline{D}; G)$ in Theorem A in the introduction, determine the action of the finite group G on X and give examples.

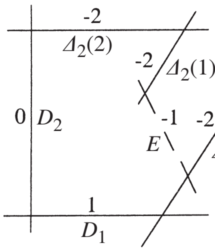


Figure 1.

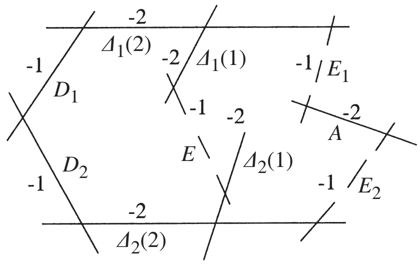


Figure 2.

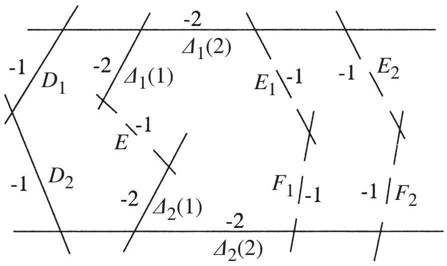


Figure 3.

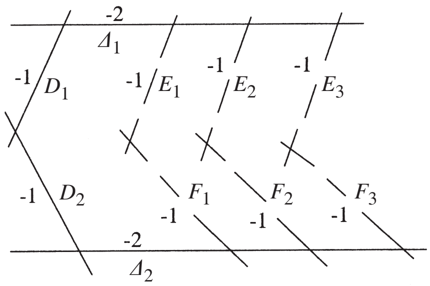


Figure 4.

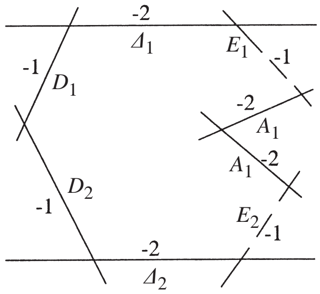


Figure 5.

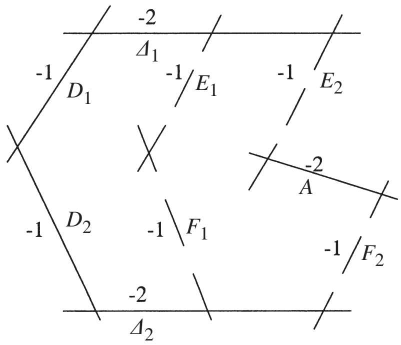


Figure 6.

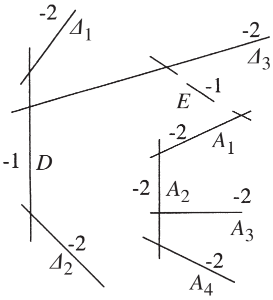


Figure 7.

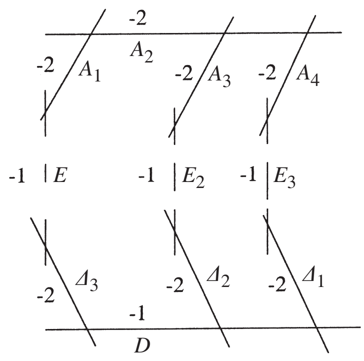


Figure 7′.

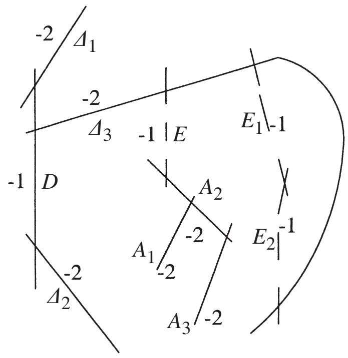


Figure 8.

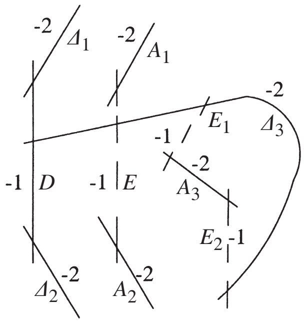


Figure 9.

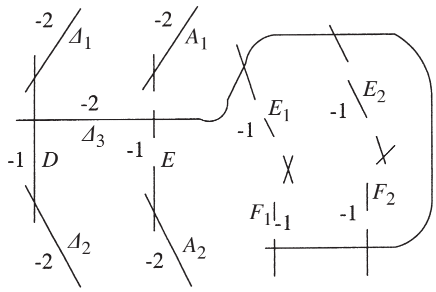


Figure 10.

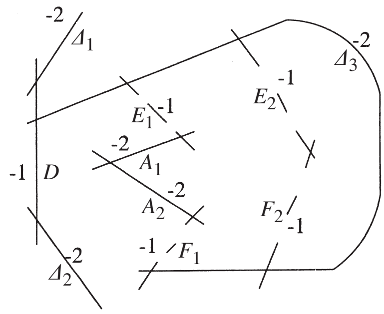


Figure 11.

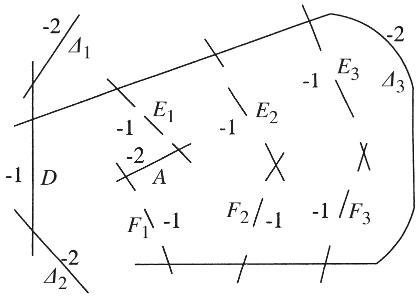


Figure 12.

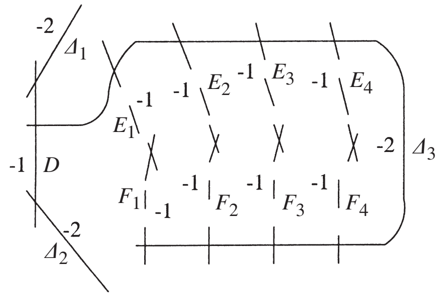


Figure 13.

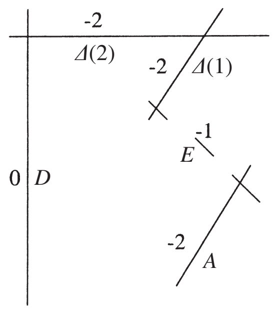


Figure 14.

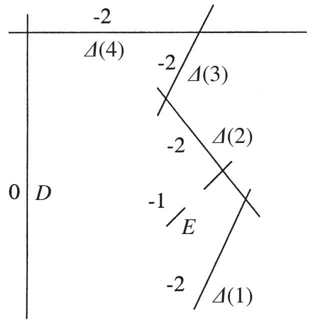


Figure 15.

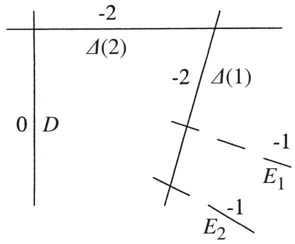


Figure 16.

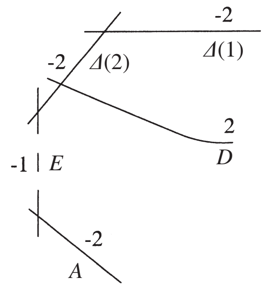


Figure 17.

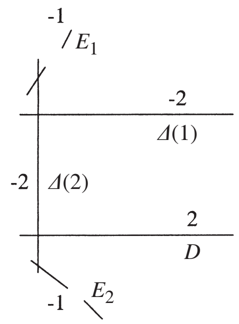


Figure 18.

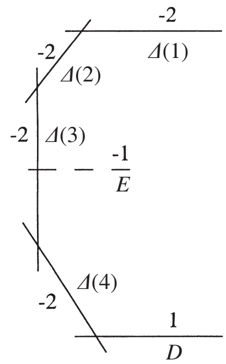
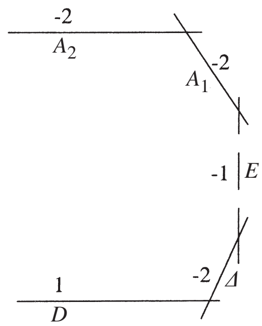


Figure 19.



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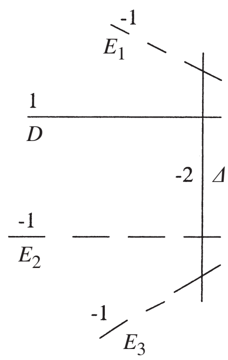

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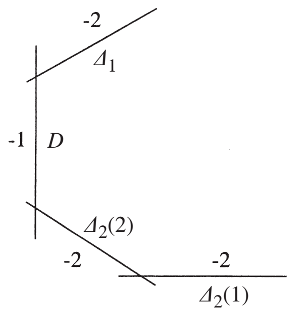

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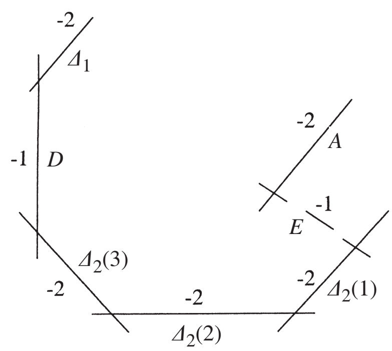

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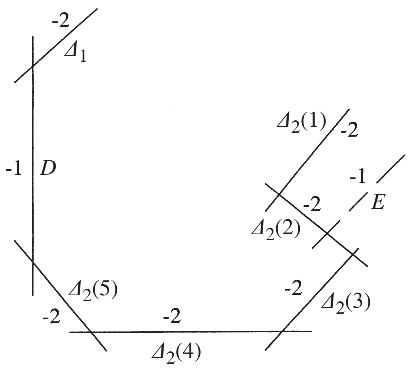

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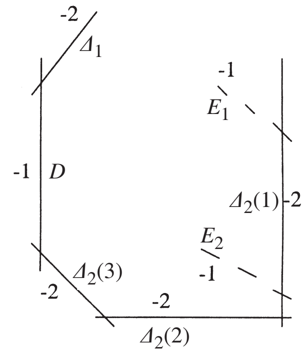

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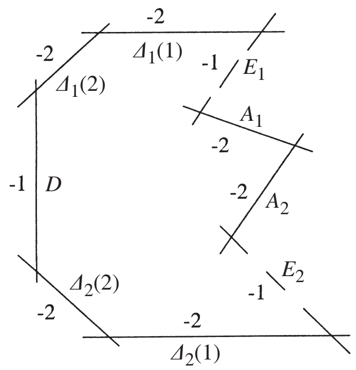

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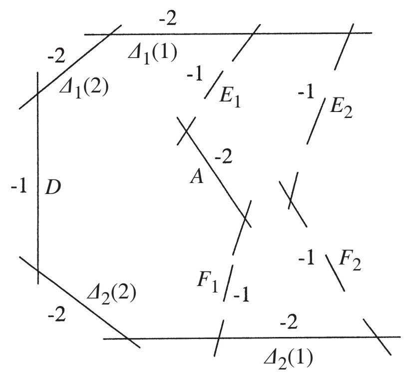

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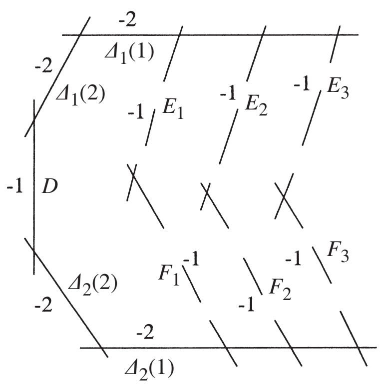

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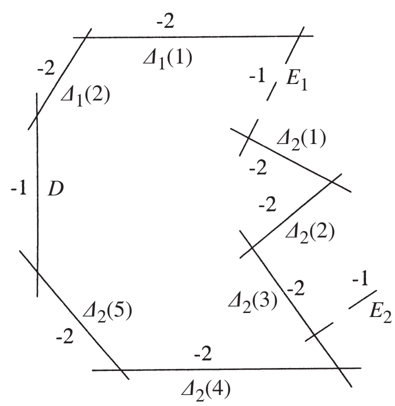

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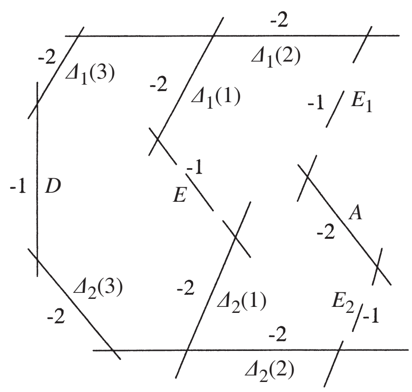

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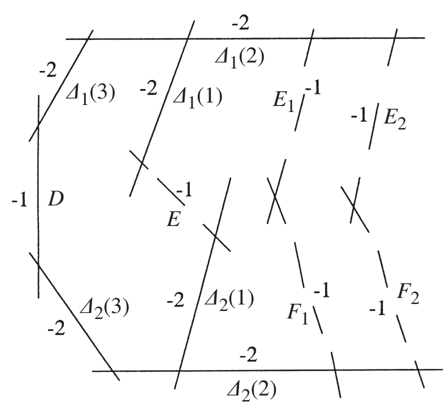

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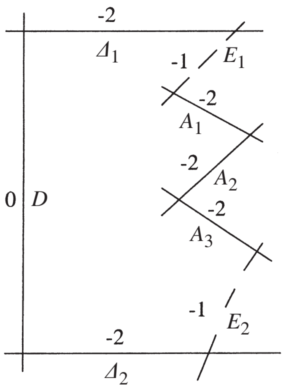

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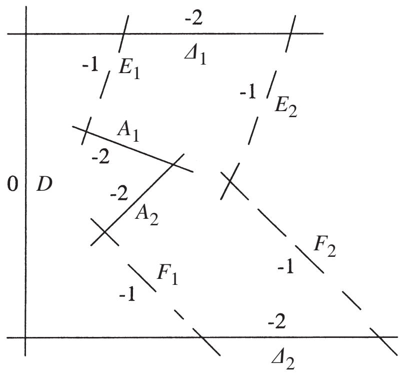

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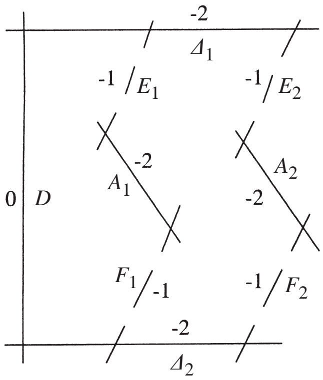

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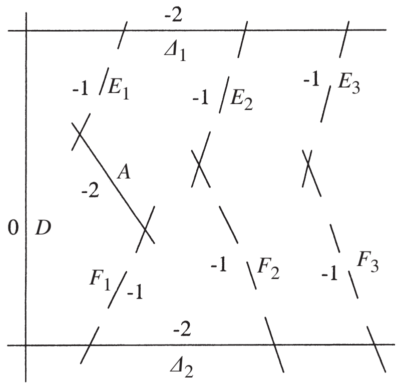

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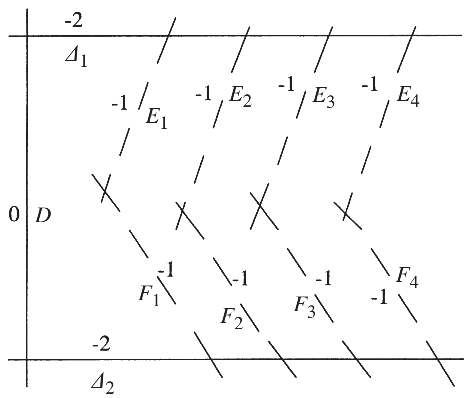

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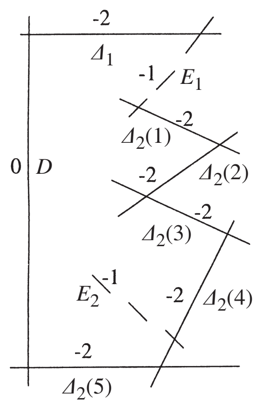
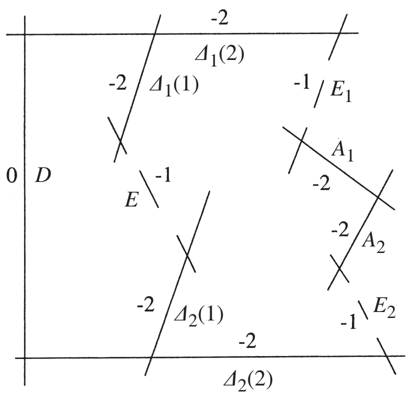


Figure 37.



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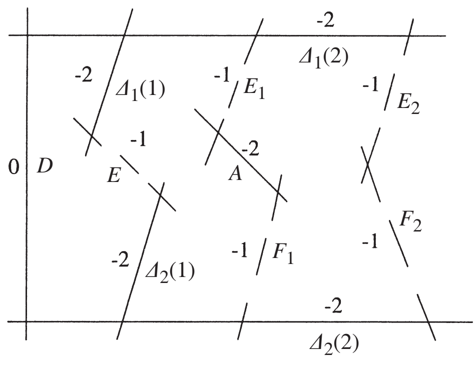

Figure 39.

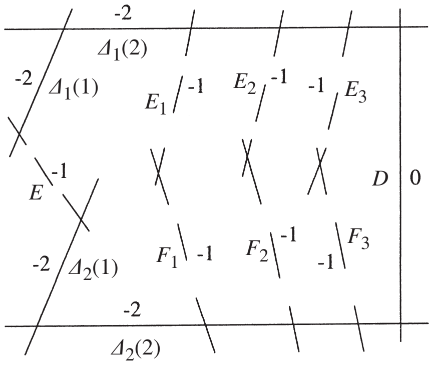

Figure 40.

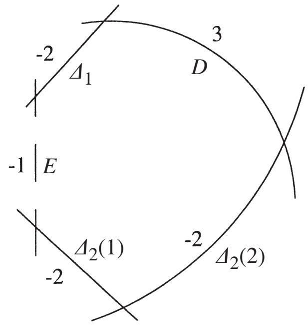

Figure 41.

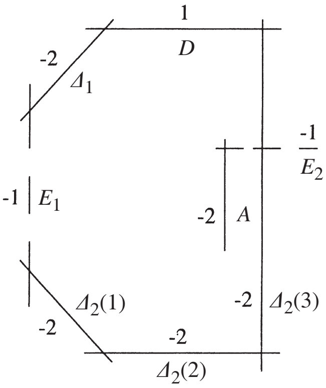

Figure 42.

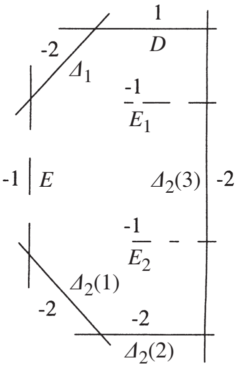

Figure 43.

FIGURE 1. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E + \Delta_2(1) + \Delta_2(2)$ to a point P. Clearly, there is an induced faithful action of G on $\boldsymbol{P}^2$ such that ψ is G-equivariant. So G is a subgroup of $PGL_2(\boldsymbol{C})$ stabilizing each component of the triangle $\psi(\Delta_1 + D_1 + D_2)$. We may assume that the three vertices of the triangle are at $[1,0,0], [0,1,0], [0,0,1]$. Then G is a subgroup of $\{\mathrm{diag}[1,b,c] \mid bc \neq 0\} \subseteq PGL_2(\boldsymbol{C})$. Conversely, any finite subgroup of $\{\mathrm{diag}[1,b,c] \mid bc \neq 0\}$ can act faithfully on this $\overline{X}$ fitting Figure 1, such that $\rho(\overline{X}/\!\!/G) = 1$ (noting that $\rho(\overline{X}) = 1$ already).

FIGURES 2–6. Let H be the (normal) subgroup of G stabilizing D_1 (and hence also D_2), and let g be an element in G switching D_1 and D_2 (see Lemma 7). Then $G = \langle g, H \rangle$. Note that H is abelian. This is because at the point $D_1 \cap D_2$, all elements of H can be diagonalized simultaneously with the same eigenvectors along the directions of D_1 and D_2.

In Figure 2, one can show that H is cyclic. Indeed, H fixes the three intersection points of the cross-section $\Delta_1(2)$ with the three singular fibres of different types, and hence $H|_{\Delta_1(2)} = \{\mathrm{id}\}$. So every h in H is diagonalized as $(1, c_h)$ at $D_1 \cap \Delta_1(2)$ with the common eigenvectors along the directions of $\Delta_1(2)$ and D_1. Thus H can be embedded in $\boldsymbol{C}^*$ via $h \mapsto c_h$ and is cyclic.

Since $\rho(\overline{X}/\!\!/\langle g \rangle) = 1$ can be easily checked, we have always $\rho(\overline{X}/\!\!/G) = 1$ so long G exists. Note that $G = \langle g \rangle \cong \boldsymbol{Z}/(2)$ is realizable in all these 5 cases (Lemma 7).

FIGURE 7. Since $\rho(\overline{X}) = 1$ and $\mathrm{Sing}\,\overline{X} = 3A_1 + D_4$, there are exactly three (-1)-curves

E, E_2, E_3 on X fitting Figure 7$'$, where there is a $\boldsymbol{P}^1$-fibration Ψ on X such that $\Gamma_0' := 2E + A_1 + (\Delta_3)_{\mathrm{red}}$, $\Gamma_1' := 2E_2 + A_3 + (\Delta_2)_{\mathrm{red}}$, $\Gamma_2' := 2E_3 + A_4 + (\Delta_1)_{\mathrm{red}}$ are all the singular fibres and A_2 and D are cross-sections of Ψ. Clearly, G permutes fibres of Ψ. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E + A_1, E_2 + A_3, E_3 + A_4, D$ to 4 points $P_1, \ldots, P_4$, respectively. Then ψ is G-equivariant; G fixes P_4 and permutes P_1, P_2, P_3. We may assume that $\psi(A_2) = \{Z = 0\}$ which is G-stable, and $P_4 = [0, 0, 1]$ which is G-fixed. Let H be the (normal) subgroup of G fixing all three points P_1, P_2, P_3 (and hence fixes the line $\{Z = 0\}$), then $H = \langle h_1 \rangle$ for some $h_1 = \mathrm{diag}[1, 1, c_1]$ of order n_1.

Let $\iota : G \to \mathrm{Aut}\{P_1, P_2, P_3\} = S_3$ be the natural homomorphism. Then $\mathrm{Im}(\iota) = S_3$, $\boldsymbol{Z}/(3), \boldsymbol{Z}/(2)$ or (1). If an element h_3 in G acts transitively on the set $\{P_1, P_2, P_3\}$, then h_3^3 acts trivially on the line $\{Z = 0\}$ and hence $h_3 = \mathrm{diag}[1, \omega, c_3]$, where $\omega = \exp(2\pi\sqrt{-1}/3)$, after the normalization that h_3 fixes two points $[1, 0, 0], [0, 1, 0]$ on the line $\{Z = 0\}$. If there is further an element h_2 in G acting as an involution on the set $\{P_1, P_2, P_3\}$, one may assume that $h_2(P_1) = P_1$ and $P_1 = [1, 1, 0]$. One can show that $h_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & c_2 \end{pmatrix}$, by using the following conditions: $h_2(P_i) = P_i$ $(i = 1, 4)$, $h_2(P_2) = P_3$, $h_2(P_3) = P_2$, $P_2 = h_3^j(P_1) = [1, \omega^j, 0]$, $P_3 = h_3^{2j}(P_1)$ $(j = 1$ or $2)$.

Suppose that $\mathrm{Im}(\iota) = \boldsymbol{Z}/(2)$ and let h_2' be in G acting as an involution on the set $\{P_1, P_2, P_3\}$. Then $h_2' = \mathrm{diag}[1, -1, c_2]$ after the normalization that h_2' fixes two points $[1, 0, 0], [0, 1, 0]$ on the line $\{Z = 0\}$.

Replacing h_3 (resp. h_2 or h_2') by its power we may assume that $\mathrm{ord}(h_3) = 3^{n_3}$ (resp. $\mathrm{ord}(h_2)$ or $\mathrm{ord}(h_2')$ is 2^{n_2}). Thus either $G = \langle h_1, h_2, h_3 \,|\, h_1 h_i = h_i h_1 (i = 2, 3), h_2^2, h_3^3, h_2^{-1} h_3 h_2 h_3 \in \langle h_1 \rangle \rangle$, or $G = \langle h_1, h_3 \,|\, h_1 h_3 = h_3 h_1, h_3^3 \in \langle h_1 \rangle \rangle$, or $G = \langle h_1, h_2' \,|\, h_1 h_2' = h_2' h_1, (h_2')^2 \in \langle h_1 \rangle \rangle$, or $G = \langle h_1 \rangle$. We have an exact sequence:

$$(1) \to \langle h_1 \rangle \to G \to G/\langle h_1 \rangle \to (1)$$

where $G/\langle h_1 \rangle = S_3$, $\boldsymbol{Z}/(3), \boldsymbol{Z}/(2)$ or (1). We may take $G = (1)$ since then $\rho(\overline{X}/\!\!/G) = \rho(\overline{X}) = 1$.

In FIGURES 8–13 below, let $Q_i := D \cap (\Delta_i)_{\mathrm{red}}$. Let $q_1 : X_1 \to X$ be the blow-up of a point R_2 $(\neq Q_i)$ on D with J_0 the exceptional curve (we may choose R_2 to be a G-fixed point if it exists, but always $R_2 \neq Q_i$). Then $-K_{X_1} = J_0 + q_1'(2D + (\Delta_1)_{\mathrm{red}} + (\Delta_2)_{\mathrm{red}} + (\Delta_3)_{\mathrm{red}})$, which is nef and big. By the Riemann-Roch theorem and the Kawamata-Viehweg vanishing theorem, $\dim|-K_{X_1}| = 1$. Let $q_0 : X_0 \to X_1$ be the blow-up of the unique base point of $|-K_{X_1}|$ (which must lie on J_0) (cf. Proposition 2 at page 40 of [**3**], or Lemma 1.7 in [**4**]) with O the exceptional divisor. Then there is an elliptic fibration $\gamma : X_0 \to \boldsymbol{P}^1$ with O the zero section and $T_0 := 2D + J_0 + (\Delta_1)_{\mathrm{red}} + (\Delta_2)_{\mathrm{red}} + (\Delta_3)_{\mathrm{red}}$ as a singular fibre (we use, by the abuse of notations, the same symbol like Δ_i to denote its proper transform on X_0) which is of type I_0^*. Let $\mathrm{Aut}_0(X_0) = \{g \in \mathrm{Aut}(X_0) \,|\, g(O) = O\}$. Then there is an induced $\mathrm{Aut}_0(X_0)$ action on X so that $q = q_1 \circ q_0 : X_0 \to X$ is $\mathrm{Aut}_0(X_0)$-equivariant. Clearly, $\mathrm{Aut}_0(X_0) = \{g \in \mathrm{Aut}(X) \,|\, g(R_2) = R_2\}$. Let T be a general fibre of the elliptic fibration on X_0. Then $\mathrm{Aut}_0(X_0)$ contains $\mathrm{Aut}_0(T) = \{g \in \mathrm{Aut}(T) \,|\, g$ fixes the point $O \cap T\} \cong \boldsymbol{Z}/(m)$ with $m = 2, 4$ or 6 (see Corollary 4.7 in [**5**] at page 321). Hence $\mathrm{Aut}_0(T)$ (and $\mathrm{Aut}_0(X_0)$) contains an involution $\sigma : t \mapsto -t$.

Let $\iota : G \to \mathrm{Aut}\{Q_1, Q_2, Q_3\} = S_3$ be the natural homomorphism. Let $H := \mathrm{Ker}(\iota)$, which then acts trivially on D; in particular, $H \subseteq \mathrm{Aut}_0(X_0)$. At the point Q_1, every element h in H has the directions of D and Δ_1 as eigenvectors with respect to the eigenvalues $1, \lambda_h$. So H can be embedded into k^* and hence H is cyclic. Note that we have an exact sequence

$$(1) \to H \to G \to G/H \to (1)$$

where $G/H = (1)$, $\boldsymbol{Z}/(2), \boldsymbol{Z}/(3)$ or S_3. Let G_0 be any finite group of $\mathrm{Aut}_0(X_0)$ containing the involution σ of a general fibre; note that σ is in the centre of $\mathrm{Aut}_0(X_0)$. We shall show that in each of Figures 8–13, we can take $G = G_0$ so that $\rho(\overline{X}/\!\!/G) = 1$.

Figure 8. In this case the elliptic fibration γ has $T_0, T_1 = B + A_1 + A_2 + A_3$ which is of type I_4, and a few irreducible fibres as singular fibers. As in Figure 10 below, considering the height pairing, we can show that $\sigma(E) = E$, $\sigma(E_1) = E_2$ and $\rho(\overline{X}/\!\!/G_0) = 1$.

Figure 9. In this case, γ has $T_0, T_i = A_i + B_i$ $(i = 1, 2, 3)$ each of which is of type I_2 or III, and a few irreducible fibres as singular fibers. As in Figure 10 below, considering the height pairing, we can show that $\sigma(E) = E$, $\sigma(E_1) = E_2$ and $\rho(\overline{X}/\!\!/G_0) = 1$.

Figure 10. In this case, γ has $T_0, T_i = A_i + B_i$ $(i = 1, 2)$, each of which is of type I_2 or III, and a few irreducible fibres as singular fibers. On the surface X_0, by the height pairing in [**18**], $\langle E, E \rangle = 2\chi(\mathcal{O}_{X_0}) + 2E.O - (1 + 1/2 + 1/2) = 0$, whence E is a torsion in $\mathrm{MW}(\gamma)$; one can see easily that E is a 2-torsion and hence $\sigma(E) = E$; also $\langle E_i, E_i \rangle = 1 = \langle F_i, F_i \rangle$, $\langle E_i, F_i \rangle = \chi(\mathcal{O}_{X_0}) + E_i.O + F_i.O - E_i.F_i - (1 + 0 + 0) = -1$, $\langle E_i, E_j \rangle = 0 = \langle E_i, F_j \rangle$ for $i \neq j$. On the surface X, since σ stabilizes the fibre $2E + A_1 + A_2$ of Φ, it permutes fibres of Φ, whence $\sigma(E_i) = E_j$ or F_j for some j. Note that in $MW(\gamma)$, $E_i + \sigma(E_i) = 0$ and hence $\langle E_i + \sigma(E_i), E_i + \sigma(E_i) \rangle = 0$. By the calculation above, we must have $\sigma(E_i) = F_i$. On the surface X, since $\mathrm{Pic}\, X$ is generated over $\boldsymbol{Q}$ by the fibre components and a 2-section Δ_3, the $\mathrm{Pic}\, \overline{X}$ is generated over $\boldsymbol{Q}$ by the images $\overline{E}, \overline{E}_i, \overline{F}_j$ of E, E_i, F_j with $2\overline{E} = \overline{E}_i + \overline{F}_i$. So it follows that $\rho(\overline{X}/\!\!/G_0) = 1$.

Figure 11. In this case, γ has $T_0, T_1 = A_1 + A_2 + B$, which is of type I_3 or IV, and a few irreducible fibres as singular fibers. On the surface X_0, we can calculate the height pairing and find that $\langle E_i + F_i, E_i + F_i \rangle = 0$; so $E_i + F_i$ is torsion and it must be zero in $MW(\gamma)$ for the latter is torsion free by [**17**] or [**10**]. So $\sigma(E_i) = -E_i = F_i$ in $MW(\gamma)$. As in Figure 10, $\rho(\overline{X}/\!\!/G_0) = 1$.

Figure 12. In this case, γ has $T_0, T_1 = A + B$ which is of type I_2 or III, and a few irreducible fibres as singular fibers. As in Figure 11, $\sigma(E_i) = F_i$ $(1 \leq i \leq 3)$ and hence as in Figure 10, $\rho(\overline{X}/\!\!/G_0) = 1$.

Figure 13. In this case, γ has T_0 and a few irreducible fibres as singular fibers. As in Figures 10 and 11, $\sigma(E_i) = F_i$ $(1 \leq i \leq 4)$ and $\rho(\overline{X}/\!\!/G_0) = 1$.

Figure 14. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E + \Delta(1) + \Delta(2)$ to a point P. We may assume that $\psi(A) = \{X = 0\}$ and $\psi(D) = \{Y = 0\}$ so that $P = [0, 0, 1]$. Then ψ is G-equivariant and $G \subseteq \{g = (a_{ij}) \in PGL_2(\boldsymbol{C}) \mid a_{21} = 0$ and g is lower triangular$\}$. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

FIGURE 15. By blowing down E, we are reduced to Figure 14. Let $P' = E \cap \Delta(2)$ which is infinitely near to the point P defined in Figure 14. Then $G \subseteq \{g = (a_{ij}) \in PGL_2(\boldsymbol{C}) \mid g(P') = P', a_{21} = 0$ and g is lower triangular$\}$. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

FIGURE 16. Let g be the element in G switching E_1 and E_2 (see Lemma 10), and let H be the (normal) subgroup of G stabilizing E_1 (and hence E_2). Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E_2 + \Delta(1) + \Delta(2)$, which is H-equivariant. As in Figure 14, $H \subseteq \{h = (a_{ij}) \in PGL_2(\boldsymbol{C}) \mid a_{21} = 0$ and h is lower triangular$\}$. Note that H is normal in G and $G = \langle g, H \rangle$.

Here is an example where $G = \langle g \rangle$ and $\mathrm{ord}(g) = 2$. Let Σ_4 be the Hirzebruch surface with the (-4)-curve M and a section B disjoint from M. Let $Y \to \Sigma_4$ be the blow-up of a point not on M. Note that $M + B$ is 2-divisible in the Picard lattice. Let $X \to Y$ be the double cover branched along $M + B$ with $\langle g \rangle = \mathrm{Gal}(X/Y)$. Let $\Delta(2)$ be the inverse on X of M, $\Delta(1)$ the proper transform on X of the fibre through the centre of the blow-up $Y \to \Sigma_4$, $E_1 + E_2$ the inverse of the exceptional curve of the same blow-up, and let D be the inverse of any general fibre. Then Figure 16 appears on this X so that $\rho(\overline{X}/\!\!/\langle g \rangle) = 1$, where $X \to \overline{X}$ is the blow-down of $\Delta(1) + \Delta(2)$.

FIGURE 17. Since E is the only (-1)-curve on X (see Figure 5 in [**19**]), E is G-stable. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E + \Delta(2) + \Delta(1)$ to a point say $P := [0, 0, 1]$. Then ψ is G-equivariant. We may also assume that $\psi(A) = \{X = 0\}$. Note that $\psi(D)$ is a conic touching A at P with order 2. Then $G \subseteq \{g = (a_{ij}) \in PGL_2(\boldsymbol{C}) \mid g(\psi(D)) = \psi(D), g$ is lower triangular$\}$. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

FIGURE 18. Note that E_1, E_2 are the only (-1)-curves on X (see Figure 6 in [**19**]) and hence G stabilizes $E_1 + E_2$. Let g be the element in G switching E_1 and E_2 (see Lemma 10) and let H be the (normal) subgroup of G stabilizing E_1 (and hence E_2). Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E_2 + \Delta(2) + \Delta(1)$ to a point say $P := [0, 0, 1]$. Then ψ is H-equivariant. H stabilizes the line $\psi(E_1)$ defined by $\{X = 0\}$ say, and also the conic $\psi(D)$ touching $\psi(E_1)$ at P with order 2. As in Figure 17, $H \subseteq \{h = (a_{ij}) \in PGL_2(\boldsymbol{C}) \mid h(\psi(D)) = \psi(D), h$ is lower triangular$\}$. Note that H is normal in G and $G = \langle g, H \rangle$.

Here is an example with $G = \langle g \rangle$ and $\mathrm{ord}(g) = 2$. Let $X \to Y$ and $\Delta(i), E_i$ be as in Figure 16, but we let D be the inverse on X of B.

FIGURE 19. Since E is the unique (-1)-curve on X (see Figure 5 in [**19**]), it is G-stable. By blowing down E, we are reduced to Figure 20. Set $P' := E \cap \Delta(3)$ which is an infinitely near point of the point P in Figure 20. Thus $G \subseteq \{g = (a_{ij}) \in PGL_2(\boldsymbol{C}) \mid g(P') = P', a_{ij} \neq 0$ only when $i = j$ or $(i, j) = (3, 1)\}$. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

FIGURE 20. Since E is the only (-1)-curve on X (see Figure 5 in [**19**]), it is G-stable. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E + A_1 + A_2$ to a point, say $P := [0, 1, 0]$. Then ψ

is G-equivariant. G fixes P and $[0,0,1]$ which is the intersection of two G-stable lines $\psi(\Delta) = \{X = 0\}$ and $\psi(D) = \{Y = 0\}$ say, whence $G \subseteq \{(a_{ij}) \in PGL_2(\boldsymbol{C}) \mid a_{ij} \neq 0$ only when $i = j$ or $(i,j) = (3,1)\}$. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

FIGURE 21. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of the E_i to the points P_i. We can show that E_i are the only (-1)-curves on X and hence ψ is G-equivariant. So $G \subseteq \mathrm{Aut}_S = \{g \in PGL_2(\boldsymbol{C}) \mid g(S) = S, g(\hat{D}) = \hat{D}\}$, where $S = \{P_1, P_2, P_3\}$ is a subset on the line $\psi(\Delta)$ and $\hat{D}$ $(= \psi(D))$ is a second line. Since $\rho(\overline{X}/\!\!/G) = 1$, the G acts on S transitively. Conversely, any finite group in Aut_S acting transitively on S can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$.

FIGURE 22. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $D + \Delta_2(2) + \Delta_2(1)$ to a point say $P := [0,0,1]$. Then ψ is G-equivariant. G fixes P and stabilizes the line $\psi(\Delta(1)) = \{X = 0\}$ say. Then $G \subseteq \{g = (a_{ij}) \in PGL_2(\boldsymbol{C}) \mid g$ is lower triangular$\}$. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

FIGURE 23. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E + A$ and $D + \Delta_2(3) + \Delta_2(2)$ to points $P := [0,1,0]$ and $[0,0,1]$ say. Then ψ is G-equivariant. G fixes P and stabilizes two lines $\psi(\Delta_2(1) = \{X = 0\}$ and $\psi(\Delta_1)) = \{Y = 0\}$ say. Then $G \subseteq \{(a_{ij}) \in PGL_2(\boldsymbol{C}) \mid a_{ij} \neq 0$ only when $i = j$ or $(i,j) = (3,1)\}$. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

FIGURE 24. By blowing down E, we are reduced to Figure 23. Set $P' := E \cap \Delta_2(2)$ which is an infinitely near point of the point P in Figure 23. Then $G \subseteq \{g = (a_{ij}) \in PGL_2(\boldsymbol{C}) \mid g(P') = P', a_{ij} \neq 0$ only when $i = j$ or $(i,j) = (3,1)\}$. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

FIGURE 25. Let g be an element in G switching E_1 and E_2 (see Lemma 11) and let H be the (normal) subgroup of G stabilizing E_1 (and hence E_2). Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $D + \Delta_1$ and $E_2 + \Delta_2(1) + \Delta_2(2)$ to two points $P_1 = [0,1,0]$ and $P_2 = [0,0,1]$ say. Then ψ is H-equivariant. H fixes P_i and stabilizes the two lines $\psi(\Delta_2(3)) = \{X = 0\}$ and $\psi(E_1) = \{Y = 0\}$ say. Then $H \subseteq \{(a_{ij}) \in PGL_2(\boldsymbol{C}) \mid a_{ij} \neq 0$ only when $i = j$ or $(i,j) = (3,1)\}$. Note that H is normal in G and $G = \langle g, H \rangle$.

Here is an example where $G = \langle g \rangle$ and $\mathrm{ord}(g) = 2$. Let M be the (-2)-curve on the Hirzebruch surface Σ_2, B a section disjoint from M and L_i are two distinct fibres. Let $p : Y \to \Sigma_2$ be the blow-up of a point on L_2 other than $L_2 \cap M$, the point $L_1 \cap M$ and its infinitely near point lying on the proper transform of M; let $\hat{E}, \hat{D}, \hat{\Delta}_1$ be irreducible curves on Y which are (the proper transforms of) the corresponding exceptional curves. Since $M + B$ is 2-divisible in the Picard lattice, there is a double cover $X \to Y$ branched along $\hat{D} + p'(M + B)$ with $\langle g \rangle = \mathrm{Gal}(X/Y)$. Let $D, \Delta_1, \Delta_2(1), \Delta_2(2), \Delta_2(3), E_1 + E_2$ be the strict inverses of $\hat{D}, L_1, L_2, \hat{\Delta}_1$ and $\hat{E}$, respectively. Then Figure 25 appears on this X so that $\rho(\overline{X}/\!\!/\langle g \rangle) = 1$, where $X \to \overline{X}$ is the blow-down of $\Delta_1 + \sum_j \Delta_2(j)$.

Figure 26. Let H be the (normal) subgroup of G stabilizing $\varDelta_1$ (and hence also all of $\varDelta_i(j)$). Blowing down D, the Figure becomes Figure 5, whence H is abelian. Thus either $G = H$, or $G = \langle g, H \rangle$ where g switches $\varDelta_1$ and $\varDelta_2$.

Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $D + \varDelta_2(2) + \varDelta_2(1)$ and $E_1 + A_1 + A_2$ to $P_1 = [1,0,0]$ and $P_2 = [0,1,0]$ say. Then ψ is H-equivariant. H fixes $P_3 = \varDelta_1(1) \cap \varDelta_1(2)$ with coordinates, say $[0,0,1]$, on $\boldsymbol{P}^2$ and also two points P_i $(i = 1,2)$ on the line $\psi(E_2) = \{Z = 0\}$. Thus $H \subseteq \{\mathrm{diag}[1,b,c] \mid bc \neq 0\}$.

Conversely, each finite group $G = H$ in $PGL_2(\boldsymbol{C})$ of the form above or $G = \langle g \rangle \cong \boldsymbol{Z}/(2)$ is realizable as a group of automorphisms on this $\overline{X}$ such that $\rho(\overline{X}/\!\!/G) = 1$. (Indeed, $\rho(\overline{X}) = 1$ already; see Lemma 7 for the second case.)

Figure 27. Let g be in G switching $\varDelta_1$ and $\varDelta_2$ (Lemma 11). Let H be the (normal) subgroup of G stabilizing $\varDelta_1$ (and hence $\varDelta_2$). As in Figure 2, we have $H = \langle h \rangle$ and $G = \langle g, h \rangle$ with $h|_{\varDelta_i(1)} = \mathrm{id}$. The case $G = \langle g \rangle \cong \boldsymbol{Z}/(2)$ is realizable (see Lemma 7; indeed Figure 27 is different from Figure 2 only in labelling).

Figure 28. Let H be the (normal) subgroup of G stabilizing $\varDelta_1$ (and hence also $\varDelta_2$). Let H_1 be the (normal) subgroup of G stabilizing E_1 and E_2 (and hence all $E_i, F_j, \varDelta_i(j)$). As in Figure 2, H is abelian and $H_1 = \langle h_1 \rangle$ with $h_1|_{\varDelta_i(1)} = \mathrm{id}$. Note that $G/H \leq \boldsymbol{Z}/(2)$ and $|H/H_1| \leq 3$; indeed, H/H_1 is abelian and acts on the set $\{E_1, E_2, E_3\}$. By Lemma 11, either $G/H = \boldsymbol{Z}/(2)$ or $G = H$ and $H/H_1 = \boldsymbol{Z}/(3)$. Each of the case $G = G/H = \boldsymbol{Z}/(2)$ and the case $G = H$ with $H/H_1 = \boldsymbol{Z}/(3)$ is realizable as a group of automorphisms on this $\overline{X}$ such that $\rho(\overline{X}/\!\!/G) = 1$ (see Figure 7 and Lemma 7, noting that the Figure becomes Figure 4 after the blow down of D).

Figure 29. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $D + \varDelta_2(5) + \varDelta_2(4), E_1 + \varDelta_2(1) + \varDelta_2(2)$ and E_2 to points say $[1,0,0], [0,1,0]$ and $[1,1,0]$ on the same line $\psi(\varDelta_2(3)) = \{Z = 0\}$. Then ψ is G-equivariant. G fixes these three points and also the intersection point $\varDelta_1(1) \cap \varDelta_1(2)$ with coordinates say $[0,0,1]$ on $\boldsymbol{P}^2$. Then $G = \langle g \rangle$ where $g = \mathrm{diag}[1,1,c]$. Conversely, any finite cyclic group can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

Figure 30. Blowing down D which is G-stable, the Figure becomes Figure 2. So either $G = \langle g, h \rangle$ or $G = \langle h \rangle$, where $h|_{\varDelta_i(2)} = \mathrm{id}$ and g switches $\varDelta_1$ and $\varDelta_2$. Each of $G = \langle g \rangle \cong \boldsymbol{Z}/(2)$ and $G = \langle h \rangle$ is realizable as a group of automorphisms on this $\overline{X}$ such that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X}) = 1$ already.)

Figure 31. Let H be the (normal) subgroup of G stabilizing $\varDelta_1$ (and hence also $\varDelta_2$). Let H_1 be the (normal) subgroup of G stabilizing E_1 (and hence all $E_i, F_j, \varDelta_i(j)$). As in Figure 2, H is abelian and $H_1 = \langle h_1 \rangle$ with $h_1|_{\varDelta_i(2)} = \mathrm{id}$. As in Figure 28, by Lemma 11, either $G/H = \boldsymbol{Z}/(2)$, or $G = H$ and $H/H_1 = \boldsymbol{Z}/(2)$. Each of the case $G = G/H \cong \boldsymbol{Z}/(2)$ and the case $G = H$ with $H/H_1 = \boldsymbol{Z}/(2)$ is realizable as a group of automorphisms on this $\overline{X}$ such that $\rho(\overline{X}/\!\!/G) = 1$ (see Lemma 7 and Figure 7).

Figure 32. Let $\psi : X \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be the blow-down of $E_1 + A_1$ and $E_2 + A_3$ to two points P_1 and P_2, respectively. Then ψ is G-equivariant. Thus G is a subgroup of $\mathrm{Aut}_S(\boldsymbol{P}^1 \times \boldsymbol{P}^1) := \{g \in \mathrm{Aut}(\boldsymbol{P}^1 \times \boldsymbol{P}^1) \mid g(S) = S, g(\hat{D}) = \hat{D}\}$, where $S = \{P_1, P_2\}$ is a set of two points on the same fibre of the first ruling and $\hat{D}$ $(= \psi(D))$ is a second fibre

of the same ruling. Let $\iota : G \to \mathrm{Aut}(S) = \boldsymbol{Z}/(2)$ be the natural homomorphism and let $H = \mathrm{Ker}(\iota)$. Then all elements of H can be diagonalized simultaneously at P_1 with the same eigenvectors along the directions of $\psi(\Delta_1)$ and $\psi(\Delta_2)$. Thus $H \subseteq \{\mathrm{diag}[b, c] \mid bc \neq 0\}$. Note that $G/H = (0)$ or $\boldsymbol{Z}/(2)$. Conversely, any finite subgroup G of $\mathrm{Aut}_S(\boldsymbol{P}^1 \times \boldsymbol{P}^1)$ can act on $\overline{X}$ faithfully with $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X}) = 1$ already.)

Figure 33. Let g be in G switching Δ_1 and Δ_2 (cf. Lemma 11). As in Figure 2, $G = \langle g, h \rangle$, where $h|_{\Delta_i} = \mathrm{id}$, and the case $G = \langle g \rangle \cong \boldsymbol{Z}/(2)$ is realizable (see Lemma 7; indeed Figure 33 is different from Figure 5 only in labelling).

Figure 34. Let $\psi : X \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be the blow-down of E_1, E_2, F_1, F_2 to four points e_1, e_2, f_1, f_2, respectively. Then ψ is G-equivariant. Let $\iota : G \to \mathrm{Aut}\{e_1, e_2, f_1, f_2\} = S_4$ be the natural homomorphism. Then $\mathrm{Im}(\iota)$ is contained in the Klein fourgroup $V = \langle (e_1e_2)(f_1f_2), (e_1f_1)(e_2f_2) \rangle$ of S_4. We assert that $\mathrm{Im}(\iota) \subseteq \langle (e_1f_2)(e_2f_1) \rangle$ is impossible. Indeed, if this assertion is false, then G permutes fibres of the $\boldsymbol{P}^1$-fibration Ψ with singular fibres $2E_1 + A_1 + \Delta_1$, $2F_2 + A_2 + \Delta_2$, where all components of $f^{-1}(\mathrm{Sing}\,\overline{X})$ are contained in fibres of Ψ; this leads to $\rho(\overline{X}/\!\!/G) \geq 2$ as in Lemma 11, which is a contradiction. So the assertion is true. Note that G is a subgroup of $\mathrm{Aut}_S(\boldsymbol{P}^1 \times \boldsymbol{P}^1) := \{g \in \mathrm{Aut}(\boldsymbol{P}^1 \times \boldsymbol{P}^1) \mid g(S) = S, g(\hat{D}) = \hat{D}\}$, where $S = \{e_1, e_2, f_1, f_2\}$ is the intersection of four fibres, two from each ruling and $\hat{D}$ $(= \psi(D))$ is a fifth fibre. By the assertion, we have:

(*) $G|_S$ equals either the Klein group V or $\langle (e_1e_2)(f_1f_2) \rangle$ or $\langle (e_1f_1)(e_2f_2) \rangle$.

Let $\iota : G \to \mathrm{Aut}\{D \cap \Delta_1, D \cap \Delta_2\} = \boldsymbol{Z}/(2)$ be the natural homomorphism. As in Figure 32, we have $H := \mathrm{Ker}(\iota) \subseteq \{\mathrm{diag}[b, c] \mid bc \neq 0\}$, and $G/H = (0)$ or $\boldsymbol{Z}/(2)$. Conversely, any finite subgroup G of $\mathrm{Aut}_S(\boldsymbol{P}^1 \times \boldsymbol{P}^1)$ satisfying (*) can act on $\overline{X}$ faithfully with $\rho(\overline{X}/\!\!/G) = 1$.

Figure 35. Let g be in G switching Δ_1 and Δ_2 (Lemma 11). Let H be the (normal) subgroup of G stabilizing Δ_1 (and hence also Δ_2). As in Figure 6, H is abelian, $G = \langle g, H \rangle$, and the case $G = \langle g \rangle \cong \boldsymbol{Z}/(2)$ is realizable (see Lemma 7; indeed Figure 35 is different from Figure 6 only in labelling).

Figure 36. Let H_1 (resp. H_2) be the (normal) subgroup of G stabilizing all E_i (resp. stabilizing Δ_1). Then both H_i are normal in G such that $G/H_2 = (0)$ or $\boldsymbol{Z}/(2)$ and $H_2/H_1 \subseteq S_4$. As in Figure 2, $H_1 = \langle h \rangle$. If $G = H_2$, then $\rho(\overline{X}/\!\!/G) = 1$ implies that H_2 acts on the set $\{E_1, \ldots, E_4\}$ transitively, i.e., H_2/H_1 is a transitive subgroup of S_4. Conversely, $\langle g \rangle \cong \boldsymbol{Z}/(2)$ can actually act on $\overline{X}$ such that $\rho(\overline{X}/\!\!/\langle g \rangle) = 1$ and $g(E_i) = F_i$ for all i (see Lemma 7, noting that Figure 36 is different from Figure 4 only in labelling).

Figure 37. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E_2 + \Delta_2(4) + \Delta_2(5)$ and $E_1 + \Delta_2(1) + \Delta_2(2)$ to points, say $P_1 = [0, 0, 1]$ and $P_2 = [0, 1, 0]$. Then ψ is G-equivariant. G fixes the P_i and the intersection of D and Δ_1 with coordinates say $[1, 0, 0]$ on $\boldsymbol{P}^2$. Set $P_1' := E_2 \cap \Delta_2(4)$. Then $G \subseteq \{g = \mathrm{diag}[1, b, c] \in PGL_2(\boldsymbol{C}) \mid g(P_1') = P_1'\}$. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X}/\!\!/G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

Figure 38. Let H be the (normal) subgroup of G stabilizing Δ_1 (and hence also Δ_2). As in Figure 2, $H = \langle h \rangle$, where $h|_{\Delta_i(2)} = \mathrm{id}$. Note that $G/H \le \boldsymbol{Z}/(2)$. The case $G = G/H \cong \boldsymbol{Z}/(2)$ is realizable (see Lemma 7; indeed, blowing down E, the Figure becomes Figure 5).

Figure 39. Let g be in G switching Δ_1 and Δ_2 (Lemma 11). As in Figure 2, $G = \langle g, h \rangle$, where $h|_{\Delta_i(2)} = \mathrm{id}$, and the case $G = \langle g \rangle \cong \boldsymbol{Z}/(2)$ is realizable (see Lemma 7; indeed Figure 39 is different from Figure 2 only in labelling).

Figure 40. Figure 40 is identical with Figure 28 with only difference in labelling.

Figure 41. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E + \Delta_2(1) + \Delta_2(2)$ to a point P_1. Then ψ is G-equivariant. G fixes P_1 and the intersection point P_2 of D with Δ_1. Thus $G \subseteq \{g \in PGL_2(\boldsymbol{C}) \mid g(P_i) = P_i (i = 1, 2), g(\hat{D}) = \hat{D}\}$, where $\hat{D}$ $(= \psi(D))$ is a conic intersecting the line $L_{P_1P_2}$ $(= \psi(\Delta_1))$ at the points P_i. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X} /\!\!/ G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

Figure 42. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of $E_1 + \Delta_2(1) + \Delta_2(2)$ and $E_2 + A$ to points $P_2 = [1, 0, 0]$ and $P_3 = [1, 1, 0]$ say. Then ψ is G-equivariant. G fixes three points P_i $(i = 1, 2, 3)$ where $P_1 = D \cap \Delta_2(3)$ with $P_1 = [0, 1, 0]$ say, all lying on the line $\psi(\Delta_2(3)) = \{Z = 0\}$, and also the intersection point $P_4 = D \cap \Delta_1$ with $P_4 = [0, 0, 1]$ say. Then $G = \langle g \rangle$ with $g = [1, 1, c]$. Conversely, any finite cyclic group can act on this $\overline{X}$ faithfully so that $\rho(\overline{X} /\!\!/ G) = 1$. (Note that $\rho(\overline{X})$ is already 1.)

Figure 43. Let $\psi : X \to \boldsymbol{P}^2$ be the blow-down of E_1, E_2 and $E + \Delta_2(1) + \Delta_2(2)$ to points P_i $(i = 1, 2, 3)$, respectively. Then ψ is G-equivariant. G fixes three points $P_3 = [1, 0, 0]$ say, $P_4 := D \cap \Delta_2(3) = [0, 1, 0]$ and $P_5 := D \cap \Delta_1 = [0, 0, 1]$. Note that $P_1, \dots, P_4$ lie on the same G-stable line $\psi(\Delta_2(3)) = \{Z = 0\}$ say. So $G \subseteq \{\mathrm{diag}[1, b, c] \mid bc \neq 0\}$. Let g be an element in G switching E_1 and E_2 (see Lemma 11). Then g switches P_1 and P_2. We may assume that $P_1 = [1, 1, 0]$. Now $g(P_1) = P_2$ and $g(P_2) = P_1$ imply that $P_2 = [1, -1, 0]$ and $g = [1, -1, c_1]$. Let H be the (normal) subgroup of G fixing P_1 (and hence P_2). Then $H = \langle h \rangle$ for some $h = \mathrm{diag}[1, 1, c_2]$. Thus $G = \langle g = \mathrm{diag}[1, -1, c_1]$, $h = \mathrm{diag}[1, 1, c_2] \rangle$. Conversely, any finite group in $PGL_2(\boldsymbol{C})$ of such form can act on this $\overline{X}$ faithfully so that $\rho(\overline{X} /\!\!/ G) = 1$.

Theorem A is a consequence of the lemmas in §1. Theorem B is proved in the arguments above. For instance, the assertion that $\kappa(\overline{X} \backslash \overline{D}) = -\infty$ in the Hypothesis (H), follows from the observation that $-(K_X + D + \Delta) = -f^*(K_{\overline{X}} + \overline{D})$ is nef and big. Indeed, from the construction of the action of G on $\overline{X}$, we see that $\rho(\overline{X} /\!\!/ G) = 1$. So the G-stable divisor $-(K_{\overline{X}} + \overline{D})$ is either numerically trivial, or ample or anti-ample (see Lemmas 1 and 2). Now the observation that $-(K_{\overline{X}} + \overline{D}).\overline{D} = -(K_X + D + \Delta).D = 2 - D.\Delta > 0$ shows that $-(K_{\overline{X}} + \overline{D})$ is ample and hence $-(K_X + D + \Delta)$ is nef and big.

Theorem C follows from the classification of the group G in §2. Indeed, for $K_X^2 \le 4$, we see that either G is a subgroup of PGL_2 as in Theorem C, or there is a sequence of subgroups of G such that the factor groups are abelian or G is as in the case of Figure 25. The easy calculation of K_X^2 is given below.

Lemma 14. *For the X in Figure m, we calculate K_X^2.*
(1) $K_X^2 = 2$, *if m is one of* 7–13, 29–31.
(2) $K_X^2 = 3$, *if m is one of* 2–3, 24, 26–28, 37–40.
(3) $K_X^2 = 4$, *if m is one of* 4–6, 23, 25, 32–36, 42–43.
(4) $K_X^2 = 5$, *if m is one of* 15, 19.
(5) $K_X^2 = 6$, *if m is one of* 1, 14, 16–18, 20–22, 41.

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