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Mod p truncated configuration spaces

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Abstract. In this paper we study the homotopy types of mod p truncated configuration spaces. In particular, we investigate a finite dimensional configuration space model for the based loop space of mod p lens space as an application.

1. Introduction.

Let $SP^d(X) = X^d/\Sigma_d$ denote the d-th symmetric product of a space X, where the symmetric group Σ_d of d letters acts on X^d by the coordinates permutations. Each element $\alpha \in SP^d(X)$ may be represented as a finite formal sum $\alpha = \sum_{j=1}^{s} d_j x_j$ ($x_j \in X$, $x_i \neq x_j$ if $i \neq j, \sum_{j=1}^{s} d_j = d$).

If $* \in X$ is a basepoint, one has inclusions $SP^d(X) \subset SP^{d+1}(X)$ given by adding a basepoint, $\alpha \mapsto \alpha + *$. We denote by $SP^\infty(X)$ the union $\bigcup_{d \geq 0} SP^d(X)$ and it is called the infinite symmetric product. For an integer $p \geq 2$, let $TP_p^\infty(X)$ denote the mod p truncated symmetric product defined by $TP_p^\infty(X) = SP^\infty(X)/\sim_p$, where the equivalence relation "$\sim_p$" is defined by $\sum_j d_j x_j \sim_p \sum_j d_j' x_j$ if $d_j \equiv d_j' \pmod p$ for any j. Recall the following result.

THEOREM 1.1 (A. Dold and R. Thom, [**2**]). *If X is a connected CW-complex, there are natural homotopy equivalences*

$$\begin{cases} SP^\infty(X) \xrightarrow{\simeq} \prod_{j \geq 1} K(H_j(X, \boldsymbol{Z}), j), \\ TP_p^\infty(X) \xrightarrow{\simeq} \prod_{j \geq 1} K(H_j(X, \boldsymbol{Z}/p), j) \end{cases}$$

such that the diagram

$$\begin{array}{ccc} SP^\infty(X) & \xrightarrow{\simeq} & \prod_{j \geq 1} K(H_j(X, \boldsymbol{Z}), j) \\ q \downarrow & & \downarrow \prod_j \pi_p^j \\ TP_p^\infty(X) & \xrightarrow{\simeq} & \prod_{j \geq 1} K(H_j(X, \boldsymbol{Z}/p), j) \end{array}$$

is homotopy commutative, where $q : SP^\infty(X) \to TP_p^\infty(X)$ denotes the natural projection and let $\pi_p^j : K(H_j(X, \boldsymbol{Z}), j) \to K(H_j(X, \boldsymbol{Z}/p), j)$ be the map induced from mod p *reduction.*

Hence the space $TP_p^\infty(X)$ can be regarded as a mod p version of $SP^\infty(X)$. In this

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paper, we shall study the corresponding result for certain type of labelled configuration spaces.

Let J be any collection of subsets of $\{1,2,\dots,n\}$ with $\mathrm{card}(\Lambda)\ge 2$ for all $\Lambda\in J$. We denote by $E_J^d(X)$ the configuration space of type J defined by

$$E_J^d(X)=\left\{(\xi_1,\dots,\xi_n)\in SP^d(X)^n:\bigcap_{j\in\Lambda}\xi_j=\varnothing\ \text{for all}\ \Lambda\in J\right\}.$$

For an integer $p\ge 2$, let $E_J^d(X;\boldsymbol{Z}/p)$ be the mod p truncated configuration space of $E_J^d(X)$ defined by $E_J^d(X;\boldsymbol{Z}/p)=E_J^d(X)/\equiv_p$, where the equivalence relation "$\equiv_p$" is defined by $(\xi_1,\dots,\xi_n)\equiv_p(\eta_1,\dots,\eta_n)$ if $\xi_j\sim_p\eta_j$ for any $1\le j\le n$. Similarly, let $\vee^J X$ be the generalized wedge of type J defined by

$$\vee^J X=\{(x_1,\dots,x_n)\in X^n:\ \text{for each}\ \Lambda\in J, x_j=*\ \text{for at least one}\ j\in\Lambda\}.$$

EXAMPLE. (i) If we take the collection $I(n)=\{\{1,2,\dots,n\}\}$ as J, then

$$\begin{cases}E_{I(n)}^d(X)=\{(\xi_1,\dots,\xi_n)\in SP^d(X)^n:\bigcap_{j=1}^n\xi_j=\varnothing\},\\ \vee^{I(n)}X=\{(x_1,\dots,x_n)\in X^n:x_j=*\ \text{for some}\ 1\le j\le n\}.\end{cases}$$

Hence, $\vee^{I(n)}X=W_n(X)$ is the n-th fat wedge of X.

(ii) If $J(n)=\{\{i,j\}:1\le i<j\le n\}$,

$$\begin{cases}E_{J(n)}^d(X)=\{(\xi_1,\dots,\xi_n)\in SP^d(X)^n:\xi_i\cap\xi_j=\varnothing\ \text{if}\ i\ne j\},\\ \vee^{J(n)}X=\vee^n X=X\vee\cdots\vee X\quad (n\ \text{times}).\end{cases}$$

Let $q_p^d:E_J^d(X)\to E_J^d(X;\boldsymbol{Z}/p)$ be the natural projection and let $E_J^d(\boldsymbol{R}^k)\to E_J^{d+1}(\boldsymbol{R}^k)$ (resp. $E_J^d(\boldsymbol{R}^k;\boldsymbol{Z}/p)\to E_J^{d+1}(\boldsymbol{R}^k;\boldsymbol{Z}/p)$) be the stabilization map defined by adding the point from the edge. Then the main results of this paper are as follows.

THEOREM 1.2. *Let $p,n\ge 2$, $k\ge 1$ be integers, and let J any collection of subsets of $\{1,2,\dots,n\}$ with $\mathrm{card}(\Lambda)\ge 2$ for all $\Lambda\in J$.*

(1) *There are natural homotopy equivalences*

$$\begin{cases}S:\lim_{d\to\infty}E_J^d(\boldsymbol{R}^k)\xrightarrow{\simeq}\Omega_0^k\vee^J K(\boldsymbol{Z},n),\\ S':\lim_{d\to\infty}E_J^d(\boldsymbol{R}^k;\boldsymbol{Z}/p)\xrightarrow{\simeq}\Omega_0^k\vee^J K(\boldsymbol{Z}/p,n),\end{cases}$$

where the limits are taken from stabilization maps.

(2) *There is a homotopy commutative diagram*

$$\begin{array}{ccc}\lim_{d\to\infty}E_J^d(\boldsymbol{R}^k) & \xrightarrow[\simeq]{S} & \Omega_0^k\vee^J K(\boldsymbol{Z},n)\\ {\scriptstyle \lim_d q_p^d}\downarrow & & \downarrow{\scriptstyle \Omega^k\vee^J\pi_p^n}\\ \lim_{d\to\infty}E_J^d(\boldsymbol{R}^k;\boldsymbol{Z}/p) & \xrightarrow[\simeq]{S'} & \Omega_0^k\vee^J K(\boldsymbol{Z}/p,n)\end{array}$$

THEOREM 1.3. *If $n\ge 3$, there is a map $E_{I(n)}^d(\boldsymbol{R};\boldsymbol{Z}/2)\to\Omega_0\boldsymbol{R}\mathrm{P}^{n-1}\simeq\Omega S^{n-1}$, which is a homotopy equivalence up to dimension $D(d,n)=(d+1)(n-2)-1$. Here we say that* *a map $f : X \to Y$ is a homotopy equivalence up to dimension D if the induced homomorphism $f_* : \pi_j(X) \to \pi_j(Y)$ is bijective when $j < D$ and surjective when $j = D$.*

We conclude with some comments on the wider significance of these results. Theorem 1.2 can be regarded as giving a simple "homotopy model" for the k-fold loop space $\Omega^k \vee^J K(G,k)$ ($G = \boldsymbol{Z}$ or $\boldsymbol{Z}/p$), and it also may be considered as one of the generalizations of the results given in [**3**]. The mod p configuration space was first studied in [**1**]. In this paper, we study another mod p labelled configuration spaces. For example, we show that $E^d_{I(n)}(\boldsymbol{R};\boldsymbol{Z}/p)$ gives the finite dimensional homotopy model for the loop space of the mod p lens space (Corollary 3.2), and we also give an improvement of the stability theorem due to J. Mostovoy [**8**] in Theorem 1.3.

2. Scanning maps.

From now on, we assume that J is a fixed collection of subsets of $\{1,2,\dots,n\}$ such that $\mathrm{card}(\Lambda) \ge 2$ for any $\Lambda \in J$.

Let $s_d : E^d_J(\boldsymbol{R}^k) \to E^{d+1}_J(\boldsymbol{R}^k)$ be the stabilization map given by adding a point from the edge ([**3**], [**4**], [**5**], [**9**]). We denote by $\lim_{d\to\infty} E^d_J(\boldsymbol{R}^k)$ the limit of stabilization maps, $E^1_J(\boldsymbol{R}^k) \xrightarrow{s_1} E^2_J(\boldsymbol{R}^k) \xrightarrow{s_2} E^3_J(\boldsymbol{R}^k) \xrightarrow{s_3} \cdots$. We can define the stabilization map $s_d : E^d_J(\boldsymbol{R}^k;\boldsymbol{Z}/p) \to E^{d+1}_J(\boldsymbol{R}^k;\boldsymbol{Z}/p)$ and $\lim_{d\to\infty} E^d_J(\boldsymbol{R}^k;\boldsymbol{Z}/p)$ in a similar way.

If $A \subset X$ is a closed subspace, let $E^d_I(X,A)$ denote the quotient space $E^d_J(X)/\sim_A$, where the equivalence relation "$\sim_A$" is given by

$$(\xi_1,\dots,\xi_n) \sim_A (\eta_1,\dots,\eta_n) \Leftrightarrow \xi_j \cap (X - A) = \eta_j \cap (X - A) \quad \text{for any } 1 \le j \le n.$$

If $A \ne \emptyset$ and $* \in A$ is a fixed basepoint, there is a natural inclusion $E^d_J(X,A) \xrightarrow{\subset} E^{d+1}_J(X,A)$ given by $(\xi_1,\dots,\xi_n) \mapsto (\xi_1 + *,\dots,\xi_n + *)$. We denote by $E_J(X,A)$ the union $E_J(X,A) = \bigcup_{d\ge 1} E^d_J(X,A)$. We can define $E_J(X,A;\boldsymbol{Z}/p)$ in a similar way.

Next, we define the scanning map $s^d_J : E^d_J(\boldsymbol{R}^k) \to \Omega^k E_J(S^k,\infty)$ as follows (cf. [**3**], [**4**], [**9**]). For each $w \in \boldsymbol{R}^k$, let $U(w) = \{x \in \boldsymbol{R}^k : \|x - w\| < \varepsilon\}$, where $\varepsilon > 0$ is fixed. Then for $\alpha = (\xi_1,\dots,\xi_n) \in E^d_J(\boldsymbol{R}^k)$, we define the map $s^d_J(\alpha) : S^k = \boldsymbol{R}^k \cup \infty \to E_J(S^k,\infty)$ by

$$z \mapsto (\xi_1 \cap \overline{U}(z),\dots,\xi_n \cap \overline{U}(z)) \in E_J(\overline{U}(z),\ \partial\overline{U}(z)) \cong E_J(S^k,\infty).$$

As $E^d_J(\boldsymbol{R}^k)$ is connected, the image of s^d_J lies in a connected component of $\Omega^k E_J(S^k,\infty)$, which we denote by $\Omega^k_d E_J(S^k,\infty)$. Hence we have a map $s^d_J : E^d_J(\boldsymbol{R}^k) \to \Omega^k_d E_J(S^k,\infty)$. Since s^J_d is compatible with s_d's, we have the scanning map $S : \lim_{d\to\infty} E^d_J(\boldsymbol{R}^k) \to \Omega^k_0 E_J(S^k,\infty)$. We can define the scanning map $S' : \lim_{d\to\infty} E^d_J(\boldsymbol{R}^k;\boldsymbol{Z}/p) \to \Omega^k_0 E_J(S^k, \infty;\boldsymbol{Z}/p)$ in a similar way.

The basic result is as follows.

Theorem 2.1. *Let $k \ge 1$ and $n, p \ge 2$ be integers and let J be any collection of subsets of $\{1,2,\dots,n\}$ such that* $\mathrm{card}(\Lambda) \ge 2$ *for each $\Lambda \in J$.*

(1) *There are natural homotopy equivalences*

$$\begin{cases} S : \lim\limits_{d\to\infty} E^d_J(\boldsymbol{R}^k) \xrightarrow{\simeq} \Omega^k_0 E_J(S^k,\infty) \\ S' : \lim\limits_{d\to\infty} E^d_J(\boldsymbol{R}^k;\boldsymbol{Z}/p) \xrightarrow{\simeq} \Omega^k_0 E_J(S^k,\infty;\boldsymbol{Z}/p) \end{cases}$$

(2) *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \lim_{d\to\infty} E_J^d(\boldsymbol{R}^k) & \xrightarrow[\simeq]{S} & \Omega_0^k E_J(S^k,\infty) \\ {\scriptstyle \lim_d q_p^d}\downarrow & & \downarrow{\scriptstyle \Omega^k q_p} \\ \lim_{d\to\infty} E_J^d(\boldsymbol{R}^k;\boldsymbol{Z}/p) & \xrightarrow[\simeq]{S'} & \Omega_0^k E_J(S^k,\infty;\boldsymbol{Z}/p) \end{array}$$

where $q_p : E_J(X,A) \to E_J(X,A;\boldsymbol{Z}/p)$ *denotes the* mod p *reduction map.*

Proof. (1) Since the second assertion is analogous to the first assertion, we only give the proof of the first part. First we consider the case $k=2$.

Let $B=(0,1)^2$ denote the unit open square in $\boldsymbol{R}^2$, and let $I^2=\bar{B}=[0,1]^2$ be the closure of B. Then we can identify $E_J^d(\boldsymbol{R}^2)\cong E_J^d(B)$ and $E_J(S^2,\infty)\simeq E_J(I^2,\partial I^2)$. The scanning map S may be decomposed into a composition of "horizontal" and "vertical" scanning maps S^H and S^V, each of which will be shown to be a homotopy equivalence.

In order to define S^V and S^H, let $\{V_t : 0<t<1\}$ be the continuous family of vertical rectangles in B defined by $V_t=\{(x,y): t-\varepsilon(t)<x<t+\varepsilon(t), 0<y<1\}$, where $\varepsilon(t)$ is a continuous function satisfying the conditions $\lim_{t\to 0}\varepsilon(t)=\lim_{t\to 1}\varepsilon(t)=0$ and $\varepsilon(t)>0$. Similarly, let $\{H_t : 0<t<1\}$ be the continuous family of horizontal rectangles in B defined by $H_t=\{(x,y): 0<x<1, t-\varepsilon(t)<y<t+\varepsilon(t)\}$.

The map S is induced (up to homotopy) from the stabilization of the map $E_J^d(B)\times I^2=E_J^d((0,1)^2)\times I^2\to E_J(I^2,\partial I^2)$ which is given by

$$((\xi_1,\xi_2),t,s)\mapsto(\xi_1,\xi_2)\cap V_t\cap H_s\in E_J^d(\overline{V}_t\cap\overline{H}_s,\partial(\overline{V}_t\cap\overline{H}_s))\cong E_J^d(I^2,\partial I^2).$$

For each closed rectangle X in $\boldsymbol{R}^2$, let σX denote the side of X which are parallel to y-axis. Let $S^{H,d}: E_J^d(B)\to\Omega E_J(I^2,\sigma I^2)=\Omega E_J(I^2,\partial I\times I)$ be the map determined by $((\xi_1,\xi_2),t)\mapsto(\xi_1,\xi_2)\cap H_t\in E_J(\overline{H}_t,\sigma\overline{H}_t)\cong E_J(I^2,\partial I\times I)$. Similarly, let $S^V: E_J(I^2,\partial I\times I)\to\Omega E_J(I^2,\partial I^2)$ be the map given by $((\xi_1,\xi_2),t)\mapsto(\xi_1,\xi_2)\cap V_t\in E_J(I^2\cap\overline{V}_t,\partial(I^2\cap\overline{V}_t))\cong E_J(I^2,\partial I^2)$. Since S is the composition of S^V and the stabilization of $S^{H,d}$, it suffices to show that S^V and the stabilization of $S^{H,d}$ are homotopy equivalences.

We begin with S^V. Up to homotopy this may be defined by $((\xi_1,\xi_2),t)\mapsto(\xi_1,\xi_2)\cap B_t$, where $B_t=\{(x,y): 0<x<1, 2t-1<y<2t\}$. Let B^* denote the rectangle $B^*=(0,1)\times(-1,2)$, and consider the commutative diagram

$$\begin{array}{ccccc} E_J(\overline{B}^*,\partial\overline{B}^*) & \xrightarrow{r_1} & E_J(\overline{B}^*,\partial\overline{B}^*\cup I^2) & \xrightarrow{\cong} & E_J(I^2,\partial I^2)^2 \\ {\scriptstyle f}\downarrow{\scriptstyle\simeq} & & & & \downarrow{\scriptstyle =} \\ \mathrm{Map}(I,E_J(E_J(I^2,\partial I^2))) & \xrightarrow{r_2} & \mathrm{Map}(\{0,1\},E_J(I^2,\partial I^2)) & \xrightarrow{\cong} & E_J(I^2,\partial I^2)^2, \end{array}$$

where r_1 and r_2 are induced from the restrictions and the vertical maps are both fibre preserving homotopy equivalences.

It follows from the Dold-Thom criterion [**2**] and [[**7**]; (3.3)] that r_1 is a quasi-fibration. Clearly r_2 is a fibration. Since the map f is induced from the scanning, its restriction to the fibre is also a homotopy equivalence. Because this restriction is just the map S^V, S^V is a homotopy equivalence.

The case of $S^{H,d}$ is similar. This is homotopic to the map given by $((\xi_1,\xi_2),t)\mapsto(\xi_1,\xi_2)\cap C_t$, where $C_t=\{(x,y):-1<x<2,2t-1<y<2t\}$. Let $C=(-1,2)\times(0,1)$, and consider the commutative diagram

$$\begin{array}{ccccc} E_J^d(\bar{C},\sigma\bar{C}) & \xrightarrow{r_1^d} & E_J(\bar{C},\sigma\bar{C}\cup B) & \xrightarrow{\cong} & E_J(I^2,\sigma I^2)^2 \\ {\scriptstyle g^d}\downarrow{\scriptstyle \simeq} & & & & \downarrow{\scriptstyle =} \\ \mathrm{Map}(I,E_J(I^2,\sigma I^2)) & \xrightarrow{r_2'} & \mathrm{Map}(\{0,1\},E_J(I^2,\sigma I^2)) & \xrightarrow{\cong} & E_J(I^2,\sigma I^2)^2, \end{array}$$

where the horizontal maps are induced from the restrictions and the vertical maps are fibre homotopy equivalences.

Clearly r_2' is a fibration. Using the method given in appendix of [**3**], we can easily see that $\pi_1(E_J^d(\boldsymbol{R}^2))$ is abelian if $d\geq 2$. Then it follows from the Dold-Thom criterion [**2**] and [[**7**]; (3.3)] that the stabilization of r_1^d is a quasi-fibration. Since the map g^d is induced from the scanning, its restriction to the fibre is equal to the map $S^{H,d}$. Hence $\lim_{d\to\infty}S^{H,d}$ is a homotopy equivalence. Therefore, S is a homotopy equivalence when $k=2$.

Next consider the general case. If $k=1$, the assertion easily follows from the above proof, and we may assume $k\geq 3$. We identify $E_J^d(\boldsymbol{R}^k)\cong E_J^d((0,1)^k)$ and $E_J(S^k,\infty)\simeq E_J(I^k,\partial I^k)$. For each $1\leq j\leq k$, let $\{I_t(j):0<t<1\}$ be the continuous family of k-dimensional cube given by

$$I_t(j)=\{(x_1,\dots,x_k):t-\varepsilon(t)<x_j<t+\varepsilon(t),0<x_i<1\ \text{ if }\ i\neq j\}.$$

Let $S_1^d:E_J^d((0,1)^k)\to\Omega E_J(I^k,\partial I\times I^{k-1})$ be the scanning map along the x_1-axis determined by $((\xi_1,\dots,\xi_n),t)\mapsto(\xi_1,\dots,\xi_n)\cap I_t(1)$. By the similar method as above, we can prove that the stabilization of S_1^d is a homotopy equivalence. Hence, $\lim_d S_1^d:\lim_{d\to\infty}E_J^d((0,1)^k)\xrightarrow{\simeq}\Omega_0E_J(I^k,\partial I\times I^{k-1})$ is a homotopy equivalence.

For each $2\leq j\leq k$, we can define $S_j:E_J(I^k,\partial I^{j-1}\times I^{k-j+1})\to\Omega E_J(I^k,\partial I^j\times I^{k-j})$ by the scanning along the x_j-axis. Then using the completely similar method of that for S^V, we can show that S_j is a homotopy equivalence. Because S is homotopic to the composite of the maps $\Omega^{k-1}S_k\circ\Omega^{k-2}S_{k-1}\circ\cdots\circ\Omega S_2\circ\lim_d S_1^d$, S is a homotopy equivalence.

(2) Let $q:SP^\infty(X)\to TP_p^\infty(X)$ be the natural projection, and let $TP_p^d(X)=q^{-1}(SP^d(X))$. Because S and S' are natural and compatible with the quotient map $SP^d(X)\to TP_p^d(X)$, the diagram (2) is homotopy commutative. □

Proposition 2.2. *There are natural homotopy equivalences*

$$\begin{cases} E_J(S^k,\infty)\xrightarrow{\simeq}\vee^J K(\boldsymbol{Z},k) \\ E_J(S^k,\infty;\boldsymbol{Z}/p)\xrightarrow{\simeq}\vee^J K(\boldsymbol{Z}/p,k) \end{cases}$$

such that the diagram

$$\begin{array}{ccc} E_J(S^k,\infty) & \xrightarrow{\simeq} & \vee^J K(\boldsymbol{Z},k) \\ {\scriptstyle q_p}\downarrow & & \downarrow{\scriptstyle \vee^J \pi_p^k} \\ E_J(S^k,\infty;\boldsymbol{Z}/p) & \xrightarrow{\simeq} & \vee^J K(\boldsymbol{Z}/p,k) \end{array}$$

is commutative up to homotopy.

Proof. Consider the inclusion map $E_J(S^k,\infty) \overset{\subset}{\to} SP^\infty(S^k,\infty)^n$. For $\varepsilon > 0$, let E_ε denote the open subspace of $E_J(S^k,\infty)$ consisting of all n-tuples $(\xi_1,\dots,\xi_n)$ such that some ξ_j is disjoint from the closed disk of radius ε about the origin. Then the radial expansion defines a deformation retract $E_\varepsilon \overset{\simeq}{\to} \vee^J SP^\infty(S^k,\infty)$. However, since $E_J(S^k,\infty)$ is the union of the E_ε for $\varepsilon > 0$, there is a homotopy equivalence $h_J : E_J(S^k,\infty) \overset{\simeq}{\to} \vee^J SP^\infty(S^k,\infty)$.

Similarly, the radial expansion also defines a homotopy equivalence $\widetilde{h_J} : E_J(S^k,\infty : \boldsymbol{Z}/p) \overset{\simeq}{\to} \vee^J TP_p^\infty(S^k,\infty)$. Since the radial expansions are compatible with the quotient map $q : SP^\infty(S^k,\infty) \to TP_p^\infty(S^k,\infty)$, there is a homotopy commutative diagram

$$\begin{array}{ccc} E_J(S^k,\infty) & \underset{\simeq}{\xrightarrow{h_J}} & \vee^J SP^\infty(S^k,\infty) \\ {\scriptstyle q_p}\downarrow & & \downarrow{\scriptstyle \vee^J q} \\ E_J(S^k,\infty;\boldsymbol{Z}/p) & \underset{\simeq}{\xrightarrow{\widetilde{h_J}}} & \vee^J TP_p^\infty(S^k,\infty). \end{array}$$

Hence it follows from Theorem 1.1 that the desired diagram is homotopy commutative. □

Proof of Theorem 1.2. The assertion easily follows from Theorem 2.1 and Proposition 2.2. □

3. The proof of Theorem 1.3.

Let $L^{n-1}(p)$ denote the mod p lens space given by $L^{n-1}(p) = S^{n-1}/(\boldsymbol{Z}/p)$.

Lemma 3.1. *For any integer $p \geq 2$, there is a fibration*

$$L^{n-1}(p) \to W_n(K(\boldsymbol{Z}/p,1)) \to (B\boldsymbol{Z}/p)^{n-1}.$$

Proof. The elementary abelian p-group $G = (\boldsymbol{Z}/p)^{n-1}$ acts on $L^{n-1}(p)$ by

$$[x_1 : \cdots : x_n] \cdot (\varepsilon_1,\dots,\varepsilon_{n-1}) = [\zeta^{\varepsilon_1}x_1 : \cdots : \zeta^{\varepsilon_{n-1}}x_{n-1} : x_n]$$

for $([x_1 : \cdots : x_n],(\varepsilon_1,\dots,\varepsilon_{n-1})) \in L^{n-1}(p) \times G$, where $\zeta = \exp(2\pi\sqrt{-1}/p)$. If Q denotes the homotopy quotient $L^{n-1}(p)//G = EG \times_G L^{n-1}(p)$, there is a fibration sequence $L^{n-1}(p) \to Q \to K(\boldsymbol{Z}/p,1)^{n-1}$. It suffices to show that there is a homotopy equivalence $Q \simeq W_n(K(\boldsymbol{Z}/p,1))$. For each $1 \leq j \leq n$, let $U_j \subset L^{n-1}(p)$ denote the open subset $U_j = \{[x_1 : \cdots : x_n] \in L^{n-1}(p) : x_j \neq 0\}$. Then each U_j is G-invariant open subspace and $L^{n-1}(p) = \bigcup_{j=1}^n U_j$. Let $E_j = EG \times_G U_j$. Since U_j is G-equivariantly contractible, $E_j \simeq BG = K(\boldsymbol{Z}/p,1)^{n-1}$. Similarly, for each $1 \leq j_1 < j_2 < \cdots < j_m \leq n$, we can show that there is a homotopy equivalence $\bigcap_{k=1}^m E_{j_k} \simeq K(\boldsymbol{Z}/p,1)^{n-m}$. If we consider the above homotopy equivalences, the inspection indicates that there is a homotopy equivalence $Q \simeq W_n(K(\boldsymbol{Z}/p,1))$. □

COROLLARY 3.2. *If* $n \geq 3$ *and* $p \geq 2$, *there is a homotopy equivalence*

$$\lim_{d\to\infty} E^d_{I(n)}(\boldsymbol{R};\boldsymbol{Z}/p) \xrightarrow{\simeq} \Omega S^{n-1}.$$

PROOF. The assertion follows from Lemma 3.1 and Theorem 1.2. □

PROOF OF THEOREM 1.3. Let $Q^d_{(n)}(\boldsymbol{R})$ be the space consisting of all n-tuples $(p_1(z),\dots,p_n(z)) \in \boldsymbol{R}[z]^n$ of monic $\boldsymbol{R}$-coefficients polynomials of degree d, such that polynomials $p_1(z),\dots,p_n(z)$ have no common real roots. We remark that there is a homotopy equivalence $f^d_n : Q^d_{(n)}(\boldsymbol{R}) \xrightarrow{\simeq} E^d_{I(n)}(\boldsymbol{R};\boldsymbol{Z}/2)$ which is compatible with stabilization maps ([**8**]). Then the assertion follows from 3.2, [[**6**]; Corollary 5] and [[**10**]; (1.3)]. □

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