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Julia sets of two permutable entire functions

By Liangwen Liao and Chung-Chun Yang

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Abstract. In this paper first we prove that if f and g are two permutable transcendental entire functions satisfying $f = f_1(h)$ and $g = g_1(h)$, for some transcendental entire function h, rational function f_1 and a function g_1, which is analytic in the range of h, then $F(g) \subset F(f)$. Then as an application of this result, we show that if $f(z) = p(z)e^{q(z)} + c$, where c is a constant, p a nonzero polynomial and q a nonconstant polynomial, or $f(z) = \int^z p(z)e^{q(z)}\,dz$, where p, q are nonconstant polynomials, such that $f(g) = g(f)$ for a nonconstant entire function g, then $J(f) = J(g)$.

1. Introduction.

Let f be a non-constant entire function, and denote by f^n the n-th iterate of f. The Fatou $F(f)$ set of f is the set of $z \in \boldsymbol{C}$ (the whole complex plane) where the family $\{f^n\}$ is normal in a neighborhood of z. Denote by $J(f)$ the complement of $F(f)$, which is called the Julia set of f. An obvious property of a Julia set for an entire or rational function f is that $J(f) = J(f^n)$. If a meromorphic function F can be expressed as $F = f \circ g$, where f and g are meromorphic functions, then f and g are called left and right factors of F, respectively. An entire or meromorphic function F is called prime (pseudo-prime) if whenever $F = f \circ g$ for some meromorphic functions f and g, then either f or g is linear (f rational or g polynomial). An entire or meromorphic function F is called left-prime if and only if whenever $F = f \circ g$ with g being transcendental then f must be linear. Moreover, we will say that a factorization is in entire sense if only entire factors are to be considered in the compositions. For more of the details, developments and related results of the factorization theory, we refer the reader to [**10**] or [**8**].

Theorem A (Baker [**1**]). *Let g be a nonlinear entire function permutable with $f(z) = ae^{bz} + c$, $(ab \neq 0, a, b, c \in \boldsymbol{C})$, then $g = f^n$ for some $n \in \boldsymbol{N}$. Hence $J(f) = J(g)$.*

Theorem B (Baker [**2**]). *If f and g are transcendental entire functions and if ∞ is neither a limit function of any subsequence of $\{f^n\}$ in a component of $F(f)$, nor of any subsequence of $\{g^n\}$ in a component of $F(g)$, then $J(f) = J(g)$.*

Theorem C (Bergweiler-Hinkkanen [**7**]). *Let f and g be two permutable transcendental entire functions. If both f and g have no wandering domains, then $J(f) = J(g)$.*

Recently Ng [**14**] obtained some results, by imposing conditions on only one of the two permutable functions.

Theorem D ([**14**]). *Let q be a nonconstant entire function and p be a polynomial*

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with at least two distinct zeros. Suppose that $f(z) = p(z)e^{q(z)}$ *is prime in entire sense. Then any nonlinear entire function* g *which permutes with* f *is of the form* $g(z) = af^n(z) + b$, *where* a *is a* k*-th root of unity and* $b \in \boldsymbol{C}$, *and hence* $J(f) = J(g)$.

THEOREM E ([**12**]). *Let* $f(z) = p(z)e^{\alpha(z)} + a$, *where* $p(z)$ *is a nonconstant polynomial and not of the form* $[p_1(z)]^n$, *where* $p_1(z)$ *a polynomial and* $n \geq 2$, $a \in \boldsymbol{C}$ *and* $\alpha(z)$ *a nonconstant entire function such that when* $p(z)$ *is linear* $= A(z-a)$ *then* $1 + (z-a)\alpha'(z)$ *has at least one but finitely many zeros. Furthermore, assume that* $f(z)$ *is pseudo-prime in entire sense. Let* g *be a nonlinear entire function permutable with* f, *then* $g(z) = a_1 f^n(z) + b_1$, *where* a_1 *is a* k*-th root of unity and* $b_1 \in \boldsymbol{C}$, *and hence* $J(f) = J(g)$.

THEOREM F ([**12**]). *Let* $f(z) = p(z)e^{q(z)} + a$, *where* p, q *are two nonconstant polynomials,* $p(z)$ *is not of the form* $p_1(z)^n$, $p_1(z)$ *a polynomial,* $n \geq 2$, *and* $a \in \boldsymbol{C}$. *If a nonlinear entire function* g *is permutable with* f, *then* $g(z) = a_1 f^n(z) + b_1$, *where* a_1 *is a* k*-th root of unity and* $b_1 \in \boldsymbol{C}$, *and hence* $J(f) = J(g)$.

In this paper, we shall derive several results which are complementary to Theorem D, Theorem E and Theorem F, and obtain a similar result for functions of the form $f(z) = \int^z p(z)e^{q(z)}\,dz$.

2. Lemmas.

LEMMA 1. *Let* f *and* g *be two entire functions. If* $g(F(f)) \subset F(f)$, *then* $F(f) \subset F(g)$.

PROOF. Let $z_0 \in F(f)$. Then there exists a neighbourhood U of z_0 such that $\overline{U} \subset F(f)$. By the assumption of the lemma, we have $g^n(U) \subset F(f)$. Therefore $g^n(z)$ will not take any value of $J(f)$ in U and hence $\{g^n\}_{n=1}^{\infty}$ is normal in U. Thus $z_0 \in F(g)$, the conclusion follows. □

LEMMA 2 ([**2**]). *Let* f *and* g *be two permutable transcendental entire functions. If* $\alpha \in F(f)$ *and there is a subsequence* f^{n_k}, *with* $n_k \to \infty$, *which has a finite limit in the component of* $F(f)$ *that contains* α, *then* $g(\alpha) \in F(f)$.

LEMMA 3 ([**7**]). *Let* f *and* g *be two permutable transcendental entire functions. If* f *does not have any wandering domains, then* $F(f) \subset F(g)$.

LEMMA 4 ([**3**]). *Let* f *be a transcendental entire function such that* $sing(f^{-1})$, *the set of singularities of* f^{-1}, *is a finite set. Then* $F(f)$ *does not have any wandering domain.*

DEFINITION AND NOTATION. Let $F(z)$ be a nonconstant entire function. An entire function $g(z)$ is called as a generalized right factor of F (denote by $g \leq F$) if there exists a function f, which is analytic on the range of g, such that $F = f \circ g$. Moreover if $h \leq f$ and $h \leq g$, then h is called a common generalized right factor of f and g.

By essentially adopting the arguments used by Eremenko-Rubel ([**9**], Theorem 1.1) in their investigations of the existence of possible common generalized right factors of two transcendental entire functions, Ng [**14**] obtained the following two results.

LEMMA 5 ([**14**], p. 133). *Let* f *and* g *be two entire functions and* $z_1, \ldots, z_k$ *be* $k \geq 2$ *distinct complex numbers such that*

$$\begin{cases} f(z_1) = f(z_2) = \cdots = f(z_k) = A, \\ g(z_1) = g(z_2) = \cdots = g(z_k) = B. \end{cases}$$

Suppose that there exist nonconstant functions f_1 and g_1 such that $f_1 \circ f = g_1 \circ g$ on $\bigcup_{i=1}^{k} \boldsymbol{U}_i$, where $\boldsymbol{U}_i$ is some open neighborhood containing z_i. If f_1 is analytic in a neighborhood of A and the order of f_1 at A is $K < k$, then there exists an entire function h (which only depends on f and g and is independent of k and z_i) with $h \le f$, $h \le g$. Moreover, among the z_is, there exist at least $m = [(k-1)/K] + 1$ distinct points $z_{n1}, \ldots, z_{nm}$ such that $h(z_{n1}) = \cdots = h(z_{nm})$.

LEMMA 6 ([**14**], p. 133). *Let f and g be two entire functions and $\{z_k\}_{k \in \boldsymbol{N}}$ be an infinite sequence of distinct complex numbers such that $f(z_k) = A$ and $g(z_k) = B$ for all $k \in \boldsymbol{N}$. Suppose that there exist nonconstant functions f_1 and g_1 such that $f_1 \circ f \equiv g_1 \circ g$ on $\bigcup_{i=1}^{\infty} \boldsymbol{U}_i$, where $\boldsymbol{U}_i$ is some open neighborhood containing z_i. If f_1 is analytic in a neighborhood of A, then there exists a transcendental entire function h with $h \le f$, $h \le g$.*

LEMMA 7 (Ng [**14**]). *Let h, k be two transcendental entire functions. Suppose that h has infinitely many zeros. Then for each $n \in \boldsymbol{N}$, there exists a zero a_n of h such that $k(z) = a_n$ has at least n distinct roots which are not zeros of h.*

LEMMA 8. *Let f, g be two permutable transcendental entire functions. If there exist a nonconstant polynomial p and an entire function k such that $p(g(z)) = k(f(z))$, then $f(F(g)) \subset F(g)$, and hence $F(g) \subset F(f)$.*

PROOF. Let $\alpha \in F(g)$. Then there exists a neighbourhood U of α such that $\overline{U} \subset F(g)$. By Lemma 2, we only need to consider the case $g^n \to \infty$ in U. Let $M = \max_{|w|=1} |k(w)|$. Since p is nonconstant polynomial, there exists a positive constant K such that $|p(z)| > M + 1$ when $|z| > K$. Since $g^n \to \infty$ in U as $n \to \infty$, there exists n_0 such that $|g^n(z)| > K$ for $n > n_0$ and $z \in U$. Thus, $|g(z)| > K$ for all $z \in g^n(U)$ $(n > n_0)$. If $f(\alpha)$ is not in $F(g)$, then, for arbitrarily large n, $\{g^n\}$ takes all values in $f(U)$, with at most one exception. Thus there exists $t = f(\zeta)$, with $\zeta \in U$, such that for some $m > n_0$

$$1 > |g^m(t)| = |g^m(f(\zeta))| = |f(g^m(\zeta))|.$$

Thus $\delta = g^m(\zeta) \in g^m(U)$, which implies $|g(\delta)| > K$, and $|f(\delta)| < 1$. Hence, we have

$$M + 1 < |p(g(\delta))| = |k(f(\delta))| \le M,$$

which is a contradiction. Thus we have that $f(\alpha) \in F(g)$. Hence $f(F(g)) \subset F(g)$ and $F(g) \subset F(f)$. □

LEMMA 9 (Bergweiler [**5**]). *If f and g are transcendental entire functions, h is a nonconstant polynomial, then $f(g) - h$ has infinitely many zeros.*

3. Main results and their proofs.

THEOREM 1. *Let f and g be two permutable transcendental entire functions. If there exist a transcendental entire functions h, a rational function f_1 and a function g_1 that is analytic in the range of h, such that $f(z) = f_1(h(z))$ and $g(z) = g_1(h(z))$, then $F(g) \subset F(f)$.*

PROOF. By Lemma 1, we only need to prove that $f(F(g)) \subset F(g)$. Let $\alpha \in F(g)$, then there exists a neighborhood U of α such that $\overline{U} \subset F(g)$. By Lemma 2, we need to consider only the case that $g^n \to \infty$ in U. By the assumption that $f = f_1(h)$, $g = g_1(h)$, f_1 has at most one pole and g_1 has at most one singular point in $\boldsymbol{C}$. Choose $a \in \boldsymbol{C}$ such that $\overline{D}(a,1) = \{w : |w-a| \le 1\}$ does not contain $f_1(\infty)$. Denote E the bounded set $f_1^{-1}(D(a,1)) = \{z : f_1(z) \in D(a,1)\}$. Furthermore, g_1 has at most one singular point in $\boldsymbol{C}$, therefore, a can be so chosen that $\overline{E}$ does not contain any singular point of g_1. Let $M = \max_{w \in \overline{E}} |g_1(w)|$. Since $g^n \to \infty$ in U as $n \to \infty$, there exists n_0 such that $|g^n(z)| > M+1$ for $n > n_0$ and $z \in U$. Thus, $|g(z)| > M+1$ for all $z \in g^n(U)$ $(n > n_0)$. If $f(\alpha)$ is not in $F(g)$, then for sufficiently large n, $\{g^n\}$ takes all values in $f(U)$, with at most one exception. Thus there exists $t = f(\zeta)$, with $\zeta \in U$, such that for some $m > n_0$

$$f(g^m(\zeta)) = g^m(f(\zeta)) = g^m(t) \in D(a,1).$$

Thus $\delta = g^m(\zeta) \in g^m(U)$ and $f(\delta) \in D(a,1)$, which implies that $h(\delta) \in E$. Hence

$$M+1 < |g(\delta)| = |g_1(h(\delta))| \le M,$$

which is a contradiction, and hence $f(\alpha) \in F(g)$. It follows that $f(F(g)) \subset F(g)$ and $F(g) \subset F(f)$. □

Combining Theorem 1 and a result of Ng ([**13**], Theorem A), we have

COROLLARY 1. *Let $f(z)$ and $g(z)$ be two permutable transcendental entire functions. If there exists a nonconstant polynomial $\Phi(x,y)$ in both x and y such that $\Phi(f(z), g(z)) \equiv 0$, then $J(f) = J(g)$.*

REMARK. This result generalizes the results of Baker [**2**], Poon-Yang [**15**] and Wang [**18**], where $\Phi(x,y)$ is chosen respectively to be $x - y - b$, $x - ay - b$ and $p(x) - q(y)$.

COROLLARY 2. *Let f and g be two permutable transcendental entire functions with $F(f) \subset F(g)$. Assume further that f is pseudo-prime. If there exists a transcendental entire functions h such that $h \le f$ and $h \le g$, then $J(f) = J(g)$.*

PROOF. By the assumption that $h \le f$ and $h \le g$, there exist functions f_1 and g_1, which are analytic in the range of h, such that

$$f = f_1(h), \quad g = g_1(h).$$

However, since f is pseudo-prime, f_1 must be a rational function with at most one pole. Hence, by Theorem 1, the conclusion follows. □

COROLLARY 3. *Let f be a transcendental entire function such that $F(f)$ has no wandering domain. Assume further that f is pseudo-prime. Let g be a nonlinear entire function permutable with f. If there exists a transcendental entire function h such that $h \le f$ and $h \le g$, then $J(f) = J(g)$.*

COROLLARY 4. *Let f and g be two permutable transcendental entire functions. Assume further that both f and g are pseudo-prime. If there exists a transcendental entire function h such that $h \le f$ and $h \le g$, then $J(f) = J(g)$.*

THEOREM 2. *Let $f(z) = p(z)e^{q(z)} + c$, where $p(z)$ is a nonzero polynomial and $q(z)$ a nonconstant polynomial. If g is a nonlinear entire function permutable with f, then $J(f) = J(g)$.*

PROOF. By Lemmas 3 and 4, we have $F(f) \subset F(g)$. Now we prove that $F(g) \subset F(f)$. There we have two cases to be dealt with.

CASE 1. $p(z)$ is a constant. If $q(z)$ is linear, then $f(z) = e^{az+b} + c$. Thus we are done by Theorem A. Now we assume that $q(z)$ a nonlinear polynomial. Then $f(z) = e^{q_1(z)} + c$, where $q_1(z) = q(z) + \text{constant}$. $(f - c) \circ g = (g - c) \circ f$ implies that either $g(z) = e^{k(z)} + c$ or $g(z) = (z - c)^n e^{k(z)} + c$ with $n \geq 1$. If $g(z) = e^{k(z)} + c$, then $q_1(g(z)) = k_1(f(z))$, where $k_1(z) = k(z) + \text{constant}$ is an entire function. Hence, the conclusion follows from Lemma 8. Now we consider the case that $g(z) = (z - c)^n \cdot e^{k(z)} + c$, $n \geq 1$. Since $\deg q \geq 2$, f' has at least one zero. If f' has a zero $\neq c$, then $f'(g)g' = g'(f)f'$ implies that $g'(f)$ has infinitely many zeros. If g' has only finitely many zeros, then it follows that there exist a zero B of f' and a zero A of g' such that $f - A$ and $g - B$ has infinitely many common zeros. By Lemma 6, it follows that there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. Thus the conclusion follows from Corollary 2. If g' has infinitely many zeros, by Lemma 7, for arbitrary large N, there exist a zero a_N of g' and a zero B of f' such that $f - a_N$ and $g - B$ has at least N distinct common zeros. It follows from Lemma 5 that there exists an entire function h such that $h \leq f$ and $h \leq g$. Furthermore, h takes some fixed value at least $[N/\deg q]$ distinct points. Since N can be arbitrarily large, so is $[N/\deg q]$, and hence h must be transcendental. By Corollary 2, the conclusion follows. If f' has only one zero c, then $f(z) = e^{A(z-c)^m + B} + c$. Noting $g(z) = (z - c)^n e^{k(z)} + c$ with $n \geq 1$, by calculating the multiplicities of the zero point c of $f'(g)g'$ and $g'(f)f'$, we conclude immediately that $n = 1$. Then $g'(z) = (1 + (z - c)k'(z))e^{k(z)}$ and all its zero must be different from c. Thus, if $1 + (z - c)k'(z)$ has a zero, then $g'(f)$ has infinitely many zeros, so does g'. Again, according to Lemma 7, this will lead to a contradiction. Finally we need to show that f and g are not permutable if $1 + (z - c)k'(z)$ has no zero. Set $1 + (z - c)k'(z) = e^{\beta(z)}$, where $\beta(z)$ is non-constant entire function. If f and g are permutable, then we have

$$A(z - c)^m e^{mk(z)} = A(z - c)^m + k_1(e^{A(z-c)^m + B} + c), \tag{1}$$

where $k_1(z) = k(z) + d$, d is a constant. Noting that $1 + (z - c)k'(z)$ has no zero, we have that $k(z)$ must be a transcendental entire function, so does $k_1(z)$. But by Lemma 9, this is impossible.

CASE 2. $p(z)$ is a nonconstant polynomial. We discuss two subcases. If $p(z)$ has at least two distinct zeros, then $p(g(z))$ has infinitely many zeros. It follows from $p(g(z))e^{q(g(z))} = (g - c) \circ f(z)$ that $(g - c) \circ f(z)$ has infinitely many zeros. Thus there exist a zero A of $p(z)$ and a zero B of $g - c$ such that $f(z) - B$ and $g(z) - A$ have infinitely many common zeros. Again by Lemma 6, there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. If $p(z)$ has only one zero, then $f(z) = (z - b)^n e^{q(z)} + c$ and $(g - b)^n e^{q(g)} = (g - c) \circ f$. Now if $g - c$ has at least two distinct zeros, by applying the same arguments as above, we conclude that there exists a

transcendental entire function h such that $h \le f$ and $h \le g$. If $g - c$ has only one zero, say a, then $g(z) = (z-a)^m e^{k(z)} + c$, where $k(z)$ is a nonconstant entire function. Thus, $f \circ g = g \circ f$ implies that $(g-b)^n e^{q(g)} = (f-a)^m e^{k(f)}$. If $b \neq c$, then $g(z) - b$ has infinitely many zeros which are also the zeros of $f(z) - a$. Hence there exists a transcendental entire function h such that $h \le f$ and $h \le g$, by Lemma 6. If $c \neq a$, we will arrive at the same conclusion. If $a = b = c$, then

$$f(z) = (z-c)^n e^{q(z)} + c, \tag{2}$$

$$g(z) = (z-c)^m e^{k(z)} + c, \tag{3}$$

where $n \ge 1$, $m \ge 1$ and $q(z)$ is a nonconstant polynomial. Thus f' has at least one and only finitely many zeros, which are different from c (a Picard exceptional value of g). Thus, $f'(g)g' = g'(f)f'$ implies that $g'(f)$ has infinitely many zeros. If g' has only finitely many zeros, it is easy to derive that there exists a transcendental entire function h such that $h \le f$ and $h \le g$. If g' has infinitely many zeros, by lemma 7, for any $N \ge n+2$, there exists a zero a_N of g' such that $f(z) = a_N$ and $g(z) = A$ has at least N common roots $z_1, z_2, \ldots, z_N$, where A is a zero of f'. Thus, by lemma 5, there exists an entire function h (which depends on f and g only) with $h \le f$, $h \le g$. Moreover, among $z_1, z_2, \ldots, z_N$, there exist at least $m = [N/(n+1)]$ distinct points at which h takes the same value. Since N as well as m can be arbitrarily large, h must be transcendental. Hence, again by Corollary 2, the conclusion follows. This also completes the proof of the theorem. □

THEOREM 3. *Let $f(z) = \int^z p(z)e^{q(z)}\,dz$, where $p(z)$ and $q(z)$ are nonconstant polynomials. If g is a nonlinear entire function permutable with f, then $J(f) = J(g)$.*

PROOF. By Lemma 4, we conclude that f has no wandering domain. It follows from Lemma 3, that $F(f) \subset F(g)$. Now we prove that $F(g) \subset F(f)$. It is obviously that $f(z)$ is pseudo-prime. Thus by Lemma 1, we only need to prove that there exists a transcendental entire function h such that $h \le f$ and $h \le g$. We need to discuss two cases.

CASE 1. $p(z)$ has two distinct zeros. Then $f'(g(z))$ has infinitely many zeros. It follows from $f'(g)g' = g'(f)f'$ that $g'(f)$ has infinitely many zeros and g' has at least one zero. If g' has only finitely many zeros, then there exist a zero A of g' and a zero B of f' such that $f(z) - A$ and $g(z) - B$ have infinitely many zeros. It follows from Lemma 6 that there exists a transcendental entire function h such that $h \le f$ and $h \le g$. If g' has infinitely many zeros, it follows from $f'(g)g' = g'(f)f'$ and Lemma 7 that for each $n \in \boldsymbol{N}$, there exist a zero a_n of g' and a zero B of f' such that $f(z) - a_n$ and $g(z) - B$ has at least n distinct common zeros. It follows from Lemma 5 that there exists an entire function h such that $h \le f$ and $h \le g$. Furthermore, h takes some fixed value at least $[n/\deg(p+1)]$ distinct points. Since n can be arbitrarily large, so is $[n/\deg(p+1)]$, and hence h must be transcendental.

CASE 2. $p(z)$ has only one zero, thus $p(z) = A(z-c)^n$, $(n \ge 1)$. If $g(z) - c$ has infinitely many zeros, it follows from $A(g(z)-c)^n e^{q(g(z))} g'(z) = Ag'(f(z))(z-c)^n e^{q(z)}$ that g' has at least one zero. If g' has only one zero, say a, then $f(z) - a$ and $g(z) - c$

has infinitely many common zeros. It follows that there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. If g' has at least two distinct zeros, it follows from $A(g(z)-c)^n e^{q(g(z))}g'(z) = Ag'(f(z))(z-c)^n e^{q(z)}$ that g' has infinitely many zeros. Thus by Lemmas 7 and 5, we have that there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$. If $g(z)-c$ has only finite many zeros, then we discuss the following three subcases.

SUBCASE 1. g' has at least two distinct zeros. It follows from $A(g(z)-c)^n e^{q(g(z))} \cdot g'(z) = Ag'(f(z))(z-c)^n e^{q(z)}$ that there exists a zero a of g' such that $f(z)-a$ and $g(z)-c$ has infinitely many common zeros. Then by Lemma 6 again, there exists a transcendental entire function h such that $h \leq f$ and $h \leq g$.

SUBCASE 2. g' has only one zero, say b, then $g'(z) = (z-b)^m e^{\alpha(z)}$. It follows that b is a Picard exceptional value. Thus $f(z) = b + p_2(z)e^{q_2(z)}$, where $p_2(z)$ and $q_2(z)$ are polynomials. It follows from Theorem 2 that $J(f) = J(g)$.

SUBCASE 3. g' has no zero, then $g'(z) = e^{\alpha(z)}$, where $\alpha(z)$ is an entire function. It follows from $(g(z)-c)^n e^{q(g(z))}e^{\alpha(z)} = e^{\alpha(f(z))}(z-c)^n e^{q(z)}$ that $g(z)-c = (z-c)e^{\beta(z)}$, where $\beta(z)$ is a nonconstant entire function. Thus

$$(f-c)\circ g = (g-c)\circ f = (f-c)e^{\beta(f)}.$$

It follows from this and Lemma 7 that $f(z)-c$ has only finitely many zeros. Thus $f(z) = c + p_2(z)e^{q_2(z)}$, where $p_2(z)$ and $q_2(z)$ are polynomials. Again by Theorem 2, we have that $J(f) = J(g)$. □

CONJECTURE. *Theorem* 3 *remains to be valid when* p *is a constant.*

THEOREM 4. *Let* $f(z) = \int^z e^{q(z)}\,dz$, *where* $q(z)$ *is a nonconstant polynomial. If* g *is a nonlinear entire function and not the form* $\int^z e^{\alpha(z)}\,dz$, *where* α *is a nonconstant entire function, then* f *and* g *are not permutable.*

PROOF. Assume that f and g are permutable. Then it follows from $e^{q(g(z))}g'(z) = g'(f(z))e^{q(z)}$ that g' has at most one zero. By assumption of Theorem, g' has one zero, hence $g'(z) = (z-c)^n e^{\alpha(z)}$. It follows from this and $e^{q(g(z))}g'(z) = g'(f(z))e^{q(z)}$ that $f(z)-c = (z-c)e^{q_1(z)}$, where $q_1(z)$ is a non-constant polynomial. Thus one can conclude that $1+(z-c)q_1'(z)$ has no zero at all. This is false which also completes the proof. □

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Liangwen LIAO
Department of Mathematics
Nanjing University
Nanjing
China
E-mail: xuliao@mail.njust.edu.cn

Chung-Chun YANG
Department of Mathematics
The Hong Kong University of Science & Technology
Kowloon
Hong Kong
E-mail: mayang@ust.hk