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Towards the classification of atoms of degenerations, I —Splitting criteria via configurations of singular fibers

Dedicated to Professor Yukio Matsumoto on the occasion of his sixtieth birthday

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Abstract. Motivated by the classification problem of atomic degenerations, in our series of papers, we make a systematic study for splitting deformations of degenerations of complex curves. We provide various new methods to construct splitting deformations, and deduce many splitting criteria of degenerations, which will be applied to the classification of atomic degenerations. Roughly, our criteria are separated into two types; in the first type the criteria are expressed in terms of the configuration of a singular fiber, and in the second type, in terms of sub-divisors of a singular fiber. In both types, our constructions are 'visible', in that we can view how the singular fiber is deformed. In the present paper, we demonstrate splitting criteria of the first type.

Introduction.

This paper constitutes one part of our series of papers on degenerations. By a degeneration, we mean a proper surjective holomorphic map $\pi : M \to \varDelta$ from a smooth complex surface M to the unit disk $\varDelta$ such that the fiber over the origin is singular and any other fiber is a smooth complex curve of genus g ($g \geq 1$). A deformation of a degeneration is called a *splitting deformation* provided that it induces a splitting of its singular fiber. We notice that it may occur that a degeneration admits no splitting deformation at all, in which case the degeneration is called *atomic*. Our main problem is to classify atomic degenerations of arbitrary genera (see [**Re**]). The classification has been known only for the very low genus cases; for the genus 1 case, by Moishezon [**Mo**], and for the genus 2 case, by Horikawa [**Ho**] together with some result of Arakawa and Ashikaga [**AA**], where these results are based on the construction of splitting deformations by the double covering method. (Note that [**Ho**] showed the existence of splittings into types I_1 modulo type 0 while splittings of type 0 are due to Corollary 4.12 of [**AA**].)

Recent progress for the genus 3 case was made by Ashikaga and Arakawa [**AA**], who obtained the classification of absolute atomic degenerations of hyperelliptic curves of genus 3, where a degeneration $\pi : M \to \varDelta$ is *absolutely atomic* provided that all degenerations with the same topological type as $\pi : M \to \varDelta$ are atomic. Their method is also based on the double covering method. Unfortunately, this method fails to work for degenerations of non-hyperelliptic curves. Some new idea is needed for constructing splitting deformations of degenerations of non-hyperelliptic curves even for the genus 3

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case (note that for the genus 2 case, all curves are hyperelliptic, but this is not the case for genus ≥ 3). In our series of papers we develop completely different methods for constructing splitting deformations, and apply them to achieve the classification of absolute atomic degenerations for the genus 3, 4 and 5 cases [**Ta**,III]. The aim of this paper is to study the relation between the configurations of singular fibers and the existence of splitting deformations. We first show that two types of degenerations are atomic.

Theorem 2.0.2. *Let $\pi : M \to \Delta$ be a degeneration of curves such that the singular fiber X is either* (I) *a reduced curve with one node*, *or* (II) *a multiple of a smooth curve of multiplicity at least* 2. *Then $\pi : M \to \Delta$ is atomic.*

(The statement for (I) may be known to experts, but for the convenience of discussions, we include it.) We remark that the proof of Theorem 2.0.2 carries over to arbitrary dimensions to show that a degeneration of type (II) is atomic, i.e. letting $\pi : M \to \Delta$ be a degeneration of compact complex manifolds of arbitrary dimension, if the singular fiber X is a multiple of a smooth complex manifold, then $\pi : M \to \Delta$ is atomic.

Next, we shall state results on existence of splitting deformations; we demonstrate several splitting criteria via the configuration of the singular fiber. Roughly, these criteria are classified into two types; the first one is in terms of some singularities on the singular fiber and the second one is in terms of the existence of irreducible components of multiplicity 1 satisfying certain properties (see the list of splitting criteria in the bottom of this introduction). We note that most of our criteria give the explicit description of splittings of singular fibers. From our criteria, we will see that many degenerations with constellar (constellation-shaped) singular fibers always admit splitting deformations (see §4 for "constellar"). Together with Theorem 2.0.2 it is interesting to know whether the following is true or not.

Conjecture 6.3.1. *A degeneration is atomic if and only if its singular fiber is either a reduced curve with one node, or a multiple of a smooth curve.*

(Actually, this conjecture seems too optimistic for higher genus cases. A more reasonable conjecture is given by replacing "atomic" by "absolutely atomic".) In order to classify atomic degenerations, the results of this paper enable us to use the induction with respect to genus g (see §6.3 for details); let Λ_g be a set of degenerations $\pi : M \to \Delta$ of curves of genus g such that

(1) the singular fiber X has a multiple node (see below) or

(2) X contains an irreducible component Θ_0 of multiplicity 1 satisfying the following condition: if $X \backslash \Theta_0$ is connected, then either genus$(\Theta_0) \geq 1$, or Θ_0 is a projective line intersecting other irreducible components at at least two points.

We note that a *multiple node* is either an intersection point of two irreducible components of the same multiplicity, or a self-intersection point of one irreducible component. As a consequence of our splitting criteria, we obtain the following.

Theorem 6.3.2. *Suppose that Conjecture* 6.3.1 *is valid for genus* $\leq g-1$. *If* $\pi : M \to \Delta$ *is a degeneration in* Λ_g, *then* π *is not atomic.*

Hence, if the assumption of this theorem is fulfilled (e.g. $g = 2$ and 3), to determine atomic degenerations of curves of genus g, it suffices to check the splittability of degenerations $\pi : M \to \Delta$ such that

(A) $X = \pi^{-1}(0)$ is stellar (star-shaped), i.e. the dual graph of X is star-shaped, or
(B) X is constellar and (B.1) X has no multiple node and (B.2) if X has an irreducible component Θ_0 of multiplicity 1, then Θ_0 is a projective line, and intersects other irreducible components of X only at one point.

In [**Ta**,III], we develop another method for constructing splitting deformations, which uses 'barkable' sub-divisors in singular fibers. This method is quite powerful and works for degenerations satisfying (A) or (B).

List of splitting criteria via configurations of singular fibers.

In the list below we notice that in some cases, two different criteria are applicable to one degeneration. Unless otherwise mentioned, a plane curve singularity always means a reduced one.

Criterion 5.1.4. *Let $\pi : M \to \Delta$ be normally minimal such that the singular fiber X has a multiple node of multiplicity at least 2. Then there exists a splitting family of $\pi : M \to \Delta$, which splits X into X_1 and X_2, where X_1 is a reduced curve with one node and X_2 is obtained from X by replacing the multiple node with a multiple annulus.*

Criterion 5.1.5. *Let $\pi : M \to \Delta$ is normally minimal such that the singular fiber X contains a multiple node (of multiplicity ≥ 1). Then $\pi : M \to \Delta$ is atomic if and only if X is a reduced curve with one node.*

Criterion 5.2.3. *Let $\pi : M \to \Delta$ be relatively minimal. Suppose that the singular fiber X has a point p such that a germ of p in X is either*
(1) a multiple of a plane curve singularity of multiplicity at least 2, or
(2) a plane curve singularity such that if it is a node, then $X \backslash p$ is not smooth.
Then $\pi : M \to \Delta$ admits a splitting family.

Criterion 6.1.1. *Let $\pi : M \to \Delta$ be normally minimal. Suppose that the singular fiber X contains an irreducible component Θ_0 of multiplicity 1 such that $X \backslash \Theta_0$ is (topologically) disconnected. Denote by $Y_1, Y_2, \ldots, Y_l$ $(l \geq 2)$ all connected components of $X \backslash \Theta_0$. Then $\pi : M \to \Delta$ admits a splitting family which splits X into $X_1, X_2, \ldots, X_l$, where X_i $(i = 1, 2, \ldots, l)$ is obtained from X by 'smoothing' $Y_1, Y_2, \ldots, \check{Y}_i, \ldots, Y_l$. Here $\check{Y}_i$ is the omission of Y_i.*

Criterion 6.2.1. *Let $\pi : M \to \Delta$ be normally minimal such that the singular fiber X contains an irreducible component Θ_0 of multiplicity 1. Let $\pi_1 : W_1 \to \Delta$ be the restriction of π to a tubular neighborhood W_1 of $X \backslash \Theta_0$ in M. Suppose that $\pi_1 : W_1 \to \Delta$ admits a splitting family Ψ_1 which splits $Y^+ := W_1 \cap X$ into $Y_1^+, Y_2^+, \ldots, Y_l^+$. Then $\pi : M \to \Delta$ admits a splitting family Ψ which splits X into $X_1, X_2, \ldots, X_l$, where X_i is obtained from Y_i^+ by gluing $\Theta_0 \backslash (W_1 \cap \Theta_0)$ along the boundary.*

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1. Preparation.

In this paper, $\Delta := \{s \in \boldsymbol{C} : |s| < 1\}$ stands for the unit disk. Let $\pi : M \to \Delta$ be a proper surjective holomorphic map from a smooth complex surface M to Δ such that $\pi^{-1}(0)$ is singular and $\pi^{-1}(s)$ for $s \neq 0$ is a smooth complex curve of genus g $(g \geq 1)$. We say that $\pi : M \to \Delta$ is a *degeneration* of complex curves of genus g with the *singular fiber* $X := \pi^{-1}(0)$. Two degenerations $\pi_1 : M_1 \to \Delta$ and $\pi_2 : M_2 \to \Delta$ are called *topologically equivalent* if there are orientation preserving homeomorphisms $H : M_1 \to M_2$ and $h : \Delta \to \Delta$ which make the following diagram commutative:

$$\begin{array}{ccc} M_1 & \xrightarrow{H} & M_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ \Delta & \xrightarrow{h} & \Delta \end{array}$$

Next, we introduce basic terminology concerned with deformations of degenerations. We set $\Delta^\dagger := \{t \in \boldsymbol{C} : |t| < \varepsilon\}$, where ε is sufficiently small. Suppose that $\mathscr{M}$ is a smooth complex 3-manifold, and $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a proper flat surjective holomorphic map. (Note: Unless we pose "flatness", a fiber of Ψ is possibly 2-dimensional, e.g. blow up of $\mathscr{M}$ at one point.) We set $M_t := \Psi^{-1}(\Delta \times \{t\})$ and $\pi_t := \Psi|_{M_t} : M_t \to \Delta \times \{t\}$. Since M is smooth and $\dim \Delta^\dagger = 1$, the composite map $\mathrm{pr}_2 \circ \Psi : \mathscr{M} \to \Delta^\dagger$ is a submersion, and so M_t is smooth. We say that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a *deformation family* of $\pi : M \to \Delta$ if $\pi_0 : M_0 \to \Delta \times \{0\}$ coincides with $\pi : M \to \Delta$. For consistency, we mainly use the notation Δ_t instead of $\Delta \times \{t\}$, and we say that $\pi_t : M_t \to \Delta_t$ is a *deformation* of $\pi : M \to \Delta$.

We introduce a special class of deformation families of a degeneration. Suppose that $\pi : M \to \Delta$ is *relatively minimal*, i.e. its singular fiber contains no (-1)-curve (exceptional curve of the first kind). A deformation family $\Psi : M \to \Delta \times \Delta^\dagger$ is said to be a *splitting deformation family* (or *splitting family*) of $\pi : M \to \Delta$ provided that for $t \neq 0$, $\pi_t : M_t \to \Delta_t$ has at least two singular fibers. In this case, we say that $\pi_t : M_t \to \Delta_t$ is a *splitting deformation* of $\pi : M \to \Delta$, and letting $X_1, X_2, \ldots, X_l$ $(l \geq 2)$ be the singular fibers of $\pi_t : M_t \to \Delta_t$, we say that *X splits into $X_1, X_2, \ldots, X_l$* where l is independent of t. (In fact, the discriminant $\mathsf{D} = \{(s,t) \in \Delta \times \Delta^\dagger : \Psi^{-1}(s,t) \text{ is singular}\}$ of Ψ is a plane curve in $\Delta \times \Delta^\dagger$ passing through the origin, and hence $\#(\mathsf{D} \cap \Delta_t)$, the number of the singular fibers of π_t, is constant for sufficiently small $t \neq 0$.) The above definition of a splitting family is too restrictive because we are rather interested in "the germ of a degeneration", and we adopt the following weaker definition. We say that $\pi : M \to \Delta$ admits a splitting family provided that for some δ $(0 < \delta < 1)$ the restriction $\pi' : M' \to \Delta' := \{|s| < \delta\}$, where $M' := \pi^{-1}(\Delta')$ and $\pi' := \pi|_{M'}$, admits a spitting family in the above sense, and if this is the case, for simplicity we use the convention to rewrite $\pi' : M' \to \Delta'$ as $\pi : M \to \Delta$.

We note that a splitting of the singular fiber induces a factorization of the topological monodromy γ of $\pi : M \to \Delta$ along the loop $\partial\Delta$ with counterclockwise orientation, where the topological monodromy is an element of the mapping class group acting on a smooth fiber (see §4). Namely take disjoint simple closed oriented loops ℓ_i in Δ_t circuiting around the points $x_i := \pi(X_i)$ $(i = 1, 2, \ldots, l)$ such that the product $\ell_1 \ell_2 \cdots \ell_l$ is homotopic to $\partial\Delta$ and their orientations coincide. Letting γ_i be the topological monodromy around X_i in $\pi_t : M_t \to \Delta_t$ along ℓ_i, then $\gamma = \gamma_1 \gamma_2 \cdots \gamma_l$.

Next we define the notion of splitting families for a degeneration $\pi : M \to \Delta$ which is *not* relatively minimal. We first introduce some notation. Let us take a sequence of blow down maps

$$M \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots \xrightarrow{f_r} M_r,$$

and degenerations $\pi_i : M_i \to \Delta$ $(i = 1, 2, \ldots, r)$ where

(1) $f_i : M_{i-1} \to M_i$ is a blow down of a (-1)-curve in M_{i-1} and the map $\pi_i : M_i \to \Delta$ is naturally induced from $\pi_{i-1} : M_{i-1} \to \Delta$, and

(2) $\pi_r : M_r \to \Delta$ is a relatively minimal.

Given a deformation family $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ of $\pi : M \to \Delta$, we shall construct a deformation family $\Psi_r : \mathscr{M}_r \to \Delta \times \Delta^\dagger$ of the relatively minimal degeneration $\pi_r : M_r \to \Delta$. First, recall that by Kodaira's Stability Theorem [**Ko2**], any (-1)-curve in a complex surface is preserved under an arbitrary deformation of the surface. Thus, there exists a family of (-1)-curves in $\mathscr{M}$. By [**FN**], we may blow down them simultaneously to obtain a deformation family $\Psi_1 : \mathscr{M}_1 \to \Delta$ of $\pi_1 : M_1 \to \Delta$. Again, by Kodaira's Stability Theorem, there exists a family of (-1)-curves in $\mathscr{M}_1$, which we blow down simultaneously to obtain a deformation family $\Psi_2 : \mathscr{M}_2 \to \Delta$ of $\pi_2 : M_2 \to \Delta$. We repeat this process and finally obtain a deformation family $\Psi_r : \mathscr{M}_r \to \Delta$ of $\pi_r : M_r \to \Delta$. Namely, given a deformation family $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ of $\pi : M \to \Delta$, we obtain a deformation family $\Psi_r : \mathscr{M}_r \to \Delta \times \Delta^\dagger$ of $\pi_r : M_r \to \Delta$. We say that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a *splitting family* of $\pi : M \to \Delta$ provided that $\Psi_r : \mathscr{M}_r \to \Delta \times \Delta^\dagger$ is a splitting family of the relatively minimal degeneration $\pi_r : M_r \to \Delta$. We say that a degeneration is *atomic* if it admits no splitting family at all.

In this paper, instead of relatively minimal degenerations, we mainly use normally minimal degenerations, because they reflect the topological type (or topological monodromies) of degenerations. See §4. We express a singular fiber as a divisor, that is, $X = \sum_i m_i \Theta_i$ where Θ_i is an irreducible component and a positive integer m_i is its multiplicity. Recall that $\pi : M \to \Delta$ is *normally minimal* if X satisfies the following conditions:

(1) the reduced curve $X_{\mathrm{red}} := \sum_i \Theta_i$ is normal crossing, and

(2) if Θ_i is a (-1)-curve, then Θ_i intersects other irreducible components at at least three points.

In this case, we also say that the singular fiber X is normally minimal.

Lemma 1.0.1. *Let $\pi : M \to \Delta$ be a normally minimal degeneration of complex curves of genus g $(g \geq 1)$. Suppose that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a deformation family of $\pi : M \to \Delta$ such that $\pi_t : M_t \to \Delta_t$ $(t \neq 0)$ has at least two normally minimal singular fibers. Then $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a splitting family of $\pi : M \to \Delta$.*

Proof. We first show the statement for the case $g \geq 2$. Let $\pi_r : M_r \to \Delta$ be the relatively minimal model of $\pi : M \to \Delta$, and let $\Psi_r : \mathscr{M}_r \to \Delta \times \Delta^\dagger$ be the deformation family of π_r, which is determined from Ψ. Suppose that Y_1 and Y_2 are normally minimal singular fibers of $\pi_t : M_t \to \Delta_t$. Then after blowing down, the image of Y_i $(i = 1, 2)$ in $M_{r,t} := \Psi_r^{-1}(\Delta \times \{t\})$ has a nontrivial topological monodromy because the topological monodromy of π_t around Y_i is nontrivial (see [**Im**] and also [**MM2**]) and a topological monodromy does not change after blowing down. So the image of Y_i in $M_{r,t}$ is a singular fiber. Hence Ψ_r is a splitting family, and thus by definition, Ψ is a splitting family. This proves the assertion for the case $g \geq 2$. When $g = 1$, this argument is also valid except that Y_1 or Y_2 is a multiple of a smooth elliptic curve, that is, a singular fiber of the form $m\Theta$, where Θ is a smooth elliptic curve and $m \geq 2$ is an integer. We note that this singular fiber has the trivial topological monodromy.

Now we consider the remaining case, that is, Y_1 or Y_2 is a multiple of a smooth elliptic curve. Notice that a multiple of a smooth elliptic curve is relatively minimal (it contains no projective line at all), and so its image in $M_{r,t}$ is also singular, yielding the proof of the assertion. □

2. Atomic degenerations.

In this section, we exhibit two types of atomic degenerations.

Theorem 2.0.2. *Let $\pi : M \to \Delta$ be a degeneration of curves such that the singular fiber X is either* (I) *a reduced curve with one node, or* (II) *a multiple of a smooth curve of multiplicity at least* 2. *Then $\pi : M \to \Delta$ is atomic.*

We notice that in the type (I), X has one or two irreducible components, in the later case, two smooth irreducible components intersecting at one point transversally. The type (II) means that X is of the form $m\Theta$, where $m \geq 2$, and Θ is a smooth curve.

Remark 2.0.3. We remark that the proof of Theorem 2.0.2 carries over to arbitrary dimensions to show that a degeneration of type (II) is atomic, i.e. letting $\pi : M \to \Delta$ be a degeneration of compact complex manifolds of arbitrary dimension, if the singular fiber X is a multiple of a smooth complex manifold, then $\pi : M \to \Delta$ is atomic.

We first demonstrate that if X is a reduced curve with one node, then $\pi : M \to \Delta$ is atomic. We prove this by contradiction. Assume that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a splitting family of π which splits X into $X_1, X_2, \ldots, X_l$ $(l \geq 2)$. We notice that a deformation of a node is either equisingular or smoothing. Hence X_i is an equisingular deformation of X, and so it is also a reduced curve with one node. Since M is diffeomorphic to M_t, we have $\chi(M) = \chi(M_t)$, where $\chi(M)$ stands for the topological Euler characteristic of M. From this equation, by the same argument as in [**BPV**] p. 97 we deduce the following relation of Euler characteristics:

$$\chi(X) - (2 - 2g) = \sum_{i=1}^{l} [\chi(X_i) - (2 - 2g)]. \tag{2.0.1}$$

Since X and $X_1, X_2, \ldots, X_l$ are reduced curves with one node, we have

$$\chi(X) = \chi(X_1) = \cdots = \chi(X_l) = 2 - 2g + 1.$$

Then (2.0.1) implies that $1 = l$, which gives the contradiction.

Note. We can also show the above statement purely analytically by the computation of $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ (cf. [**Pa1**]). In fact, if X splits into $X_1, X_2, \dots, X_l$ $(l \geq 2)$, then the node (A_1-singularity) of X splits into l nodes. However, an A_1-singularity does not admit any splitting. This gives a contradiction.

3. The Proof of Theorem 2.0.2 for the type (II).

Next, we shall demonstrate that if X is a multiple $m\Theta$ of a smooth curve Θ, then $\pi : M \to \Delta$ is atomic. The proof is quite intricate and long, so we separate the statement into several claims to clarify the main step of the proof; for a deformation $\pi_t : M_t \to \Delta_t$ of $\pi : M \to \Delta$, we first construct an unramified covering $p_t : \tilde{M}_t \to M_t$, and then show that the Stein factorization of $\pi_t \circ p_t$ factors through a smooth family over a disk.

Preparation. First, we construct an unramified cyclic m-covering of M. For this purpose, we consider a line bundle $L = \mathcal{O}_M(\Theta)$ on M. Notice that $L^{\otimes m} \cong \mathcal{O}_M$, because $m\Theta$ is the principal divisor defined by the holomorphic function π. We set $F_s := \pi^{-1}(s)$ (so $F_0 = m\Theta$). Then L has the following properties:

(i) For $s \neq 0$, the restriction $L|_{F_s}$ is the trivial bundle on F_s, and
(ii) the restriction $L|_\Theta$ is a line bundle on Θ such that $(L|_\Theta)^{\otimes m} \cong \mathcal{O}_\Theta$.

Next, we take an open covering $M = \bigcup_\alpha U_\alpha$, and let $U_\alpha \times \boldsymbol{C}$ be local trivializations of L with coordinates $(z_\alpha, \zeta_\alpha) \in U_\alpha \times \boldsymbol{C}$. We take a non-vanishing holomorphic section $\tau = \{\tau_\alpha\}$ of $L^{\otimes(-m)} \cong \mathcal{O}_M$. Equations $\tau_\alpha(z_\alpha)\zeta_\alpha^m + 1 = 0$ define a smooth hypersurface $\tilde{M}$ in L because these equations are compatible with the transition functions of L. The map $f : \tilde{M} \to M$ given by $f(z_\alpha, \zeta_\alpha) = z_\alpha$ is an unramified cyclic m-covering. From the properties of the line bundle L,

(i) for $s \neq 0$, $f^{-1}(F_s)$ has m connected components such that each connected component is diffeomorphic to F_s, and
(ii) $\tilde{\Theta} := f^{-1}(\Theta)$ is connected, and $f|_{\tilde{\Theta}} : \tilde{\Theta} \to \Theta$ is an unramified cyclic m-covering.

In order to show that $\pi : M \to \Delta$ is atomic, we shall prove that for an arbitrary deformation family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ of π, the map $\pi_t : M_t \to \Delta_t$ has a unique singular fiber, and it is of the form $m\Theta_t$, where Θ_t is diffeomorphic to Θ. For this purpose, we first construct an unramified cyclic covering of $\mathcal{M}$; notice that $\mathcal{M}$ is diffeomorphic to $M \times \Delta^\dagger$, and the map $\tilde{M} \times \Delta^\dagger \to M \times \Delta^\dagger$, $(x,t) \mapsto (f(x), t)$ is an unramified cyclic m-covering. Thus we have an unramified cyclic m-covering $\rho : \tilde{\mathcal{M}} \to \mathcal{M}$, where we give the complex structure on $\tilde{\mathcal{M}}$ induced from that on $\mathcal{M}$ by ρ. (This is possible, because ρ is unramified.) By construction, setting $\tilde{M}_t := \rho^{-1}(M_t)$, the restriction $p_t : \tilde{M}_t \to M_t$ of ρ to $\tilde{M}_t$ is also an unramified cyclic m-covering. Applying the Stein factorization to the map $\pi_t \circ p_t : \tilde{M}_t \to \Delta_t$, we obtain a commutative diagram

$$\begin{array}{ccc} \tilde{M}_t & \xrightarrow{p_t} & M_t \\ {\scriptstyle \tilde{\pi}_t}\big\downarrow & & \big\downarrow{\scriptstyle \pi_t} \\ \tilde{\Delta}_t & \xrightarrow{\bar{p}_t} & \Delta_t, \end{array} \tag{3.0.2}$$

where (1) $\tilde{\Delta}_t$ is a smooth curve and $\bar{p}_t$ is an m-covering, and (2) $\tilde{\pi}_t : \tilde{M}_t \to \tilde{\Delta}_t$ is a proper surjective map such that all fibers are (topologically) connected. Note that the

Stein Factorization Theorem asserts that since $\tilde{M}_t$ is normal, $\tilde{\Delta}_t$ is also normal (e.g. [**GR**] p. 213). As is well known, any normal curve, hence $\tilde{\Delta}_t$, is smooth. We also notice that since p_t is a cyclic covering, $\bar{p}_t$ is also a cyclic covering. Indeed, let $\tilde{\gamma}$ be a generator of the covering transformation group of p_t, and for $x \in \tilde{\Delta}_t$, set $F_x := \tilde{\pi}_t^{-1}(x)$. From the commutativity of the above diagram, $\tilde{\gamma}(F_x)$ for arbitrary x is also a fiber of $\tilde{\pi}_t$, say F_y where $y \in \tilde{\Delta}_t$. Defining an automorphism γ on $\tilde{\Delta}_t$ by $\gamma x = y$, then γ generates the covering transformation group of $\bar{p}_t$, implying that $\bar{p}_t$ is a cyclic covering.

Proof. After the above preparation, we prove Theorem 2.0.2 for the type (II). The key ingredients of the proof are the following two claims, which together imply that the Stein factorization (3.0.2) is nothing but the stable reduction of $\pi_t : M_t \to \Delta_t$. In what follows, we always assume that $|t|$ is sufficiently small. □

Claim A. $\tilde{\pi}_t : \tilde{M}_t \to \tilde{\Delta}_t$ is a smooth family, i.e. all fibers of $\tilde{\pi}_t$ are smooth.

Claim B. $\tilde{\Delta}_t$ is an open disk.

Assuming Claims A and B for a moment, we will verify that $\pi_t : M_t \to \Delta_t$ has only one singular fiber, and it is of the form $m\Theta_t$. First, we note the following.

Lemma 3.0.4. *Suppose that $p : \tilde{\Delta} \to \Delta$ is a cyclic m-covering, where $\tilde{\Delta}$ and Δ are open unit disks. Then the covering transformation group fixes exactly one point in $\tilde{\Delta}$, and p is given by the map $z \mapsto z^m$ possibly after coordinate change.*

Proof. Let $\gamma : \tilde{\Delta} \to \tilde{\Delta}$ be a generator of the covering transformation group. Then γ is an element of $\mathrm{Aut}(\tilde{\Delta})$, which is isomorphic to the fractional linear transformation group $PSL_2(\boldsymbol{R})$ of the unit disk (Poincaré disk). From $\gamma^m = 1$, the transformation γ is an elliptic element. Thus it fixes exactly one point in $\tilde{\Delta}$, and γ is of the form $z \mapsto e^{2\pi i/m} z$ possibly after coordinate change. Thus $p : \tilde{\Delta} \to \Delta$ is given by $z \mapsto z^m$. □

Now we complete the proof of the theorem. By Claim A, $\tilde{\pi}_t : \tilde{M}_t \to \tilde{\Delta}_t$ is a smooth family. Let $\tilde{\gamma}_t$ be a generator of the covering transformation group of the unramified cyclic m-covering $\tilde{M}_t \to M_t$. By the construction of the Stein factorization of $\pi_t \circ p_t$, the transformation $\tilde{\gamma}_t$ determines a generator γ_t of the covering transformation group of the cyclic m-covering $\tilde{\Delta}_t \to \Delta_t$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{M}_t & \xrightarrow{\tilde{\gamma}_t} & \tilde{M}_t \\ {\scriptstyle \tilde{\pi}_t}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_t} \\ \tilde{\Delta}_t & \xrightarrow{\gamma_t} & \tilde{\Delta}_t \end{array} \tag{3.0.3}$$

Namely, the pair $(\tilde{\gamma}_t, \gamma_t)$ generates an equivariant $\boldsymbol{Z}_m$-action on $\tilde{\pi}_t : \tilde{M}_t \to \tilde{\Delta}_t$, and $\pi_t : M_t \to \Delta_t$ is the quotient of $\tilde{\pi}_t : \tilde{M}_t \to \tilde{\Delta}_t$ by this action. Recall that Δ_t is a disk, while by Claim B, $\tilde{\Delta}_t$ is also a disk. Noting that any open disk is biholomorphic to the unit one, we apply Lemma 3.0.4 to the cyclic m-covering $\tilde{\Delta}_t \to \Delta_t$, and see that γ_t fixes exactly one point, say $\tilde{x}_t$ on $\tilde{\Delta}_t$. From the commutativity of the diagram (3.0.3), we have

LEMMA 3.0.5. *The $\tilde{\gamma}_t$-action on $\tilde{M}_t$ stabilizes precisely one fiber $\tilde{\Theta}_t := \tilde{\pi}_t^{-1}(\tilde{x}_t)$, and except this fiber, this action cyclically permutes the m fibers in each orbit.*

$$\begin{array}{ccc} \tilde{\Theta}_t \subset \tilde{M}_t & \xrightarrow{\tilde{\gamma}_t} & \tilde{\Theta}_t \subset \tilde{M}_t \\ {\scriptstyle \tilde{\pi}_t}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_t} \\ \tilde{x}_t \in \tilde{\Delta}_t & \xrightarrow{\gamma_t} & \tilde{x}_t \in \tilde{\Delta}_t \end{array}$$

As $\pi_t : M_t \to \Delta_t$ is the quotient of the smooth family $\tilde{\pi}_t : \tilde{M}_t \to \tilde{\Delta}_t$ by the equivariant $\boldsymbol{Z}_m$-action, it follows from Lemma 3.0.5 that $\pi_t : M_t \to \Delta_t$ has a unique singular fiber over the point $x_t := \bar{p}_t(\tilde{x}_t)$. This fiber is a multiple of a smooth curve, because $\tilde{M}_t \to M_t$ is unramified cyclic, so in particular, the $\boldsymbol{Z}_m$-action on $\tilde{\Theta}_t$ is an unramified cyclic action. Namely, the singular fiber of $\pi_t : M_t \to \Delta_t$ is $m\Theta_t$, where Θ_t is the image of $\tilde{\Theta}_t$ under the quotient map (the multiplicity equals the order m of the $\tilde{\gamma}_t$-action on $\tilde{\Theta}_t$). Finally, we claim that Θ_t diffeomorphic to Θ. In fact, the restriction of Ψ to $\bigcup_t \Theta_t$ is a smooth family over the 'underlying' reduced curve $\mathsf{D}_{\mathrm{red}}$ of the discriminant of Ψ. (Note that $\mathsf{D}_{\mathrm{red}}$ is a disk. See Remark 3.1.3 below.) By Ehresmann's Theorem, any fiber Θ_t is diffeomorphic to $\Theta_0 = \Theta$. Thus, assuming Claims A and B, we proved Theorem 2.0.2, and so it remains to demonstrate these claims.

3.1. Proof of Claim A.

We start with preparation. Let $X_1, X_2, \dots, X_{\mathsf{d}}$ be the singular fibers of $\pi_t : M_t \to \Delta_t$, and set $x_i := \pi_t(X_i)$. We introduce notation associated to the basic diagram:

$$\begin{array}{ccc} \tilde{M}_t & \xrightarrow{p_t} & M_t \\ {\scriptstyle \tilde{\pi}_t}\downarrow & & \downarrow{\scriptstyle \pi_t} \\ \tilde{\Delta}_t & \xrightarrow{\bar{p}_t} & \Delta_t \end{array} \tag{3.1.1}$$

We set $\bar{p}_t^{-1}(x_i) := \{\tilde{x}_i^{(1)}, \tilde{x}_i^{(2)}, \dots, \tilde{x}_i^{(N_i)}\}$, and let r_i be the ramification index of $\tilde{x}_i^{(j)}$ (so $\bar{p}_t : z \mapsto z^{r_i}$ around $\tilde{x}_i^{(j)}$), where we remark that r_i does not depend on j because $\bar{p}_t : \tilde{\Delta}_t \to \Delta_t$ is a cyclic covering. Since the covering degree of $\bar{p}_t : \tilde{\Delta}_t \to \Delta_t$ is m, we have

$$m = r_i \cdot \#(\bar{p}_t^{-1}(x_i)) = r_i N_i. \tag{3.1.2}$$

For a fiber $\tilde{X}_i^{(j)} := \tilde{\pi}_t^{-1}(\tilde{x}_i^{(j)})$, we write $\tilde{X}_i^{(j)} = \tilde{a}_i \tilde{Y}_i^{(j)}$, where $\tilde{a}_i$ is a positive integer and $\tilde{Y}_i^{(j)}$ is not a multiple divisor, i.e. gcd{coefficients of $\tilde{Y}_i^{(j)}$} $= 1$. Note that $\tilde{a}_i$ does not depend on j, because $\bar{p}_t : \tilde{\Delta}_t \to \Delta_t$ is a cyclic covering. Next, recalling that X_i is a singular fiber of $\pi_t : M_t \to \Delta_t$, we write $X_i = a_i Y_i$, where a_i is a positive integer and Y_i is not a multiple divisor. Notice that

$$(\bar{p}_t \circ \tilde{\pi}_t)^{-1}(x_i) = r_i \tilde{a}_i \tilde{Y}_i^{(j)}. \tag{3.1.3}$$

As p_t is unramified, the fiber of $\pi_t \circ p_t : \tilde{M}_t \to \Delta_t$ over the point x_i is a multiple fiber of multiplicity a_i. (The fiber $(\pi_t \circ \bar{p}_t)^{-1}(x_i)$ is not connected; there are N_i connected components.) Thus from the commutativity of the diagram (3.1.1), together with (3.1.3), we have

$$a_i = r_i \tilde{a}_i. \tag{3.1.4}$$

We notice

LEMMA 3.1.1. $m\tilde{a}_i = N_i a_i$.

PROOF. Indeed, $m\tilde{a}_i = r_i N_i \tilde{a}_i = a_i N_i$, where the first and second equalities follow from (3.1.2) and (3.1.4) respectively. □

Next, we note that if there is a singular fiber of $\tilde{\pi}_t$, then it is a fiber over some $\tilde{x}_i^{(j)}$. Indeed, if $\tilde{X}$ is a singular fiber of $\tilde{\pi}_t$, then the image $p_t(\tilde{X})$ is a singular fiber of π_t. Therefore, to prove Claim A, it is enough to demonstrate that for any $\tilde{x}_i^{(j)}$, the fiber $\tilde{X}_i^{(j)} = \tilde{\pi}_t^{-1}(\tilde{x}_i^{(j)})$ is smooth.

Now we shall show that all $\tilde{X}_i^{(j)}$ are smooth. Although the proof is involved, the essential part of the idea is to relate the singular fibers of $\pi_t \circ p_t$ and the singular fiber of $\pi_0 \circ p_0$. Namely, using the diagram

$$\tilde{\mathscr{M}} \xrightarrow{\rho} \mathscr{M} \xrightarrow{\Psi} \Delta \times \Delta^\dagger,$$

we relate the singular fibers of the following two diagrams ('embedded' in the above diagram) by taking the limit $t \to 0$:

$$\tilde{M}_t \xrightarrow{p_t} M_t \xrightarrow{\pi_t} \Delta_t \quad \text{and} \quad \tilde{M}_0 \xrightarrow{p_0} M_0 \xrightarrow{\pi_t} \Delta_0.$$

STEP 1. We consider the discriminant $\mathsf{D} \subset \Delta \times \Delta^\dagger$ of Ψ; it is a plane curve in $\Delta \times \Delta^\dagger$ through $(0,0)$ defined as the locus where the rank of the differential $d\Psi$ is not maximal. Topologically, D is

$$\{(s,t) \in \Delta \times \Delta^\dagger : \Psi^{-1}(s,t) \text{ is singular}\},$$

but possibly non-reduced. For our discussion, we rather use the underlying reduced curve $\mathsf{D}_{\mathrm{red}}$ of D. By the Weierstrass Preparation Theorem, $\mathsf{D}_{\mathrm{red}}$ is defined by a Weierstrass polynomial

$$s^{\mathsf{n}} + c_{\mathsf{n}-1}(t)s^{\mathsf{n}-1} + c_{\mathsf{n}-2}(t)s^{\mathsf{n}-2} + \cdots + c_0(t) = 0, \tag{3.1.5}$$

where $c_i(t)$ is a holomorphic function with $c_i(0) = 0$. By the definition of underlying reduced curves, this equation contains no multiple root, in other words, the discriminant $\Delta(t)$ of the above Weierstrass polynomial does not vanish identically (but possibly vanishes for some t). Now we claim that $\mathsf{n} = \mathsf{d}$, where d is the number of the singular fibers of $\pi_t : M_t \to \Delta_t$. Indeed, when $t = 0$, (3.1.5) is $s^{\mathsf{n}} = 0$, which clearly has a multiple root, so $\Delta(0) = 0$. Since zeroes of the holomorphic function $\Delta(t)$ are isolated, $\Delta(t)$ does not vanish for sufficiently small t $(t \neq 0)$. Consequently, (3.1.5) has n distinct roots, and so π_t has precisely n singular fibers, implying that $\mathsf{n} = \mathsf{d}$. This verifies the claim, and we have

$$\mathsf{D}_{\mathrm{red}} = \{s^{\mathsf{d}} + c_{\mathsf{d}-1}(t)s^{\mathsf{d}-1} + c_{\mathsf{d}-2}(t)s^{\mathsf{d}-2} + \cdots + c_0(t) = 0\}. \tag{3.1.6}$$

Next, we define a ramified d-covering $\boldsymbol{\phi} : \mathsf{D}_{\mathrm{red}} \to \Delta^{\dagger}$ by $(s,t) \mapsto t$. Then

$$\boldsymbol{\phi}^{-1}(t) = \begin{cases} \mathsf{d} \text{ distinct points} & \text{for } t \neq 0 \\ \text{a multiple point } s^{\mathsf{d}} = 0 & \text{for } t = 0. \end{cases}$$

Step 2. To relate the singular fibers of $\pi_t \circ p_t$ and $\pi_0 \circ p_0$, we consider the hypersurface $\tilde{\mathscr{H}} := (\Psi \circ \rho)^{-1}(\mathsf{D}_{\mathrm{red}})$ in the complex 3-manifold $\tilde{\mathscr{M}}$. For the remainder of the proof, to emphasize the parameter t, we use 'precise' notation $\tilde{X}_{i,t}^{(j)}$ instead of $\tilde{X}_i^{(j)}$ etc. Notice that

$$\mathscr{H} \cap \tilde{M}_t = \begin{cases} \text{the disjoint union of all } \tilde{X}_{i,t}^{(j)} & \text{for } t \neq 0 \\ \mathsf{d}m\tilde{\Theta} & \text{for } t = 0, \end{cases} \tag{3.1.7}$$

where we can see $\mathscr{H} \cap \tilde{M}_0 = \mathsf{d}m\tilde{\Theta}$ as follows. Since $\pi_0^{-1}(0) = m\Theta$ and p_0 is unramified (locally biholomorphic), we have $(\pi_0 \circ p_0)^{-1}(0) = m\tilde{\Theta}$, hence the fiber of $\pi_0 \circ p_0$ over the multiple point $s^{\mathsf{d}} = 0$ is $\mathsf{d}m\tilde{\Theta}$, and so $\mathscr{H} \cap \tilde{M}_0 = \mathsf{d}m\tilde{\Theta}$.

By the first equation of (3.1.7), our goal is to show that $\mathscr{H} \cap \tilde{M}_t$ is smooth for all $t \neq 0$. To demonstrate this, fixing an arbitrary point $y \in \tilde{\Theta}$ $(= p_0^{-1}(\Theta))$, we take local coordinates (z_1, z_2, t) around y in $\tilde{M}$ such that $z_1 = t = 0$ locally defines $\tilde{\Theta}$. (Note: By the definition of deformation families, $\mathrm{pr}_2 \circ \Psi : \mathscr{M} \to \Delta \times \Delta^{\dagger} \to \Delta^{\dagger}$ is a submersion. Since ρ is unramified, $\mathrm{pr}_2 \circ \Psi \circ \rho : \tilde{\mathscr{M}} \to \Delta^{\dagger}$ is also a submersion. By the Implicit Function Theorem, we may 'lift' $t \in \Delta^{\dagger}$ to a coordinate of $\tilde{\mathscr{M}}$.)

Let $f(z_1, z_2, t) = 0$ be a defining equation of $\tilde{\mathscr{H}}$ around y in $\tilde{\mathscr{M}}$. For later discussion, we use the notation $f_t(z_1, z_2)$ instead of $f(z_1, z_2, t)$. By the first equation of (3.1.7), $\tilde{\mathscr{H}} \cap M_t = \coprod_{i=1}^{\mathsf{d}} (\coprod_{j=1}^{N_i} \tilde{X}_{i,t}^{(j)})$ (disjoint union) and $\tilde{X}_{i,t}^{(j)} = a_i \tilde{Y}_{i,t}^{(j)}$, so we have a factorization

$$f_t = \prod_{i=1}^{\mathsf{d}} f_{i,t}^{a_i}, \quad \text{where } f_{i,t} = \prod_{j=1}^{N_i} g_{i,t}^{(j)}, \tag{3.1.8}$$

and $g_{i,t}^{(j)} = 0$ defines $\tilde{Y}_{i,t}^{(j)}$ locally. By the second equation of (3.1.7), $f_0(z_1, z_2) = z_1^{\mathsf{d}m}$, hence setting $t = 0$ in (3.1.8), we have

$$z_1^{\mathsf{d}m} = f_0 = \prod_{i=1}^{\mathsf{d}} f_{i,0}^{a_i}, \tag{3.1.9}$$

and so we may express $g_{i,0}^{(j)}(z_1, z_2) = z_1^{d_i^{(j)}} \cdot u_i^{(j)}(z_1, z_2)$, where $d_i^{(j)}$ is a positive integer and $u_i^{(j)}$ is a non-vanishing holomorphic function. By the comparison of the degrees of z_1 in (3.1.9), we have

$$\mathsf{d}m = \sum_{i=1}^{\mathsf{d}} a_i (d_i^{(1)} + d_i^{(2)} + \cdots + d_i^{(N_i)}). \tag{3.1.10}$$

Now we show the key lemma.

Lemma 3.1.2. *$\tilde{a}_i = d_i^{(1)} = d_i^{(2)} = \cdots = d_i^{(N_i)} = 1$ for $i = 1, 2, \ldots, \mathsf{d}$.*

Proof. From (3.1.10), we note

$$\mathsf{d}m = \sum_{i=1}^{\mathsf{d}} a_i(d_i^{(1)} + d_i^{(2)} + \cdots + d_i^{(N_i)}) \tag{3.1.11}$$
$$\geq \sum_{i=1}^{\mathsf{d}} a_i N_i$$
$$= \sum_{i=1}^{\mathsf{d}} \tilde{a}_i m,$$

where the second inequality follows from $d_i^{(1)}, d_i^{(2)}, \ldots, d_i^{(N_i)} \geq 1$ and the last equality follows from Lemma 3.1.1. Thus we have $\mathsf{d}m \geq \sum_{i=1}^{\mathsf{d}} \tilde{a}_i m$, which implies that $\tilde{a}_1 = \tilde{a}_2 = \cdots = \tilde{a}_{\mathsf{d}} = 1$, and this inequality is an equality. In particular, (3.1.11) is also an equality, and so $d_i^{(1)} = d_i^{(2)} = \cdots = d_i^{(N_i)} = 1$. This completes the proof. $\square$

Now, it is immediate to complete the proof of Claim A. From $\tilde{a}_i = 1$, we have $\tilde{X}_{i,t}^{(j)} = \tilde{Y}_{i,t}^{(j)}$. On the other hand, from $\tilde{d}_i^{(j)} = 1$, $\tilde{Y}_{i,0}^{(j)}$ is smooth, because it is locally defined by $z_1 \cdot u_i^{(j)}(z_1, z_2) = 0$. Thus for sufficiently small t, $\tilde{Y}_{i,t}^{(j)}$ is smooth, and so $\tilde{X}_{i,t}^{(j)} = \tilde{Y}_{i,t}^{(j)}$ is smooth. This completes the proof of Claim A.

Remark 3.1.3. If $\mathsf{d} = 1$, i.e. $\pi_t : M_t \to \Delta_t$ has only one singular fiber, then $\mathsf{D}_{\mathrm{red}} = \{s + c_0(t) = 0\}$ (see (3.1.6)) is a disk in $\Delta \times \Delta^{\dagger}$.

(It may be worth to point out that we did not use the Stein factorization of the map $\Psi \circ \rho$; it factors through a normal surface S, which possibly has a singularity. In contrast, the Stein factorization for a map from a normal total space to a curve necessarily factors through a *smooth* curve, which is much easier to be treated. See [**GR**] p. 213 for details.)

3.2. Proof of Claim B.

We shall show Claim B asserting that $\tilde{\Delta}_t$ is a disk, for which we will apply a topological argument. By shrinking $M_t, \tilde{M}_t, \Delta_t$ and $\tilde{\Delta}_t$, we regard them as closed manifolds with boundary. We first take diffeomorphisms $\phi_t : M_0 \to M_t$ and $\bar{\phi}_t : \partial\Delta_0 \to \partial\Delta_t$ which make the following diagram commute (for the existence of ϕ_t, see Lemma 3.3.2 below):

$$\begin{tikzcd} \partial M_0 \arrow[r, "\phi_t"] \arrow[d, "\pi_0"'] & \partial M_t \arrow[d, "\pi_t"] \\ \partial\Delta_0 \arrow[r, "\bar{\phi}_t"] & \partial\Delta_t \end{tikzcd}$$

(Namely, the restriction of $\bar{\phi}_t$ to the boundary ∂M_0 is fiber-preserving.) Recall that we constructed $p_t : \tilde{M}_t \to M_t$ from $p_0 : \tilde{M}_0 \to M_0$ via the diffeomorphism $\phi_t : M_0 \to M_t$. Hence there is a natural diffeomorphism $\Phi_t : \tilde{M}_0 \to \tilde{M}_t$, which is a lifting of ϕ_t (that is, $\Phi_t \circ p_t = p_0 \circ \phi_t$), and the restriction of Φ_t to $\partial\tilde{M}_0$ is fiber-preserving, i.e. the following diagram commutes

$$\begin{array}{ccc} \partial\tilde{M}_0 & \xrightarrow{\Phi_t} & \partial\tilde{M}_t \\ {\scriptstyle\tilde{\pi}_0}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}_t} \\ \partial\tilde{\varDelta}_0 & \xrightarrow{\bar{\Phi}_t} & \partial\tilde{\varDelta}_t, \end{array}$$

where $\bar{\Phi}_t$ is a diffeomorphism. Now we fix a fiber $C_0 := \tilde{\pi}_0^{-1}(y_0)$, where $y_0 \in \partial\tilde{\varDelta}_0$, and let $\iota_0 : C_0 \hookrightarrow \tilde{M}_0$ be the natural embedding. Then $C_t := \Phi_t(C_0)$ is a fiber of $\tilde{\pi}_t$ over $y_t := \bar{\Phi}_t(y_0) \in \partial\tilde{\varDelta}_t$, and let $\iota_t : C_t \hookrightarrow \tilde{M}_t$ be the natural embedding:

$$\begin{array}{ccc} C_0 \subset \partial\tilde{M}_0 & \xrightarrow{\Phi_t} & C_t \subset \partial\tilde{M}_t \\ {\scriptstyle\tilde{\pi}_0}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}_t} \\ y_0 \in \partial\tilde{\varDelta}_t & \xrightarrow{\bar{\Phi}_t} & y_t \in \partial\tilde{\varDelta}_t \end{array}$$

After this preparation, we can demonstrate that $\tilde{\varDelta}_t$ is a disk. Note that $\tilde{\varDelta}_t$ is a *real* compact surface with a connected boundary isomorphic to S^1. (By the construction of $\tilde{M}_t$, the boundary $\partial\tilde{M}_t$ is connected, and so $\partial\tilde{\varDelta}_t$ is connected.) Thus if the genus of $\tilde{\varDelta}_t$ is g, then $\tilde{\varDelta}_t$ is homotopically equivalent to the bouquet $S^1 \vee S^1 \vee \cdots \vee S^1$ of $2g$ circles, and so

$$\pi_1(\tilde{\varDelta}_t) = \underbrace{\boldsymbol{Z} * \boldsymbol{Z} * \cdots * \boldsymbol{Z}}_{2g}, \quad \text{the free group of rank } 2g.$$

Hence it suffices to show that $\pi_1(\tilde{\varDelta}_t) = 1$. For this, we first take the homotopy exact sequence associated to the differentiable fiber bundle $\tilde{\pi}_0 : \tilde{M}_0 \to \tilde{\varDelta}_0$, where we note that by Ehresmann's Theorem, a smooth family is a differentiable fiber bundle:

$$\pi_2(\tilde{\varDelta}_0) \to \pi_1(C_0) \xrightarrow{\iota_{0*}} \pi_1(\tilde{M}_0) \to \pi_1(\tilde{\varDelta}_0) \to 1$$

Next, noting that from Claim A, $\tilde{\pi}_t : \tilde{M}_t \to \tilde{\varDelta}_t$ is a differentiable fiber bundle, so we may take the homotopy exact sequence associated to it:

$$\pi_2(\tilde{\varDelta}_t) \to \pi_1(C_t) \xrightarrow{\iota_{t*}} \pi_1(\tilde{M}_t) \to \pi_1(\tilde{\varDelta}_t) \to 1$$

The following commutative diagram relates the above two homotopy exact sequences:

$$\begin{array}{ccccccccc} \pi_2(\tilde{\varDelta}_0) & \longrightarrow & \pi_1(C_0) & \xrightarrow{\iota_{0*}} & \pi_1(\tilde{M}_0) & \longrightarrow & \pi_1(\tilde{\varDelta}_0) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ \pi_2(\tilde{\varDelta}_t) & \longrightarrow & \pi_1(C_t) & \xrightarrow{\iota_{t*}} & \pi_1(\tilde{M}_t) & \longrightarrow & \pi_1(\tilde{\varDelta}_t) & \longrightarrow & 1, \end{array} \tag{3.2.1}$$

where the vertical arrows are induced by Φ_t. Since $\tilde{\varDelta}_0$ is a disk, we have $\pi_1(\tilde{\varDelta}_0) = \pi_2(\tilde{\varDelta}_0) = 1$, and so ι_{0*} is an isomorphism. Two vertical arrows are also isomorphisms, because they are induced by the diffeomorphism Φ_t. From the commutativity of the diagram (3.2.1), we see that ι_{t*} is an isomorphism, and so the exactness of the bottom horizontal sequence of (3.2.1) implies that $\pi_1(\tilde{\varDelta}_t) = 1$ and so $\tilde{\varDelta}_t$ is a disk.

3.3. Supplement: Construction of diffeomorphisms.

In this subsection, by a fiber bundle we always mean a differentiable one. Suppose that $\Psi : \mathscr{M} \to \varDelta \times \varDelta^\dagger$ is a deformation family of $\pi : M \to \varDelta$. Note that the restriction

$\pi_t|_{\partial M_t} : \partial M_t \to \partial \varDelta_t$ is a fiber bundle. The following lemma may be known to the geometers, but for the convenience of the reader, we include the proof. (Hereafter, for consistency, we denote $\pi_0 : M_0 \to \varDelta_0$ instead of $\pi : M \to \varDelta$.)

Lemma 3.3.1. *There exists a diffeomorphism $\phi_t : M_0 \to M_t$ such that the restriction $\phi_t|_{\partial M_0}$ preserves fibers, that is, there exists a diffeomorphism $\bar{\phi}_t : \partial \varDelta_0 \to \partial \varDelta_t$ which makes the following diagram commute*:

$$\begin{array}{ccc} \partial M_0 & \xrightarrow{\phi_t} & \partial M_t \\ {\scriptstyle \pi_0}\big\downarrow & & \big\downarrow{\scriptstyle \pi_t} \\ \partial \varDelta_0 & \xrightarrow{\bar{\phi}_t} & \partial \varDelta_t \end{array}$$

Warning. Although the *restriction* of ϕ_t to the boundary ∂M_0 commutes with maps π_0 and π_t, this is not the case for ϕ_t itself.

Proof. For simplicity, we assume that $\varDelta$ is the unit disk. We choose $r_1, r_2 \in \boldsymbol{R}$ so that $0 < r_2 < r_1 < 1$, and define an open covering $\varDelta \times \varDelta^\dagger = U_{\text{in}} \cup U_{\text{out}}$, where

$$U_{\text{in}} := \{(s,t) \in \varDelta \times \varDelta^\dagger : |s| < r_1\}, \quad U_{\text{out}} := \{(s,t) \in \varDelta \times \varDelta^\dagger : |s| > r_2\}.$$

We then take an open covering $\mathscr{M} = \mathscr{M}_{\text{in}} \cup \mathscr{M}_{\text{out}}$, where $\mathscr{M}_{\text{in}} := \Psi^{-1}(U_{\text{in}})$ and $\mathscr{M}_{\text{out}} := \Psi^{-1}(U_{\text{out}})$. Taking r_1 sufficiently close to 1, we assume that $\mathscr{M}_{\text{out}}$ contains no singular fiber, i.e. the restriction $\Psi_{\text{out}} := \Psi|_{\mathscr{M}_{\text{out}}}$ is a fiber bundle. In particular, Ψ_{out} is a submersion. Hence there exists a vector field v_{out} on $\mathscr{M}_{\text{out}}$ such that

$$d\Psi_{\text{out}}(v_{\text{out}}) = \frac{\partial}{\partial t}. \tag{3.3.1}$$

Similarly, we set $\Psi_{\text{in}} := \Psi|_{\mathscr{M}_{\text{in}}}$. By the definition of deformation families, the composite map $\text{pr}_2 \circ \Psi_{\text{in}} : \mathscr{M}_{\text{in}} \to \varDelta^\dagger$ is a fiber bundle with smooth complex surfaces as fibers, and so a submersion. Thus there exists a vector field v_{in} on $\mathscr{M}_{\text{in}}$ such that

$$d(\text{pr}_2 \circ \Psi_{\text{in}})(v_{\text{in}}) = \frac{\partial}{\partial t}. \tag{3.3.2}$$

Notice that in (3.3.1), $\partial/\partial t$ is a vector field on $\varDelta \times \varDelta^\dagger$, while in (3.3.2), it is a vector field on $\varDelta^\dagger$. We shall 'patch' two vector fields v_{in} and v_{out} by a partition of unity, and define a vector field v on $\mathscr{M}$; we first define open subsets $U'_{\text{in}} \subset U_{\text{in}}$ (resp. $U'_{\text{out}} \subset U_{\text{out}}$) as follows. Take $r'_1, r'_2 \in \boldsymbol{R}$ satisfying $0 < r'_1 < r_2 < r_1 < r'_2 < 1$, and set

$$U'_{\text{in}} := \{(s,t) \in \varDelta \times \varDelta^\dagger : |s| < r'_1\}, \quad U'_{\text{out}} := \{(s,t) \in \varDelta \times \varDelta^\dagger : |s| > r'_2\}.$$

Notice that $U'_{\text{in}} \cap U'_{\text{out}} = \varnothing$. Now we put $\mathscr{M}'_{\text{in}} := \Psi^{-1}(U'_{\text{in}})$ and $\mathscr{M}'_{\text{out}} := \Psi^{-1}(U'_{\text{out}})$. Then $\mathscr{M}'_{\text{in}} \cap \mathscr{M}'_{\text{out}} = \varnothing$. Using a partition of unity, we can construct a vector field v on $\mathscr{M}$ such that

$$v = \begin{cases} v_{\text{in}} & \text{on } \mathscr{M}'_{\text{in}} \\ v_{\text{out}} & \text{on } \mathscr{M}'_{\text{out}}. \end{cases}$$

Finally, we integrate the vector field v on $\mathscr{M}$ to obtain a one-parameter family of diffeomorphisms $\phi_t : M_0 \to M_t$ with the desired property. $\square$

4. Topological monodromies and singular fibers.

Before we the proceed to state splitting criteria, we briefly review the relation between the topological monodromies and the configurations of singular fibers (see [**MM2**] and [**Ta**,II] for details). First, we recall the topological monodromy of a degeneration $\pi : M \to \Delta$. For this purpose, it is convenient to consider M and Δ as manifolds with boundary, so Δ is the closed unit disk. We write $\partial\Delta = \{e^{i\theta} : 0 \le \theta \le 2\pi\}$, and set $C_\theta := \pi^{-1}(e^{i\theta})$. Using a partition of unity, we construct a vector field v on ∂M such that $d\pi(v) = \partial/\partial\theta$. Then the integration of v yields a one-parameter family of diffeomorphisms $h_\theta : C_0 \to C_\theta$ (see Figure 1). In particular, $h_{2\pi}$ is a self-homeomorphism of C_0. Setting $h := h_{2\pi}$, we refer to h as the *topological monodromy* of $\pi : M \to \Delta$.

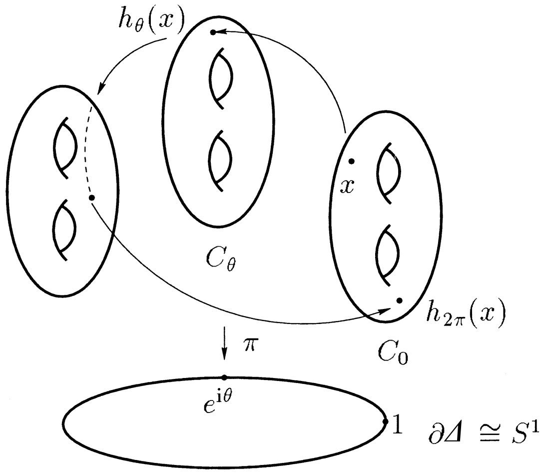


Figure 1.

Topological monodromies are very special homeomorphisms; they are either periodic or pseudo-periodic (see [**MM2**], and also [**Im**]). Recall that a homeomorphism h of a curve C is (1) *periodic* if for some positive integer m, h^m is isotopic to the identity, and (2) *pseudo-periodic* if for some loops $l_1, l_2, \dots, l_n$ on C, the restriction h on $C \backslash \{l_1, l_2, \dots, l_n\}$ is periodic. (In [**MM2**], periodic homeomorphisms are considered to be special cases of pseudo-periodic homeomorphisms by taking $\{l_1, l_2, \dots, l_n\} = \emptyset$. However for our discussion it is convenient to distinguish periodic homeomorphisms from pseudo-periodic ones.) According to whether the topological monodromy is periodic or pseudo-periodic, the singular fiber is *stellar* (star-shaped) or *constellar* (constellation-shaped). In some sense, a constellar singular fiber is obtained by 'bonding' stellar ones (see [**MM2**] and [**Ta**,II]).

Remark 4.0.2. Based on a topological argument, Matsumoto and Montesinos [**MM2**] showed that the configuration of the singular fiber of a degeneration is completely determined by its topological monodromy. In [**Ta**,II], we gave an algebro-geometric proof for their results, and clarified the relation between topological monodromies and quotient singularities.

Now the followings are the simplest examples for periodic and pseudo-periodic homeomorphisms respectively:

Example 4.0.3 (Periodic). h is an unramified periodic homeomorphism, that is, the quotient map $C \to C/\langle h \rangle$ is an unramified cyclic covering.

Example 4.0.4 (Pseudo-periodic). h is a right Dehn twist along a simple closed loop l on C, so the restriction of h to $C \backslash l$ is isotopic to the identity.

A degeneration with the topological monodromy in Example 4.0.3 has a singular fiber $m\Theta$, where m is the order of h, and Θ is a smooth curve which is the quotient of C by the action of h. On the other hand, the singular fiber of a degeneration with the topological monodromy in Example 4.0.3 is a reduced curve with one node (this node is obtained by 'pinching' l on C). By Theorem 2.0.2, both of these degenerations are atomic. Namely, all degenerations with the simplest topological monodromies are atomic. To the contrary, if the topological monodromy is 'complicated', what can we say about splittability? In this case, the singular fiber is also complicated, so the reader may imagine that they are not atomic (complicated objects should not be atoms!). In the later half of this paper, we will show that this guess is true.

5. Splitting criteria via configurations, I.

In this and subsequent sections, we will give splitting criteria of degenerations in terms of configurations of their singular fibers. As a consequence of these criteria, we will see that many degenerations with constellar singular fibers always admit splitting families. We point out that these criteria are powerful for determining atomic degenerations by induction with respect to genus g (see §6.3 for details).

In the discussion below, we often use the realization of M as the graph of π; for a degeneration $\pi : M \to \Delta$, the graph of π is defined by

$$\mathrm{Graph}(\pi) = \{(x, s) \in M \times \Delta : \pi(x) - s = 0\}.$$

Of course, $\mathrm{Graph}(\pi)$ is a smooth hypersurface in $M \times \Delta$, and M is canonically isomorphic to $\mathrm{Graph}(\pi)$ by $x \in M \mapsto (x, \pi(x)) \in M \times \Delta$. Under this isomorphism, the map $\pi : M \to \Delta$ corresponds to the projection $(x, s) \in \mathrm{Graph}(\pi) \mapsto s \in \Delta$. In the discussion below, we identify $\mathrm{Graph}(\pi)$ with M via the canonical isomorphism, and we write M instead of $\mathrm{Graph}(\pi)$.

5.1. Criterion in terms of nodes.

In this subsection, we shall provide splitting criteria in terms of some singularity on the singular fiber.

Definition 5.1.1. Let m be a positive integer. A singularity analytically isomorphic to

$$V_m := \{(x, y) \in \boldsymbol{C}^2 : x^m y^m = 0\}$$

is called a *multiple node of multiplicity m*.

Note that when $m \geq 2$, V_m is non-reduced. By abuse of terminology, we also say that the origin of V_m is a multiple node.

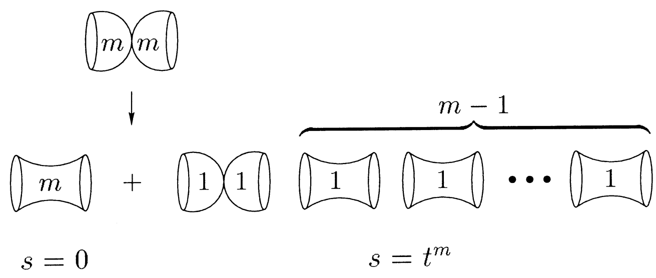
We consider a hypersurface $\mathscr{M} := \{(x, y, s, t) \in \boldsymbol{C}^4 : (xy + t)^m - s = 0\}$, and define a holomorphic map $\Psi : \mathscr{M} \to \boldsymbol{C}^2$ by $(x, y, s, t) \mapsto (s, t)$. Clearly, $\Psi^{-1}(0, 0) = V_m$, and so Ψ is a two-parameter deformation family of V_m. Next, we shall compute the discriminant of Ψ. Since

$$\frac{\partial \Psi}{\partial x} = my(xy+t)^{m-1}, \quad \frac{\partial \Psi}{\partial y} = mx(xy+t)^{m-1},$$

we have $\partial\Psi/\partial x = \partial\Psi/\partial y = 0$ if and only if either (1) $x = y = 0$ or (2) $xy + t = 0$. We note that $t^m - s = 0$ for (1), and $s = 0$ for (2).

Lemma 5.1.2. *The discriminant of Ψ consists of curves $s = t^m$ and $s = 0$ in $\boldsymbol{C}^2$. To be explicit, for $t \neq 0$,*

(1) *$\Psi^{-1}(t^m, t)$ is the disjoint union of $m-1$ annuli and a node,*
(2) *$\Psi^{-1}(0, t)$ is a multiple of an annulus of multiplicity m.*



Figrue 2.

Proof. The fiber $\Psi^{-1}(t^m, t)$ $(t \neq 0)$ is defined by

$$xy[(xy)^{m-1} + {}_mC_1(xy)^{m-2}t + \cdots + {}_mC_i(xy)^{m-i-1}t^i + \cdots + {}_mC_1 t^{m-1}] = 0.$$

This equation factorizes as $xy\prod_{i=1}^{m-1}(xy + \alpha_i t) = 0$, where $\alpha_i = e^{2\pi\sqrt{-1}i/m} - 1$ $(i = 1, 2, \ldots, m-1)$ are the solutions of

$$X^{m-1} + {}_mC_1 X^{m-2} + \cdots + {}_mC_i X^{m-i-1} + \cdots + {}_mC_1 = 0,$$

that is, nonzero solutions of $(X+1)^m - 1 = 0$. Hence $\Psi^{-1}(t^m, t)$ $(t \neq 0)$ is the disjoint union of a node $xy = 0$ and $m-1$ annuli $xy + \alpha_i = 0$ $(i = 1, 2, \ldots, m-1)$. On the other hand, $\Psi^{-1}(0, t) = \{(xy+t)^m = 0\}$ is a multiple annulus of multiplicity m. $\square$

Before proceeding we introduce a terminology.

Definition 5.1.3. A domain is called *annular* if it is biholomorphic to a domain of the form

$$U = \{(x, y) \in \boldsymbol{C}^2 : r < |x| < 1, |y| < 1, |xy| < d\},$$

where r $(0 < r < 1)$ is sufficiently close to 1 and d satisfies $0 < d < 1$.

We note that an annular domain U is diffeomorphic to the product of an annulus and a disk.

Criterion 5.1.4. *Let $\pi : M \to \Delta$ be normally minimal such that the singular fiber X has a multiple node p of multiplicity at least 2. Then there exists a splitting family of $\pi : M \to \Delta$, which splits X into X_1 and X_2, where X_1 is a reduced curve with one node and*

X_2 is obtained from X by replacing the multiple node p with a multiple annulus (see Figure 6 for example).

Proof. If necessary, shrink M and Δ, i.e. consider $\pi : M \to \Delta = \{|s| < \delta\}$ for some δ $(0 < \delta < 1)$, and take an open covering $M = W_0 \cup W_1$ (see Figure 3) such that

(1) W_0 is an open set around p (hence $W_0 \cap X$ is the multiple node), given by

$$\{(x, y) \in \boldsymbol{C}^2 : |x| < 1, |y| < 1, |x^m y^m| < \delta\} \quad \text{(see Figure 4)},$$

(2) $W_1 \cap X$ is 'outside' the multiple node, and

(3) $\pi(W_0) = \pi(W_1) = \Delta$.

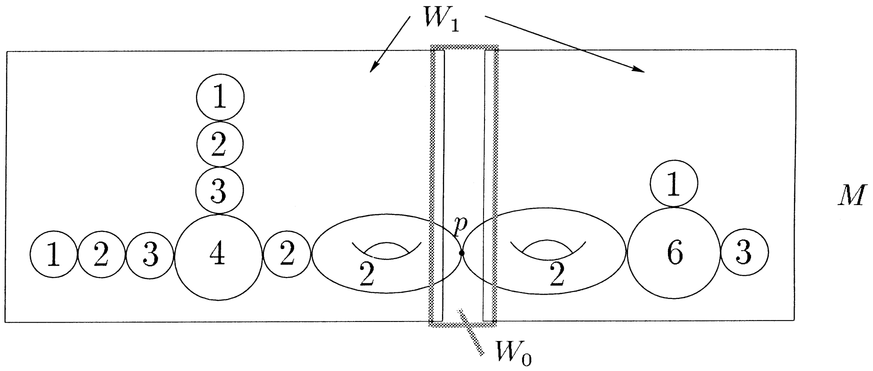


Figure 3.

Notice that the boundary of W_0 consists of two connected components respectively defined by $|x| = 1$ and $|y| = 1$. (Each component is a solid torus.) Taking r $(0 < r < 1)$ sufficiently close to 1, we consider open sets:

$$U = \{(x, y) \in W_0 : r < |x| < 1\}, \quad V = \{(x, y) \in W_0 : r < |y| < 1\},$$

and then U and V are neighborhoods of boundary components of W_0. See Figure 4. Notice that U and V are annular domains (take $d = \delta^{1/m}$ in Definition 5.1.3), and $\pi(U) = \pi(V) = \Delta$.

For the subsequent argument, we write $U = U_0$ and $V = V_0$, and then M is obtained by patching W_0 and W_1 along $U_0 \subset W_0$ and $U_1 \subset W_1$ and along $V_0 \subset W_0$ and $V_1 \subset W_1$ where U_1 and V_1 are annular domains in W_1. We rewrite $(x, y) = (z_\beta, \zeta_\beta) \in U_0$, then we have $\pi(z_\beta, \zeta_\beta) = z_\beta^m \zeta_\beta^m$. Next, we take coordinates $(z_\alpha, \zeta_\alpha) \in U_1$. Then

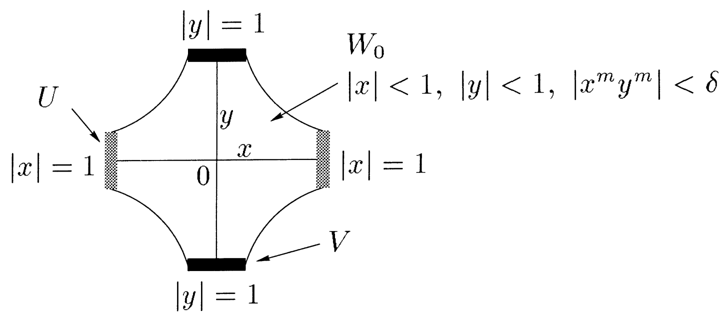


Figure 4. U and V are respectively described by the gray and black bold lines (in the real 2-dimensional figure, two gray lines are disconnected, but they are in fact connected; the same for two black lines).

$\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha^m f_\alpha(z_\alpha, \zeta_\alpha)$, where f_α is a non-vanishing holomorphic function. As $\pi(z_\alpha, \zeta_\alpha) = \pi(z_\beta, \zeta_\beta)$, we have

$$\zeta_\alpha^m f_\alpha(z_\alpha, \zeta_\alpha) = z_\beta^m \zeta_\beta^m.$$

Note that the holomorphic function $z_\beta^m \zeta_\beta^m$ on the right has an m-th root $z_\beta \zeta_\beta$, which is a single-valued function. Thus $\zeta_\alpha^m f_\alpha$ also has a *single valued* m-th root function $\zeta_\alpha f_\alpha^{1/m}$ such that $\zeta_\alpha f_\alpha^{1/m} = z_\beta \zeta_\beta$. Rewriting $\zeta_\alpha f_\alpha^{1/m}$ by ζ_α, the gluing map of W_0 and W_1 along U_0 and U_1 is of the form

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = z_\beta \zeta_\beta,$$

where $\phi_{\alpha\beta}$ is holomorphic. Similarly, we may assume that the gluing map along V_0 and V_1 is also of this form.

Now we define a smooth hypersurface $\mathscr{M}_0$ in $W_0 \times \Delta \times \Delta^\dagger$ by

$$\{(z_\beta, \zeta_\beta, s, t) \in W_0 \times \Delta \times \Delta^\dagger : (z_\beta \zeta_\beta + t)^m - s = 0\}.$$

We also define a smooth hypersurface $\mathscr{M}_1$ in $W_1 \times \Delta \times \Delta^\dagger$ by

$$\{(x, s, t) \in W_1 \times \Delta \times \Delta^\dagger : \pi(x) - s = 0\}.$$

Let $\Psi_i : \mathscr{M}_i \to \Delta \times \Delta^\dagger$ $(i = 0, 1)$ be the natural projection. From Lemma 5.1.2, for $t \neq 0$,

$$\text{(5.1.1)} \qquad \Psi_0^{-1}(s, t) = \begin{cases} \text{disjoint union of } m-1 \text{ annuli and a node}, & s = t^m, \\ \text{a multiple annulus of multiplicity } m, & s = 0. \end{cases}$$

On the other hand, we have

$$\text{(5.1.2)} \qquad \Psi_1^{-1}(s, t) = \begin{cases} X \cap W_1, & s = 0, \\ \text{smooth}, & \text{otherwise}. \end{cases}$$

Now we glue $\mathscr{M}_0$ with $\mathscr{M}_1$ by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = z_\beta \zeta_\beta + t$$

along $U_0 \times \Delta^\dagger$ and $U_1 \times \Delta^\dagger$, and similarly along $V_0 \times \Delta^\dagger$ and $V_1 \times \Delta^\dagger$. Note that this gluing map transforms the defining equation of $\mathscr{M}_0$ to that of $\mathscr{M}_1$, and we obtain a complex 3-manifold $\mathscr{M}$. However we notice that the boundary of $\mathscr{M}_0$ is not patched with that of $\mathscr{M}_1$, causing that the natural projection $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is not proper. (A fiber on the boundary of $\mathscr{M}$ is a non-compact curve with boundary. See Figure 5.)

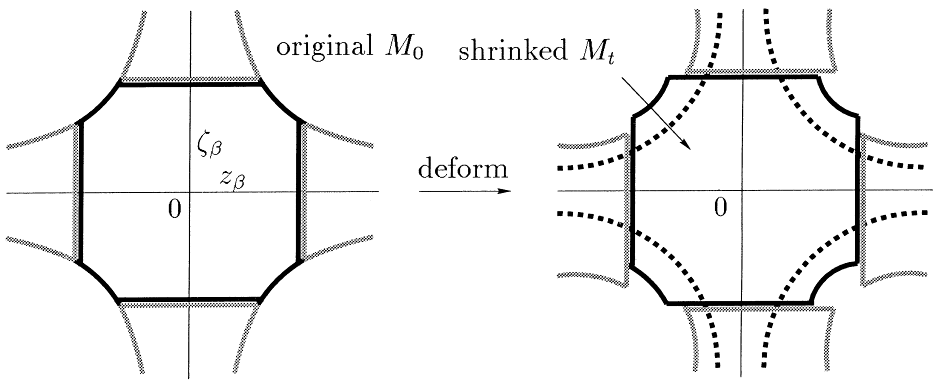


Figure 5. The region surrounded by dashed lines is M_t in a shrinked $\mathscr{M}$.

To avoid this situation, we need to retake smaller Δ. First note that

$$|z_\beta \zeta_\beta| - |t| \leq |z_\beta \zeta_\beta + t|.$$

Assuming that (z_β, ζ_β) lies on the boundary, i.e. $|z_\beta \zeta_\beta| = \delta^{1/m}$, together with $|t| < \varepsilon$, we have $\delta^{1/m} - \varepsilon \leq |z_\beta \zeta_\beta + t|$ (recall the radius ε of $\Delta^\dagger$ is very small). Choose a positive number δ' satisfying $\delta' < \delta^{1/m} - \varepsilon$ and set $\Delta = \{|s| < \delta'\}$. Since $\delta' < |z_\beta \zeta_\beta + t|$, we may shrink $\mathscr{M}$ so that the natural projection $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is proper. (M_t in the shrinked $\mathscr{M}$ is described in Figure 5).

Finally we show that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a splitting family. Consider two fibers $X_1 = \Psi^{-1}(t^m, t)$ and $X_2 = \Psi^{-1}(0, t)$. (X_1 and X_2 are fibers of $\pi_t : M_t \to \Delta_t$.) From (5.1.1) and (5.1.2), X_1 is a reduced curve with one node, and X_2 is obtained from X by replacing the multiple node with a multiple annulus, and no other singular fibers. As both of X_1 and X_2 are normally minimal, it follows from Lemma 1.0.1 that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a splitting family, which splits X into X_1 and X_2. $\square$

The above construction of $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ also works for the case where p is a multiple node of multiplicity 1. But $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is *not* necessarily a splitting family of $\pi : M \to \Delta$. This is exactly the case when $X \backslash \{p\}$ is smooth, i.e. X is a reduced curve with one node. In which case, $X_2 = \Psi^{-1}(0, t)$ is a smooth fiber (in fact, π is atomic by Theorem 2.0.2). Except this case, $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a splitting family

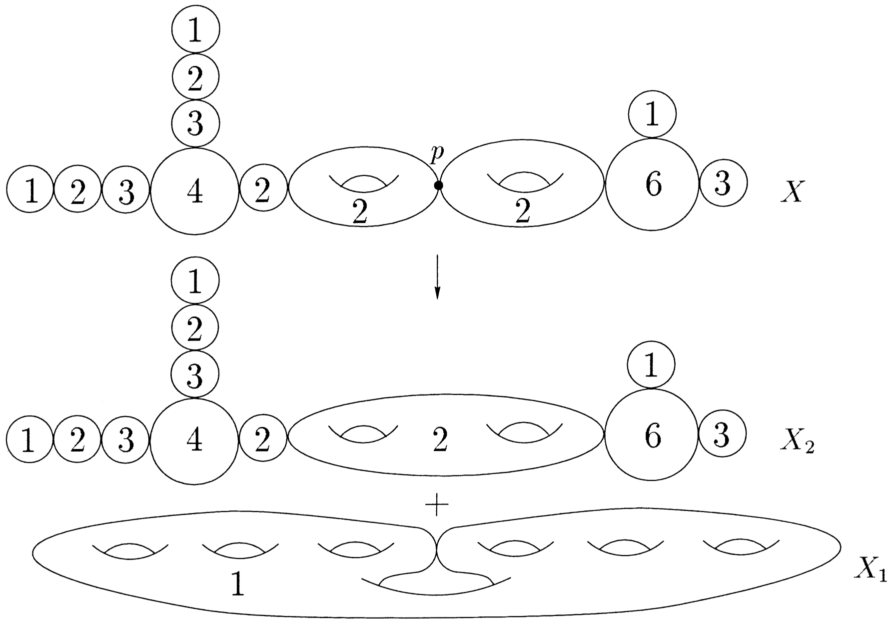


Figure 6. An example for Criterion 5.1.4. The vanishing cycles of the multiple node $p \in X$ are two disjoint loops on a smooth fiber, and topologically X_1 is obtained by pinching one of them.

of $\pi : M \to \Delta$, which splits X into X_1 and X_2, where X_1 is a reduced curve with one node, and X_2 is obtained from X by replacing the reduced node with an annulus. Combined this result with Criterion 5.1.4, we have the following criterion.

CRITERION 5.1.5. *Let $\pi : M \to \Delta$ is normally minimal such that the singular fiber X*

contains a multiple node (*of multiplicity* $m \geq 1$). *Then* $\pi : M \to \Delta$ *is atomic if and only if* X *is a reduced curve with one node.*

REMARK 5.1.6. In the construction of Ψ in Criterion 5.1.4, we only used one multiple node. When X has n multiple nodes p_i $(i = 1, 2, \dots, n)$ of multiplicity m_i, we can generalize the construction in Criterion 5.1.4 to construct a splitting family of $\pi : M \to \Delta$ such that $\pi_t : M_t \to \Delta_t$ contains singular fibers X_i $(i = 1, 2, \dots, n)$, which is obtained from X by replacing the multiple node p_i with the multiple annulus of multiplicity m_i.

5.2. Criterion in terms of plane curve singularities.

In this subsection, we always suppose that $\pi : M \to \Delta$ is relatively minimal (not necessarily normally minimal). We will exhibit a splitting criterion in terms of plane curve singularities on X. We begin by introducing some terminology. Assume that the origin of $V := \{(x, y) \in \boldsymbol{C}^2 : F(x, y) = 0\}$ is a plane curve singularity. (Throughout this paper, *a plane curve singularity always means a reduced one*.) For a positive integer m, setting

$$V_m := \{(x, y) \in \boldsymbol{C}^2 : F(x, y)^m = 0\},$$

we say that V_m is a *multiple plane curve singularity of multiplicity m*. Instead of V_m, we mainly use the notation mV, expressing V_m as a divisor.

PROPOSITION 5.2.1. *Suppose that there exists a point* $p \in X$ *such that a germ of* p *in* X *is a multiple of a plane curve singularity and the multiplicity* m *is at least* 2. *Then* $\pi : M \to \Delta$ *admits a splitting family.*

PROOF. The idea of the proof is similar to that of Criterion 5.1.4; first take an open covering $M = W_0 \cup W_1$ with the following properties (see Figure 7):

(1) W_0 is a small neighborhood of p, so $W_0 \cap X$ is a germ of the multiple plane curve singularity mV,

(2) $W_1 \cap X$ is 'outside' mV,

(3) $\pi(W_0) = \pi(W_1) = \Delta$, and the intersection of W_0 and W_1 consists of disjoint annular domains (see Definition 5.1.3).

To simplify the discussion, we only consider the case where the intersection of W_0 and W_1 is one annular domain. So M is obtained by patching W_0 and W_1 along a pair of annular domains, say, $U_0 \subset W_0$ and $U_1 \subset W_1$. (The argument below works for the general case).

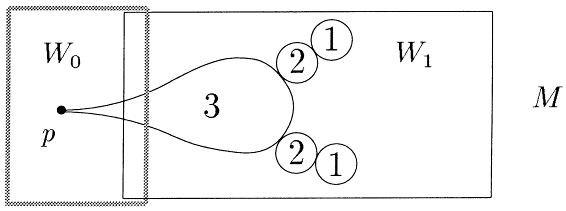


Figure 7.

Take coordinates $(z_\beta, \zeta_\beta) \in W_0$. Then $\pi(z_\beta, \zeta_\beta) = F(z_\beta, \zeta_\beta)^m$ where $F(z_\beta, \zeta_\beta) = 0$ defines the plane curve singularity V. Next, we take coordinates $(z_\alpha, \zeta_\alpha) \in U_1$, then $\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha^m u_\alpha(z_\alpha, \zeta_\alpha)^m$ for some non-vanishing holomorphic function u_α. Rewriting $\zeta_\alpha u_\alpha^{1/m}$ by ζ_α, we have $\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha^m$. Since $\pi(z_\alpha, \zeta_\alpha) = \pi(z_\beta, \zeta_\beta)$, we have $\zeta_\alpha^m = F(z_\beta, \zeta_\beta)^m$. As in the proof of Criterion 5.1.4, possibly after coordinate change, we have $\zeta_\alpha = F(z_\beta, \zeta_\beta)$. So the gluing map of W_0 and W_1 along U_0 and U_1 is of the form

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = F(z_\beta, \zeta_\beta),$$

where $\phi_{\alpha\beta}$ is holomorphic. Next, we take a non-equisingular deformation of V:

$$V_t : F(z_\beta, \zeta_\beta) + G(z_\beta, \zeta_\beta, t) = 0, \quad \text{where } G \text{ is holomorphic and } G(z_\beta, \zeta_\beta, 0) = 0.$$

For example, if V is a node (A_1-singularity), take $G(z_\beta, \zeta_\beta, t) := t$, and otherwise take a *Morsification* of V, i.e. V_t $(t \neq 0)$ has only nodes (A_1-singularities). Here recall that any isolated hypersurface singularity always admits a Morsification. See, for example Dimca [**Di**] p. 82. Next, we define a smooth hypersurface $\mathcal{M}_0$ in $W_0 \times \Delta \times \Delta^\dagger$ by

$$\{(z_\beta, \zeta_\beta, s, t) \in M_0 \times \Delta \times \Delta^\dagger : (F(z_\beta, \zeta_\beta) + G(z_\beta, \zeta_\beta, t))^m - s = 0\}.$$

Similarly, we define a smooth hypersurface $\mathcal{M}_1$ in $W_1 \times \Delta \times \Delta^\dagger$ by

$$\{(x, s, t) \in W_1 \times \Delta \times \Delta^\dagger : \pi(x) - s = 0\}.$$

We glue $\mathcal{M}_0$ with $\mathcal{M}_1$ along $U_0 \times \Delta^\dagger$ and $U_1 \times \Delta^\dagger$ by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = F(z_\beta, \zeta_\beta) + G(z_\beta, \zeta_\beta, t),$$

which yields a complex 3-manifold $\mathcal{M}$. We then shrink $\mathcal{M}$ in such a way that the natural projection $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is proper (Remark 5.2.2 below). Notice that the fiber $X_1 := \Psi^{-1}(0, t)$ is a singular fiber, which is obtained from X by replacing the multiple plane curve singularity mV with mV_t. (To describe other singular fibers, it is necessary to compute the discriminant of the family $(F(z_\beta, \zeta_\beta) + G(z_\beta, \zeta_\beta, t))^m - s = 0$.) Since $\pi : M \to \Delta$ is relatively minimal, $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family. $\square$

Remark 5.2.2. Since $G(z_\beta, \zeta_\beta, 0) = 0$, we may assume that $|G(z_\beta, \zeta_\beta, t)| < \gamma$ for $|t| < \varepsilon$ where γ is sufficiently small. Then for (z_β, ζ_β) on the boundary ∂M (hence $|F(z_\beta, \zeta_\beta)^m| = \delta = \text{radius}(\Delta)$), the following inequalities hold:

$$\delta^{1/m} - \gamma < |F(z_\beta, \zeta_\beta)| - |G(z_\beta, \zeta_\beta, t)| \leq |F(z_\beta, \zeta_\beta) + G(z_\beta, \zeta_\beta, t)|.$$

Taking a positive integer δ' satisfying $\delta' < \delta^{1/m} - \gamma$, we have

$$\delta' < |F(z_\beta, \zeta_\beta) + G(z_\beta, \zeta_\beta, t)|.$$

We retake smaller $\Delta = \{|s| < \delta'\}$. As in the proof of Criterion 5.1.4, we may shrink $\mathcal{M}$ so that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is proper.

In the assumption of the above proposition, if we replace $m \geq 2$ by $m = 1$, what can we say about the splittability of $\pi : M \to \Delta$? Also in this case, the above construction works, and we obtain a splitting family, except the case where p is a node and $X \backslash p$

is smooth, i.e. X is a reduced curve with one node (this is an atomic case). Combined with Proposition 5.2.1, we have the following result.

CRITERION 5.2.3. *Let $\pi : M \to \Delta$ be relatively minimal. Suppose that the singular fiber X has a point p such that a germ of p in X is either*

(1) *a multiple of a plane curve singularity of multiplicity at least* 2, *or*

(2) *a plane curve singularity such that if it is a node, then $X \backslash p$ is not smooth. Then $\pi : M \to \Delta$ admits a splitting family.*

6. Splitting criteria via configurations, II.

In this section, we shall present another type of splitting criteria in terms of existence of an irreducible component of multiplicity 1 satisfying a certain property.

6.1. Criterion in terms of connected components.

CRITERION 6.1.1. *Let $\pi : M \to \Delta$ be normally minimal. Suppose that the singular fiber X contains an irreducible component Θ_0 of multiplicity* 1 *such that $X \backslash \Theta_0$ is* (*topologically*) *disconnected. Denote by $Y_1, Y_2, \dots, Y_l$ $(l \geq 2)$ all connected components of $X \backslash \Theta_0$. Then $\pi : M \to \Delta$ admits a splitting family which splits X into $X_1, X_2, \dots, X_l$, where X_i $(i = 1, 2, \dots, l)$ is obtained from X by 'smoothing' $Y_1, Y_2, \dots, \check{Y}_i, \dots, Y_l$* (see Figure 8 for example). *Here $\check{Y}_i$ is the omission of Y_i.*

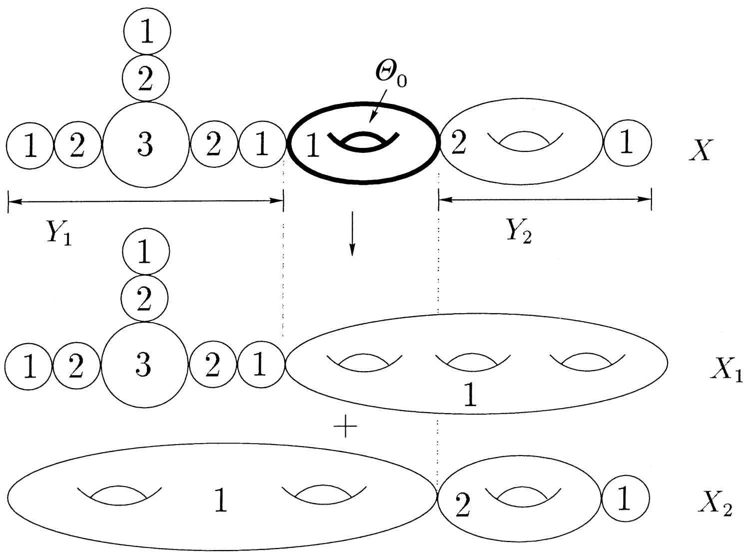


Figure 8. An example for Criterion 6.1.1.

PROOF. To avoid complicated notation, we only show the statement for the case where Y_i and Θ_0 intersects only at one point p_i. (The construction below works for the general case.) We take an open covering $M = W_0 \cup W_1 \cup \cdots \cup W_l$ (see Figure 9) such that

(1) $W_i \cap X = Y_i \cup D_i$, where $D_i \subset \Theta_0$ is a disk around p_i,

(2) $W_0 \cap X = \Theta_0 \backslash \{D_1' \cup D_2' \cup \cdots \cup D_l'\}$, where D_i' is a disk satisfying $p_i \in D_i' \subset D_i$,

(3) $\pi(W_1) = \pi(W_2) = \cdots = \pi(W_l) = \Delta$ and the intersection of W_0 and W_i $(i = 1, 2, \ldots, l)$ is an annular domain (see Definition 5.1.3). So M is reconstructed by patching W_0 with W_i $(i = 1, 2, \ldots, l)$ along annular domains $U_0 \subset W_0$ and $U_i \subset W_i$.

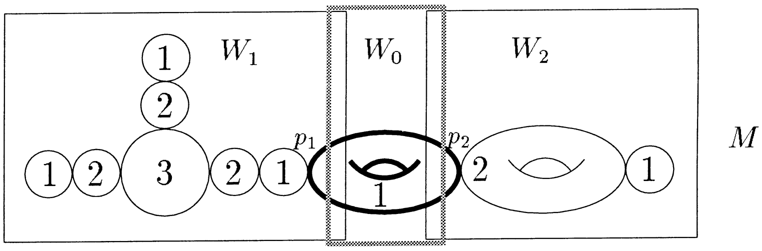


Figure 9.

Here we choose W_i so that D_i (and so D_i') are sufficiently small.

For simplicity, we set $Y_i^+ := Y_i \cup D_i$ and $\Theta_0^- := \Theta_0 \backslash \{D_1' \cup D_2' \cup \cdots \cup D_l'\}$ (see Figure 10 for the case of Figure 9).

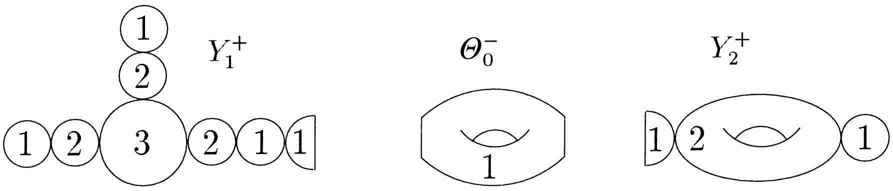


Figure 10.

Now we shall construct a splitting family of π in the following steps: First, construct complex 3-manifolds $\mathscr{M}_i$ $(i = 0, 1, \ldots, l)$ together with holomorphic maps $\Psi_i : \mathscr{M}_i \to \Delta \times \Delta^\dagger$. Secondly, glue $\mathscr{M}_i$ together to construct a complex 3-manifold $\mathscr{M}$ so that Ψ_i $(i = 0, 1, \ldots, l)$ determine a holomorphic map $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$. Finally, we will show that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a splitting family of π.

Step 1. Construction of complex 3-manifolds $\mathscr{M}_0, \mathscr{M}_1, \ldots, \mathscr{M}_l$.

We put $\mu := e^{2\pi i/l}$, and consider a smooth hypersurface $\mathscr{M}_i$ in $W_i \times \Delta \times \Delta^\dagger$ $(i = 1, 2, \ldots, l)$ defined by

$$\{(x, s, t) \in W_i \times \Delta \times \Delta^\dagger : \pi(x) - s + \mu^i t = 0\}. \tag{6.1.1}$$

Let $\Psi_i : \mathscr{M}_i \to \Delta \times \Delta^\dagger$ be the natural projection. Then for $t \neq 0$, we have

$$\Psi_i^{-1}(s, t) = \begin{cases} Y_i^+, & s = \mu^i t, \\ \text{smooth}, & \text{otherwise}. \end{cases} \tag{6.1.2}$$

Next, we consider a smooth hypersurface $\mathscr{M}_0$ in $W_0 \times \Delta \times \Delta^\dagger$ defined by

$$\{(x, s, t) \in W_0 \times \Delta \times \Delta^\dagger : \pi(x) - s = 0\}.$$

Let $\Psi_0 : \mathcal{M}_0 \to \Delta \times \Delta^\dagger$ be the natural projection. Then for $t \neq 0$, we have

$$\Psi_0^{-1}(s,t) = \begin{cases} \Theta_0^-, & s = 0, \\ \text{smooth}, & \text{otherwise}. \end{cases} \tag{6.1.3}$$

(Note that Θ_0^- is also smooth.)

STEP 2. Gluing $\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_l$ together.

Let $(z_\alpha, \zeta_\alpha) \in U_0$ and $(z_\beta, \zeta_\beta) \in U_i$ be coordinates of annular domains. Denote by m_i the multiplicity of the irreducible component intersecting Θ_0 at p_i. As the multiplicity of Θ_0 is 1, we may write

$$\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha f_\alpha(z_\alpha, \zeta_\alpha), \quad \pi(z_\beta, \zeta_\beta) = z_\beta^{m_i} \zeta_\beta g_\beta(z_\beta, \zeta_\beta),$$

where f_α and g_β are non-vanishing holomorphic functions. Rewriting $\zeta_\alpha f_\alpha$ by ζ_α (resp. $\zeta_\beta g_\beta$ by ζ_β), we have $\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha$ (resp. $\pi(z_\beta, \zeta_\beta) = z_\beta^{m_i}\zeta_\beta$). Since $\pi(z_\alpha, \zeta_\alpha) = \pi(z_\beta, \zeta_\beta)$, we obtain a relation $\zeta_\alpha = z_\beta^{m_i}\zeta_\beta$. Hence the gluing map of W_0 and W_i along U_0 and U_i is of the form

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = z_\beta^{m_i}\zeta_\beta.$$

Next we glue $\mathcal{M}_0$ with $\mathcal{M}_i$ $(i = 1, 2, \dots, l)$ along $U_0 \times \Delta^\dagger$ and $U_i \times \Delta^\dagger$ by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = z_\beta^{m_i}\zeta_\beta + \mu^i t,$$

which yields a complex 3-manifold $\mathcal{M}$ because this map transforms the defining equation of $\mathcal{M}_i$ to that of $\mathcal{M}_0$.

Shrink $\mathcal{M}$ so that the natural projection $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$, $(x, s, t) \mapsto (s, t)$, is proper. (Use a similar argument to the proof of Criterion 5.1.4 and Remark 5.2.2.) We claim that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family. Indeed, from (6.1.2) and (6.1.3), for $t \neq 0$,

$$\Psi^{-1}(s,t) = \begin{cases} X_i, & s = \mu^i t, \\ \text{smooth}, & \text{otherwise}, \end{cases}$$

where X_i is obtained from X by smoothing $Y_1^+, Y_2^+, \dots, \check{Y}_i^+, \dots, Y_l^+$. As X_i is normally minimal, it follows from Lemma 1.0.1 that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family which splits X into $X_1, X_2, \dots, X_l$. This verifies our assertion. (Note: the discriminant of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is $\prod_{i=1}^l (s - \mu^i t) = 0$.) $\square$

From the above construction, we can deduce some property of topological monodromies. Let γ be the topological monodromy of $\pi : M \to \Delta$ along a loop $\partial\Delta$ with counterclockwise orientation, and γ_i be the topological monodromy around X_i in $\pi_t : M_t \to \Delta_t$ along the oriented loops ℓ_i in a symmetric position as in Figure 11. Then we have a relation $\gamma = \gamma_1\gamma_2\cdots\gamma_l$. Moreover the following holds.

PROPOSITION 6.1.2. *The topological monodromies $\gamma_1, \gamma_2, \dots, \gamma_l$ commute.*

PROOF. To show this, we slightly modify the above construction of $\Psi : \mathcal{M} \to \Delta \times$

$\varDelta^{\dagger}$; let σ be an arbitrary permutation of the set $\{1, 2, \dots, l\}$. Instead of $\mathscr{M}_i$, we define $\mathscr{M}_{\sigma,i}$ as follows (cf. (6.1.1)):

$$\mathscr{M}_{\sigma,i} := \{(x, s, t) \in W_i \times \varDelta \times \varDelta^{\dagger} : \pi(x) - s + \mu^{\sigma(i)} t = 0\},$$

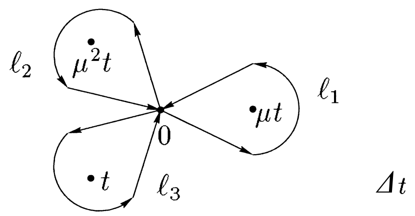


Figure 11. The choice of loops ($l = 3$ case).

while we take $\mathscr{M}_0$ as in the above construction:

$$\{(x, s, t) \in W_0 \times \varDelta \times \varDelta^{\dagger} : \pi(x) - s = 0\}.$$

Then we glue $\mathscr{M}_0$ with $\mathscr{M}_{\sigma,i}$ $(i = 1, 2, \dots, l)$ by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = z_\beta^{m_i} \zeta_\beta + \mu^{\sigma(i)} t,$$

and obtain a complex 3-manifold $\mathscr{M}_\sigma$. The natural projection $\Psi_\sigma : \mathscr{M}_\sigma \to \varDelta \times \varDelta^{\dagger}$ is also splitting family which splits X into $X_1, X_2, \dots, X_l$. But $X_1, X_2, \dots, X_l$ appears in the order $X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(l)}$, hence we have a relation

$$\gamma = \gamma_{\sigma(1)} \gamma_{\sigma(2)} \cdots \gamma_{\sigma(l)}.$$

Since σ is an arbitrary permutation, it follows that $\gamma_1, \gamma_2, \dots, \gamma_l$ commute. □

Remark 6.1.3. In the construction of Ψ in Criterion 6.1.1, we used only one irreducible component of multiplicity 1. As is clear from the construction, we can similarly construct a splitting family by using several irreducible components $\Theta_0^{(1)}, \Theta_0^{(2)}, \dots, \Theta_0^{(n)}$ of multiplicity 1 *simultaneously* provided that $X \backslash \{\Theta_0^{(1)} \cup \Theta_0^{(2)} \cup \dots \cup \Theta_0^{(n)}\}$ is disconnected. More generally, in some cases, we can construct a splitting family by 'mixing up' all constructions in this paper.

6.2. Inductive criterion.

Let $\pi : M \to \varDelta$ be normally minimal such that its singular fiber X contains an irreducible component Θ_0 of multiplicity 1. We suppose that $X \backslash \Theta_0$ is connected. Also in this case, we have some splitting criterion. To state our result, we need to introduce some notation. Let $Y := X \backslash \Theta_0$ and let $p_1, p_2, \dots, p_n$ be the intersection points of Θ_0 with other irreducible components of X. Take an open covering $M = W_0 \cup W_1$ (see Figure 12) such that

(1) $W_1 \cap X = Y \cup D_1 \cup D_2 \cup \dots \cup D_n$, where $D_i \subset \Theta_0$ is a disk around p_i,
(2) $W_0 \cap X = \Theta_0 \backslash \{D_1' \cup D_2' \cup \dots \cup D_n'\}$, where D_i' is a disk satisfying $p_i \in D_i' \subset D_i$,
(3) $\pi(W_0) = \pi(W_1) = \varDelta$ and the intersection of W_0 and W_1 is a disjoint union of annular domains.

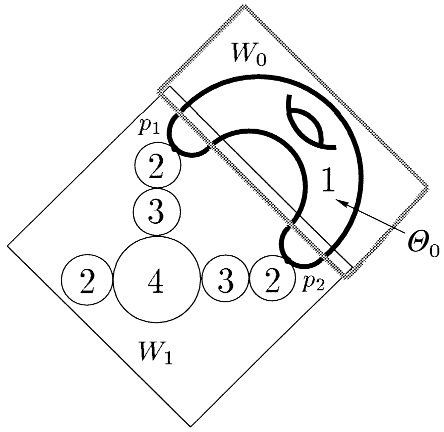


Figure 12.

Here, we choose W_1 so that D_i (and so D_i') are sufficiently small. For simplicity, we set

$$Y^+ := Y \cup D_1 \cup D_2 \cup \cdots \cup D_n, \quad \Theta_0^- := \Theta_0 \backslash \{D_1' \cup D_2' \cup \cdots \cup D_n'\} \quad \text{(Figure 13)}.$$

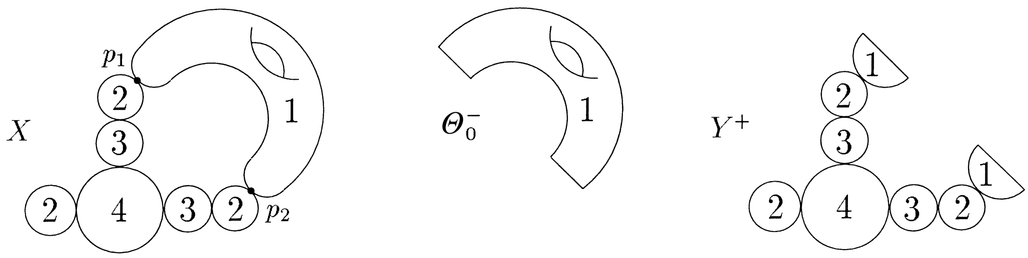


Figure 13.

Criterion 6.2.1. *Let $\pi : M \to \Delta$ be normally minimal such that the singular fiber X contains an irreducible component Θ_0 of multiplicity* 1. *Let $\pi_1 : W_1 \to \Delta$ be the restriction of π to a tubular neighborhood W_1 of $X \backslash \Theta_0$ in M. Suppose that $\pi_1 : W_1 \to \Delta$ admits a splitting family Ψ_1 which splits $Y^+ := W_1 \cap X$ into $Y_1^+, Y_2^+, \ldots, Y_l^+$. Then $\pi : M \to \Delta$ admits a splitting family Ψ which splits X into $X_1, X_2, \ldots, X_l$, where X_i is obtained from Y_i^+ by gluing Θ_0^- along the boundary.*

Note. We note that $\pi_1 : W_1 \to \Delta$ is a degeneration of curves with boundary, for which we may also define the notion of splitting families in the same way as for degenerations of compact curves.

Proof. Note that M is obtained by patching W_0 and W_1 along annular domains U_0 and U_i where U_i is an annular domain near p_i. As in the proof of Criterion 6.1.1, we take coordinates $(z_\alpha, \zeta_\alpha) \in U_0$ with $\pi(z_\alpha, \zeta_\alpha) = \zeta_\alpha$, and also $(z_\beta, \zeta_\beta) \in U_i$ with $\pi(z_\beta, \zeta_\beta) = z_\beta^{m_i}\zeta_\beta$ such that the gluing map of U_0 and U_i is of the form

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = z_\beta^{m_i}\zeta_\beta,$$

where $\phi_{\alpha\beta}$ is holomorphic. Now, letting $\Psi_1 : \mathscr{M}_1 \to \Delta \times \Delta^\dagger$ be the splitting family of $\pi_1 : W_1 \to \Delta$ in the assumption, we consider a map $\tilde{\pi}_1 := \mathrm{pr}_1 \circ \Psi_1 : \mathscr{M}_1 \to \Delta$, and then realize $\mathscr{M}_1$ as the graph of $\tilde{\pi}_1$:

$$\mathscr{M}_1 = \{(x,s,t) \in W_1 \times \Delta \times \Delta^\dagger : \tilde{\pi}_1(x,t) - s = 0\}.$$

Notice that $\tilde{\pi}_1(x,0) = \pi_1(x)$, hence we may express $\tilde{\pi}_1(x,t) = \pi_1(x) + h_1(x,t)$, where h_1 is a holomorphic function satisfying $h_1(x,0) = 0$. Next, we define a smooth hypersurface $\mathscr{M}_0$ in $W_0 \times \Delta \times \Delta^\dagger$ by

$$\mathscr{M}_0 = \{(x,s,t) \in W_0 \times \Delta \times \Delta^\dagger : \pi(x) - s = 0\}.$$

Finally we glue $\mathscr{M}_0$ with $\mathscr{M}_1$ along $U_0 \times \Delta^\dagger$ and $U_i \times \Delta^\dagger$ by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = z_\beta^{m_i}\zeta_\beta + h_1(z_\beta, \zeta_\beta),$$

and we obtain a complex 3-manifold $\mathscr{M}$. Shrink $\mathscr{M}$ in such a way that the natural projection $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is proper. (Use a similar argument to the proof of Criterion 5.1.4 and Remark 5.2.2.) We claim that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a splitting family of π. Indeed, if the fiber Y_k^+ of Ψ_1 over a point $x_k \in \Delta_t$ is singular, then by construction, $\Psi^{-1}(x_k)$ is obtained by gluing Y_k^+ with Θ_0^- along the boundary, and hence it is singular. $\square$

From $\pi_1 : W_1 \to \Delta$ in Criterion 6.2.1, we shall construct a degeneration $\pi' : M' \to \Delta$ of *compact* curves, whose singular fiber X' is obtained from Y^+ by replacing the disk D_i $(i = 1, 2, \ldots, n)$ with a projective line (see Figure 14), after that, we will restate Criterion 6.2.1 in terms of this degeneration. First, letting E_i be a disk, we glue W_1 with $E_i \times \Delta$ $(i = 1, 2, \ldots, n)$ along U_i by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, \zeta_\beta), \quad \zeta_\alpha = z_\beta^{m_i}\zeta_\beta,$$

where $\phi_{\alpha\beta}$ is as in the above proof, $(z_\alpha, \zeta_\alpha) \in E_i \times \Delta$ and $(z_\beta, \zeta_\beta) \in U_i$. Then we obtain a complex surface M'. Define a map $\pi' : M' \to \Delta$ by $\pi'|_{W_1} = \pi$, and $\pi'|_{E_i \times \Delta}(z_\alpha, \zeta_\alpha) = \zeta_\alpha$. By construction, the singular fiber of π' is obtained from Y^+ by replacing D_i $(i = 1, 2, \ldots, n)$ with a projective line; two disks D_i and E_i are glued to become a projective line. Then Criterion 6.2.1 is restated as follows:

Criterion 6.2.1$'$. *If $\pi' : M' \to \Delta$ admits a splitting family, then $\pi : M \to \Delta$ also admits a splitting family.* (Note: By construction, the converse is true.)

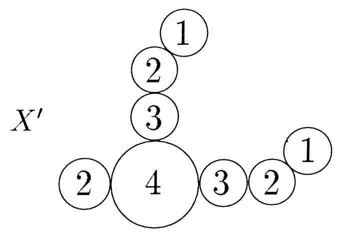


Figure 14.

Let g (resp. g') be the genus of a smooth fiber of $\pi : M \to \Delta$ (resp. $\pi' : M' \to \Delta$). Except the case where Θ_0 is a projective line intersecting other irreducible components at only one point, we have $g' < g$, and so $\pi' : M' \to \Delta$ is a degeneration of curves of lower genus. Indeed, let Θ_0 intersect other irreducible components at n points. By a topological consideration, it is easy to see that

$$g = g' + (n-1) + \mathrm{genus}(\Theta_0). \tag{6.2.1}$$

Hence we have $g' < g$, unless Θ_0 is a projective line intersecting other irreducible components at only one point.

6.3. Consequence of splitting criteria.

In this subsection, we assume that any degeneration is normally minimal. The splitting criteria obtained in this paper altogether imply that if the singular fiber X is constellar, then in many cases, $\pi : M \to \Delta$ admits a splitting family. Taking into account Theorem 2.0.2, it is interesting to know whether the following conjecture is true or not (cf. Conjecture 6.3.1$'$ below):

CONJECTURE 6.3.1. *A degeneration is atomic if and only if its singular fiber is either a reduced curve with one node, or a multiple of a smooth curve.*

(This conjecture is valid for the genus 1 and 2 cases: for the genus 1 case, any atomic fiber is either a rational curve with one node, or a multiple of a smooth elliptic curves by [**Mo**], and for the genus 2 case, any atomic fiber is a reduced curve with one node by [**Ho**] together with [**AA**].) Now we shall deduce a useful theorem from our splitting criteria. Let Λ_g be a set of degenerations $\pi : M \to \Delta$ of curves of genus g such that

(1) the singular fiber X has a multiple node (here we exclude the case where X is a reduced curve with only one node), or

(2) X contains an irreducible component Θ_0 of multiplicity 1 satisfying the following condition: if $X \backslash \Theta_0$ is connected, then either genus$(\Theta_0) \geq 1$, or Θ_0 is a projective line intersecting other irreducible components at at least two points.

As a consequence of our splitting criteria, we obtain the following.

THEOREM 6.3.2. *Suppose that Conjecture* 6.3.1 *is valid for genus* $\leq g-1$. *If* $\pi : M \to \Delta$ *is a degeneration in* Λ_g, *then* π *is not atomic.*

PROOF. First, by Criterion 5.1.5, if the singular fiber contains a multiple node, then π admits a splitting family. Next, suppose that X contains an irreducible component Θ_0 of multiplicity 1. If $X \backslash \Theta_0$ is not connected, then $\pi : M \to \Delta$ has a splitting family (Criterion 6.1.1). On the other hand, if $X \backslash \Theta_0$ is connected, then under the assumption of this theorem, we can apply Criterion 6.2.1$'$, and see that $\pi : M \to \Delta$ admits a splitting family except the case where Θ_0 is a projective line and it intersects other irreducible components at only one point (cf. (6.2.1)). Hence the assertion follows. $\square$

Thus if the assumption of this theorem is fulfilled (for example, $g = 3$), to determine atomic degenerations of curves of genus g, it is enough to investigate the splittability for degenerations $\pi : M \to \Delta$ such that either

(A) $X = \pi^{-1}(0)$ is stellar, or

(B) X is constellar and (B.1) X has no multiple node and (B.2) if X has an irreducible component Θ_0 of multiplicity 1, then Θ_0 is a projective line intersecting other irreducible components of X only at one point.

In the terminology of [**Ta**,II], a singular fiber in (B) is obtained by 'bonding' stellar singular fibers such that any bonding of two branches is either (-1)-bonding, or 0-bonding of two branches with the same multiplicity at least 2. See also [**MM2**]. For these cases, we can apply another method (construction of splitting families via barkable sub-divisors), which is developed in [**Ta**,III].

Discussion and open problems.

For higher genus cases, Conjecture 6.3.1 seems too optimistic. It is more reasonable to replace 'atomic' with 'absolutely atomic', where a degeneration $\pi : M \to \Delta$ is called *absolutely atomic* if all degenerations with the same topological type as $\pi : M \to \Delta$ are atomic (for example, when X is a reduced curve with one node or a multiple of a smooth curve. See Theorem 2.0.2).

Conjecture 6.3.1$'$. *A degeneration is absolutely atomic if and only if its singular fiber is either a reduced curve with one node, or a multiple of a smooth curve.*

In [**Ta**,III] we showed that this is valid for $g \leq 5$. We also point out that we can show an analogous statement to Theorem 6.3.2 by the same argument.

Theorem 6.3.2$'$. *Suppose that Conjecture* 6.3.1$'$ *is valid for genus* $\leq g-1$. *If* $\pi : M \to \Delta$ *is a degeneration in* Λ_g, *then* π *is not absolutely atomic.*

It is plausible that for higher genus cases, there may be an atomic degeneration which is *not* absolutely atomic. However, no examples have been known, and so we ask

Problem 6.3.3. *Do there exist two degenerations* $\pi_1 : M_1 \to \Delta$ *and* $\pi_2 : M_2 \to \Delta$ *with the same topological type such that* π_1 *is atomic while* π_2 *is not*?

Note that for the genus ≥ 2 case, there are degenerations with the same singular fiber, but with different topological types [**MM2**]. Taking this into account, it is natural ask the following problem analogous to Problem 6.3.3.

Problem 6.3.4. *Do there exist two degenerations* $\pi_1 : M_1 \to \Delta$ *and* $\pi_2 : M_2 \to \Delta$ *with the same singular fiber but with different topological types such that* π_1 *is atomic while* π_2 *is not*?

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