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Minimally fine limits at infinity for p-precise functions

Dedicated to Professor Hisako Watanabe on the occasion of her 60th birthday

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Abstract. Our aim in this paper is to discuss minimally fine limits at infinity for locally p-precise functions in the half space of $\boldsymbol{R}^n$ with vanishing boundary limits. We also give a measure condition for a set to be minimally thin at infinity.

1. Introduction.

Let u be a nonnegative superharmonic function on $D = \{x = (x_1, \dots, x_{n-1}, x_n) \in \boldsymbol{R}^n;\ x_n > 0\}$, where $n \geqq 2$. Then it is known (cf. Lelong-Ferrand [**5**]) that u is uniquely decomposed as

$$u(x) = ax_n + \int_D G(x, y)\, d\mu(y) + \int_{\partial D} P(x, y)\, d\nu(y),$$

where a is a nonnegative number, μ (resp. ν) is a nonnegative measure on D (resp. ∂D), G is the Green function for D and P is the Poisson kernel for D. The first author [**9**, Theorem 1] showed that if $0 \leqq \beta \leqq 1$, then

$$\lim_{|x| \to \infty,\, x \in D - E} x_n^{-\beta} |x|^{\beta - 1} (u(x) - ax_n) = 0$$

with a set E in D which is β-minimally thin at infinity; if in addition $\int_D y_n^\gamma\, d\mu(y) < \infty$ $(1 - n \leqq \gamma < 1)$, then

$$\lim_{|x| \to \infty,\, x \in D - E'} x_n^{-\beta} |x|^{n + \gamma - (2 - \beta)} \int_D G(x, y)\, d\mu(y) = 0$$

with a suitable exceptional set $E' \subset D$. For related results, we also refer the reader to Essén-Jackson [**3**, Theorem 4.6], Aikawa [**1**], and Miyamoto-Yoshida [**7**].

Our main aim in this paper is to establish the analogue of these results for locally p-precise functions u in D satisfying

$$\int_D |\nabla u(x)|^p x_n^\gamma\, dx < \infty, \tag{1}$$

where ∇ denotes the gradient, $1 < p < \infty$ and $-1 < \gamma < p - 1$ (see Ohtsuka [**14**] and Ziemer [**15**] for locally p-precise functions).

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We denote by $\boldsymbol{D}^{p,\gamma}$ the space of all locally p-precise functions on D satisfying (1), by $\boldsymbol{D}_0^{p,\gamma}$ the space of all functions $u \in \boldsymbol{D}^{p,\gamma}$ having vertical limit zero at almost every boundary point of D, and by $\boldsymbol{HD}^{p,\gamma}$ the space of all harmonic functions on D in $\boldsymbol{D}^{p,\gamma}$. According to Riesz decomposition (cf. Deny-Lions [**2**]), $u \in \boldsymbol{D}^{p,\gamma}$ is represented as

$$u = u_0 + h, \tag{2}$$

where $u_0 \in \boldsymbol{D}_0^{p,\gamma}$ and $h \in \boldsymbol{HD}^{p,\gamma}$. We show that the decomposition is unique (see Theorem 2 below). We give the integral representations of u_0 and h, and then discuss minimally fine limits at infinity for functions in $\boldsymbol{D}_0^{p,\gamma}$.

For this purpose, consider the kernel function

$$k_{\beta,\gamma}(x,y) = x_n^{1-\beta} y_n^{-\gamma/p} |x-y|^{1-n} |\bar{x}-y|^{-1},$$

where $\bar{x} = (x_1, \dots, x_{n-1}, -x_n)$ for $x = (x_1, \dots, x_{n-1}, x_n)$. To evaluate the size of exceptional sets, we use the capacity

$$C_{k_{\beta,\gamma},p}(E;G) = \inf \int_D g(y)^p \, dy,$$

where E is a subset of an open set G in D and the infimum is taken over all nonnegative measurable functions g such that $g = 0$ outside G and

$$\int_D k_{\beta,\gamma}(x,y) g(y) \, dy \geqq 1 \quad \text{for all } x \in E.$$

We say that $E \subset D$ is (minimally) $(k_{\beta,\gamma}, p)$-thin at infinity if

$$\sum_{i=1}^{\infty} 2^{-i(n+\gamma-(1-\beta)p)} C_{k_{\beta,\gamma},p}(E_i; D_i) < \infty, \tag{3}$$

where $E_i = \{x \in E : 2^i \leqq |x| < 2^{i+1}\}$ and $D_i = \{x \in D : 2^{i-1} < |x| < 2^{i+2}\}$.

Our main aim in this paper is to establish the following theorem (cf. [**9**, Theorem 1]).

Theorem 1. *Let* $p > 1$, $-1 < \gamma < p-1$, $(1-\beta)p - n < \gamma$ *and* $0 \leqq \beta \leqq 1$. *If* $u \in \boldsymbol{D}_0^{p,\gamma}$, *then there exists a set* $E \subset D$ *such that* E *is* $(k_{\beta,\gamma}, p)$*-thin at infinity and*

$$\lim_{|x| \to \infty,\, x \in D - E} x_n^{-\beta} |x|^{(n+\gamma-(1-\beta)p)/p} u(x) = 0.$$

Next we are concerned with the measure condition on minimally thin sets. For a measurable set $E \subset \boldsymbol{R}^n$, denote by $|E|$ the Lebesgue measure of E.

Proposition 1. *Let* $0 \leqq \beta < 1$ *and* $-1 < \gamma < p-1$. *If* (3) *holds, then*

$$\sum_{i=1}^{\infty} \left(\frac{|E_i|}{|B_i|} \right)^{(1-(1-\beta)/n)p} < \infty,$$

where $E_i = E \cap B_{i+1} - B_i$ *with* $B_i = B(0, 2^i) \cap D$.

Finally we give an example of minimally thin sets. For a nondecreasing function φ on $\boldsymbol{R}^1$ such that $0 < \varphi(2t) \leqq M\varphi(t)$ for $t > 0$ with a positive constant M, we set

$$T_\varphi = \{x = (x', x_n); 0 < x_n < \varphi(|x'|)\}.$$

PROPOSITION 2 (cf. Aikawa [**1**, Proposition 5.1]). *Let* $0 < \beta \leqq 1$ *and* $p(1-\beta) - 1 < \gamma < p - 1$. *Assume further that*

$$\lim_{r \to \infty} \frac{\varphi(r)}{r} = 0. \tag{4}$$

Then T_φ *is* $(k_{\beta,\gamma}, p)$*-thin at infinity if and only if*

$$\int_1^\infty \left(\frac{\varphi(t)}{t}\right)^{p(-1+\beta)+\gamma+1} \frac{dt}{t} < \infty. \tag{5}$$

For example, $\varphi(r) = r[\log(1+r)]^{-\delta}$ satisfies (5), when $\delta\{p(-1+\beta)+\gamma+1\} > 1$.

In the final section we discuss fine limits at infinity for locally p-precise functions, as an extension of Kurokawa-Mizuta [**4**].

2. Riesz decomposition.

Let $u \in \boldsymbol{D}^{p,\gamma}$. If $1 \leqq q < p$ and $q < p/(1+\gamma)$, then Hölder's inequality gives

$$\int_G |\nabla u(x)|^q \, dx \leqq \left(\int_G x_n^{-\gamma q/(p-q)} \, dx\right)^{1-q/p} \left(\int_G |\nabla u(x)|^p x_n^\gamma \, dx\right)^{q/p} < \infty$$

for any bounded open set $G \subset D$. Hence we can find a locally q-precise extension $\bar{u}$ to $\boldsymbol{R}^n$ such that $\bar{u}(x', x_n) = u(x', x_n)$ for $x_n > 0$ and $\bar{u}(x', x_n) = u(x', -x_n)$ for $x_n < 0$.

For fixed $\xi \in D$, we take $r > 0$ such that $B = B(\xi, r) \subset D$. Then, in view of [**11**, Lemma 1], $u \in \boldsymbol{D}^{p,\gamma}$ is represented as

$$\begin{aligned} u(x) &= c_n \sum_{i=1}^n \int_B \frac{x_i - y_i}{|x-y|^n} \frac{\partial \bar{u}}{\partial y_i}(y) \, dy \\ &\quad + c_n \sum_{i=1}^n \int_{\boldsymbol{R}^n - B} \left(\frac{x_i - y_i}{|x-y|^n} - \frac{\xi_i - y_i}{|\xi - y|^n}\right) \frac{\partial \bar{u}}{\partial y_i}(y) \, dy + A \\ &= c_n \sum_{i=1}^n \int_D \left(\frac{x_i - y_i}{|x-y|^n} - \frac{\bar{x}_i - y_i}{|\bar{x} - y|^n}\right) \frac{\partial u}{\partial y_i}(y) \, dy \\ &\quad + 2c_n \sum_{i=1}^n \int_D \left(\frac{\bar{x}_i - y_i}{|\bar{x}-y|^n} - \frac{\bar{\xi}_i - y_i}{|\bar{\xi} - y|^n}\right) \frac{\partial u}{\partial y_i}(y) \, dy \\ &\quad - c_n \sum_{i=1}^n \int_{D-B} \left(\frac{\xi_i - y_i}{|\xi-y|^n} - \frac{\bar{\xi}_i - y_i}{|\bar{\xi} - y|^n}\right) \frac{\partial u}{\partial y_i}(y) \, dy \\ &\quad + c_n \sum_{i=1}^n \int_B \frac{\bar{\xi}_i - y_i}{|\bar{\xi}-y|^n} \frac{\partial u}{\partial y_i}(y) \, dy + A \\ &= u_0(x) + h(x) \end{aligned} \tag{6}$$

for almost every $x \in D$, where $u_0 \in \boldsymbol{D}_0^{p,\gamma}$, $h \in \boldsymbol{HD}^{p,\gamma}$, A is a constant determined by u and $\xi \in D$.

Lemma 1 (cf. [**8**, Theorem 1]). *If* $u \in \boldsymbol{D}^{p,\gamma}$, *then the vertical limit*

$$\lim_{x_n \to 0+} u(x', x_n)$$

exists for almost every $x' \in \boldsymbol{R}^{n-1}$; *and moreover,*

$$u_0(x) = c_n \sum_{i=1}^{n} \int_D \left(\frac{x_i - y_i}{|x - y|^n} - \frac{\bar{x}_i - y_i}{|\bar{x} - y|^n} \right) \frac{\partial u}{\partial y_i}(y)\, dy \in \boldsymbol{D}_0^{p,\gamma}.$$

Lemma 2. *If* $u \in \boldsymbol{D}_0^{p,\gamma} \cap \boldsymbol{HD}^{p,\gamma}$, *then* u *is equal to zero.*

Proof. We first note that if $1 \leqq q < p$ and $q < p/(1+\gamma)$, then we can find a locally q-precise extension $\tilde{u}$ to $\boldsymbol{R}^n$ such that $\tilde{u}(x', x_n) = u(x', x_n)$ for $x_n > 0$ and $\tilde{u}(x', x_n) = -u(x', -x_n)$ for $x_n < 0$ as was remarked above. We shall show that $\Delta \tilde{u} = 0$ in the weak sense. For this purpose, let $\varphi \in C_0^\infty(\boldsymbol{R}^n)$. Since u is harmonic in D, we note by Green's formula that

$$\int \tilde{u}(x) \Delta \varphi(x)\, dx = \lim_{\varepsilon \to 0+} \{I(\varepsilon) + J(\varepsilon)\},$$

where

$$I(\varepsilon) = -\int_{\boldsymbol{R}^{n-1}} u(x', \varepsilon) \left\{ \frac{\partial \varphi}{\partial x_n}(x', \varepsilon) + \frac{\partial \varphi}{\partial x_n}(x', -\varepsilon) \right\} dx'$$

and

$$J(\varepsilon) = \int_{\boldsymbol{R}^{n-1}} \frac{\partial u}{\partial x_n}(x', \varepsilon) \{\varphi(x', \varepsilon) - \varphi(x', -\varepsilon)\}\, dx'$$

for $\varepsilon > 0$. Here note that

$$\liminf_{\varepsilon \to 0+} \varepsilon \int_{\{x' \in \boldsymbol{R}^{n-1} : |x'| < R\}} \left| \frac{\partial u}{\partial x_n}(x', \varepsilon) \right| dx' = 0$$

for $R > 0$. Since $u \in \boldsymbol{D}_0^{p,\gamma}$,

$$\begin{aligned} \int_{\{x' \in \boldsymbol{R}^{n-1} : |x'| < R\}} |u(x', \varepsilon)|\, dx' &= \int_{\{x' \in \boldsymbol{R}^{n-1} : |x'| < R\}} \left| \int_0^\varepsilon \frac{\partial u}{\partial x_n}(x', t)\, dt \right| dx' \\ &\leqq \int_{\{x = (x', x_n) : |x'| < R,\, 0 < x_n < \varepsilon\}} \left| \frac{\partial u}{\partial x_n}(x) \right| dx, \end{aligned}$$

which implies that

$$\lim_{\varepsilon \to 0+} \int_{\{x' \in \boldsymbol{R}^{n-1} : |x'| < R\}} |u(x', \varepsilon)|\, dx' = 0.$$

Thus we have

$$\lim_{\varepsilon \to 0+} I(\varepsilon) = 0$$

and

$$\liminf_{\varepsilon\to 0+} J(\varepsilon) = 0,$$

from which it follows that $\tilde{u}(x)$ is harmonic in $\boldsymbol{R}^n$ in the weak sense. Since $u \in \boldsymbol{D}^{p,\gamma}$, we can apply [**11**, Lemma 2] to see that $\tilde{u}(x)$ is constant, so that u is equal to zero by the assumption. □

In view of Lemma 2, we have the following result.

THEOREM 2. *The Riesz decomposition* (2) *is unique*.

3. Proof of Theorem 1.

In view of Theorem 2, we see that $u \in \boldsymbol{D}_0^{p,\gamma}$ is represented as

$$u(x) = c_n \sum_{i=1}^{n} \int_D \left(\frac{x_i - y_i}{|x-y|^n} - \frac{\bar{x}_i - y_i}{|\bar{x}-y|^n} \right) \frac{\partial u}{\partial y_i}(y)\, dy$$

for almost every $x \in D$.

We prepare the following result.

LEMMA 3. *There exists a positive constant M such that*

$$\left| \frac{x_i - y_i}{|x-y|^n} - \frac{\bar{x}_i - y_i}{|\bar{x}-y|^n} \right| \leqq M \frac{x_n}{|x-y|^{n-1}|\bar{x}-y|}$$

for $x = (x_1, \dots, x_n)$ *and* $y = (y_1, \dots, y_n)$ *in* D.

PROOF. First note that

$$|x-y|^{-n} - |\bar{x}-y|^{-n} \leqq \frac{n(|\bar{x}-y|^2 - |x-y|^2)}{|x-y|^n|\bar{x}-y|(|\bar{x}-y| + |x-y|)} \leqq \frac{4nx_ny_n}{|x-y|^n|\bar{x}-y|^2}.$$

In case $i = n$, we have

$$\begin{aligned}
\left| \frac{x_n - y_n}{|x-y|^n} - \frac{\bar{x}_n - y_n}{|\bar{x}-y|^n} \right| &= |(x_n - y_n)(|x-y|^{-n} - |\bar{x}-y|^{-n}) + 2x_n|\bar{x}-y|^{-n}| \\
&\leqq 4n|x-y|x_ny_n|x-y|^{-n}|\bar{x}-y|^{-2} + 2x_n|\bar{x}-y|^{-n} \\
&\leqq (4n+2)x_n|x-y|^{1-n}|\bar{x}-y|^{-1}.
\end{aligned}$$

In case $1 \leqq i \leqq n-1$, we have

$$\begin{aligned}
\left| \frac{x_i - y_i}{|x-y|^n} - \frac{\bar{x}_i - y_i}{|\bar{x}-y|^n} \right| &= |(x_i - y_i)(|x-y|^{-n} - |\bar{x}-y|^{-n})| \\
&\leqq 4n|x-y|x_ny_n|x-y|^{-n}|\bar{x}-y|^{-2} \\
&\leqq 4nx_n|x-y|^{1-n}|\bar{x}-y|^{-1}.
\end{aligned}$$

Hence the required result follows. □

Throughout this paper, let M denote various constants independent of the variables in question.

In view of Lemma 3,

$$|u(x)| \leqq Mx_n \int_D |x-y|^{1-n}|\bar{x}-y|^{-1}|\nabla u(y)|\,dy$$
$$= Mx_n^{\beta} \int_D k_{\beta,\gamma}(x,y)\{y_n^{\gamma/p}|\nabla u(y)|\}\,dy \tag{7}$$

for almost every $x \in D$.

Here we prepare the following lemma.

Lemma 4. *For a nonnegative measurable function* $f \in L^p(D)$, *we set*

$$U(x) = \int_D k_{\beta,\gamma}(x,y)f(y)\,dy, \quad x \in D.$$

If $p > 1$, $-1 < \gamma < p-1$, $(1-\beta)p - n < \gamma$ *and* $0 \leqq \beta \leqq 1$, *then there exists a set* E *such that* E *is* $(k_{\beta,\gamma}, p)$*-thin at infinity and*

$$\lim_{|x|\to\infty,\, x\in D-E} |x|^{(n+\gamma-(1-\beta)p)/p}U(x) = 0.$$

Proof. For fixed $x \in \boldsymbol{R}^n$, $x \neq 0$, we write

$$U(x) = \int_{G_1} k_{\beta,\gamma}(x,y)f(y)\,dy + \int_{G_2} k_{\beta,\gamma}(x,y)f(y)\,dy + \int_{G_3} k_{\beta,\gamma}(x,y)f(y)\,dy$$
$$= U_1(x) + U_2(x) + U_3(x),$$

where

$$G_1 = \{y \in D : |y| \geqq 2|x|\},$$
$$G_2 = \{y \in D : |y| \leqq |x|/2\},$$
$$G_3 = \{y \in D : |x|/2 \leqq |y| \leqq 2|x|\}.$$

From Hölder's inequality, we obtain

$$U_1(x) = \int_{G_1} k_{\beta,\gamma}(x,y)f(y)\,dy$$
$$\leqq Mx_n^{1-\beta}\left(\int_{G_1} |y|^{-np'}y_n^{-\gamma p'/p}\,dy\right)^{1/p'}\left(\int_{G_1} f(y)^p\,dy\right)^{1/p}$$
$$\leqq Mx_n^{1-\beta}|x|^{-(n+\gamma)/p}\left(\int_{G_1} f(y)^p\,dy\right)^{1/p},$$

where $1/p + 1/p' = 1$. Hence we have

$$\lim_{|x|\to\infty} |x|^{(n+\gamma-(1-\beta)p)/p}U_1(x) = 0.$$

For any $r > 0$, we set

$$f = f\chi_{B(0,r)} + f\chi_{D-B(0,r)} = f_1 + f_2,$$

where χ_E denotes the characteristic function of a Borel set $E \in \mathbf{R}^n$. From Hölder's inequality, we have

$$\begin{aligned} U_2(x) &\leqq M x_n^{1-\beta}|x|^{-n} \int_{G_2} y_n^{-\gamma/p} f(y)\,dy \\ &\leqq M x_n^{1-\beta}|x|^{-n} \int_{G_2} y_n^{-\gamma/p} f_1(y)\,dy + M x_n^{1-\beta}|x|^{-n} \left(\int_{G_2} y_n^{-\gamma p'/p}\,dy\right)^{1/p'} \left(\int_{G_2} f_2(y)^p\,dy\right)^{1/p} \\ &\leqq M x_n^{1-\beta}|x|^{-n} \int_{G_2} y_n^{-\gamma/p} f_1(y)\,dy + M x_n^{1-\beta}|x|^{-(n+\gamma)/p} \left(\int_{G_2} f_2(y)^p\,dy\right)^{1/p} \\ &\leqq M x_n^{1-\beta}|x|^{-n} \int_{B(0,r)\cap D} y_n^{-\gamma/p} f(y)\,dy + M x_n^{1-\beta}|x|^{-(n+\gamma)/p} \left(\int_{D-B(0,r)} f(y)^p\,dy\right)^{1/p}, \end{aligned}$$

so that

$$|x|^{(n+\gamma-(1-\beta)p)/p} U_2(x) \leqq M|x|^{(n+\gamma)/p-n} \int_{B(0,r)\cap D} y_n^{-\gamma/p} f(y)\,dy + M\left(\int_{D-B(0,r)} f(y)^p\,dy\right)^{1/p}.$$

Hence we obtain

$$\limsup_{|x|\to\infty} |x|^{(n+\gamma-(1-\beta)p)/p} U_2(x) \leqq M\left(\int_{D-B(0,r)} f(y)^p\,dy\right)^{1/p}$$

for every $r > 0$, which implies that the left hand side is equal to zero.

Since $f \in L^p(D)$, we can find a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty} a_i = \infty$ and

$$\sum_{i=1}^{\infty} a_i \int_{D_i} f(y)^p\,dy < \infty;$$

recall $D_i = \{y \in D : 2^{i-1} < |y| < 2^{i+2}\}$. Consider the sets

$$E_i = \{x \in D : 2^i \leqq |x| < 2^{i+1},\ U_3(x) \geqq a_i^{-1/p} 2^{-i(n+\gamma-(1-\beta)p)/p}\}$$

for $i = 1, 2, \ldots.$ If $x \in E_i$, then

$$\begin{aligned} a_i^{-1/p} &\leqq 2^{i(n+\gamma-(1-\beta)p)/p} U_3(x) \\ &\leqq 2^{i(n+\gamma-(1-\beta)p)/p} \int_{D_i} k_{\beta,\gamma}(x,y) f(y)\,dy, \end{aligned}$$

so that it follows from the definition of $C_{k_{\beta,\gamma},p}$ that

$$C_{k_{\beta,\gamma},p}(E_i; D_i) \leqq a_i 2^{i(n+\gamma-(1-\beta)p)} \int_{D_i} f(y)^p\,dy.$$

Define $E = \bigcup_{i=1}^{\infty} E_i$. Then $E \cap B(0, 2^{i+1}) - B(0, 2^i) = E_i$ and

$$\sum_{i=1}^{\infty} 2^{-i(n+\gamma-(1-\beta)p)} C_{k_{\beta,\gamma},p}(E_i; D_i) < \infty.$$

Clearly,

$$\lim_{|x|\to\infty,\, x\in D-E} |x|^{(n+\gamma-(1-\beta)p)/p} U_3(x) = 0.$$

Thus the proof of the lemma is completed. □

Remark 1. The proof of Lemma 4 shows that

$$\int_D (1+|y|)^{-n} y_n^{-\gamma/p} f(y)\, dy < \infty,$$

which is equivalent to the condition that $U(x) \not\equiv \infty$. Hence, for $x \in D$, $U(x) = \infty$ if and only if

$$\int_{B(x,r)} k_{\beta,\gamma}(x,y) f(y)\, dy = \infty \quad \text{whenever } 0 < r < x_n.$$

We have the following result.

Lemma 5 (cf. [**12**]). *For a set $E \subset D$, $C_{k_{\beta,\gamma},p}(E; D) = 0$ if and only if E is of $(1,p)$-capacity zero, which means that $C_{k_1,p}(E \cap B(0,r); B(0,2r)) = 0$ for every $r > 0$, where $k_1(x,y) = |x-y|^{1-n}$.*

Lemma 5 implies that inequality (7) holds for every $x \in D$ except that of a set of $C_{k_{\beta,\gamma},p}$-capacity zero. Now Theorem 1 follows from Lemma 4.

4. Proof of Proposition 1.

Let g be a nonnegative measurable function such that $g = 0$ outside D_i and

$$\int_D k_{\beta,\gamma}(x,y) g(y)\, dy \geqq 1$$

for every $x \in E_i$. Then we have by Fubini's theorem

$$\begin{aligned} |E_i| &\leqq \int_{E_i} \left(\int_{B_{i+2}} k_{\beta,\gamma}(x,y) g(y)\, dy \right) dx \\ &= \int_{B_{i+2}} g(y) y_n^{-\gamma/p} \left(\int_{E_i} x_n^{1-\beta} |x-y|^{1-n} |\bar{x}-y|^{-1}\, dx \right) dy. \end{aligned}$$

Take $r \geqq 0$ such that $|B(0,r)| = |E_i|$, that is,

$$\sigma_n r^n = |E_i|$$

with σ_n denoting the volume of the unit ball. Here note that if $y \in D$, then

$$\int_{E_i} x_n^{1-\beta}|x-y|^{1-n}|\bar{x}-y|^{-1}\,dx \leqq \int_{E_i} |x-y|^{1-n-\beta}\,dx$$
$$\leqq \int_{B(y,r)} |x-y|^{1-n-\beta}\,dx$$
$$\leqq Mr^{1-\beta} = M|E_i|^{(1-\beta)/n}.$$

Therefore we obtain by Hölder's inequality

$$|E_i| \leqq M|E_i|^{(1-\beta)/n}\int_{B_{i+2}} g(y)y_n^{-\gamma/p}\,dy$$
$$\leqq M|E_i|^{(1-\beta)/n}\left(\int_{B_{i+2}} g(y)^p\,dy\right)^{1/p}\left(\int_{B_{i+2}} y_n^{-\gamma p'/p}\,dy\right)^{1/p'}$$
$$\leqq M|E_i|^{(1-\beta)/n}\left(\int_{B_{i+2}} g(y)^p\,dy\right)^{1/p} 2^{i(-\gamma/p+n/p')}.$$

Hence it follows from the definition of $C_{k_{\beta,\gamma},p}$ that

$$|E_i|^{(1-(1-\beta)/n)p} \leqq M2^{i(np-(n+\gamma))}C_{k_{\beta,\gamma},p}(E_i;D_i),$$

which yields

$$\sum_{i=1}^{\infty}\left(\frac{|E_i|}{|B_i|}\right)^{(1-(1-\beta)/n)p} < \infty.$$

5. Proof of Proposition 2.

By the definition of $C_{k_{\beta,\gamma},p}$, we obtain the next results.

Lemma 6 (cf. [**10**, Lemma 4]). *For $r>0$ and a Borel set E in D, let $rE = \{rx : x \in E\}$. Then*

$$C_{k_{\beta,\gamma},p}(rE;rG) = r^{n+\gamma-(1-\beta)p}C_{k_{\beta,\gamma},p}(E;G).$$

Lemma 7 (cf. [**12**, Lemma 2.2]). *Let G, G_1 and G_2 be bounded open sets in D such that $\overline{G}\cap D \subset G_1 \cap G_2$. Then*

$$C_{k_{\beta,\gamma},p}(E;G_1) \sim C_{k_{\beta,\gamma},p}(E;G_2)$$

whenever $E \subset G$, that is, there exist $M_1, M_2 > 0$ such that

$$M_1C_{k_{\beta,\gamma},p}(E;G_1) \leqq C_{k_{\beta,\gamma},p}(E;G_2) \leqq M_2C_{k_{\beta,\gamma},p}(E;G_1)$$

whenever $E \subset G$.

For $r>0$ and $s>0$, set

$$S(r,s) = \{x = (x',x_n) \in \boldsymbol{R}^{n-1}\times\boldsymbol{R}^1 : |x'|<r, 0<x_n<s\}.$$

LEMMA 8. *Let* $(1-\beta)p-1<\gamma<p-1$ *and* $0<\beta\leqq 1$. *Then there exist* $M_1, M_2>0$ *such that*

$$M_1 s^{p(-1+\beta)+\gamma+1}\leqq C_{k_{\beta,\gamma},p}(S(1,s);B(0,2)\cap D)\leqq M_2 s^{p(-1+\beta)+\gamma+1}$$

whenever $0<s\leqq 1$.

PROOF. To prove the first inequality, let g be a nonnegative measurable function such that $g=0$ outside $B(0,2)$ and

$$\int_D k_{\beta,\gamma}(z,y)g(y)\,dy\geqq 1$$

for every $z\in S(1,s)$. Then we have by Fubini's theorem

$$\begin{aligned}\int_{S(1,s)}dz &\leqq \int_{S(1,s)}\left(\int_D k_{\beta,\gamma}(z,y)g(y)\,dy\right)dz\\ &=\int_D g(y)y_n^{-\gamma/p}\left(\int_{S(1,s)} z_n^{1-\beta}|z-y|^{1-n}|\bar{z}-y|^{-1}\,dz\right)dy.\end{aligned}$$

For $z=(z',z_n)$ and $y=(y',y_n)$, set

$$a=|z_n-y_n|\quad\text{and}\quad b=|z_n+y_n|.$$

Here note that

$$\begin{aligned}I&\equiv\int_{S(1,s)}z_n^{1-\beta}|z-y|^{1-n}|\bar{z}-y|^{-1}\,dz\\
&\leqq M\int_0^s z_n^{1-\beta}\left(\int_{|z'|\leqq 1}(|z'-y'|+|z_n-y_n|)^{1-n}(|z'-y'|+|z_n+y_n|)^{-1}\,dz'\right)dz_n\\
&=M\int_0^s z_n^{1-\beta}\left(\int_0^3(r+a)^{1-n}(r+b)^{-1}r^{n-2}\,dr\right)dz_n\\
&\leqq M\int_0^s z_n^{1-\beta}\left(a^{1-n}b^{-1}\int_0^a r^{n-2}\,dr+b^{-1}\int_a^b r^{-1}\,dr+\int_b^\infty r^{-2}\,dr\right)dz_n\\
&\leqq M\int_0^s z_n^{1-\beta}b^{-1}\left(1+\log\frac{b}{a}\right)dz_n\\
&=My_n^{1-\beta}\int_0^{s/y_n}t^{1-\beta}(1+t)^{-1}\left(1+\log\left|\frac{1+t}{1-t}\right|\right)dt.\end{aligned}$$

If $y_n>2s$, then

$$I\leqq My_n^{1-\beta}\int_0^{s/y_n}t^{1-\beta}\,dt=My_n^{1-\beta}(s/y_n)^{2-\beta}=My_n^{-1}s^{2-\beta}.$$

If $y_n<2^{-1}s$, then

$$I \leqq M y_n^{1-\beta} \int_0^2 t^{1-\beta}\left(1+\log\left|\frac{1+t}{1-t}\right|\right) dt + M y_n^{1-\beta} \int_2^{s/y_n} t^{-\beta}\, dt$$
$$\leqq M y_n^{1-\beta} + M y_n^{1-\beta} (s/y_n)^{1-\beta} \log(s/y_n)$$
$$\leqq M s^{1-\beta} \log(s/y_n).$$

Since $I \leqq M y_n^{1-\beta}$ when $2^{-1}s \leqq y_n \leqq 2s$, we have by Hölder's inequality

$$\int_{S(1,s)} dz \leqq M s^{2-\beta} \int_{\{y:s<y_n<2\}} g(y) y_n^{-\gamma/p-1}\, dy$$
$$+ M s^{1-\beta} \int_{\{y:0<y_n<s\}} g(y) y_n^{-\gamma/p} \log\{1+(s/y_n)\}\, dy$$
$$\leqq M s^{2-\beta} \left(\int_D g(y)^p\, dy\right)^{1/p} \left(\int_s^2 y_n^{(-\gamma/p-1)p'}\, dy_n\right)^{1/p'}$$
$$+ M s^{1-\beta} \left(\int_D g(y)^p\, dy\right)^{1/p} \left(\int_0^s y_n^{-\gamma p'/p} [\log\{1+(s/y_n)\}]^{p'}\, dy_n\right)^{1/p'}$$
$$\leqq M s^{2-\beta-(\gamma+1)/p} \left(\int_D g(y)^p\, dy\right)^{1/p}.$$

Taking the infimum over all such g, we arrive at the first inequality.

To prove the second inequality, take δ such that

$$(\gamma/p - 1 <)\ -1/p < \delta < \gamma/p - (1-\beta). \tag{8}$$

Define

$$f(y) = \begin{cases} y_n^{\delta}, & \text{if } y = (y', y_n) \in S(1,s), \\ 0, & \text{otherwise.} \end{cases}$$

If $x = (x', x_n) \in S(1,s)$, then

$$\int_{S(1,s)} k_{\beta,\gamma}(x,y) f(y)\, dy \geqq M x_n^{1-\beta} \int_0^{x_n/2} \left(\int_0^1 (r+x_n)^{-n} r^{n-2}\, dr\right) y_n^{-\gamma/p+\delta}\, dy_n$$
$$\geqq M x_n^{-\beta} \int_0^{x_n/2} y_n^{\delta-\gamma/p}\, dy_n$$
$$= M x_n^{1-\beta+\delta-\gamma/p} \geqq M s^{1-\beta+\delta-\gamma/p}$$

since $\gamma/p - 1 < \delta$ and $1-\beta+\delta-\gamma/p < 0$ by (8). Hence it follows from the definition of $C_{k_{\beta,\gamma},p}$ that

$$C_{k_{\beta,\gamma},p}(S(1,s); B(0,2)\cap D) \leqq M s^{(-1+\beta-\delta+\gamma/p)p} \int_D f(y)^p\, dy = M s^{p(-1+\beta)+\gamma+1},$$

which completes the proof. □

For a nondecreasing function φ on $\boldsymbol{R}^1$ such that $0 < \varphi(2t) \leqq \varphi(t)$ for $t > 0$, we set

$$T_\varphi = \{x = (x', x_n) : 0 < x_n < \varphi(|x'|)\},$$

$$T_{\varphi,i} = T_\varphi \cap B_{i+1} - B_i, \quad B_i = B(0, 2^i) \cap D.$$

Lemma 9. *Let $0 < \beta \leqq 1$ and $p(1-\beta) - 1 < \gamma < p - 1$. Assume that φ is as in Proposition 2. Then there exist $N_1, N_2 > 0$ independent of i such that*

$$N_1 \left(\frac{\varphi(2^i)}{2^i}\right)^{p(-1+\beta)+\gamma+1} \leqq 2^{-i(n+\gamma-(1-\beta)p)} C_{k_{\beta,\gamma},p}(T_{\varphi,i}; D_i) \leqq N_2 \left(\frac{\varphi(2^i)}{2^i}\right)^{p(-1+\beta)+\gamma+1}.$$

Proof. Let $\xi_i = (2^i + 2^{i-2}, 0, \dots, 0)$. Define

$$T_i' = \{x = (x', x_n) : |(x', 0) - \xi_i| < 2^{i-2}, 0 < x_n < \varphi(2^i)\}$$

and

$$T_i'' = S(2^{i+1}, \varphi(2^{i+1})) \cap B_{i+1} - B_i.$$

If $\varphi(2^{i+1}) < 2^{i-2}$, then $T_i' \subset T_{\varphi,i} \subset T_i''$. On the other hand we have by Lemmas 6 and 7

$$C_{k_{\beta,\gamma},p}(T_i'; D_i) \sim 2^{(i-2)(n+\gamma-(1-\beta)p)} C_{k_{\beta,\gamma},p}(S(1, \varphi(2^i)/2^{i-2}); B_1)$$

and

$$C_{k_{\beta,\gamma},p}(T_i''; D_i) \sim 2^{(i+1)(n+\gamma-(1-\beta)p)} C_{k_{\beta,\gamma},p}(S(1, \varphi(2^{i+1})/2^{i+1}); B_1).$$

Thus, in view of (4), Lemma 8 gives Lemma 9 readily. □

By using Lemma 9 we can prove Proposition 2.

6. Fine limits of p-precise functions.

In this section we are concerned with fine limits at infinity for functions in $\boldsymbol{D}^{p,\gamma}$. For this purpose, consider the kernel function

$$k_\gamma(x, y) = |x - y|^{1-n} |y_n|^{-\gamma/p}$$

and the capacity

$$C_{k_\gamma,p}(E; G) = \inf \int_D g(y)^p \, dy,$$

where E is a subset of an open set G in $\boldsymbol{R}^n$ and the infimum is taken over all nonnegative measurable functions g such that $g = 0$ outside G and

$$\int k_\gamma(x, y) g(y) \, dy \geqq 1 \quad \text{for all } x \in E.$$

We say that $E \subset D$ is (k_γ, p)-thin at infinity if

$$\sum_{i=1}^{\infty} 2^{-i(n+\gamma-p)} C_{k_\gamma,p}(E_i; R_i) < \infty, \tag{9}$$

where $E_i = \{x \in E : 2^i \leqq |x| < 2^{i+1}\}$ and $R_i = \{x \in \boldsymbol{R}^n : 2^{i-1} < |x| < 2^{i+2}\}$.

THEOREM 3. *Let $p > 1$, $-1 < \gamma < p-1$ and $n+\gamma-p > 0$. If $u \in \boldsymbol{D}^{p,\gamma}$, then there exist a set $E \subset D$ and a number A such that E is (k_γ, p)-thin at infinity and*

$$\lim_{|x|\to\infty,\, x\in D-E} |x|^{(n+\gamma-p)/p}\{u(x) - A\} = 0. \tag{10}$$

REMARK 2. If $n+\gamma-p = 0$, then (10) is replaced by

$$\lim_{|x|\to\infty,\, x\in D-E} (\log|x|)^{-1/p'}\{u(x) - A\} = 0.$$

THEOREM 4. *Let $p > 1$, $-1 < \gamma < p-1$ and $n+\gamma-p > 0$. If $h \in \boldsymbol{HD}^{p,\gamma}$, then*

$$\lim_{|x|\to\infty,\, x\in D} x_n^{(n+\gamma-p)/p}\{h(x) - A\} = 0$$

for some number A.

For proofs of these results, we first note that $u \in \boldsymbol{D}^{p,\gamma}$ is represented as

$$u(x) = c_n \sum_{i=1}^{n} \int \frac{x_i - y_i}{|x-y|^n} \frac{\partial \bar{u}}{\partial y_i}(y)\, dy + A$$

for every $x \in D - E'$, where A is a constant and $C_{k_\gamma,p}(E'; D) = 0$. As in the proof of Theorem 1, we write for $x \in D$

$$\begin{aligned} u(x) - A &= c_n \sum_{i=1}^{n} \int_{G_1} \frac{x_i - y_i}{|x-y|^n} \frac{\partial \bar{u}}{\partial y_i}(y)\, dy + c_n \sum_{i=1}^{n} \int_{G_2} \frac{x_i - y_i}{|x-y|^n} \frac{\partial \bar{u}}{\partial y_i}(y)\, dy \\ &\quad + c_n \sum_{i=1}^{n} \int_{G_3} \frac{x_i - y_i}{|x-y|^n} \frac{\partial \bar{u}}{\partial y_i}(y)\, dy \\ &= u_1(x) + u_2(x) + u_3(x), \end{aligned}$$

where $G_1 = \{y \in \boldsymbol{R}^n : |y| \geqq 2|x|\}$, $G_2 = \{y \in \boldsymbol{R}^n : |y| \leqq |x|/2\}$ and $G_3 = \{y \in \boldsymbol{R}^n : |x|/2 \leqq |y| \leqq 2|x|\}$. We can prove as in the proof of Theorem 1 that

$$\lim_{|x|\to\infty,\, x\in D} |x|^{(n+\gamma-p)/p}\{u_1(x) + u_2(x)\} = 0$$

and

$$\lim_{|x|\to\infty,\, x\in D-E''} |x|^{(n+\gamma-p)/p} u_3(x) = 0$$

with a set $E'' \subset D$ satisfying (9). Thus Theorem 3 is derived.

Next suppose $u \in \boldsymbol{HD}^{p,\gamma}$. To prove Theorem 4, we note that

$$\sum_{i=1}^{n} \int_{B(x, x_n/2)} \frac{x_i - y_i}{|x-y|^n} \frac{\partial \bar{u}}{\partial y_i}(y)\, dy = 0$$

for $x \in D$, because u is harmonic in D. Hence we obtain by Hölder's inequality

$$|u_3(x)| \leqq M \int_{G_3} (x_n + |x - y|)^{1-n} |\nabla \bar{u}(y)|\, dy$$
$$\leqq M x_n^{(p-n-\gamma)/p} \left(\int_{G_3} |\nabla \bar{u}(y)|^p y_n^\gamma\, dy \right)^{1/p},$$

which yields

$$\lim_{|x| \to \infty,\, x \in D} x_n^{(n+\gamma-p)/p} u_3(x) = 0.$$

Now Theorem 4 is proved.

Remark 3. Let $p > 1$, $-1 < \gamma < p - 1$ and $n + \gamma - p > 0$. Then we can find a function $h \in \boldsymbol{HD}^{p,\gamma}$ such that

$$\limsup_{|x| \to \infty,\, x \in D} |x|^{(n+\gamma-p)/p} h(x) = \infty \tag{11}$$

and

$$\lim_{|x| \to \infty,\, x \in D} x_n^{(n+\gamma-p)/p} h(x) = 0. \tag{12}$$

Let $e_j = (2^j, 0, \dots, 0)$ and consider

$$f(y) = \sum_{j=1}^{\infty} 2^{-j(n+\gamma)/p} |y - e_j|^{-\varepsilon} \chi_{B(e_j, 2^{j-2}) - D}(y),$$

where $1 < \varepsilon < (n + \gamma)/p$ and χ_E denotes the characteristic function of E. Then

$$\int f(y)^p |y_n|^\gamma\, dy = M \sum_j 2^{-j(n+\gamma)} 2^{-j(\varepsilon p - n - \gamma)} < \infty.$$

Now define

$$h(x) = \int_{\boldsymbol{R}^n - D} \frac{x_n - y_n}{|x - y|^n} f(y)\, dy.$$

Note that $h \in \boldsymbol{D}^{p,\gamma}$ (cf. [**8**, Lemma 6]) and h is harmonic in D. We see that (12) holds by the proofs of Theorems 3 and 4. Moreover,

$$\liminf_{x \to e_j} h(x) \geqq \int_{\boldsymbol{R}^n - D} \frac{(e_j)_n - y_n}{|e_j - y|^n} f(y)\, dy = \infty$$

for each j, which proves (11). Thus h has all the required conditions.

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