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A construction of a family of full compact minimal surfaces in 4-dimensional flat tori

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Abstract. In this paper, we will construct a family of full compact oriented minimal surfaces of genus 3 with degenerate Gauss map in 4-dimensional flat tori, which are not holomorphic with respect to any complex structure of the tori.

1. Introduction.

A little over a hundred years ago, H. A. Schwarz constructed triply-periodic minimal surfaces in 3-space ([**11**]). This provided us with the first compact minimal surface in a flat real 3-torus. He solved the Plateau problem of a 4-sided polygonal curve in $\boldsymbol{R}^3$ and constructed a minimal surface via reflections across the boundary edges. In 1970, A. Schoen described 17 triply-periodic minimal surfaces in $\boldsymbol{R}^3$ including and inspired by five such examples by Schwarz ([**10**]). Based on Schoen's manuscript, Karcher has written a short treatise on some of the examples of Schoen ([**5**]). Karcher's manuscript includes computer graphics images of some these examples as well as some related interesting new examples. T. Nagano and B. Smyth extended Schwarz' idea to the higher dimensional case and abstractly constructed compact minimal surfaces in flat n-tori ($n \geq 3$) ([**7**], [**8**]). There are, however, few concrete examples in this case.

On the other hand, M. Micallef has shown that an oriented stable minimal surface in a 4-dimensional flat torus is always holomorphic with respect to some orthogonal complex structure on the torus ([**6**]).

Now we consider two problems: (1) Construct concrete examples of full compact minimal surfaces via Weierstrass representation in some flat 4-tori. (2) Prove that the stability in Micallef's theorem is essential *i.e.* there exists a full minimal surface which is not holomorphic with respect to any complex structure of the 4-tori. In this paper, we obtain a family of full compact oriented minimal surfaces of genus 3 with degenerate Gauss map in some 4-tori, which are not holomorphic with respect to any complex structure of the tori.

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2. Survey.

In this section, we review some fundamental results. A lattice Λ in a real vector space $\boldsymbol{R}^n$ is a discrete subgroup of maximal rank in $\boldsymbol{R}^n$. A flat tori is a quotient $\boldsymbol{R}^n/\Lambda$

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with a lattice Λ of $\boldsymbol{R}^n$ and the metric induced from the standard Euclidean metric on $\boldsymbol{R}^n$. Let $\{u_1, \dots, u_m\}$ $(m \geq n)$ be a sequence of vectors which span $\boldsymbol{R}^n$. In general, $\{u_1, \dots, u_m\}$ are not lattice vectors.

Proposition 2.1 ([**2**, section 6]). *$\{u_1, \dots, u_m\}$ are lattice vectors if and only if there exist lattice vectors $\{v_1, \dots, v_n\}$ such that*

$$(v_1, v_2, \dots, v_n) = (u_1, u_2, \dots, u_m) G_1,$$
$$(u_1, u_2, \dots, u_m) = (v_1, v_2, \dots, v_n) G_2,$$

where G_1 is an (m, n)-matrix and G_2 is an (n, m)-matrix whose components are integers.

An n-periodic minimal surface properly immersed in $\boldsymbol{R}^n$ corresponds to a minimal immersion f of a compact oriented surface M into a flat tori $\boldsymbol{R}^n/\Lambda$. With the induced conformal structure, M is a compact Riemann surface and f is a conformal minimal immersion. Thus, we will treat the case where the source manifold is a compact Riemann surface and the target manifold is a flat tori. The fundamental theorem of minimal surfaces in flat tori is given by

Theorem 2.1 (Generalized Weierstrass Representation). *If $f : M \to \boldsymbol{R}^n/\Lambda$ is a conformal minimal immersion then, after a translation, f can be represented by*

$$f(p) = \Re \int_{p_0}^{p} (\omega_1, \omega_2, \dots, \omega_n)^T \quad \operatorname{Mod} \Lambda,$$

where $p_0 \in M$, superscript T means the transposed matrix and $\{\omega_1, \omega_2, \dots, \omega_n\}$ are holomorphic differentials on M satisfying

(1) *$\{\omega_1, \omega_2, \dots, \omega_n\}$ has no common zeros,*
(2) *$\sum_{i=1}^{n} \omega_i^2 = 0$ (conformal condition),*
(3) *Period matrix $\Omega := \{\Re \int_\gamma (\omega_1, \omega_2, \dots, \omega_n)^T \mid \gamma \in H_1(M, \boldsymbol{Z})\}$ is a sublattice of Λ (periodic condition).*

Conversely, every minimal surface in $\boldsymbol{R}^n/\Lambda$ is obtained by the above construction.

Next, we review the theory of Gauss map ([**3**]). Let $G_{n,2}$ denote the Grassmannian of oriented 2-planes in $\boldsymbol{R}^n$. Let $f : M \to \boldsymbol{R}^n/\Lambda$ be an immersion of a compact Riemann surface into a flat tori. The generalized Gauss map $G : M \to G_{n,2}$ is defined by $G(p) = f_*(T_pM)$, which is obtained by parallel translation of the tangent space T_pM to the origin of $\boldsymbol{R}^n$. Recall that $G_{n,2}$ is identified with the quadric $Q_{n-2} \subset \boldsymbol{C}P^{n-1}$ defined by $\{[w] \in \boldsymbol{C}P^{n-1} \mid w \cdot w = \sum_i (w^i)^2 = 0\}$, where $\cdot$ is the complex bilinear inner product. If z is a local complex coordinate on M, then $f_z(p)$ is a homogeneous coordinate for $G(p)$. If the Gauss image lies in a hyperplane of $\boldsymbol{C}P^{n-1}$, that is, if there exists a non-zero vector $A \in \boldsymbol{C}^n$ (and can be considered $A \in \boldsymbol{C}P^{n-1}$) such that $A \cdot f_z \equiv 0$, we call the Gauss map is degenerate. We can normalize this non-zero vector $A \in \boldsymbol{C}P^{n-1}$ as follows.

Lemma 2.1 ([**3**, p. 28, Proposition 2.4]). *To each point $A = (a_1, \dots, a_n) \in \boldsymbol{C}P^{n-1}$, $n \geq 3$, one may assign a real number t lying in the interval $0 \leq t \leq 1$ with the following properties*:

(I) *A is equivalent under the action of $SO(n)$ to $(t, i, 0, \dots, 0)$;*
(II) *$t = 0 \Leftrightarrow A$ is a real vector (i.e. $(a_1, \dots, a_n) = \lambda(r_1, \dots, r_n)$, $\lambda \in \boldsymbol{C}$, $r_i \in \boldsymbol{R}$, $i = 1, \dots, n$.)*
(III) *$t = 1 \Leftrightarrow A \in Q_{n-2}$.*
(IV) *if t, t' correspond to vectors A, A', then A and A' are equivalent under $SO(n)$ if and only if $t = t'$.*

The nature of a minimal surface with a degenerate Gauss map depends strongly on the nature of the hyperplane containing its Gauss image, or equivalently on the nature of vector A.

In the case $n = 4$, Weierstrass representation can be reduced to the following.

Theorem 2.2 ([**3**, Theorem 4.7]). *If $f : M \to \boldsymbol{R}^4/\Lambda$ is a conformal minimal immersion with degenerate Gauss map, then f can be represented by*

$$\Re \int_{p_0}^{p} \left(1, it, \frac{1}{2}\left(\frac{-1+t^2}{F} + F\right), \frac{i}{2}\left(\frac{-1+t^2}{F} - F\right)\right)^T \omega,$$

where F is a meromorphic function on M, $t \in [0, 1]$ is a constant, ω is a holomorphic differential on M such that

$$\left\{\omega, it\omega, \frac{1}{2}\left(\frac{-1+t^2}{F} + F\right)\omega, \frac{i}{2}\left(\frac{-1+t^2}{F} - F\right)\omega\right\}$$

are holomorphic differentials on M and

$$\Omega = \left\{\Re \int_{\gamma} \left(1, it, \frac{1}{2}\left(\frac{-1+t^2}{F} + F\right), \frac{i}{2}\left(\frac{-1+t^2}{F} - F\right)\right)^T \omega \mid \gamma \in H_1(M, \boldsymbol{Z})\right\}$$

is a sublattice of Λ. Conversely, every minimal surface in $\boldsymbol{R}^4/\Lambda$ with degenerate Gauss map is obtained by the above construction.

Remark 2.1. In the case of $t = 0$, the above minimal surface lies in $\boldsymbol{R}^3/\Lambda$. In the case of $t = 1$, f is holomorphic with respect to some complex structure of the tori (see Proposition 4.6 b) and the proof of Theorem 4.7 in [**3**]) (We now consider the surface that satisfy the periodic condition).

3. Construction.

Let M be the hyperelliptic Riemann surface defined by the equation

$$w^2 = z^8 + 14z^4 + 1.$$

Then M is of genus 3 and can be considered as a branched double cover of the z-plane, the branch points occuring at the 8 points $ae^{k\pi i/4}$ and $a^{-1}e^{k\pi i/4}$ where $a = (1 + \sqrt{3}/\sqrt{2})$ and $k \in \{1, 3, 5, 7\}$. Identifying the z-plane with S^2 via stereographic projection, we have these branch points at corners $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ of a cube inscribed inside S^2. It is well-known that we can write out a basis for the holomorphic differential on M by $\{dz/w, z(dz/w), z^2(dz/w)\}$.

Since we immerse M minimally into some 4-torus, its Gauss image lies in $\boldsymbol{CP}^3$. Thus the Gauss map degenerates (p. 50 in [**3**]) and we can use the Weierstrass representation given by Theorem 2.2. We substitute $F = z$, $\omega = z(dz/w)$ and obtain $f : M \to \boldsymbol{R}^4$ by

$$p \mapsto \Re \int^p \Phi = \Re \int^p \left(z, itz, \frac{-1 + t^2 + z^2}{2}, \frac{i(-1 + t^2 - z^2)}{2} \right)^T \frac{dz}{w}. \tag{4}$$

In the case $t = 0$, the minimal surface (4) thus obtained is the Schwarz surface and in the case $t = 1$, f is holomorphic with respect to some complex structure of the tori. Since

$$\Phi = \left(z, itz, \frac{-2 + t^2}{4}(1 - z^2) + \frac{t^2}{4}(1 + z^2), i\frac{t^2}{4}(1 - z^2) + i\frac{t^2 - 2}{4}(1 + z^2) \right)^T \frac{dz}{w},$$

we can use Marty's periodic calculations (p. 185 in [**9**]). We consider the closed curves $\gamma_1 \cup \gamma_2$, $\gamma_3 \cup \gamma_4$:

$$\gamma_1(t) = (z(t), w(t)) = (-ti, w(t)) \quad \text{where } t \in [-\infty, 0] \text{ and } w(t) > 0,$$

$$\gamma_2(t) = (t, w(t)) \quad \text{where } t \in [0, \infty] \text{ and } w(t) > 0,$$

$$\gamma_3(t) = (-ti, w(t)) \quad \text{where } t \in [-1, 1] \text{ and } w(t) > 0,$$

$$\gamma_4(t) = (e^{ti}, w(t)) \quad \text{where } t \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \text{ and } w(0) > 0,$$

and we can verify

$$\Re \int_{\gamma_1 \cup \gamma_2} \Phi = \left(\frac{A}{2}, 0, \frac{t^2}{4}B, -\frac{t^2}{4}B \right)^T$$

$$\Re \int_{\gamma_3 \cup \gamma_4} \Phi = \left(0, -\frac{t}{2}B, 0, \frac{t^2 - 2}{4}A - \frac{t^2}{4}B \right)^T,$$

where

$$A := \int_0^\infty \frac{dt}{\sqrt{1 - t^2 + t^4}} \quad \text{and} \quad B := \int_0^\infty \frac{dt}{\sqrt{1 + t^2 + t^4}}.$$

Note that $A > B$.

A homology basis can be obtained by applying the rotation $(z, w) \mapsto (iz, w)$ to $\gamma_1 \cup \gamma_2$, $\gamma_3 \cup \gamma_4$. We obtain the period matrix Ω defined by

$$\Omega = \begin{pmatrix} \frac{A}{2} & 0 & -\frac{A}{2} & 0 & \frac{A}{2} & 0 \\ 0 & -\frac{t}{2}B & 0 & -\frac{t}{2}B & 0 & -\frac{t}{2}B \\ \frac{t^2}{4}B & 0 & \frac{t^2}{4}B & \frac{t^2 - 2}{4}A + \frac{t^2}{4}B & -\frac{t^2}{4}B & 0 \\ -\frac{t^2}{4}B & \frac{t^2 - 2}{4}A - \frac{t^2}{4}B & \frac{t^2}{4}B & 0 & \frac{t^2}{4}B & -\frac{t^2 - 2}{4}A + \frac{t^2}{4}B \end{pmatrix}.$$

Now we choose

$$t^2 = \frac{A/2}{(m/2n)B + (A-B)/4} \in (0,1) \quad \left(i.e.\ \frac{m}{n}\frac{t^2}{2}B + \frac{t^2-2}{4}A - \frac{t^2}{4}B = 0\right), \tag{5}$$

for $m/n \in \boldsymbol{N}$ sufficiently large. We may assume m and n are prime (*i.e.* $(m,n)=1$). Now we consider two cases, n is even or odd.

First, let n be even. Since $(m,n)=1$, there exist integers x and y such that

$$nx + my = 1. \tag{6}$$

Note that m and y are odd. Since $(m,n)=1$, we obtain $(m,n/2)=1$. Thus there exist integers x' and y' such that

$$\frac{n}{2}x' + my' = 1. \tag{7}$$

Moreover as $(m,2)=1$, there exist integers x'' and y'' such that

$$2x'' + my'' = 1. \tag{8}$$

Now we consider three matrices:

$$\Lambda_{even} := \begin{pmatrix} A/2 & 0 & 0 & 0 \\ 0 & tB & (y/2)tB & 0 \\ 0 & 0 & (1/n)(t^2/2)B & (1/n)(t^2/2)B \\ 0 & 0 & 0 & -(1/n)(t^2/2)B \end{pmatrix},$$

$$G_1 := \begin{pmatrix} my'' - (n/2)my''(x+y) + x'' & 0 & x+y & X \\ -(n/2)y''(1-(n/2)x) & -1 & 0 & (1-(n/2)x)y' + (n/2)x'y''(1-(n/2)x) \\ -(n/2)my''(x+y) & 0 & x+y & my'(x+y) + (n/2)mx'y''(x+y) \\ (n/2)myy'' & 0 & -y & -myy' - (n/2)mx'yy'' \\ x'' & 0 & 0 & -x'x'' \\ (n^2/4)xy'' & -1 & 0 & -(n/2)xy' - (n^2/4)xx'y'' \end{pmatrix},$$

where $X = x' + my'(x+y) - mx'y'' + (n/2)mx'y''(x+y) - x'x''$,

$$G_2 := \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -(n/2)x & -(n/2)y & -(n/2)(x+y) & 0 & (n/2)x - 1 \\ 0 & -m & n & n-m & 0 & m \\ n/2 & m & -n/2 & 0 & -n/2 & -m \end{pmatrix}.$$

Then, we directly obtain

$$\Lambda_{even} = \Omega G_1, \quad \Omega = \Lambda_{even} G_2, \tag{9}$$

by (5), (6), (7) and (8).

Next, consider the case where n is odd. Since $(m,n)=1$, there exist integers x and y such that

$$nx + my = 1. \tag{10}$$

Since $(n,2)=1$, there exist integers x' and y' such that

$$2x' + ny' = 1. \tag{11}$$

Moreover as $(2m, n) = 1$, there exist integers x'' and y''

$$nx'' + 2my'' = 1. \tag{12}$$

Now we consider three matrices:

$$\Lambda_{odd} := \begin{pmatrix} A & 0 & 0 & (x''/2)A \\ 0 & (t/2)B & 0 & 0 \\ 0 & 0 & (1/n)(t^2/2)B & x''(t^2/4)B \\ 0 & 0 & 0 & -(1/n)(t^2/4)B \end{pmatrix},$$

$$G_1' := \begin{pmatrix} 1 & -(m-n)y' & x+y+(m-n)yy' & x''-(m-n)y'y'' \\ 0 & -x' & x'y & y''-x'y'' \\ 0 & -(m-n)y' & x+y+(m-n)yy' & -(m-n)y'y'' \\ 0 & -ny' & -y+nyy' & -ny'y'' \\ 1 & 0 & 0 & 0 \\ 0 & -x' & x'y & -x'y'' \end{pmatrix},$$

$$G_2' := \begin{pmatrix} my'' & -mx'' & -my'' & 0 & 1-my'' & mx'' \\ 0 & -1 & 0 & -1 & 0 & -1 \\ nmy'' & -nmx'' & n(1-my'') & n-m & -nmy'' & nmx'' \\ n & 2m & -n & 0 & -n & -2m \end{pmatrix}.$$

Then, we directly obtain

$$\Lambda_{odd} = \Omega G_1', \quad \Omega = \Lambda_{odd} G_2', \tag{13}$$

by (5), (10), (11) and (12).

Thus, by Proposition 2.1, (9) and (13), we can construct a family of full compact oriented minimal surfaces in some 4-dimensional flat tori

$$f : M \to \boldsymbol{R}^4/\Lambda_{even}$$

$$f : M \to \boldsymbol{R}^4/\Lambda_{odd}$$

via (4).

REMARK 3.1. By Remark 2.1 and (5), these minimal surfaces are not holomorphic with respect to any complex structure of the tori. Moreover, non-holomorphicity follows from Theorem 2.3 in [**1**].

REMARK 3.2. By (5), the parameter $t \in (0,1)$ of minimal surfaces with degenerate Gauss map can be taken densely in $(0,1)$ because $\boldsymbol{Q}$ is dense in $\boldsymbol{R}$. Thus, we construct a dense family of compact oriented minimal surfaces of genus 3 with degenerate Gauss map in some 4-tori, which are not holomorphic with respect to any complex structure of the tori.

REMARK 3.3. The Schwarz surface have the associated minimal surface f_θ for a dense set of angles $\theta \in S^1$. This fact can be proved by the symmetries of a cube (Theorem 6.2 in [**4**]). The lattice of the minimal surface defined by (4) is not a cube. Thus, it is difficult to see that they have the associated minimal surface.

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