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Some subsemigroups of extensions of C^*-algebras

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Abstract. In this paper we investigate the structure of the subsemigroup generated by the inner automorphisms in $\mathbf{Ext}(\mathscr{Q}, \boldsymbol{K})$. As an application, we give a new point of view to the example of J. Plastiras, which are two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ satisfying $\mathfrak{A} \not\cong \mathfrak{B}$ and $\boldsymbol{M}_2 \otimes \mathfrak{A} \cong \boldsymbol{M}_2 \otimes \mathfrak{B}$.

0. Introduction.

J. Plastiras exhibited an example which is a pair of C^*-algebras such that $\mathfrak{A} \not\cong \mathfrak{B}$ and $\boldsymbol{M}_2 \otimes \mathfrak{A} \cong \boldsymbol{M}_2 \otimes \mathfrak{B}$ ([**5**], [**6**]). They are constructed as extensions of $\mathscr{Q}$ by $\boldsymbol{K}$, where $\boldsymbol{K}$ is the C^*-algebra of compact operators and $\mathscr{Q}$ is the quotient C^*-algebra of all the bounded linear operators $\boldsymbol{B}$ by $\boldsymbol{K}$. So they are not nuclear. For a class of nuclear C^*-algebras, we can construct such a pair of C^*-algebras using the classification result for them by K-theory ([**2**], [**3**]). In [**7**], T. Sakamoto constructs such a pair of non-nuclear C^*-algebras.

In this paper, we consider the family of special extensions of $\mathscr{Q}$ by $\boldsymbol{K}$ which contains Plastiras' examples. Our aim is to investigate their semigroup structure and to show that the datum for this semigroup is the useful invariant for them as like as K-theoretic datum for some nuclear C^*-algebras.

1. Preliminaries and Main result.

Here we give fundamental facts of extension theory along [**1**] and [**8**]. Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space. We denote by $\boldsymbol{B}$ (resp. $\boldsymbol{K}$) a C^*-algebra $\boldsymbol{B}(\mathscr{H})$ (resp. $\boldsymbol{K}(\mathscr{H})$) of bounded linear operators (resp. compact operators) on $\mathscr{H}$. We also denote by $\mathscr{Q}$ a C^*-algebra $\boldsymbol{B}(\mathscr{H})/\boldsymbol{K}(\mathscr{H})$. Let A, B and C be C^*-algebras and α (resp. β) a *-homomorphism from A to B (resp. from B to C). We call a short exact sequence E as below an extension of C by A:

$$E : 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

that is, α is injective, β is surjective and $\operatorname{Im}\alpha = \operatorname{Ker}\beta$. Then there exists a *-homomorphism σ from B to the multiplier C^*-algebra $M(A)$ of A with $\sigma \circ \alpha = \iota$, i.e.,

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$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \| & & \downarrow{\scriptstyle\sigma} \\ A & \xrightarrow[\iota]{} & M(A) \end{array},$$

where ι is the canonical inclusion map from A to $M(A)$. The Busby invariant for this extension E is defined as the *-homomorphism τ_E from C to $M(A)/A$ given by

$$\tau_E(c) = \pi \circ \sigma(b),$$

where b is a lift of c through β and π is the quotient map from $M(A)$ to $M(A)/A$. It is known that τ_E is characterized by the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau_E} & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota} & M(A) & \xrightarrow{\pi} & M(A)/A & \longrightarrow & 0 \end{array}.$$

We remark that, if we define the pull-back C^*-algebra PB and the map ψ as follows:

$$\mathrm{PB} = \{(x, c) \in M(A) \oplus C \mid \pi(x) = \tau_E(c)\}$$

$$\psi : B \ni b \mapsto (\sigma(b), \beta(b)) \in \mathrm{PB},$$

then B is isomorphic to PB for the isomorphism ψ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\psi} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & \mathrm{PB} & \longrightarrow & C & \longrightarrow & 0 \end{array}.$$

Let

$$E_1 : 0 \to A \to B_1 \to C \to 0$$

$$E_2 : 0 \to A \to B_2 \to C \to 0$$

be extensions and τ_i the Busby invariant for E_i $(i = 1, 2)$. We call E_1 and E_2 strongly equivalent when there is a unitary $u \in M(A)$ such that $\tau_2(c) = \pi(u)\tau_1(c)\pi(u)^*$ for all $c \in C$, equivalently there are a unitary $v \in M(A)$ and a *-isomorphism γ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Ad(v)} & & \downarrow{\scriptstyle\gamma} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B_2 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is commutative. Then we denote $E_1 \approx E_2$ or $\tau_1 \approx \tau_2$. Let $\mathrm{Ext}(C, A)$ be a set of extensions of C by A. We denote by $\mathbf{Ext}(C, A)$ the set $\mathrm{Ext}(C, A)/\approx$ of strongly equivalent classes of $\mathrm{Ext}(C, A)$. When A satisfies $A \cong M_2(A)$, $\mathbf{Ext}(C, A)$ becomes an abelian semigroup. The addition of $[E_1]$ and $[E_2] \in \mathbf{Ext}(C, A)$ is defined by the equivalent class of the extension which is corresponding to the Busby invariant

$$\tau_1 \oplus \tau_2 : C \to M(A)/A \oplus M(A)/A \hookrightarrow M(\boldsymbol{M}_2(A))/\boldsymbol{M}_2(A) \cong M(A)/A.$$

In this paper, we consider the extension semigroup $\mathbf{Ext}(\mathscr{Q}, \boldsymbol{K})$. We denote by π the canonical quotient map from $\boldsymbol{B}$ onto $\mathscr{Q}$. Let α be an inner *-automorphism of $\mathscr{Q}$. Then we can see that α is the Busby invariant for an extension $E \in \mathrm{Ext}(\mathscr{Q}, \boldsymbol{K})$. We denote by $\mathscr{G}$ a subsemigroup of $\mathbf{Ext}(\mathscr{Q}, \boldsymbol{K})$ generated by extensions corresponding to all the inner *-automorphisms of $\mathscr{Q}$.

The inner *-automorphism α has the form $\alpha(\cdot) = u^* \cdot u$ for some unitary $u \in \mathscr{Q}$. Let $V \in \boldsymbol{B}$ be a lift of u, that is, $\pi(V) = u$. Then V is a Fredholm operator, and we put $n = \mathrm{Index}\, V \in \boldsymbol{Z}$. Let $S (\in \boldsymbol{B})$ be a unilateral shift. We remark $\mathrm{Index}\, S = -1$. We define a *-automorphism $\tau(n)$ of $\mathscr{Q}$ by

$$\tau(n)(x) = \pi(S)^n x \pi(S^*)^n, \quad x \in \mathscr{Q}.$$

Then there exists a unitary $U \in \boldsymbol{B}$ such that $VS^n = U|VS^n|$, i.e., $u\pi(S)^n = \pi(U)$. So we have that α is strongly equivalent to $\tau(n)$, that is, $[\alpha] = [\tau(n)]$.

Let G be a restricted direct product of non-negative integers $\boldsymbol{Z}_{\geq 0}$ except $\mathbf{0}$, i.e.,

$$\begin{aligned} G &= \coprod_{\boldsymbol{Z}} \boldsymbol{Z}_{\geq 0} \backslash \{\mathbf{0}\} \\ &= \{g = (m(k))_{k \in \boldsymbol{Z}} \mid m(k) \in \boldsymbol{Z}_{\geq 0}, 0 < \#\{k \in \boldsymbol{Z} \mid m(k) \neq 0\} < \infty\}, \end{aligned}$$

where # denotes the cardinal number of set. By the above fact, we can define the surjective semigroup homomorphism τ from G to $\mathscr{G}$ as follows:

$$\tau(g) = \left[\bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{m(k)} \tau(k)\right] = \sum_{k \in \boldsymbol{Z}} m(k)[\tau(k)],$$

where $g = (m(k))_{k \in \boldsymbol{Z}} \in G$.

We define a map φ from G to $\boldsymbol{N}$ and a map ψ from G to $\boldsymbol{Z}$ as follows: for $g = (m(k))_{k \in \boldsymbol{Z}} \in G$,

$$\begin{aligned} \varphi(g) &= \sum_{k \in \boldsymbol{Z}} m(k), \\ \psi(g) &= \sum_{k \in \boldsymbol{Z}} k m(k). \end{aligned}$$

We introduce two notations as follows: for $l \in \boldsymbol{Z}$ and $g = (m(k))_{k \in \boldsymbol{Z}} \in G$,

$$\begin{aligned} l \cdot g &= ((l \cdot g)(k))_{k \in \boldsymbol{Z}} \in G \\ l + g &= (m(l + k))_{k \in \boldsymbol{Z}} \in G, \end{aligned}$$

where

$$(l \cdot g)(k) = \begin{cases} m(s) & k = ls \\ 0 & \text{otherwise} \end{cases}.$$

Then we can easily get

$$\varphi(l \cdot g) = \varphi(g), \quad \psi(l \cdot g) = l\psi(g),$$
$$\varphi(l + g) = \varphi(g) \quad \text{and} \quad \psi(l + g) = \psi(g) + l\varphi(g).$$

For $g = (m(k))_{k \in \boldsymbol{Z}} \in G$, we define a C^*-subalgebra $\mathscr{A}(g)$ of $\boldsymbol{B}(\bigoplus_{\varphi(g)} \mathscr{H}) \cong \boldsymbol{M}_{\varphi(g)}(\boldsymbol{B})$ as follows:

$$\mathscr{A}(g) = \left\{ \bigoplus_{k \in \boldsymbol{Z}} (\overbrace{S^k T S^{*k} \oplus \cdots \oplus S^k T S^{*k}}^{m(k)}) \mid T \in \boldsymbol{B} \right\} + \boldsymbol{K}\left(\bigoplus_{\varphi(g)} \mathscr{H}\right),$$

where S^k (resp. $(S^*)^k$) means $(S^*)^{-k}$ (resp. S^{-k}) for a negative integer k. Let $\iota(g)$ be a injective *-homomorphism from $\boldsymbol{K}$ to $\mathscr{A}(g)$ which is obtained by a composition of a natural isomorphism of $\boldsymbol{K}$ to $\boldsymbol{K}(\bigoplus_{\varphi(g)} \mathscr{H})$ and the canonical inclusion map of $\boldsymbol{K}(\bigoplus_{\varphi(g)} \mathscr{H})$ into $\mathscr{A}(g)$. We define a surjective *-homomorphism $\pi(g)$ from $\mathscr{A}(g)$ to $\mathscr{Q}$ as follows:

$$\pi(g)\left(\bigoplus_{k \in \boldsymbol{Z}} (\overbrace{S^k T S^{*k} \oplus \cdots \oplus S^k T S^{*k}}^{m(k)}) + K \right) = \pi(T),$$

where $K \in \boldsymbol{K}(\bigoplus_{\varphi(g)} \mathscr{H})$. Then we have the following extension:

$$E(g) : 0 \longrightarrow \boldsymbol{K} \xrightarrow{\iota(g)} \mathscr{A}(g) \xrightarrow{\pi(g)} \mathscr{Q} \longrightarrow 0,$$

and its Busby invariant coincides with

$$\bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{m(k)} \tau(k).$$

Then we have the following statement and this is our main result:

Theorem 1.1. *For $g, h \in G$ and $n \in \boldsymbol{N}$, we have the following*:

(1) $\tau(g) = \tau(h) \Leftrightarrow \varphi(g) = \varphi(h)$ *and* $\psi(g) = \psi(h)$, *that is,*

$$\mathscr{G} \ni \tau(g) \mapsto (\varphi(g), \psi(g)) \in \boldsymbol{N} \times \boldsymbol{Z}$$

gives a semigroup isomorphism from $\mathscr{G}$ onto $\boldsymbol{N} \times \boldsymbol{Z}$.

(2) $\mathscr{A}(g) \cong \mathscr{A}(h) \Leftrightarrow \varphi(g) = \varphi(h)$ *and* $\psi(g) \equiv \psi(h) \bmod \varphi(g)$.

(3) $\mathscr{A}(g) \otimes \boldsymbol{M}_n \cong \mathscr{A}(n \cdot g)$.

We give the proof of theorem in the next section.

Corollary 1.2. *For any $n \in \boldsymbol{N}$ and $n \geq 2$, there exist $g, h \in G$ such that $\mathscr{A}(g) \otimes \boldsymbol{M}_k$ is not isomorphic to $\mathscr{A}(h) \otimes \boldsymbol{M}_k$ for any $1 \leq k \leq n-1$ and $\mathscr{A}(g) \otimes \boldsymbol{M}_n$ is isomorphic to $\mathscr{A}(h) \otimes \boldsymbol{M}_n$.*

Proof. We choose g and h such that

$$\varphi(g) = \varphi(h) = n, \quad \psi(g) = 0 \quad \text{and} \quad \psi(h) = 1.$$

Then we have $\psi(k \cdot g) = 0 < \psi(k \cdot h) = k < n$ for any $k = 1, 2, \dots, n-1$ and $\varphi(n \cdot g) = \varphi(n \cdot h) = n$, $\psi(n \cdot g) \equiv \psi(n \cdot h) \equiv 0 \mod n$. This implies that $\mathscr{A}(g)$ and $\mathscr{A}(h)$ satisfy the required property. $\square$

For $g = (m(k))_{k \in \boldsymbol{Z}}$, $h = (n(k))_{k \in \boldsymbol{Z}} \in G$ with

$$m(k) = \begin{cases} 2 & k = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad n(k) = \begin{cases} 1 & k = 0, 1 \\ 0 & \text{otherwise} \end{cases},$$

we have $\varphi(g) = \varphi(h) = 2$, $\psi(g) = 0$, $\psi(h) = 1$. It follows that $\mathscr{A}(g) \otimes \boldsymbol{M}_2 \cong \mathscr{A}(h) \otimes \boldsymbol{M}_2$, but $\mathscr{A}(g)$ is not isomorphic to $\mathscr{A}(h)$. This example is the same one given by J. Plastiras.

2. Proof of Theorem.

Lemma 2.1. *The K_0-group $K_0(\mathscr{A}(g))$ for $\mathscr{A}(g)$ is isomorphic to $\boldsymbol{Z}/\varphi(g)\boldsymbol{Z}$.*

Proof. Let $g = (m(k))_{k \in \boldsymbol{Z}} \in G$. From the short exact sequence of C^*-algebras

$$0 \longrightarrow \boldsymbol{K} \xrightarrow{\iota(g)} \mathscr{A}(g) \xrightarrow{\pi(g)} \mathscr{Q} \longrightarrow 0,$$

we can get the exact sequence of K-groups of C^*-algebras as follows:

$$\begin{array}{ccccc} K_0(\boldsymbol{K}) & \xrightarrow{\iota(g)_*} & K_0(\mathscr{A}(g)) & \xrightarrow{\pi(g)_*} & K_0(\mathscr{Q}) \\ {\scriptstyle \delta_1}\uparrow & & & & \downarrow \\ K_1(\mathscr{Q}) & \xleftarrow[\pi(g)_*]{} & K_1(\mathscr{A}(g)) & \xleftarrow[\iota(g)_*]{} & K_1(\boldsymbol{K}) \end{array}.$$

Since $K_0(\mathscr{Q}) = \{0\}$, we have

$$K_0(\mathscr{A}(g)) \cong K_0(\boldsymbol{K})/\delta_1(K_1(\mathscr{Q})).$$

It is known that $K_0(\boldsymbol{K}) \cong K_1(\mathscr{Q}) \cong \boldsymbol{Z}$ and the class of P (resp. $\pi(S)$) is a generator of $K_0(\boldsymbol{K})$ (resp. $K_1(\mathscr{Q})$), where $P \in \boldsymbol{B}$ is a projection of rank one and $S \in \boldsymbol{B}$ is a unilateral shift.

We put $P_n = 1 - S^n S^{*n}$ $(n = 1, 2, \dots)$ and define a unitary $W(k) \in \boldsymbol{M}_2(\boldsymbol{B})$ as follows: for $k \geq 0$,

$$\begin{aligned} W(k) &= \begin{pmatrix} S(1 - P_k) & P_{k+1} \\ -P_k & (1 - P_k)S^* \end{pmatrix} \\ &= \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}^k \begin{pmatrix} S & P_1 \\ 0 & S^* \end{pmatrix} \begin{pmatrix} S^* & 0 \\ 0 & S^* \end{pmatrix}^k + \begin{pmatrix} 0 & P_k \\ -P_k & 0 \end{pmatrix}, \end{aligned}$$

and for $k < 0$,

$$W(k) = \begin{pmatrix} S & P_1 \\ 0 & S^* \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}^k \begin{pmatrix} S & P_1 \\ 0 & S^* \end{pmatrix} \begin{pmatrix} S^* & 0 \\ 0 & S^* \end{pmatrix}^k + \begin{pmatrix} 0 & P_1 \\ 0 & 0 \end{pmatrix}.$$

Then we have

$$W = \bigoplus_{k \in \boldsymbol{Z}} \overbrace{W(k) \oplus \cdots \oplus W(k)}^{m(k)}$$

is unitary in $\boldsymbol{M}_2(\mathscr{A}(g))$ and

$$\pi(g) \otimes \mathrm{id}_2(W) = \begin{pmatrix} \pi(S) & 0 \\ 0 & \pi(S^*) \end{pmatrix}.$$

By the definition of δ_1, we have

$$\delta_1([\pi(S)]) = [W^*(1_{\mathscr{A}(g)} \oplus 0_{\mathscr{A}(g)})W] - [1_{\mathscr{A}(g)} \oplus 0_{\mathscr{A}(g)}].$$

By the calculation

$$\begin{aligned} &[\begin{pmatrix} S(1-P_k) & P_{k+1} \\ -P_k & (1-P_k)S^* \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S(1-P_k) & P_{k+1} \\ -P_k & (1-P_k)S^* \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] \\ &= [\begin{pmatrix} 1-P_k & 0 \\ 0 & P_{k+1} \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] = [P] \end{aligned}$$

and

$$\begin{aligned} &[\begin{pmatrix} S & P_1 \\ 0 & S^* \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S & P_1 \\ 0 & S^* \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] \\ &= [\begin{pmatrix} 1 & 0 \\ 0 & P_1 \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] = [P], \end{aligned}$$

it follows that

$$\delta_1([\pi(S)]) = \varphi(g)[P].$$

This means that

$$K_0(\mathscr{A}(g)) \cong \boldsymbol{Z}/\varphi(g)\boldsymbol{Z}. \qquad \square$$

For $g = (m(k))_{k \in \boldsymbol{Z}}$, we can choose integers $k_1 < k_2 < \cdots < k_l$ such that

$$\{k \in \boldsymbol{Z} \mid m(k) \neq 0\} = \{k_1, k_2, \ldots, k_l\}.$$

We remark that, if we put

$$\begin{aligned} &m_1 = \cdots = m_{m(k_1)} = k_1, \quad m_{m(k_1)+1} = \cdots = m_{m(k_1)+m(k_2)} = k_2, \ldots, \\ &m_{m(k_1)+\cdots+m(k_{l-1})+1} = \cdots = m_{\varphi(g)} = k_l, \end{aligned}$$

then we have $\psi(g) = \sum_{j=1}^{\varphi(g)} m_j$ and

$$\begin{aligned} \mathscr{A}(g) &= \left\{ \bigoplus_{k \in \boldsymbol{Z}} (\overbrace{S^k T S^{*k} \oplus \cdots \oplus S^k T S^{*k}}^{m(k)}) \mid T \in \boldsymbol{B} \right\} + \boldsymbol{K}\left(\bigoplus_{\varphi(g)} \mathscr{H} \right) \\ &= \left\{ \bigoplus_{j=1}^{\varphi(g)} S^{m_j} T (S^*)^{m_j} \mid T \in \boldsymbol{B} \right\} + \boldsymbol{K}\left(\bigoplus_{\varphi(g)} \mathscr{H} \right). \end{aligned}$$

Lemma 2.2. *For any $n \in \boldsymbol{Z}$ and $g \in G$, we have*

$$\mathscr{A}(g) \cong \mathscr{A}(n+g).$$

Proof. It is sufficient to show that $\mathscr{A}(g) \cong \mathscr{A}(1+g)$. Using the above notation and $\varphi(g) = \varphi(1+g)$ and $\psi(1+g) = \psi(g) + \varphi(g)$, we have

$$\begin{aligned}\mathscr{A}(1+g) &= \left\{ \bigoplus_{j=1}^{\varphi(g)} S^{m_j+1} T (S^*)^{m_j+1} \mid T \in \boldsymbol{B} \right\} + \boldsymbol{K}\left(\bigoplus_{\varphi(g)} \mathscr{H} \right) \\ &= \left\{ \bigoplus_{j=1}^{\varphi(g)} S^{m_j} STS^* (S^*)^{m_j} \mid T \in \boldsymbol{B} \right\} + \boldsymbol{K}\left(\bigoplus_{\varphi(g)} \mathscr{H} \right).\end{aligned}$$

Clearly $\mathscr{A}(1+g) \subset \mathscr{A}(g)$. Remarking the fact $\boldsymbol{B} \subset S\boldsymbol{B}S^* + \boldsymbol{K}$, we have $\mathscr{A}(1+g) = \mathscr{A}(g)$. □

Lemma 2.3. *The class of the unit of $\mathscr{A}(g)$ is equal to $\psi(g)[P]$ in $K_0(\mathscr{A}(g))$, where P is a minimal projection of $\mathscr{A}(g)$.*

Proof. By the above lemma, we can see

$$\begin{aligned}\mathscr{A}(g) &= \left\{ \bigoplus_{j=1}^{\varphi(g)} S^{m_j} T (S^*)^{m_j} \mid T \in \boldsymbol{B} \right\} + \boldsymbol{K}\left(\bigoplus_{\varphi(g)} \mathscr{H} \right) \\ &= \left\{ \bigoplus_{j=1}^{\varphi(g)} S^{n+m_j} T (S^*)^{n+m_j} \mid T \in \boldsymbol{B} \right\} + \boldsymbol{K}\left(\bigoplus_{\varphi(g)} \mathscr{H} \right),\end{aligned}$$

for $n \in \boldsymbol{N}$ with $n + m_1 > 0$. Since $1 \in \boldsymbol{B}$ is equivalent to some orthogonal projections $Q_1, Q_2, \ldots, Q_{\varphi(g)}$ such that $1 = Q_1 + Q_2 + \cdots + Q_{\varphi(g)}$, $S^k S^{*k}$ is equivalent to $S^k Q_i S^{*k}$ for positive integer k. So we have

$$\begin{aligned}[1_{\mathscr{A}(g)}] &\in K_0(\mathscr{A}(g)) \\ &= \left[\bigoplus_{j=1}^{\varphi(g)} S^{n+m_j} (S^*)^{n+m_j} \right] + \sum_{j=1}^{\varphi(g)} (n+m_j)[P] \\ &= \varphi(g) \left[\bigoplus_{j=1}^{\varphi(g)} S^{n+m_j} (S^*)^{n+m_j} \right] + (n\varphi(g) + \psi(g))[P].\end{aligned}$$

This implies $[1_{\mathscr{A}(g)}] = \psi(g)[P]$ in $K_0(\mathscr{A}(g))$. □

Before to prove theorem 1.1, we note that

$$(\mathscr{Q} \otimes 1_n)' \cap \mathscr{Q} \otimes \boldsymbol{M}_n = 1_{\mathscr{Q}} \otimes \boldsymbol{M}_n.$$

Indeed, it is known that a unital simple C^*-algebra has a trivial center and the Calkin algebra $\mathscr{Q}$ is simple. This implies the above fact.

Proof of Theorem 1.1. (1) First we assume that $\tau_g = \tau_h$. Then the fact $\mathscr{A}(g) \cong \mathscr{A}(h)$ implies $\varphi(g) = \varphi(h)$ by lemma 2.1. We use the notation $\tau(g) = [\bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{m(k)} \tau_k]$,

$\tau(h) = [\bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{n(k)} \tau_k]$ and $S_g = \bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{m(k)} \pi(S)^k$, $S_h = \bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{n(k)} \pi(S)^k \in \mathscr{Q} \otimes \boldsymbol{M}_{\varphi(g)}$. Then $\tau(g) = \tau(h)$ means that there exist a unitary U in $\boldsymbol{B} \otimes \boldsymbol{M}_{\varphi(g)}$ such that

$$S_g(x \otimes 1_{\varphi(g)})S_g^* = (\pi \otimes \mathrm{id}_{\varphi(g)}(U))^* S_h(x \otimes 1_{\varphi(g)})S_h^*(\pi \otimes \mathrm{id}_{\varphi(g)}(U))$$

for all $x \in \mathscr{Q}$. Since $S_h^*(\pi \otimes \mathrm{id}_{\varphi(g)}(U))S_g \in (\mathscr{Q} \otimes 1_{\varphi(g)})'$, we have $S_h^*(\pi \otimes \mathrm{id}_{\varphi(g)}(U))S_g \in 1_{\mathscr{Q}} \otimes \boldsymbol{M}_{\varphi(g)}$. So $S_h^*(\pi \otimes \mathrm{id}_{\varphi(g)}(U))S_g$ have a unitary lift in $1_{\boldsymbol{B}} \otimes \boldsymbol{M}_{\varphi(g)}$. This means $0 = \mathrm{Index}((\bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{n(k)} S^k)^* U (\bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{m(k)} S^k) = -\psi(h) + \psi(g)$, that is, $\psi(g) = \psi(h)$. Conversely we assume that $\varphi(g) = \varphi(h)$ and $\psi(g) = \psi(h)$. Then we have $\mathrm{Index}((\bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{n(k)} S^k)(\bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{m(k)} S^k)^* = 0$. So there exists a unitary U in $\boldsymbol{B} \otimes \boldsymbol{M}_{\varphi(g)}$ such that $S_h S_g^* = \pi \otimes \mathrm{id}_{\varphi(g)}(U)$. This implies that $\tau(g) = \tau(h)$.

(2) First we assume that $\mathscr{A}_g \cong \mathscr{A}_h$. By lemma 2.1 and lemma 2.3, it is immediately found that $\varphi(g) = \varphi(h)$ and $\psi(g) \equiv \psi(h) \mod \varphi(g)$. Conversely we assume that $\varphi(g) = \varphi(h)$ and $\psi(g) = \psi(h) + n\varphi(g)$ for $n \in \boldsymbol{Z}$. Then we have $\tau(g) = \tau(n+g)$. This implies $\mathscr{A}(g) \cong \mathscr{A}(n+g) \cong \mathscr{A}(h)$ by lemma 2.2.

(3) Suppose that $\mathscr{A}(g)$ is the following form:

$$\mathscr{A}(g) = \left\{ \bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{m(k)} S^k T S^{*k} \mid T \in \boldsymbol{B} \right\} + \boldsymbol{K}\left(\bigoplus_{\varphi(g)} \mathscr{H} \right).$$

Therefore we can regard $\mathscr{A}(g) \otimes \boldsymbol{M}_n$ as the following:

$$\mathscr{A}(g) \otimes \boldsymbol{M}_n = \left\{ \bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{m(k)} (S^k \otimes \mathbf{1}_n) T' (S^{*k} \otimes \mathbf{1}_n) \mid T' \in \boldsymbol{B}\left(\bigoplus_n \mathscr{H} \right) \right\} + \boldsymbol{K}\left(\bigoplus_{n\varphi(g)} \mathscr{H} \right).$$

This means that $\mathscr{A}(g) \otimes \boldsymbol{M}_n \cong \mathscr{A}(n \cdot g)$. $\square$

For $g \in G$, we define the C^*-algebra

$$\mathscr{A}(g) = \left\{ \bigoplus_{k \in \boldsymbol{Z}} \bigoplus_{m(k)} S^k T S^{*k} \mid T \in \boldsymbol{B}(\mathscr{H}) \right\} + \boldsymbol{K}\left(\bigoplus_{\varphi(g)} \mathscr{H} \right).$$

Then we can see that the essential commutant $\mathrm{EC}(\mathscr{A}(g))$ of $\mathscr{A}(g)$ becomes an AF-algebra and $\pi \otimes \mathrm{id}_{\varphi(g)}(\mathrm{EC}(\mathscr{A}(g)))$ is isomorphic to $\boldsymbol{M}_{\varphi(g)}(\boldsymbol{C})$. Since $\mathscr{A}(g)$ and $\mathscr{A}(h)$ contain the algebra of compact operators, the isomorphism from $\mathscr{A}(g)$ to $\mathscr{A}(h)$ deduces the isomorphism from $\mathrm{EC}(\mathscr{A}(g))$ to $\mathrm{EC}(\mathscr{A}(h))$. It is known that isomorphism classes of AF-algebras are classified up by the K-theoretic datum. In this case, we can see

$$\begin{aligned}&(K_0(\mathrm{EC}(\mathscr{A}(g))), K_0(\mathrm{EC}(\mathscr{A}(g)))_+, [1]_{K_0})\\ &\quad = (\boldsymbol{Z} \oplus \boldsymbol{Z}, (\{0\} \oplus \boldsymbol{Z}_{\geq 0}) \cup (\boldsymbol{Z}_{>0} \oplus \boldsymbol{Z}), (\varphi(g), \psi(g))).\end{aligned}$$

We remark that, for any integer $k \in \boldsymbol{Z}$, the following groups are order isomorphic (and preserving the order unit):

$$(\boldsymbol{Z} \oplus \boldsymbol{Z}, (\{0\} \oplus \boldsymbol{Z}_{\geq 0}) \cup (\boldsymbol{Z}_{>0} \oplus \boldsymbol{Z}), (\varphi(g), \psi(g))),$$

$$(\boldsymbol{Z} \oplus \boldsymbol{Z}, (\{0\} \oplus \boldsymbol{Z}_{\geq 0}) \cup (\boldsymbol{Z}_{>0} \oplus \boldsymbol{Z}), (\varphi(g), \psi(g) + k\varphi(g))).$$

This means that the K-theoretic datum is a complete invariant for the family $\{\mathscr{A}(g) \mid g \in G\}$ of non-nuclear C^*-algebras.

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