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The behaviour of dimension functions on unions of closed subsets

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Abstract. We discuss the behaviour of (transfinite) dimension functions, in particular, Cmp and trInd, on finite and countable unions of closed subsets in separable metrizable spaces.

1. Introduction.

It is well known that there exist (transfinite) dimension functions d such that $d(X_1 \cup X_2) > \max\{dX_1, dX_2\}$ even if the subspaces X_1 and X_2 are closed in the union $X_1 \cup X_2$.

Let $\mathscr{K}$ be a class of topological spaces, β, α be ordinals such that $\beta < \alpha$, and X be a space from $\mathscr{K}$ with $dX = \alpha$ which is the union of finitely many closed subsets with $d \leq \beta$. Define $m(X, d, \beta, \alpha) = \min\{k : X = \bigcup_{i=1}^{k} X_i$, where X_i is closed in X and $dX_i \leq \beta\}$, $m_{\mathscr{K}}(d, \beta, \alpha) = \min\{m(X, d, \beta, \alpha) : X \in \mathscr{K}$ and $m(X, d, \beta, \alpha)$ exists$\}$ and $M_{\mathscr{K}}(d, \beta, \alpha) = \sup\{m(X, d, \beta, \alpha) : X \in \mathscr{K}$ and $m(X, d, \beta, \alpha)$ exists$\}$.

We will say that $m_{\mathscr{K}}(d, \beta, \alpha)$ and $M_{\mathscr{K}}(d, \beta, \alpha)$ do not exist if there is no space X from $\mathscr{K}$ with $dX = \alpha$ which is the union of finitely many closed subsets with $d \leq \beta$. It is evident that either $m_{\mathscr{K}}(d, \beta, \alpha)$ and $M_{\mathscr{K}}(d, \beta, \alpha)$ satisfy $2 \leq m_{\mathscr{K}}(d, \beta, \alpha) \leq M_{\mathscr{K}}(d, \beta, \alpha) \leq \infty$ or they do not exist.

Two natural problems arise.

Problem 1. Determine the values of $m_{\mathscr{K}}(d, \beta, \alpha)$ and $M_{\mathscr{K}}(d, \beta, \alpha)$ for given $\mathscr{K}$, d, β, α.

Problem 2. Find a (transfinite) dimension function d having for given pair $2 \leq k \leq l \leq \infty$, $m_{\mathscr{K}}(d, \beta, \alpha) = k$ and $M_{\mathscr{K}}(d, \beta, \alpha) = l$.

Let $\mathscr{C}$ be the class of metrizable compact spaces and $\mathscr{P}$ be the class of separable completely metrizable spaces. By trind (trInd) we denote Hurewicz's (Smirnov's) transfinite extension of ind (Ind) and Cmp is the large inductive compactness degree introduced by de Groot. We shall recall their definitions in the next section. Let $\alpha = \lambda(\alpha) + n(\alpha)$ be the natural decomposition of the ordinal $\alpha \geq 0$ into the sum of a limit ordinal $\lambda(\alpha)$ (observe that $\lambda(\text{an integer} \geq 0) = 0$) and a nonnegative integer $n(\alpha)$. Let $\beta < \alpha$ be ordinals, put $p(\beta, \alpha) = (n(\alpha) + 1)/(n(\beta) + 1)$ and $q(\beta, \alpha) =$ the smallest integer $\geq p(\beta, \alpha)$. In section 2 of this article we prove

Theorem 1. (i) *Let $0 \leq \beta < \alpha$ be finite ordinals. Then*

$$m_{\mathscr{P}}(\mathrm{Cmp}, \beta, \alpha) = q(\beta, \alpha) \quad \textit{and} \quad M_{\mathscr{P}}(\mathrm{Cmp}, \beta, \alpha) = \infty.$$

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(ii) *Let* $\beta < \alpha$ *be infinite ordinals. Then we have*

$$m_{\mathscr{C}}(\mathrm{trInd}, \beta, \alpha) = \begin{cases} q(\beta, \alpha), & \text{if } \lambda(\beta) = \lambda(\alpha), \\ \text{does not exist}, & \text{otherwise}. \end{cases}$$

$$M_{\mathscr{C}}(\mathrm{trInd}, \beta, \alpha) = \begin{cases} \infty, & \text{if } \lambda(\beta) = \lambda(\alpha), \\ \text{does not exist}, & \text{otherwise}. \end{cases}$$

Theorem 2. (i) *For every finite* $\alpha \geq 1$ *there exists a space* $X_\alpha \in \mathscr{P}$ *such that*

(a) $\mathrm{Cmp}\, X_\alpha = \alpha$;

(b) $X_\alpha = \bigcup_{i=1}^{\infty} Y_i$, *where each* Y_i *is closed in* X_α *and* $\mathrm{Cmp}\, Y_i \leq 0$;

(c) $X_\alpha \neq \bigcup_{i=1}^{m} Z_i$, *where each* Z_i *is closed in* X_α *and* $\mathrm{Cmp}\, Z_i \leq \alpha - 1$, *and m is any integer* ≥ 1.

(ii) *For every infinite* α *with* $n(\alpha) \geq 1$ *there exists a space* $X_\alpha \in \mathscr{C}$ *such that*

(a) $\mathrm{trInd}\, X_\alpha = \alpha$;

(b) $X_\alpha = \bigcup_{i=1}^{\infty} Y_i$, *where each* Y_i *is closed in* X_α *and finite-dimensional*;

(c) $X_\alpha \neq \bigcup_{i=1}^{m} Z_i$, *where each* Z_i *is closed in* X_α *and* $\mathrm{trInd}\, Z_i \leq \alpha - 1$, *and m is any integer* ≥ 1.

In sections 3 and 4, we introduce and study new dimension functions: the additive compactness degree $\mathrm{Cmp}_\cup$ and the transfinite additive inductive dimension functions $\mathrm{ind}_\cup$ and $\mathrm{Ind}_\cup$. In connection with the previous part we prove

Theorem 3. (i) *Let* $0 \leq \beta < \alpha$ *be finite ordinals. Then*

$$m_{\mathscr{P}}(\mathrm{Cmp}_\cup, \beta, \alpha) = M_{\mathscr{P}}(\mathrm{Cmp}_\cup, \beta, \alpha) = q(\beta, \alpha).$$

(ii) *Let* $\beta < \alpha$ *be infinite ordinals. Then we have* $m_{\mathscr{C}}(\mathrm{ind}_\cup, \beta, \alpha) = m_{\mathscr{C}}(\mathrm{Ind}_\cup, \beta, \alpha) = M_{\mathscr{C}}(\mathrm{ind}_\cup, \beta, \alpha) = M_{\mathscr{C}}(\mathrm{Ind}_\cup, \beta, \alpha) = q(\beta, \alpha)$, *if* $\lambda(\beta) = \lambda(\alpha)$ *and they do not exist otherwise.*

Our terminology follows [**E**] and [**AN**].

2. Evaluations of $m_{\mathscr{K}}(d, \beta, \alpha)$ and $M_{\mathscr{K}}(d, \beta, \alpha)$.

All spaces in this paper are separable and metrizable except those considered in Remark 3. Hence, outside Remark 3, a space means a separable metrizable space. The notation $X \sim Y$ means that the spaces X and Y are homeomorphic. At first we consider the following construction.

Step 1. Let X be a space without isolated points and P a countable dense subset of X. Consider Alexandroff's dublicate $D = X \cup X^1$ of X, where each point of X^1 is clopen in D. Remove from D those points of X^1 which do not correspond to any point from P. Denote the obtained space by $L(X, P)$. Observe that $L(X, P)$ is the disjoint union of X with the countable dense subset P^1 of $L(X, P)$ consisting of points from X^1 corresponding to the points from P. The space $L(X, P)$ is separable and metrizable. It will be compact if X is compact. Put $L_1(X, P) = L(X, P)$. Assume that X is a completely metrizable space (recall that the increment $bX \backslash X$ in any compactification bX of X is an F_σ-set in bX). Observe that $L(bX, P)$ is a compactification of $L(X, P)$ and the increment $L(bX, P) \backslash L(X, P)$ $(\sim bX \backslash X)$ is an F_σ-set in $L(bX, P)$. Hence $L(X, P)$ is also completely metrizable.

STEP 2. Let X be a space with a countable subset R consisting of isolated points. Let Y be a space. Substitute each point of R in X by a copy of Y. The obtained set W has the natural projection $\mathrm{pr} : W \to X$. Define the topology on W as the smallest topology such that the projection pr is continuous and each copy of Y has its original topology as a subspace of this new space. The obtained space is denoted by $L(X, R, Y)$. It is separable and metrizable and it will be compact (completely metrizable) if X and Y are the same. Moreover $L(X, R, Y)$ is the disjoint union of the closed subspace $X \backslash R$ of X (which we will call *basic* for the space $L(X, R, Y)$) and countably many clopen copies of Y.

STEP 3. Let X be a space without isolated points and P be a countable dense subset of X. Define $L_n(X, P) = L(L_1(X, P), P^1, L_{n-1}(X, P))$, $n \geq 2$. Observe that for any open subset O of $L_n(X, P)$ meeting the basic subset X of $L_n(X, P)$ there is a copy of $L_{n-1}(X, P)$ contained in O. Put $L_*(X, P) = \{*\} \cup \bigoplus_{n=1}^{\infty} L_n(X, P)$. (Here by $\{*\} \cup \bigoplus_{i=1}^{\infty} X_i$ we mean the one-point extension of the free union $\bigoplus_{i=1}^{\infty} X_i$ such that a neighborhood base at the point $*$ consists of the sets $\{*\} \cup \bigoplus_{i=k}^{\infty} X_i$, $k = 1, 2, \ldots$). Observe that $L_*(X, P)$ is separable and metrizable, and it contains a copy of $L_q(X, P)$ for each q. $L_*(X, P)$ will be compact (completely metrizable) if X is the same.

All our dimension functions d are assumed to be monotone with respect to closed subsets and $d(\text{a point}) \leq 0$.

LEMMA 1. *Let d be a dimension function and X be a space without isolated points which cannot be written as the union of $k \geq 1$ closed subsets with $d \leq \alpha$, where α is an ordinal. Let also P be a countable dense subset of X. Then*

(a) *for every q we have $L_q(X, P) \neq \bigcup_{i=1}^{qk} X_i$, where each X_i is closed in $L_q(X, P)$ and $dX_i \leq \alpha$;*

(b) *$L_*(X, P) \neq \bigcup_{i=1}^{m} X_i$, where each X_i is closed in $L_*(X, P)$ and $dX_i \leq \alpha$, and m is any integer ≥ 1.*

PROOF. (a) Apply induction. Suppose that $L_q(X, P) = (X_1 \cup \cdots \cup X_k) \cup (X_{k+1} \cup \cdots \cup X_{qk})$, where each X_i is closed in $L_q(X, P)$ and $dX_i \leq \alpha$. Consider the open set $O = L_q(X, P) \backslash (X_1 \cup \cdots \cup X_k)$. Observe that $O \subset \bigcup_{i=1}^{(q-1)k} X_{k+i}$, O meets the basic subset X of $L_q(X, P)$ and so contains a copy of $L_{q-1}(X, P)$. This implies that $L_{q-1}(X, P) = \bigcup_{i=1}^{(q-1)k} Y_i$, where each Y_i is closed in $L_q(X, P)$ and $dY_i \leq \alpha$. This contradiction proves (a).

(b) follows directly from (a). □

All our classes $\mathscr{K}$ of topological spaces are assumed to be monotone with respect to closed subsets and closed under operations $L(,)$ and $L(, ,)$.

LEMMA 2. *Let $\mathscr{K}$ be a class of topological spaces, α be an ordinal ≥ 0 and d be a dimension function such that $dL(L(S, P), P^1, T) \leq \alpha$ for any S, T from $\mathscr{K}$ with $dS \leq \alpha$, $dT \leq \alpha$ and any P. Let $X \in \mathscr{K}$ be such that $X = \bigcup_{i=1}^{k} X_i$, where each X_i is closed in X, without isolated points and $dX_i \leq \alpha$. Let also P_i be a countable dense subset of X_i for each i. Then for each q the space $L_q(X, \bigcup_{i=1}^{k} P_i)$ exists and is the union of k^q closed subsets with $d \leq \alpha$.*

PROOF. Observe that X is a space without isolated points and the countable set $P = \bigcup_{i=1}^{k} P_i$ is dense in X. Hence the space $L_q(X, P)$ exists for each q. We prove Lemma 2 by induction. Consider the case $q = 1$. Observe that $L_1(X, P) = \bigcup_{i=1}^{k} L_1(X_i, P_i)$, where each $L_1(X_i, P_i)$ is closed in $L_1(X, P)$. Moreover for each i, $L_1(X_i, P_i) = L(L(X_i, P_i), P_i^1, Y)$, where Y is a singleton with $dY \le 0 \le \alpha$. By the property of d, $dL_1(X_i, P_i) \le \alpha$. Assume now that $L_{q-1}(X, P) = Y_1 \cup \cdots \cup Y_{k^{q-1}}$, where each Y_i is closed in $L_{q-1}(X, P)$ and $dY_i \le \alpha$. Observe that $L_q(X, P) = L(L(X_1, P_1), P_1^1, Y_1) \cup \cdots \cup L(L(X_k, P_k), P_k^1, Y_1) \cup \cdots \cup L(L(X_1, P_1), P_1^1, Y_{k^{q-1}}) \cup \cdots \cup L(L(X_k, P_k), P_k^1, Y_{k^{q-1}})$. Note that all k^q terms in the right side of this representation of $L_q(X, P)$ are closed in $L_q(X, P)$ and with $d \le \alpha$ because of the property of d. □

We will say that a dimension function d satisfies *the sum theorem of type A* if for any X being the union of two closed subspaces X_1 and X_2 with $dX_i \le \alpha_i$, where each α_i is finite and ≥ 0, we have $dX \le \alpha_1 + \alpha_2 + 1$. A space X is *completely decomposable in the sense of the dimension function d* if $dX = \alpha$, where α is an integer ≥ 1, and $X = \bigcup_{i=1}^{\alpha+1} X_i$, where each X_i is closed in X and $dX_i = 0$. Observe that if this space X belongs to a class $\mathscr{K}$ of topological spaces then $m_{\mathscr{K}}(d, \beta, \alpha) \le m(X, d, \beta, \alpha) \le \alpha + 1$ for each β with $0 \le \beta < \alpha$.

We will say that a transfinite dimension function d satisfies *the sum theorem of type A_{tr}* if for any X being the union of two closed subspaces X_1 and X_2 with $dX_i \le \alpha_i$ and $\alpha_2 \ge \alpha_1$ we have $dX \le \alpha_2$, if $\lambda(\alpha_1) < \lambda(\alpha_2)$, and $dX \le \alpha_2 + n(\alpha_1) + 1$, if $\lambda(\alpha_1) = \lambda(\alpha_2)$. A space X is *completely decomposable in the sense of the transfinite dimension function d* if $dX = \alpha$, where α is an infinite ordinal with $n(\alpha) \ge 1$, and $X = \bigcup_{i=1}^{n(\alpha)+1} X_i$, where each X_i is closed in X and $dX_i = \lambda(\alpha)$. Observe that if this space X belongs to a class $\mathscr{K}$ of topological spaces then $m_{\mathscr{K}}(d, \beta, \alpha) \le m(X, d, \beta, \alpha) \le n(\alpha) + 1$ for each β with $\lambda(\alpha) \le \beta < \alpha$.

To every space X one assigns *the large inductive compactness degree* Cmp as follows.

(i) $\operatorname{Cmp} X = -1$ if and only if X is compact;

(ii) $\operatorname{Cmp} X = 0$ if and only if there is a base $\mathscr{B}$ for the open sets of X such that the boundary $\operatorname{Bd} U$ is compact for each U in $\mathscr{B}$;

(iii) $\operatorname{Cmp} X \le \alpha$, where α is an integer ≥ 1, if for each pair of disjoint closed subsets A and B of X there exists a partition C between A and B in X such that $\operatorname{Cmp} C \le \alpha - 1$;

(iv) $\operatorname{Cmp} X = \alpha$ if $\operatorname{Cmp} X \le \alpha$ and $\operatorname{Cmp} X > \alpha - 1$;

(v) $\operatorname{Cmp} X = \infty$ if $\operatorname{Cmp} X > \alpha$ for every positive integer α.

Recall also the definitions of *the transfinite inductive dimenions* trind and trInd.

(i) $\operatorname{trInd} X = -1$ if and only if $X = \varnothing$;

(ii) $\operatorname{trInd} X \le \alpha$, where α is an ordinal ≥ 0, if for each pair of disjoint closed subsets A and B of X there exists a partition C between A and B in X such that $\operatorname{trInd} C < \alpha$;

(iii) $\operatorname{trInd} X = \alpha$ if $\operatorname{trInd} X \le \alpha$ and $\operatorname{trInd} X \le \beta$ holds for no $\beta < \alpha$;

(iv) $\operatorname{trInd} X = \infty$ if $\operatorname{trInd} X \le \alpha$ holds for no ordinal α.

The definition of trind is obtained by replacing the set A in (ii) with a point of X.

Remark 1. (i) Note that Cmp satisfies the sum theorem of type A ([**ChH**, Theorem 2.2]) and for each integer $\alpha \geq 1$ there exists a separable completely metrizable space C_α with $\operatorname{Cmp} C_\alpha = \alpha$ which is completely decomposable in the sense of Cmp ([**ChH**, Theorem 3.1]). For the convenience of the reader, we recall that $C_\alpha = \{0\} \times ([0,1]^\alpha \backslash (0,1)^\alpha) \cup \bigcup_{i=1}^\infty \{x_i\} \times [0,1]^\alpha \subset I^{\alpha+1}$, where $\{x_i\}_{i=1}^\infty$ is a sequence of real numbers such that $0 < x_{i+1} < x_i \leq 1$ for all i and $\lim_{i\to\infty} x_i = 0$. Note that the closed subsets in the decomposition of C_α can be assumed without isolated points.

(ii) Note also that trInd satisfies the sum theorem of type A_{tr} ([**E**, Theorem 7.2.7]) and for each infinite ordinal α with $n(\alpha) \geq 1$ there exists a metrizable compact space S^α (Smirnov's compactum) with $\operatorname{trInd} S^\alpha = \alpha$ which is completely decomposable in the sense of trInd ([**Ch**, Lemma 3.5]). Recall that Smirnov's compacta $S^0, S^1, \ldots, S^\alpha, \ldots$, $\alpha < \omega_1$, are defined by transfinite induction: S^0 is the one-point space, $S^\alpha = S^\beta \times [0,1]$ for $\alpha = \beta + 1$, and if α is a limit ordinal, then $S^\alpha = \{*_\alpha\} \cup \bigoplus_{\beta<\alpha} S^\beta$ is the one-point compactification of the free union of all the previously defined S^β's, where $*_\alpha$ is the compactifying point. Note that the closed subsets in the decomposition of S^α can be assumed without isolated points.

(iii) Observe that trind satisfies another sum theorem. Namely, for any X being the union of two closed subspaces X_1 and X_2 with $\operatorname{trind} X_i \leq \alpha_i$ and $\alpha_2 \geq \alpha_1$ we have $\operatorname{trind} X \leq \alpha_2$, if $\lambda(\alpha_1) < \lambda(\alpha_2)$, and $\operatorname{trind} X \leq \alpha_2 + 1$, if $\lambda(\alpha_1) = \lambda(\alpha_2)$ [**Ch**, Theorem 3.9].

Proposition 1. (i) *Let $\mathscr{K}$ be a class of topological spaces, d be a dimension function satisfying the sum theorem of type A, α be an integer ≥ 1 and X be a space from $\mathscr{K}$ with $dX = \alpha$ which is completely decomposable in the sense of d. Then for any integer $0 \leq \beta < \alpha$ we have $m_{\mathscr{K}}(d,\beta,\alpha) = m(X,d,\beta,\alpha) = q(\beta,\alpha)$.*

(ii) *Let $\mathscr{K}$ be a class of topological spaces, d be a transfinite dimension function satisfying the sum theorem of type A_{tr}, α be an infinite ordinal with $n(\alpha) \geq 1$ and X be a space from $\mathscr{K}$ with $dX = \alpha$ which is completely decomposable in the sense of d. Then for any infinite ordinal $\beta < \alpha$ we have $m_{\mathscr{K}}(d,\beta,\alpha) = m(X,d,\beta,\alpha) = q(\beta,\alpha)$ if $\lambda(\beta) = \lambda(\alpha)$ and $m_{\mathscr{K}}(d,\beta,\alpha)$ does not exist otherwise.*

Proof. We prove only (ii) as (i) is similar. The case $\lambda(\beta) < \lambda(\alpha)$ is clear from the properties of d, and we assume $\lambda(\beta) = \lambda(\alpha)$. Observe that if a space Z from $\mathscr{K}$ with $dZ = \alpha$ can be written as the union $\bigcup_{i=1}^k Y_i$, where each Y_i is closed in Z and $dY_i \leq \beta$, by the properties of d, $\alpha \leq \lambda(\beta) + kn(\beta) + k - 1$ so that $k \geq (n(\alpha)+1)/(n(\beta)+1) = p(\beta,\alpha)$. Hence $m(Z,d,\beta,\alpha) \geq q(\beta,\alpha)$ and thereby $m_{\mathscr{K}}(d,\beta,\alpha) \geq q(\beta,\alpha)$. Now observe that the space X can be written as the union of $q(\beta,\alpha)(n(\beta)+1)$ closed sets each with $d = \lambda(\alpha)$. Hence, by the properties of d, X can be written as the union of $q(\beta,\alpha)$ closed subsets each with $d \leq \lambda(\alpha) + n(\beta) = \beta$. Thus $m_{\mathscr{K}}(d,\beta,\alpha) \leq m(X,d,\beta,\alpha) \leq q(\beta,\alpha)$ and, therefore, $m_{\mathscr{K}}(d,\beta,\alpha) = m(X,d,\beta,\alpha) = q(\beta,\alpha)$. □

The *deficiency* def is defined in the following way: For a space X,

$$\operatorname{def} X = \min\{\dim(Y \backslash X)\colon\ Y \text{ is a metrizable compactification of } X\}.$$

Recall that $\operatorname{Cmp} X \leq \operatorname{def} X$ and $\operatorname{def} X = 0$ if and only if $\operatorname{Cmp} X = 0$.

Lemma 3. (i) $\operatorname{def} L(L(X,P),P^1,Y) = \max\{\operatorname{def} X, \operatorname{def} Y\}$ *for any, X,P,Y. In particular, we have* $\operatorname{Cmp} L(L(X,P),P^1,Y) \leq 0$ *if* $\operatorname{Cmp} X \leq 0$ *and* $\operatorname{Cmp} Y \leq 0$.

(ii) $\operatorname{trInd} L(L(X,P),P^1,Y) = \max\{\operatorname{trInd} X, \operatorname{trInd} Y\}$ *for any compacta* X, Y *and any* P.

Proof. (i) Let bX and bY be metrizable compactifications of X and Y respectively such that $\dim(bX\setminus X) = \operatorname{def} X$ and $\dim(bY\setminus Y) = \operatorname{def} Y$. Observe that the space $L(L(bX,P),P^1,bY)$ is a compactification of $L(L(X,P),P^1,Y)$ and the increment $Z = L(L(bX,P),P^1,bY)\setminus L(L(X,P),P^1,Y)$ is the union of countably many closed subsets, one of which is homeomorphic to $bX\setminus X$ and the others are homeomorphic to $bY\setminus Y$. So by the countable sum theorem for dim we get that $\dim Z = \max\{\dim(bX\setminus X), \dim(bY\setminus Y)\} = \max\{\operatorname{def} X, \operatorname{def} Y\}$. Hence $\operatorname{def} L(L(X,P),P^1,Y) \le \max\{\operatorname{def} X, \operatorname{def} Y\}$, thereby $\operatorname{def} L(L(X,P),P^1,Y) = \max\{\operatorname{def} X, \operatorname{def} Y\}$.

(ii) At first let us prove the statement when Y is a singleton. Observe that in this case $L(L(X,P),P^1,Y) = L(X,P)$. Consider two disjoint closed subsets A and B of $L(X,P)$. Recall that $L(X,P)$ contains a copy of X. Choose a partition C between $A\cap X$ and $B\cap X$ in X. Extend the partition to a partition C_1 between A and B in $L(X,P)$. Consider another partition C_2 between A and B in $L(X,P)$ which is "thin" (i.e. $\operatorname{Int}_{L(X,P)} C_2 = \emptyset$) and is in C_1. Observe that $C_2 \subset C$. Hence $\operatorname{trInd} L(X,P) = \operatorname{trInd} X$.

Now let us consider the general case. Assume that A and B are disjoint closed subsets in $L(L(X,P),P^1,Y)$. Recall that there is the natural continuous projection $\operatorname{pr} : L(L(X,P),P^1,Y) \to L(X,P)$. Consider the closed subsets $\operatorname{pr} A$ and $\operatorname{pr} B$ of $L(X,P)$. If they are disjoint, choose a partition C_2 between $\operatorname{pr} A$ and $\operatorname{pr} B$ in $L(X,P)$ like in the previous part. Observe that $\operatorname{pr}^{-1} C_2$ is a partition between A and B in $L(L(X,P),P^1,Y)$ such that $\operatorname{pr}^{-1} C_2$ is homeomorphic to a closed subset of C. Assume now that $\operatorname{pr} A\cap \operatorname{pr} B \ne \emptyset$. Note that $Q^1 = \operatorname{pr} A\cap \operatorname{pr} B$ is finite and $L(L(X,P),P^1,Y)$ is the free union of $L(L(X,(P\setminus Q)),P^1\setminus Q^1,Y)$, where Q is the finite subset of P corresponding to Q^1 and finitely many copies of Y. Choose a partition between A and B in X and a partition between A and B in each of the copies of Y corresponding to points of Q. It follows from the foregoing discussion that the free union of these partitions constitutes a partition in $L(L(X,P),P^1,Y)$ between A and B. We conclude that $\operatorname{trInd} L(L(X,P),P^1,Y) = \max\{\operatorname{trInd} X, \operatorname{trInd} Y\}$. □

Proof of Theorem 1.

(i) Because of Remark 1 and Proposition 1, we need only establish that $M_{\mathscr{P}}(\mathrm{Cmp},\beta,\alpha) = \infty$. Consider the space $C_\alpha = \bigcup_{i=1}^{\alpha+1} X_i$, where each X_i is closed in X, without isolated points and $\mathrm{Cmp}\, X_i = 0$, from Remark 1. Let P_i be a countable dense subset of X_i. Put $P = \bigcup_{i=1}^{\alpha+1} P_i$. Recall that $\operatorname{def} C_\alpha = \alpha$ ([**ChH**, Theorem 3.1]). So by Lemma 3 for any integer q we have $\operatorname{def} L_q(C_\alpha,P) = \alpha$ and hence $\mathrm{Cmp}\, L_q(C_\alpha,P) = \alpha$. Observe that by Lemmas 2 and 3, we get that the completely metrizable space $L_q(C_\alpha,P)$ is the union of $(\alpha+1)^q$ many closed subspaces with $\mathrm{Cmp} \le 0$. Hence $m(L_q(C_\alpha,P), \mathrm{Cmp},\beta,\alpha) \le (\alpha+1)^q$. Since Cmp satisfies the sum theorem of type A, C_α cannot be represented as α-many closed subsets with $\mathrm{Cmp} \le 0$. By Lemma 1 we have $m(L_q(C_\alpha,P),\mathrm{Cmp},\beta,\alpha) \ge q\alpha \ge q$. Since $\lim_{q\to\infty} q = \infty$ we get $M_{\mathscr{P}}(\mathrm{Cmp},\beta,\alpha) = \infty$.

(ii) By similar arguments as in the proof of (i) one can prove $M_{\mathscr{C}}(\operatorname{trInd},\beta,\alpha) = \infty$, if $\lambda(\beta) = \lambda(\alpha)$; and does not exist otherwise. □

PROOF OF THEOREM 2. (i) Put $X_\alpha = \{*\} \cup \bigoplus_{i=1}^{\infty} L_i(C_\alpha, P)$. Observe that X_α is completely metrizable and is the union of countably many closed subspaces with $\mathrm{Cmp} \le 0$. Since $\mathrm{def}\, X_\alpha = \alpha$, we have $\mathrm{Cmp}\, X_\alpha = \alpha$. Now observe that $\lim_{i\to\infty} m(L_i(C_\alpha, P), \mathrm{Cmp}, \alpha - 1, \alpha) = \infty$. Hence X_α cannot be written as the finite union of closed subsets with $\mathrm{Cmp} \le \alpha - 1$.

(ii) Put $X_\alpha = \{*\} \cup \bigoplus_{i=1}^{\infty} L_i(S^\alpha, P)$. Observe that X_α is compact and is the union of countably many finite-dimensional closed subspaces (recall that S^α and therefore $L_i(S^\alpha, P)$ have the same property). Since for each i, $\mathrm{trInd}\, L_i(S^\alpha, P) = \alpha$, we have $\mathrm{trInd}\, X_\alpha = \alpha$. Now observe that $\lim_{i\to\infty} m(L_i(S^\alpha, P), \mathrm{trInd}, \alpha - 1, \alpha) = \infty$. Hence X_α cannot be written as the finite union of closed subsets with $\mathrm{trInd} \le \alpha - 1$. □

REMARK 2. Let Q be the set of rational numbers of the closed interval $[0,1]$. Recall that for the spaces $X = Q \times [0,1]^n$ and $Y = ([0,1] \backslash Q) \times I^n$ we have $\mathrm{Cmp}\, X = \mathrm{def}\, X = \mathrm{Cmp}\, Y = \mathrm{def}\, Y = n$ ([**AN**, p.18 and p.56]). It is easy to observe that X satisfies points (a)–(c) of Theorem 2 (i). However, X is not completely metrizable. Note that Y is completely metrizable and satisfies points (a) and (c) of Theorem 2 (i) but not (b). Observe that Smirnov's compactum S^α with $n(\alpha) \ge 1$ satisfies points (a) and (b) of Theorem 2 (ii) but not (c). Note also that any Cantor manifold Z with $\mathrm{trInd}\, Z = \alpha$, where α is an infinite ordinal with $n(\alpha) \ge 1$, (see for such spaces for example in [**O**]) satisfies points (a) and (c) of Theorem 2 (ii) but not (b).

Let d be a (transfinite) dimension function. A space X with $dX \ne \infty$ is said *to have property* $(*)_d$ if for every open nonempty subset O of the space X there exists a closed in X subset $F \subset O$ with $dF = dX$.

Observe that the spaces X, Y from Remark 2 have property $(*)_{\mathrm{Cmp}}$ and Z has property $(*)_{\mathrm{trInd}}$.

PROPOSITION 2. *Let X be a completely metrizable space with $dX \ne \infty$. Then $X \ne \bigcup_{i=1}^{\infty} X_i$, where each X_i is closed in X and $dX_i < dX$ if and only if there exists a closed subspace Y of X such that* (i) *$dY = dX$ and* (ii) *Y has the property $(*)_d$.*

PROOF. ($\Leftarrow$) Assume $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is closed in X and $dX_i < dX$. It suffices to obtain a contradiction. Consider $Y = \bigcup_{i=1}^{\infty} (X_i \cap Y)$. Observe that each $Y_i = (X_i \cap Y)$ is closed in Y and $dY_i \le dX_i < dX = dY$. By Baire's theorem there exists a natural number i such that $\mathrm{Int}\, Y_i \ne \emptyset$. By property $(*)_d$, there is a closed in Y (in X as well) subset $F \subset \mathrm{Int}\, Y_i$ with $dF = dY$. Now, by the monotonicity of d, we have $dF \le dY_i < dY$, a contradiction.

($\Rightarrow$) Let $\mathcal{G}$ be the family of all open sets of X that can be written as a countable union of closed subsets of X with $d < dX$. Because X is separable metrizable, $G = \bigcup \mathcal{G} \in \mathcal{G}$. Let $Y = X \backslash G$. By the assumed property of X, we have $dY = dX$. Consider now an open set U of Y. Then $U = Y \cap V$ for some open set V of X. If U contains no closed subset of X with $d = dX$ then clearly $V = U \cup (V \backslash Y)$ is the countable union of closed subsets of X with $d < dX$. Hence $V \in \mathcal{G}$, $V \subset G$ and so $U = \emptyset$. Thus, Y has property $(*)_d$. □

REMARK 3. This remark concerns non-metrizable compact spaces. Using the construction of Lokucievskij's example ([**E**, p.140]), Chatyrko, Kozlov and Pasynkov [**ChKP**, Remark 3.15 (b)] presented for each $n = 3, 4, \ldots$ a compact Hausdorff space X_n

such that $\operatorname{ind} X_n = 2$ and $m(X_n, \operatorname{ind}, 1, 2) = n$. Hence it is clear that $m_{\mathscr{N}}(\operatorname{ind}, 1, 2) = 2$ and $M_{\mathscr{N}}(\operatorname{ind}, 1, 2) = \infty$, where $\mathscr{N}$ is the class of compact Hausdorff spaces. In [**K**] Kotkin constructed a compact Hausdorff space X with $\operatorname{ind} X = 3$ which is the union of three one-dimensional in the sense of ind closed subspaces. Hence, $m_{\mathscr{N}}(\operatorname{ind}, 1, 3) = 3$ and $m_{\mathscr{N}}(\operatorname{ind}, 2, 3) = 2$. Filippov in [**F**] presented for every n a compact Hausdorff space F_n with $\operatorname{ind} F_n = n$, which is the union of finitely many one-dimensional in the sense of ind closed subspaces, thereby $m_{\mathscr{N}}(\operatorname{ind}, k, n) < \infty$ for each $1 \leq k < n$. By the sum theorem from Remark 1 (iii) for ind which is valid in fact for all regular spaces, one can get that $m_{\mathscr{N}}(\operatorname{ind}, 1, n) \geq 2^{n-2} + 1$ for each n.

3. The dimension function $\mathrm{Cmp}_{\cup}$.

We begin with the definition of the additive large compactness degree $\mathrm{Cmp}_{\cup}$.

Definition 1. Let X be a space. Then we define the additive large compactness degree $\mathrm{Cmp}_{\cup}$ as follows.

(i) $\mathrm{Cmp}_{\cup} X = -1$ if and only if X is compact.

(ii) $\mathrm{Cmp}_{\cup} X = 0$ if and only if $\mathrm{Cmp}\, X = 0$.

(iii) $\mathrm{Cmp}_{\cup} X \leq \alpha$, where α is an integer ≥ 1, if $X = \bigcup_{i=1}^{\alpha+1} Z_i$, where each Z_i is closed in X and $\mathrm{Cmp}\, Z_i \leq 0$.

(iv) $\mathrm{Cmp}_{\cup} X = \alpha$ if and only if $\mathrm{Cmp}_{\cup} X \leq \alpha$ and the inequality $\mathrm{Cmp}_{\cup} X \leq \beta$ holds for no $\beta < \alpha$.

(v) $\mathrm{Cmp}_{\cup} X = \infty$ if and only if $\mathrm{Cmp}_{\cup} X \leq \alpha$ holds for no integer α.

The following properties of $\mathrm{Cmp}_{\cup}$ are evident.

(1) $\mathrm{Cmp}_{\cup}$ is monotone with respect to closed subspaces and $\mathrm{Cmp}\, X \leq \mathrm{Cmp}_{\cup} X$ for any X.

(2) $\mathrm{Cmp}_{\cup}$ satisfies the sum theorem of type A.

(3) $\mathrm{Cmp}_{\cup} C_\alpha = \alpha$, where C_α is described in Remark 1.

(4) Every space X with $\mathrm{Cmp}_{\cup} X = \alpha \geq 1$ is completely decomposable in the sense of $\mathrm{Cmp}_{\cup}$.

Proof of Theorem 3 (i).

By the above properties of $\mathrm{Cmp}_{\cup}$ and Proposition 1, we have $m(X, \mathrm{Cmp}_{\cup}, \beta, \alpha) = q(\beta, \alpha)$ for every space X with $\mathrm{Cmp}_{\cup} X = \alpha \geq 1$. Hence, $m_{\mathscr{P}}(\mathrm{Cmp}_{\cup}, \beta, \alpha) = M_{\mathscr{P}}(\mathrm{Cmp}_{\cup}, \beta, \alpha) = q(\beta, \alpha)$. □

Let α be an integer ≥ 1 and P be a countable dense subset of C_α. Then it can be proved similarly as in the proof of Theorem 1 that for every integer $q \geq 2$ we have $\operatorname{def} L_q(C_\alpha, P) = \alpha < q\alpha \leq \mathrm{Cmp}_{\cup} L_q(C_\alpha, P) \leq (\alpha+1)^q$. Observe also that for the spaces X, Y from Remark 2 and $L_*(C_\alpha, P)$ we have $\mathrm{Cmp}_{\cup} X = \mathrm{Cmp}_{\cup} Y = \mathrm{Cmp}_{\cup} L_*(C_\alpha, P) = \infty$, although $\operatorname{def} X = \operatorname{def} Y = n$ and $\operatorname{def} L_*(C_\alpha, P) = \alpha$.

Question. Is it true that $\operatorname{def} X \leq \mathrm{Cmp}_{\cup} X$ for any X?

4. The dimensions $\operatorname{ind}_{\cup}$ and $\operatorname{Ind}_{\cup}$.

Definition 2. Let X be a space. Then we define the additive transfinite inductive dimension $\operatorname{ind}_{\cup}$ as follows.

(i) $\operatorname{ind}_{\cup} X = -1$ if and only if $X = \emptyset$.

(ii) $\operatorname{ind}_{\cup} X \leq \alpha$, where α is an ordinal ≥ 0, if

(a) for every point $x \in X$ and any neighborhood Ox there exists a neighborhood Ux such that $Ux \subset Ox$ and $\operatorname{ind}_{\cup} \operatorname{Bd} Ux < \alpha$ (i.e. $\operatorname{ind}_{\cup} \operatorname{Bd} Ux \leq \beta$ for some $\beta < \alpha$), if α is finite or limit; or

(b) $X = \bigcup_{i=1}^{n(\alpha)+1} Z_i$, where each Z_i is closed in X and $\operatorname{ind}_{\cup} Z_i \leq \lambda(\alpha)$, if α is infinite and $n(\alpha) \geq 1$.

Remark 4. It is easy to prove by induction the following facts.

(i) For any ordinal α, if A is a closed subset of a space X and $\operatorname{ind}_{\cup} X \leq \alpha$ then $\operatorname{ind}_{\cup} A \leq \alpha$.

(ii) For any ordinal β, α with $\beta < \alpha$, if $\operatorname{ind}_{\cup} X \leq \beta$ then $\operatorname{ind}_{\cup} X \leq \alpha$.

Definition 2 (continued).

(iii) $\operatorname{ind}_{\cup} X = \alpha$ if and only if $\operatorname{ind}_{\cup} X \leq \alpha$ and the inequality $\operatorname{ind}_{\cup} X \leq \beta$ holds for no $\beta < \alpha$.

(iv) $\operatorname{ind}_{\cup} X = \infty$ if and only if $\operatorname{ind}_{\cup} X \leq \alpha$ holds for no ordinal number α.

Similarly one can define the function $\operatorname{Ind}_{\cup}$, i.e., $\operatorname{Ind}_{\cup}$ is defined by replacing the arbitrary point $x \in X$ in Definition 2 (ii) (a) with an arbitrary closed subset A of X.

Observe that for any space X and $\alpha \leq \omega_0$, $\operatorname{ind}_{\cup} X = \alpha$ if and only if $\operatorname{trind} X = \alpha$, and $\operatorname{Ind}_{\cup} X = \alpha$ if and only if $\operatorname{trInd} X = \alpha$.

Proposition 3. Let X be a space. Then

(i) $\operatorname{ind}_{\cup} X \leq \operatorname{Ind}_{\cup} X$;

(ii) $\operatorname{ind}_{\cup} A \leq \operatorname{ind}_{\cup} X$, if $A \subset X$, and $\operatorname{Ind}_{\cup} A \leq \operatorname{Ind}_{\cup} X$, if A is a closed subset of X;

(iii) $\operatorname{trind} X \leq \operatorname{ind}_{\cup} X$, and $\operatorname{trInd} X \leq \operatorname{Ind}_{\cup} X$.

Proof. Let us check only the first inequality of (iii). Apply induction on $\operatorname{ind}_{\cup} X = \alpha$. Let $\alpha \geq \omega_0$. If α is limit, then we consider a point $x \in X$ and any neighborhood Ox. By Definition 2, there exists a neighborhood Ux such that $Ux \subset Ox$ and $\operatorname{ind}_{\cup} \operatorname{Bd} Ux < \alpha$. By induction the inequality $\operatorname{trind} \operatorname{Bd} Ux < \alpha$ holds. So we have $\operatorname{trind} X \leq \alpha$. If $n(\alpha) \geq 1$ then $X = \bigcup_{i=1}^{n(\alpha)+1} Z_i$, where each Z_i is closed in X and $\operatorname{ind}_{\cup} Z_i \leq \lambda(\alpha)$. By induction $\operatorname{trind} Z_i \leq \lambda(\alpha)$. Recall that trind satisfies the sum theorem of type A. Hence $\operatorname{trind} X \leq \alpha$. □

Observe that the space $X = \bigoplus_{n=1}^{\infty} I^n$ has $\operatorname{ind}_{\cup} X = \omega_0$ and $\operatorname{Ind}_{\cup} X = \infty$.

The following technical lemma can be found in [**ChH**, Lemma 2.1].

Lemma 4. *Let X be a space such that $X = X_1 \cup X_2$, where each X_i is closed in X, and A, B be two closed disjoint subsets of X such that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset$, $i = 1, 2$. Choose a partition C_1 in X_1 between $A \cap X_1$ and $B \cap X_1$ such that $X_1 \backslash C_1 = U_1 \cup V_1$, where U_1, V_1 are open in X_1 and disjoint, and $A \cap X_1 \subset U_1$, $B \cap X_1 \subset V_1$. Choose also a partition C_2 in X_2 between $A \cap X_2$ and $((C_1 \cup V_1) \cup B) \cap X_2$ such that $X_2 \backslash C_2 = U_2 \cup V_2$, where U_2, V_2 are open in X_2 and disjoint, and $A \cap X_2 \subset U_2$, $((C_1 \cup V_1) \cup B) \cap X_2 \subset V_2$. Then the set $C = X \backslash (((U_1 \backslash X_2) \cup U_2) \cup (V_1 \cup (V_2 \backslash X_1)))$ is a partition in X between A and B such that $C \subset C_1 \cup C_2 \cup (X_1 \cap X_2)$.*

Proposition 4. *Let d be one of the functions* $\operatorname{ind}_{\cup}$ *or* $\operatorname{Ind}_{\cup}$. *Let a space X be the* *union of two closed subspaces* X_1 *and* X_2 *such that* $dX_1 \leq \alpha_1$, $dX_2 \leq \alpha_2$ *and* $\alpha_1 \leq \alpha_2$. *Then*

(i)

$$dX \leq \begin{cases} \alpha_2, & \text{if } \lambda(\alpha_1) < \lambda(\alpha_2), \\ \alpha_2 + n(\alpha_1) + 1, & \text{if } \lambda(\alpha_1) = \lambda(\alpha_2). \end{cases}$$

i.e. d *satisfies the sum theorem of type* A_{tr}.

(ii) *If* $d(X_1 \cap X_2) < \lambda(\alpha_2)$, *then* $dX \leq \alpha_2$.

PROOF. In the proof of (i) let us consider only the case of $\mathrm{ind}_{\cup}$. Apply induction on α_2. Let $\alpha_2 \geq \omega_0$. Consider the case when α_2 is limit. If $\alpha_1 < \alpha_2$ (i.e. $\lambda(\alpha_1) < \lambda(\alpha_2) = \alpha_2$), one easily sees that every point of X has arbitrarily small neighborhood U with $\mathrm{ind}_{\cup}(\mathrm{Bd}\, U \cap X_2) < \alpha_2$. Then, by the inductive hypothesis, $\mathrm{ind}_{\cup} \mathrm{Bd}\, U < \alpha_2$ and therefore $\mathrm{ind}_{\cup} X \leq \alpha_2$. If $\alpha_1 = \alpha_2$, then, by Definition 2, we have $\mathrm{ind}_{\cup} X \leq \alpha_2 + 1$. Now assume $n(\alpha_2) \geq 1$. Then, by Definition 2, $X_2 = \bigcup_{i=1}^{n(\alpha_2)+1} Z_i^{(2)}$, where each $Z_i^{(2)}$ is closed in X and $\mathrm{ind}_{\cup} Z_i^{(2)} \leq \lambda(\alpha_2)$. If $\lambda(\alpha_1) < \lambda(\alpha_2)$ put $Y_i = X_1 \cup Z_i^{(2)}$. By induction, $\mathrm{ind}_{\cup} Y_i \leq \lambda(\alpha_2)$ for each i. Observe that $X = \bigcup_{i=1}^{n(\alpha_2)+1} Y_i$ and hence $\mathrm{ind}_{\cup} X \leq \alpha_2$. If $\lambda(\alpha_1) = \lambda(\alpha_2)$, then we have $X_1 = \bigcup_{i=1}^{n(\alpha_1)+1} Z_i^{(1)}$, where $Z_i^{(1)}$ is closed in X and $\mathrm{ind}_{\cup} Z_i^{(1)} \leq \lambda(\alpha_1)$. It is clear that $\mathrm{ind}_{\cup} X \leq \alpha_2 + n(\alpha_1) + 1$.

In the proof of (ii), let us also consider only the case of $\mathrm{ind}_{\cup}$. Apply induction on α_2. Let $\alpha_2 \geq \omega_0$. Consider the case when α_2 is limit. If $x \in X_1 \backslash X_2$ or $x \in X_2 \backslash X_1$ then one can easily find a neighborhood Ux such that $Ux \subset Ox$ and $\mathrm{ind}_{\cup} \mathrm{Bd}\, Ux < \alpha_2$. Let now $x \in X_1 \cap X_2$ and A be a closed subset of X such that $x \notin A$ and $A \cap X_i \neq \emptyset$ for every i. Choose a partition C_1 in X_1 between the point x and the set $A \cap X_1$ such that $\mathrm{ind}_{\cup} C_1 < \alpha_2$. Let $X_1 \backslash C_1 = U_1 \cup V_1$, where U_1, V_1 are open in X_1 and disjoint, and $x \in U_1$. Choose a partition C_2 in X_2 between the point x and the set $((C_1 \cup V_1) \cup A) \cap X_2$ such that $\mathrm{ind}_{\cup} C_2 < \alpha_2$. Put $Y = C_1 \cup C_2 \cup (X_1 \cap X_2)$. Observe that by (i), the inequality $\mathrm{ind}_{\cup} Y < \alpha_2$ holds. By Lemma 4, there exists a partition C between the point x and the set A such that $C \subset Y$. So $\mathrm{ind}_{\cup} C < \alpha_2$. Now assume $n(\alpha_2) \geq 1$. Then by Definition 2, $X_k = \bigcup_{i=1}^{n(\alpha_2)+1} Z_i^{(k)}$, where $Z_i^{(k)}$ is closed in X and $\mathrm{ind}_{\cup} Z_i^{(k)} \leq \lambda(\alpha_2)$, $k = 1, 2$. Put $Y_i = Z_i^{(1)} \cup Z_i^{(2)}$ for each i. Observe that $\mathrm{ind}_{\cup}(Z_i^{(1)} \cap Z_i^{(2)}) < \lambda(\alpha_2)$. By induction, the inequality $\mathrm{ind}_{\cup} Y_i \leq \lambda(\alpha_2)$ holds. Observe that $X = \bigcup_{i=1}^{n(\alpha_2)+1} Y_i$. Hence $\mathrm{ind}_{\cup} X \leq \alpha_2$. □

PROPOSITION 5. *Let* X *be a space. Then* $\mathrm{ind}_{\cup} X \leq \omega_0 \cdot \mathrm{trind}\, X$. *In particular*, $\mathrm{trind}\, X < \infty$ (ω_1) *if and only if* $\mathrm{ind}_{\cup} X < \infty$ (ω_1).

PROOF. We need to prove only the inequality $\mathrm{ind}_{\cup} X \leq \omega_0 \cdot \mathrm{trind}\, X$. Apply induction on $\mathrm{trind}\, X = \alpha$. Let $\alpha \geq \omega_0$ and $\mathscr{B} = \{U_i\}_{i=1}^{\infty}$ be a countable base for the space X such that $\mathrm{trind}\, \mathrm{Bd}\, U_i = \alpha_i < \alpha$ for every i. By induction we have $\mathrm{ind}_{\cup} \mathrm{Bd}\, U_i \leq \omega_0 \cdot \alpha_i < \omega_0 \cdot \alpha$ for every i. Observe that the ordinal number $\omega_0 \cdot \alpha$ is limit. Hence, by Definition 2, we have $\mathrm{ind}_{\cup} X \leq \omega_0 \cdot \alpha$. □

PROPOSITION 6. *Let* X *be a space such that* $\mathrm{trInd}\, X < \infty$. *Then*, $\mathrm{Ind}_{\cup} X = \mathrm{ind}_{\cup} X$.

PROOF. We need to prove only the inequality $\mathrm{ind}_{\cup} X \geq \mathrm{Ind}_{\cup} X$. Apply induction on $\mathrm{ind}_{\cup} X = \alpha$. Assume that $\alpha \geq \omega_0$. By [**E**, Theorem 7.1.25] there is a compact subspace K of X such that $\operatorname{trInd} K < \infty$ and $\operatorname{Ind} F < \infty$ for each closed subspace F of X disjoint from K. If α is limit then there exists a countable base $\mathscr{B} = \{U_i\}_{i=1}^{\infty}$ for X such that $\operatorname{ind}_{\cup} \operatorname{Bd} U_i = \alpha_i < \alpha$ for every i. Consider a pair A, B of closed disjoint subsets of X. If one of them is disjoint from K then we can easily choose a partition C between A and B which is disjoint from K and hence $\operatorname{Ind} C < \infty$. Suppose now that $A \cap K \neq \emptyset$ and $B \cap K \neq \emptyset$. Choose a finite covering $\{U_{i_k}\}_{k=1}^{m}$ of $A \cap K$ by elements from $\mathscr{B}$ such that $\operatorname{Cl}(U_{i_k}) \cap B = \emptyset$ for every k. Observe that the set $D = A \setminus \bigcup_{k=1}^{m} U_{i_k}$ is disjoint from K. So we can find a neighborhood O of D such that $\operatorname{Cl}(O) \cap (K \cup B) = \emptyset$. Hence $\operatorname{Ind} \operatorname{Bd} O < \infty$. Observe that the set $U = O \cup \bigcup_{k=1}^{m} U_{i_k}$ is a neighborhood of A such that $\operatorname{Cl}(U) \cap B = \emptyset$ and $\operatorname{Bd} U \subset \operatorname{Bd} O \cup \bigcup_{k=1}^{m} \operatorname{Bd} U_{i_k}$. By Proposition 4 (i), we have $\operatorname{ind}_{\cup} \operatorname{Bd} U < \alpha$. Hence by the inductive assumption and Definition 2, $\operatorname{Ind}_{\cup} X \leq \alpha$.

Now assume $n(\alpha) \geq 1$. Then $X = \bigcup_{i=1}^{n(\alpha)+1} Z_i$, where each Z_i is closed in X and $\operatorname{ind}_{\cup} Z_i \leq \lambda(\alpha)$. By induction we have that $\operatorname{Ind}_{\cup} Z_i \leq \lambda(\alpha)$. Hence $\operatorname{Ind}_{\cup} X \leq \alpha$ by Definition 2. □

Corollary 1. *For every compact space X, $\operatorname{Ind}_{\cup} X = \operatorname{ind}_{\cup} X$.*

Proof. It suffices to check this equality when $\operatorname{trInd} X = \infty$. Then $\operatorname{trind} X = \infty$ too ([**E**, Corollary 7.1.32]). By Proposition 3, we have $\operatorname{ind}_{\cup} X = \infty$. Hence $\operatorname{Ind}_{\cup} X = \operatorname{ind}_{\cup} X$. □

Recall that the notation $\alpha(+)\beta$ means the natural sum of the ordinals [**KM**].

Proposition 7. *Let X_i be a space with $\operatorname{ind}_{\cup} X_i \leq \alpha_i \geq 0$, $i = 1, 2$. Then $\operatorname{ind}_{\cup}(X_1 \times X_2) \leq \alpha_1(+)\alpha_2 + n(\alpha_1) \cdot n(\alpha_2)$.*

Proof. Let $\gamma = \alpha_1(+)\alpha_2$. Apply induction on γ. Assume that $\gamma \geq \omega_0$. If γ is limit then both α_1 and α_2 are limit (recall that 0 is limit). Consider a point p and the rectangular neighborhood $U \times V$ of p such that $\operatorname{ind}_{\cup} \operatorname{Bd} U = \beta_1 < \alpha_1$ and $\operatorname{ind}_{\cup} \operatorname{Bd} V = \beta_2 < \alpha_2$. Observe that $\operatorname{Bd}(U \times V) = (\operatorname{Bd} U \times \operatorname{Cl}(V)) \cup (\operatorname{Cl}(U) \times \operatorname{Bd} V)$ and $\beta_1(+)\alpha_2 + n(\beta_1) \cdot n(\alpha_2) < \gamma$, $\alpha_1(+)\beta_2 + n(\alpha_1) \cdot n(\beta_2) < \gamma$. By the induction and Proposition 4 (i), we have $\operatorname{ind}_{\cup} \operatorname{Bd}(U \times V) < \gamma$. Hence $\operatorname{ind}_{\cup}(X_1 \times X_2) \leq \gamma = \alpha_1(+)\alpha_2$.

Now let $n(\gamma) \geq 1$. Observe that $n(\gamma) = n(\alpha_1) + n(\alpha_2)$. Let $n(\alpha_1) \geq 1$. Then $X_1 = \bigcup_{i=1}^{n(\alpha_1)+1} Z_i^{(1)}$, where each $Z_i^{(1)}$ is closed in X_1 and $\operatorname{ind}_{\cup} Z_i^{(1)} \leq \lambda(\alpha_1)$. If $n(\alpha_2) = 0$, then by induction we have $\operatorname{ind}_{\cup}(Z_i^{(1)} \times X_2) \leq \lambda(\alpha_1)(+)\alpha_2$. Observe that $\lambda(\alpha_1)(+)\alpha_2$ is limit and $X_1 \times X_2 = \bigcup_{i=1}^{n(\alpha_1)+1} (Z_i^{(1)} \times X_2)$. So $\operatorname{ind}_{\cup}(X_1 \times X_2) \leq \lambda(\alpha_1)(+)\alpha_2 + n(\alpha_1) = \alpha_1(+)\alpha_2$. If $n(\alpha_2) \geq 1$, then $X_2 = \bigcup_{i=1}^{n(\alpha_2)+1} Z_i^{(2)}$, where each $Z_i^{(2)}$ is closed in X_2 and $\operatorname{ind}_{\cup} Z_i^{(2)} \leq \lambda(\alpha_2)$. Observe that in this case we have $X_1 \times X_2 = \bigcup_{i=1}^{n(\alpha_1)+1} \bigcup_{j=1}^{n(\alpha_2)+1} (Z_i^{(1)} \times Z_j^{(2)})$ and $(n(\alpha_1) + 1) \cdot (n(\alpha_2) + 1) = n(\alpha_1) + n(\alpha_2) + n(\alpha_1) \cdot n(\alpha_2) + 1$, and we can apply induction. □

Recall that Smirnov's compactum S^{α} described in Remark 1, where α is an infinite ordinal $< \omega_1$, is the union $\bigcup_{i=1}^{n(\alpha)+1} Z_i$, where each Z_i is closed in S^{α}, and for any k with $0 \leq k \leq n(\alpha)$ we have $\operatorname{trInd}(\bigcup_{i=1}^{k+1} Z_i) = \lambda(\alpha) + k$ ([**Ch**, Lemma 3.5]). Moreover we can assume that each Z_i is the disjoint union of $[0,1]^{n(\alpha)}$ (a point in the case $n(\alpha) = 0$) and countably many clopen compacta S^{β_i}, $i = 1, 2, \ldots$, with $\beta_i < \lambda(\alpha)$ such that for any point $x \in [0,1]^{n(\alpha)}$ we have $\operatorname{ind}_x Z_i < \infty$. Then we have the following.

Proposition 8. *For any α and any k with $0 \le k \le n(\alpha)$ we have* $\mathrm{ind}_\cup(\bigcup_{i=1}^{k+1} Z_i) = \lambda(\alpha) + k$, *where each Z_i is the subspace of S^α described above. In particular,* $\mathrm{ind}_\cup S^\alpha = \alpha$.

Proof. It suffices to prove the inequality $\mathrm{ind}_\cup Z_i \le \lambda(\alpha)$ for every infinite $\alpha < \omega_1$ and each i. If α is limit, then by induction we have $\mathrm{ind}_\cup S^\alpha \le \alpha$. If $n(\alpha) \ge 1$, then for each i the inequality $\mathrm{ind}_\cup Z_i \le \lambda(\alpha)$ is valid by induction and the construction of Z_i. Now by Definition 2, we get the inequality $\mathrm{ind}_\cup(\bigcup_{i=1}^{k+1} Z_i) \le \lambda(\alpha) + k$ for any α and any k with $0 \le k \le n(\alpha)$. □

Corollary 2. *For any infinite ordinal number α with $n(\alpha) \ge 1$ there exists a compact space X_α with $\mathrm{ind}_\cup X_\alpha = \alpha$ such that for any non-negative integers p, q with $p + q = n(\alpha) - 1$ there exist closed subsets $X_{\alpha,p}$ and $X_{\alpha,q}$ of X_α with $X_\alpha = X_{\alpha,p} \cup X_{\alpha,q}$, $\mathrm{ind}_\cup X_{\alpha,p} = \lambda(\alpha) + p$ and $\mathrm{ind}_\cup X_{\alpha,q} = \lambda(\alpha) + q$.*

Proof of Theorem 3 (ii).

Observe that every space X with $\mathrm{ind}_\cup X = \alpha$, where α is an infinite ordinal with $n(\alpha) \ge 1$, is completely decomposable in the sense of $\mathrm{ind}_\cup$. So by Proposition 1, we have $m(X, \mathrm{ind}_\cup, \beta, \alpha) = q(\beta, \alpha)$ for every space X with $\mathrm{ind}_\cup X = \alpha$. Hence $m_{\mathscr{C}}(\mathrm{ind}_\cup, \beta, \alpha) = M_{\mathscr{C}}(\mathrm{ind}_\cup, \beta, \alpha) = q(\beta, \alpha)$. Observe that by Corollary 1, we have also $m_{\mathscr{C}}(\mathrm{Ind}_\cup, \beta, \alpha) = M_{\mathscr{C}}(\mathrm{Ind}_\cup, \beta, \alpha) = q(\beta, \alpha)$. □

Recall the definition of *D-dimension* D introduced by Henderson [**H**].

One assigns $D(\emptyset) = -1$, and for every space X one defines $D(X)$ as the smallest ordinal number α such that there exists a closed cover $\{A_\beta\}_{\beta \le \lambda(\alpha)}$ of the space X satisfying the following conditions:

(i) The union $\bigcup\{A_\beta : \delta \le \beta \le \lambda(\alpha)\}$ is closed for every $\delta \le \lambda(\alpha)$.

(ii) For every $x \in X$ the set $\{\beta \le \lambda(\alpha) : x \in A_\beta\}$ has a largest element.

(iii) $\dim A_\beta < \infty$ for every $\beta < \lambda(\alpha)$, and $\dim A_{\lambda(\alpha)} \le n(\alpha)$.

If no such ordinal exists, one assigns $D(X) = \infty$.

Recall also that $D(S^\alpha) = \alpha$, $\alpha < \omega_1$, and for any X which is the union of two closed subspaces X_1 and X_2 we have $D(X) = \max\{D(X_1), D(X_2)\}$.

Remark 5. Write $S^{\omega_0+4} = \bigcup_{i=1}^5 Z_i$, as in the paragraph preceding Proposition 8. By the sum theorem for D we can assume that $D(Z_1) = \omega_0 + 4$. Put $Y = \bigcup_{i=1}^4 Z_i$. Observe that $D(Y) = \omega_0 + 4$. By the sum theorem for trind (Remark 1 (iii)), $\mathrm{trind}\, Y \le \omega_0 + 2$. Furthermore, in view of Proposition 8, $\mathrm{trInd}\, Y = \mathrm{ind}_\cup Y = \omega_0 + 3$. Note that the first example of a compact space with different transfinite dimensions trind, trInd and D was presented by Luxemburg [**L**].

Remark 6. Let $n(\alpha) \ge 1$ and $S^\alpha = \bigcup_{i=1}^{n(\alpha)+1} Z_i$, as in the paragraph preceding Proposition 8. We can assume that each Z_i is without isolated points. Choose a dense subset P_i of Z_i for each i. Put $P = \bigcup_{i=1}^{n(\alpha)+1} P_i$. Then for every integer $q \ge 2$ we have $\mathrm{trInd}\, L_q(S^\alpha, P) = \alpha < \lambda(\alpha) + qn(\alpha) \le \mathrm{ind}_\cup L_q(S^\alpha, P) \le \lambda(\alpha) + (n(\alpha) + 1)^q$. Indeed, the equality follows from Lemma 3 (ii), the second inequality follows from Lemma 1 and Proposition 8. The last inequality follows from Lemma 2 because the analogue of Lemma 3 (ii) for $\mathrm{ind}_\cup$ is readily seen to be valid in the case of limit ordinals.

Note also that $\mathrm{trInd}\, L_*(S^{\omega_0+1}, P) = \omega_0 + 1$, but $\mathrm{Ind}_\cup L_*(S^{\omega_0+1}, P) = \omega_0 + \omega_0$.

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