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Spherical rigidities of submanifolds in Euclidean spaces

Dedicated to Professor Buchin Su for his 100th birthday

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Abstract. In this paper, we study n-dimensional complete immersed submanifolds in a Euclidean space $\boldsymbol{E}^{n+p}$. We prove that if M^n is an n-dimensional compact connected immersed submanifold with nonzero mean curvature H in $\boldsymbol{E}^{n+p}$ and satisfies either:

(1) $S \leq \dfrac{n^2H^2}{n-1}$, or

(2) $n^2H^2 \leq \dfrac{(n-1)R}{n-2}$,

then M^n is diffeomorphic to a standard n-sphere, where S and R denote the squared norm of the second fundamental form of M^n and the scalar curvature of M^n, respectively.

On the other hand, in the case of constant mean curvature, we generalized results of Klotz and Osserman [**11**] to arbitrary dimensions and codimensions; that is, we proved that the totally umbilical sphere $S^n(c)$, the totally geodesic Euclidean space $\boldsymbol{E}^n$, and the generalized cylinder $S^{n-1}(c) \times \boldsymbol{E}^1$ are only n-dimensional ($n > 2$) complete connected submanifolds M^n with constant mean curvature H in $\boldsymbol{E}^{n+p}$ if $S \leq n^2H^2/(n-1)$ holds.

1. Introduction.

It is well known by Nash that every finite dimensional Riemannian manifold possesses an isometric embedding into a Euclidean space of a sufficiently high dimension. Therefore, research of submanifolds in a Euclidean space $\boldsymbol{E}^{n+p}$ of $n+p$ dimensions requires some additional conditions. In this paper, we shall agree that a submanifold means an immersed submanifold. A classical theorem of Hadamard states that a compact connected orientable hypersurface in $\boldsymbol{E}^{n+1}$ with positive sectional curvature is diffeomorphic to a standard sphere $S^n(c)$. This result was generalized by Van Heijenoort [**18**] and Sacksteder [**15**]. They proved that an n-dimensional complete connected orientable hypersurface M^n in $\boldsymbol{E}^{n+1}$ is a boundary of a convex body in $\boldsymbol{E}^{n+1}$ if every sectional curvature of M^n is non-negative and at least one is positive. In particular, they proved that an n-dimensional locally convex (that is, the second fundamental form is semi-definite) compact connected orientable hypersurface M^n in $\boldsymbol{E}^{n+1}$ is diffeomorphic to $S^n(c)$. In [**6**] and [**7**], Chern and Lashof studied the total curvature of an n-dimensional compact connected orientable submanifold in $\boldsymbol{E}^{n+p}$. They showed that the total cur-

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vature of an n-dimensional compact connected orientable submanifold in $\boldsymbol{E}^{n+p}$ is not less than $2c_{n+p-1}$, and also that, if the equality holds, then M^n is diffeomorphic to $S^n(c)$, where c_{n+p-1} is the volume of the unit sphere $S^{n+p-1}(1)$. Recently, using a theorem introduced by Lawson and Simons in [**12**], Shiohama and Xu [**17**] proved that an n-dimensional connected orientable complete submanifold M^n in $\boldsymbol{E}^{n+p}$ is homeomorphic to $S^n(c)$ if $n > 3$ and $\sup_{M^n}(S - (n^2H^2/(n-1))) < 0$. It is clear that this condition $\sup_{M^n}(S - (n^2H^2/(n-1))) < 0$ yields that the mean curvature is nonzero at each point of M^n and M^n is compact by Myers theorem. In this paper, we shall prove a stronger result under a weaker condition than the one in [**17**]. That is, we first prove the following:

MAIN THEOREM 1. *An n-dimensional compact connected submanifold M^n with everywhere nonzero mean curvature H in $\boldsymbol{E}^{n+p}$ is diffeomorphic to a sphere $S^n(c)$ if one of the following conditions is satisfied*:

(1) $S \le \dfrac{n^2H^2}{n-1}$,

(2) $n^2H^2 \le \dfrac{(n-1)R}{n-2}$,

where S and R denote the squared norm of the second fundamental form of M^n and the scalar curvature of M^n, respectively.

On the other hand, Klotz and Osserman [**11**] proved that a complete orientable surface M^2 with constant mean curvature H and non-negative Gaussian curvature is isometric to a totally umbilical sphere $S^2(c)$, a totally geodesic plane $\boldsymbol{E}^2$, or cylinder $\boldsymbol{E}^1 \times S^1(c)$. It is well known that the Gaussian curvature is non-negative if and only if $S \le n^2H^2/(n-1)$ holds in the case of $n = 2$. Next, we shall generalize the result due to Klotz and Osserman to higher dimensions and higher codimensions under the same condition of constant mean curvature.

MAIN THEOREM 2. *Let M^n be an n-dimensional $(n > 2)$ complete connected submanifold with constant mean curvature H in $\boldsymbol{E}^{n+p}$. If $S \le n^2H^2/(n-1)$ is satisfied, then M is isometric to the totally umbilical sphere $S^n(c)$, the totally geodesic Euclidean space $\boldsymbol{E}^n$, or the generalized cylinder $S^{n-1}(c) \times \boldsymbol{E}^1$, where S denotes the squared norm of the second fundamental form of M^n.*

REMARK. The result due to Klotz and Osserman [**11**] was extended by the author and Nonaka [**5**] to higher dimensions and higher codimensions under the stronger condition that the mean curvature vector is parallel.

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2. Preliminaries.

Let $\boldsymbol{E}^{n+p}$ be an $(n+p)$-dimensional Euclidean space and M^n an n-dimensional connected submanifold in $\boldsymbol{E}^{n+p}$. We choose a local field of orthonormal frames

$\{e_1, \dots, e_{n+p}\}$ adapted to $\boldsymbol{E}^{n+p}$ and dual coframes $\{\omega_1, \dots, \omega_{n+p}\}$ in such a way that, restricted to the submanifold M^n, $\{e_1, \dots, e_n\}$ are tangent to M^n. Let $\{\omega_{AB}\}$ denote the connection forms of $\boldsymbol{E}^{n+p}$. The canonical forms $\{\omega_A\}$ and connection forms $\{\omega_{AB}\}$ restricted to M^n are also denoted by the same symbols. We then have

$$\omega_\alpha = 0, \quad \alpha = n+1, \dots, n+p. \tag{2.1}$$

We see that $e_1, \dots, e_n$ is a local field of orthonormal frames adapted to the induced Riemannian metric on M^n and $\omega_1, \dots, \omega_n$ is a local field of its dual coframes on M^n. It follows from (2.1) and Cartan's Lemma that

$$\omega_{\alpha i} = \sum_{j=1}^{n} h_{ij}^{\alpha} \omega_j, \quad h_{ij}^{\alpha} = h_{ji}^{\alpha}. \tag{2.2}$$

Second fundamental form II and mean curvature vector $\boldsymbol{h}$ of M^n are defined by

$$II = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} h_{ij}^{\alpha} \omega_i \omega_j e_\alpha, \tag{2.3}$$

$$\boldsymbol{h} = \frac{1}{n} \sum_{\alpha=n+1}^{n+p} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right) e_\alpha. \tag{2.4}$$

The mean curvature H of M^n is defined by

$$H = \frac{1}{n} \sqrt{\sum_{\alpha=n+1}^{n+p} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2}. \tag{2.5}$$

Let $S = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2$ denote the squared norm of the second fundamental form of M^n. The connection form of M^n is characterized by the structure equations

$$d\omega_i = -\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$d\omega_{ij} = -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l=1}^{n} R_{ijkl} \omega_k \wedge \omega_l, \tag{2.7}$$

$$R_{ijkl} = \sum_{\alpha=n+1}^{n+p} (h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha}) \tag{2.8}$$

where R_{ijkl} represents components of the curvature tensor of M^n. Letting R_{ij} and R denote components of the Ricci curvature and the scalar curvature of M^n, respectively, we obtain from (2.8):

$$R_{jk} = \sum_{\alpha=n+1}^{n+p} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} h_{jk}^{\alpha} - \sum_{i=1}^{n} h_{ik}^{\alpha} h_{ji}^{\alpha} \right), \tag{2.9}$$

$$R = n^2 H^2 - S. \tag{2.10}$$

We also have

$$d\omega_{\alpha\beta} = -\sum_{\gamma=n+1}^{n+p} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \frac{1}{2}\sum_{i,j=1}^{n} R_{\alpha\beta ij}\omega_i \wedge \omega_j, \tag{2.11}$$

$$R_{\alpha\beta ij} = \sum_{l=1}^{n}(h_{il}^{\alpha}h_{lj}^{\beta} - h_{jl}^{\alpha}h_{li}^{\beta}). \tag{2.12}$$

By taking the exterior differentiation of (2.2) and defining h_{ijk}^{α} by

$$\sum_{k=1}^{n} h_{ijk}^{\alpha}\omega_k = dh_{ij}^{\alpha} - \sum_{k=1}^{n} h_{ik}^{\alpha}\omega_{kj} - \sum_{k=1}^{n} h_{jk}^{\alpha}\omega_{ki} - \sum_{\beta=n+1}^{n+p} h_{ij}^{\beta}\omega_{\beta\alpha}, \tag{2.13}$$

we obtain Codazzi equation by straightforward computation:

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha} = h_{jik}^{\alpha}. \tag{2.14}$$

We take the exterior differentiation of (2.13) and define h_{ijkl}^{α} by

$$\sum_{l=1}^{n} h_{ijkl}^{\alpha}\omega_l = dh_{ijk}^{\alpha} - \sum_{l=1}^{n} h_{ljk}^{\alpha}\omega_{li} - \sum_{l=1}^{n} h_{ilk}^{\alpha}\omega_{lj} - \sum_{l=1}^{n} h_{ijl}^{\alpha}\omega_{lk} - \sum_{\beta=n+1}^{n+p} h_{ijk}^{\beta}\omega_{\beta\alpha}. \tag{2.15}$$

Then, Ricci formula for the second fundamental form is given by

$$h_{ijkl}^{\alpha} - h_{ijlk}^{\alpha} = \sum_{m=1}^{n} h_{mj}^{\alpha}R_{mikl} + \sum_{m=1}^{n} h_{im}^{\alpha}R_{mjkl} + \sum_{\beta=n+1}^{n+p} h_{ij}^{\beta}R_{\beta\alpha kl}. \tag{2.16}$$

The Laplacian Δh_{ij}^{α} of h_{ij}^{α} is defined by

$$\Delta h_{ij}^{\alpha} = \sum_{k=1}^{n} h_{ijkk}^{\alpha}.$$

From the Codazzi equation (2.14) and the Ricci formula (2.16), we obtain for any α, $n+1 \leq \alpha \leq n+p$,

$$\begin{aligned} \Delta h_{ij}^{\alpha} &= \sum_{k=1}^{n} h_{kijk}^{\alpha} \\ &= \sum_{k=1}^{n} h_{kkij}^{\alpha} + \sum_{k,m=1}^{n} h_{km}^{\alpha}R_{mijk} + \sum_{k,m=1}^{n} h_{mi}^{\alpha}R_{mkjk} + \sum_{k=1}^{n}\sum_{\beta=n+1}^{n+p} h_{ki}^{\beta}R_{\beta\alpha jk}. \end{aligned} \tag{2.17}$$

The following Generalized Maximum Principle of Omori [**14**] and Yau [**21**] will be used in section 3.

Generalized Maximum Principle (Omori [**14**] and Yau [**21**]). *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $f \in C^2(M)$*

a function bounded from above on M^n. Then for any $\varepsilon > 0$, there exists a point $p \in M^n$ such that

$$f(p) \geq \sup f - \varepsilon, \quad \|\mathrm{grad}\, f\|(p) < \varepsilon, \quad \Delta f(p) < \varepsilon.$$

3. The reduction of codimensions.

In this section, we shall prove the following:

THEOREM 3.1. *Let M^n be an n-dimensional submanifold with everywhere nonzero mean curvature H in $\boldsymbol{E}^{n+p}$ which satisfies one of the subsequent conditions. Then M^n lies in an $(n+1)$-dimensional totally geodesic submanifold congruent to $\boldsymbol{E}^{n+1}$ of $\boldsymbol{E}^{n+p}$ if $S \leq n^2H^2/(n-1)$ holds*:

(1) *M^n is compact.*

(2) *M^n is complete and the mean curvature of M^n is constant.*

S denotes the squared norm of the second fundamental form of M^n.

PROOF. Since the mean curvature of M^n is nonzero at each point of M^n, we know that $e_{n+1} = \boldsymbol{h}/H$ is a unit normal vector field defined globally on M^n. Hence, M^n is orientable. We define S_1 and S_2 as

$$S_1 = \sum_{i,j=1}^{n} (h_{ij}^{n+1} - H\delta_{ij})^2, \quad S_2 = \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2, \tag{3.1}$$

respectively. Then, S_1 and S_2 are functions defined on M^n globally, which do not depend on the choice of the orthonormal frame $\{e_1, \dots, e_n\}$. Also,

$$S - nH^2 = S_1 + S_2. \tag{3.2}$$

From the definition of mean curvature vector $\boldsymbol{h}$, we know that $nH = \sum_{i=1}^{n} h_{ii}^{n+1}$ and $\sum_{i=1}^{n} h_{ii}^{\alpha} = 0$ for $n+2 \leq \alpha \leq n+p$ on M^n. Setting $H_\alpha = (h_{ij}^{\alpha})$ and defining $N(A) = \mathrm{trace}({}^tAA)$ for $n \times n$-matrix A, by making use of a direct computation we have, from (2.12) and the Gauss equation (2.8),

$$\begin{aligned}\sum_{\alpha=n+2}^{n+p} \sum_{i,j,k,l=1}^{n} h_{ij}^{\alpha} h_{kl}^{\alpha} R_{lijk} = {} & \sum_{\alpha=n+2}^{n+p} \mathrm{trace}(H_{n+1}H_\alpha)^2 - \sum_{\alpha=n+2}^{n+p} [\mathrm{trace}(H_{n+1}H_\alpha)]^2 \\ & + \sum_{\alpha,\beta=n+2}^{n+p} \mathrm{trace}(H_\alpha H_\beta)^2 - \sum_{\alpha,\beta=n+2}^{n+p} [\mathrm{trace}(H_\alpha H_\beta)]^2,\end{aligned}$$

$$\begin{aligned}\sum_{\alpha=n+2}^{n+p} \sum_{i,j,k,l=1}^{n} h_{ij}^{\alpha} h_{li}^{\alpha} R_{lkjk} = {} & nH \sum_{\alpha=n+2}^{n+p} \mathrm{trace}(H_{n+1}H_\alpha^2) \\ & - \sum_{\alpha=n+2}^{n+p} \mathrm{trace}(H_{n+1}^2 H_\alpha^2) - \sum_{\alpha,\beta=n+2}^{n+p} \mathrm{trace}(H_\alpha H_\beta H_\beta H_\alpha),\end{aligned}$$

and

$$\sum_{\alpha,\beta=n+1}^{n+p}\sum_{i,j,k=1}^{n} h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk} = \sum_{\alpha,\beta=n+1}^{n+p}\operatorname{trace}(H_\alpha H_\beta)^2 - \sum_{\alpha,\beta=n+1}^{n+p}\operatorname{trace}(H_\alpha H_\beta H_\beta H_\alpha).$$

Hence, we conclude from the formula (2.17) in section 2, that

$$(3.3)\qquad \frac{1}{2}\Delta S_2 = \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}$$
$$= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + nH\sum_{\alpha=n+2}^{n+p}\operatorname{trace}(H_{n+1}H_\alpha^2) - \sum_{\alpha=n+2}^{n+p}[\operatorname{trace}(H_{n+1}H_\alpha)]^2$$
$$- \sum_{\alpha,\beta=n+2}^{n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha,\beta=n+2}^{n+p}[\operatorname{trace}(H_\alpha H_\beta)]^2$$
$$+ \sum_{\alpha=n+2}^{n+p}\operatorname{trace}(H_{n+1}H_\alpha)^2 - \sum_{\alpha=n+2}^{n+p}\operatorname{trace}(H_{n+1}^2 H_\alpha^2).$$

According to the following Lemma 3.1 and the definition of S_2, we obtain

$$(3.4)\qquad - \sum_{\alpha,\beta=n+2}^{n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha,\beta=n+2}^{n+p}[\operatorname{trace}(H_\alpha H_\beta)]^2 \geq -\frac{3}{2}S_2^2.$$

Lemma 3.1 (see [**13**]). *For symmetric matrices* $A_1,\dots,A_q$ $(q \geq 1)$, *put* $S_{\alpha\beta} = \operatorname{trace}(A_\alpha A_\beta)$, $S_0 = \sum_{\alpha=1}^{q} S_{\alpha\alpha}$, *and* $N(A_\alpha) = \operatorname{trace}({}^tA_\alpha A_\alpha)$. *Then*

$$\sum_{\alpha,\beta=1}^{q} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta=1}^{q} S_{\alpha\beta}^2 \leq \frac{3}{2}S_0^2.$$

Since $e_{n+1} = \boldsymbol{h}/H$, we have $\operatorname{trace}(H_\alpha) = 0$ for $\alpha = n+2,\dots,n+p$ and $\operatorname{trace}(H_{n+1}) = nH$.

$$- \sum_{\alpha=n+2}^{n+p}\{\operatorname{trace}(H_{n+1}H_\alpha)\}^2 + \sum_{\alpha=n+2}^{n+p}\operatorname{trace}(H_{n+1}H_\alpha)^2 - \sum_{\alpha=n+2}^{n+p}\operatorname{trace}(H_{n+1}^2 H_\alpha^2)$$
$$= \sum_{\alpha=n+2}^{n+p}[-\{\operatorname{trace}(H_{n+1}H_\alpha)\}^2 + \operatorname{trace}(H_{n+1}H_\alpha)^2 - \operatorname{trace}(H_{n+1}^2 H_\alpha^2)]$$
$$= \sum_{\alpha=n+2}^{n+p}[-\{\operatorname{trace}\{(H_{n+1} - HI)H_\alpha\}\}^2$$
$$+ \operatorname{trace}\{(H_{n+1} - HI)H_\alpha\}^2 - \operatorname{trace}\{(H_{n+1} - HI)^2 H_\alpha^2\}],$$

where I denotes the unit matrix.

For a fixed α, $n+2 \leq \alpha \leq n+p$, we can take a local orthonormal frame field $\{e_1,\dots,e_n\}$ such that $h_{ji}^{\alpha} = \lambda_i^{\alpha}\delta_{ij}$. Thus, we have $\sum_{i=1}^{n}\lambda_i^{\alpha} = 0$ and $\operatorname{trace} H_\alpha^2 = \sum_{i=1}^{n}(\lambda_i^{\alpha})^2$. Let $B = H_{n+1} - HI = (b_{ij})$. We have $b_{ij} = b_{ji}$ for any $i,j = 1,\dots,n$, $\sum_{i=1}^{n} b_{ii} = 0$ and $\sum_{i,j=1}^{n} b_{ij}^2 = S_1$.

$$\begin{aligned}
-[\operatorname{trace}\{(H_{n+1}-HI)H_\alpha\}]^2 &+ \operatorname{trace}\{(H_{n+1}-HI)H_\alpha\}^2 - \operatorname{trace}\{(H_{n+1}-HI)^2H_\alpha^2\}\\
&= -\{\operatorname{trace}(BH_\alpha)\}^2 + \operatorname{trace}(BH_\alpha)^2 - \operatorname{trace}(B^2H_\alpha^2)\\
&= -\left(\sum_{i=1}^n b_{ii}\lambda_i^\alpha\right)^2 + \sum_{i=1}^n b_{ij}^2\lambda_i^\alpha\lambda_j^\alpha - \sum_{i=1}^n b_{ij}^2(\lambda_i^\alpha)^2.
\end{aligned}$$

Clearly, λ_i^α and b_{ij} for $i, j = 1, \dots, n$ satisfy the conditions in (1) of Lemma in the Appendix, which is algebraic; a proof of it can be found in [**3**]. For the reader's convenience, we shall give the proof in the Appendix. We obtain

$$\begin{aligned}
-[\operatorname{trace}\{(H_{n+1}-HI)H_\alpha\}]^2 &+ \operatorname{trace}\{(H_{n+1}-HI)H_\alpha\}^2\\
&- \operatorname{trace}\{(H_{n+1}-HI)^2H_\alpha^2\} \geq -S_1 \operatorname{trace} H_\alpha^2.
\end{aligned}$$

Since the two sides of the above inequality do not depend on the choice of local orthonormal frame fields, we have

$$\begin{aligned}
(3.5)\qquad \sum_{\alpha=n+2}^{n+p} [-[\operatorname{trace}\{(H_{n+1}-HI)H_\alpha\}]^2&\\
+ \operatorname{trace}\{(H_{n+1}-HI)H_\alpha\}^2 &- \operatorname{trace}\{(H_{n+1}-HI)^2H_\alpha^2\}]\\
\geq -S_1 \sum_{\alpha=n+2}^{n+p} \operatorname{trace} H_\alpha^2 &= -S_1S_2.
\end{aligned}$$

Making use of the same assertion as above, we obtain, for fixed α, $n+2 \leq \alpha \leq n+p$,

$$\operatorname{trace}\{(H_{n+1}-HI)H_\alpha^2\} = \sum_{i=1}^n b_{ii}(\lambda_i^\alpha)^2.$$

From (2) and (3) of Lemma in the Appendix, we obtain

$$\operatorname{trace}\{(H_{n+1}-HI)H_\alpha^2\} \geq -\frac{n-2}{\sqrt{n(n-1)}}\sqrt{S_1} \operatorname{trace} H_\alpha^2.$$

Hence, we conclude

$$\begin{aligned}
(3.6)\quad nH\sum_{\alpha=n+2}^{n+p} \operatorname{trace}(H_{n+1}H_\alpha^2) &= nH\sum_{\alpha=n+2}^{n+p} \operatorname{trace}\{(H_{n+1}-HI)H_\alpha^2\} + nH^2\sum_{\alpha=n+2}^{n+p} \operatorname{trace} H_\alpha^2\\
&= nH\sum_{\alpha=n+2}^{n+p} \operatorname{trace}\{(H_{n+1}-HI)H_\alpha^2\} + nH^2S_2\\
&\geq nH^2S_2 - \sqrt{\frac{n}{n-1}}(n-2)H\sqrt{S_1}S_2.
\end{aligned}$$

From (3.3), (3.4), (3.5), and (3.6), we have

$$\begin{aligned}(3.7)\quad \frac{1}{2}\Delta S_2 &\geq \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \left(nH^2 - \sqrt{\frac{n}{n-1}}(n-2)H\sqrt{S_1} - S_1 - \frac{3}{2}S_2\right)S_2 \\ &\geq \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \left(nH^2 - \frac{n(n-2)}{2(n-1)}H^2 - \frac{n-2}{2}S_1 - S_1 - \frac{3}{2}S_2\right)S_2 \\ &= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \left(nH^2 - \frac{n(n-2)}{2(n-1)}H^2 + \frac{n^2H^2}{2} - \frac{n}{2}S + \frac{(n-3)}{2}S_2\right)S_2 \\ &= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \left\{\frac{n}{2}\left(\frac{n^2H^2}{n-1} - S\right) + \frac{(n-3)}{2}S_2\right\}S_2 \\ &\geq \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \left\{\frac{(n-3)}{2}S_2\right\}S_2 \geq 0.\end{aligned}$$

When M^n is compact, from Stokes formula we obtain

$$(3.8)\qquad \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 = 0$$

on M^n; and all inequalities are equalities. Hence, we have $S_2 \equiv 0$ for $n > 3$. When $n = 3$, we obtain

$$S_2 \equiv 0 \quad \text{or} \quad S \equiv \frac{n^2H^2}{n-1} \quad \text{and} \quad \sqrt{\frac{n}{n-1}}H \equiv \sqrt{S_1}.$$

From $S = S_1 + S_2 + nH^2$, we also infer that $S_2 \equiv 0$.

When M^n is complete and the mean curvature is constant, from the condition $S \leq n^2H^2/(n-1)$ and from (2.9) we know that the Ricci curvature of M^n is bounded from below. Applying the Generalized Maximum Principle of Omori [**14**] and Yau [**21**] stated in section 2 to the function S_2, we find that there exists a sequence $\{p_k\} \subset M^n$ such that

$$(3.9)\qquad \lim_{k\to\infty} S_2(p_k) = \sup S_2 \quad \text{and} \quad \lim_{k\to\infty} \sup \Delta S_2(p_k) \leq 0.$$

Since $S \leq n^2H^2/(n-1)$, we know that $\{h_{ij}^{\alpha}(p_k)\}$, for any $i, j = 1, 2, \dots, n$ and any $\alpha = n+1, \dots, n+p$, is a bounded sequence. Hence, we can assume $\lim_{k\to\infty} h_{ij}^{\alpha}(p_k) = \tilde{h}_{ij}^{\alpha}$; if necessary, we can take a subsequence. From (3.7) and (3.9), by obtaining the limit of (3.7), we know that all inequalities are equalities. Hence, $\sup S_2 = 0$ for $n > 3$. When $n = 3$, if $\sup S_2 \neq 0$, we know $\lim_{k\to\infty}(n^2H^2/(n-1) - S)(p_k) = 0$ and $\lim_{k\to\infty}\sqrt{n/(n-1)}H(p_k) = \lim_{k\to\infty}\sqrt{S_1(p_k)}$. Let $\lim_{k\to\infty} H(p_k) = \tilde{H}$, $\lim_{k\to\infty} S(p_k) = \tilde{S}$ and $\lim_{k\to\infty} S_1(p_k) = \tilde{S}_1$. Then, we have $n^2\tilde{H}^2/(n-1) = \tilde{S}$, $(n/(n-1))\tilde{H}^2 = \tilde{S}_1$ and $\tilde{S} = \sup S_2 + \tilde{S}_1 + n\tilde{H}^2 = \tilde{S} + \sup S_2$. This is impossible. Hence, we obtain $\sup S_2 = 0$. That is, $S_2 = 0$ on M^n. From (3.7), we have

$$(3.10)\qquad \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 = 0$$

on M^n. Thus, we infer $S_2 \equiv 0$ and (3.10) holds on M^n under the assumption of Theorem 3.1.

From (2.13), we have, for any $\alpha \neq n+1$,

$$\sum_{i,k=1}^{n} h_{iik}^{\alpha}\omega_k = -nH\omega_{\alpha n+1}.$$

Hence, (3.10) yields $\omega_{\alpha n+1} = 0$ for any α. Thus, we know that e_{n+1} is parallel in the normal bundle $T^{\perp}(M^n)$ of M^n. Hence, if we denote by N_1 the normal subbundle spanned by $e_{n+2}, e_{n+3}, \dots, e_{n+p}$ of the normal bundle of M^n, then M^n is totally geodesic with respect to N_1. Since e_{n+1} is parallel in the normal bundle, we know that the normal subbundle N_1 is invariant under parallel translation with respect to normal connection of M^n. Then, from Theorem 1 in [**20**], we conclude that M^n lies in a totally geodesic submanifold congruent to $\boldsymbol{E}^{n+1}$ of $\boldsymbol{E}^{n+p}$. This completes our proof. □

4. Proof of Main Theorems.

This section presents a proof of our Main Theorems.

Proof of Main Theorem 1. From Gauss equation (2.10), we have $R = n^2H^2 - S$. Hence, we know that these two conditions in Main Theorem 1 are equivalent to each other. Thus, we shall only prove Main Theorem 1 under the condition $S \leq n^2H^2/(n-1)$. From Theorem 3.1, we know that M^n lies in a totally geodesic submanifold $\boldsymbol{E}^{n+1}$ of $\boldsymbol{E}^{n+p}$. We denote by H' the mean curvature of M^n in $\boldsymbol{E}^{n+1}$. Since $\boldsymbol{E}^{n+1}$ is totally geodesic in $\boldsymbol{E}^{n+p}$, we have $H = H'$; that is, the mean curvature H' of M^n in $\boldsymbol{E}^{n+1}$ is the same as in $\boldsymbol{E}^{n+p}$. We also know that the squared norm S' of the second fundamental form of M^n in $\boldsymbol{E}^{n+1}$ is the same as in $\boldsymbol{E}^{n+p}$. Hence, $S' \leq n^2(H')^2/(n-1)$ and $H' \neq 0$. We choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$ for $i, j = 1, 2, \dots, n$; h_{ij} and λ_i denote components of the second fundamental form and principal curvatures of M^n in $\boldsymbol{E}^{n+1}$, respectively. Thus, we obtain

$$\sum_{i=1}^{n}(\lambda_i)^2 \leq \frac{(\sum_{i=1}^{n}\lambda_i)^2}{n-1}.$$

From Lemma 4.1 in Chen [**1**, p. 56], we have, for any i, j,

$$\lambda_i\lambda_j \geq 0.$$

Hence, we know that the principal curvatures are non-negative on M^n because the mean curvature is nonzero at each point of M^n. Namely, M^n is locally convex. Therefore, M^n is diffeomorphic to $S^n(c)$ from the result obtained by Van Heijenoort [**18**] and Sacksteder [**15**]. This completes the proof of Main Theorem 1. □

Proof of Main Theorem 2. Since mean curvature H is constant, we have $H = 0$ or $H > 0$. In the case of $H = 0$, we have $S = 0$ on M^n, since $S \leq n^2H^2/(n-1)$ holds. Therefore, we know that M^n is totally geodesic. Hence, M^n is isometric to the hyperplane $\boldsymbol{E}^n$. Next, we assume $H > 0$. Thus $e_{n+1} = \boldsymbol{h}/H$ is a unit normal vector field defined globally on M^n. Hence, M^n is orientable. From the proof of Theorem 3.1, we know that the unit normal vector field $e_{n+1} = \boldsymbol{h}/H$ is parallel in the normal bundle of M^n. Since the mean curvature is constant on M^n, we conclude that the mean curvature vector $\boldsymbol{h} = He_{n+1}$ is also parallel in the normal bundle of M^n. From results obtained by the author and Nonaka [**5**], we know that Main Theorem 2 is true. This completes the proof of Main Theorem 2. □

5. Appendix.

In the Appendix, we shall prove the following:

LEMMA.

(1) *Let $a_1, \dots, a_n$ and b_{ij} for $i, j = 1, \dots, n$ be real numbers satisfying $\sum_{i=1}^n a_i = 0$, $\sum_{i=1}^n b_{ii} = 0$, $\sum_{i,j=1}^n b_{ij}^2 = b$, and $b_{ij} = b_{ji}$ for $i, j = 1, \dots, n$. Then*

$$-\left(\sum_{i=1}^n b_{ii}a_i\right)^2 + \sum_{i,j=1}^n b_{ij}^2 a_i a_j - \sum_{i,j=1}^n b_{ij}^2 a_i^2 \geq -\sum_{i=1}^n a_i^2 b. \tag{5.1}$$

(2) *Let b_i for $i = 1, \dots, n$ be real numbers satisfying $\sum_{i=1}^n b_i = 0$ and $\sum_{i=1}^n b_i^2 = B$. Then*

$$\sum_{i=1}^n b_i^4 - \frac{B^2}{n} \leq \frac{(n-2)^2}{n(n-1)} B^2. \tag{5.2}$$

(3) *Let a_i and b_i for $i = 1, \dots, n$ be real numbers satisfying $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i^2 = a$. Then*

$$\sum_{i=1}^n a_i b_i^2 \geq -\sqrt{\sum_{i=1}^n b_i^4 - \frac{(\sum_{i=1}^n b_i^2)^2}{n}}\sqrt{a}. \tag{5.3}$$

PROOF. In order to prove (1), we consider the function

$$f(x_{ij}) = -\left(\sum_{i=1}^n x_{ii}a_i\right)^2 - \frac{1}{2}\sum_{i,j=1}^n x_{ij}^2 (a_j - a_i)^2, \tag{5.4}$$

subject to the constraint conditions

$$\sum_{i=1}^n x_{ii} = 0 \quad \text{and} \quad \sum_{i,j=1}^n x_{ij}^2 = b. \tag{5.5}$$

Making use of Lagrangian multipliers, we shall calculate the minimum of the function $f(x_{ij})$ with constraint conditions (5.5). Let

$$g = f(x_{ij}) + \lambda \sum_{i=1}^n x_{ii} + \mu\left(\sum_{i,j=1}^n x_{ij}^2 - b\right),$$

where λ and μ are the Lagrangian multipliers. We have

$$g = -\left(\sum_{i=1}^{n} a_i x_{ii}\right)^2 - \frac{1}{2}\sum_{i,j=1}^{n} x_{ij}^2 (a_j - a_i)^2 + \lambda \sum_{i=1}^{n} x_{ii} + \mu\left(\sum_{i,j=1}^{n} x_{ij}^2 - b\right).$$

If f attains its minimum f_0 at some point (x_{ij}), we have

$$-2\sum_{i=1}^{n} a_i x_{ii} a_j + \lambda + 2\mu x_{jj} = 0, \quad \text{for } j = 1, \dots, n, \tag{5.6}$$

$$-x_{ij}(a_j - a_i)^2 + 2\mu x_{ij} = 0, \quad \text{for } i \neq j. \tag{5.7}$$

Hence,

$$-\left(\sum_{i=1}^{n} a_i x_{ii}\right)^2 + \mu \sum_{j=1}^{n} x_{jj}^2 = 0,$$

$$-\frac{1}{2}\sum_{i,j=1}^{n} x_{ij}^2 (a_j - a_i)^2 + \mu \sum_{i,j=1,i\neq j}^{n} x_{ij}^2 = 0.$$

Thus,

$$f_0 = -\mu b.$$

From (5.6) and $\sum_{i=1}^{n} a_i = 0$, we obtain $\lambda = 0$ and

$$\left(\mu - \sum_{j=1}^{n} a_j^2\right)\sum_{i=1}^{n} x_{ii} a_i = 0,$$

$$\mu \sum_{j=1}^{n} x_{jj}^2 - \left(\sum_{i=1}^{n} x_{ii} a_i\right)^2 = 0.$$

If $\sum_{i=1}^{n} x_{ii} a_i \neq 0$, we have $\mu = \sum_{j=1}^{n} a_j^2$. Hence,

$$f_0 = -\mu b = -\sum_{j=1}^{n} a_j^2 b.$$

If $\sum_{i=1}^{n} x_{ii} a_i = 0$, we have $\mu \sum_{j=1}^{n} x_{jj}^2 = 0$. $\mu = 0$ yields $f_0 = 0$. If $\mu \neq 0$, we have $\sum_{j=1}^{n} x_{jj}^2 = 0$. Hence, $b = 0$ or there exists $i \neq j$ such that $x_{ij} \neq 0$. From (5.7), we obtain

$$2\mu = (a_i - a_j)^2 \leq 2\sum_{j=1}^{n} a_j^2.$$

Therefore,

$$f_0 \geq -\sum_{j=1}^{n} a_j^2 b.$$

Since $\sum_{i=1}^{n} a_i = 0$, $\sum_{i=1}^{n} b_{ii} = 0$, $\sum_{i,j=1}^{n} b_{ij}^2 = b$, and $b_{ij} = b_{ji}$ for $i, j = 1, \dots, n$ hold, we have

$$-\left(\sum_{i=1}^{n} b_{ii}a_i\right)^2 + \sum_{i,j=1}^{n} b_{ij}^2 a_i a_j - \sum_{i,j=1}^{n} b_{ij}^2 a_i^2 = -\left(\sum_{i=1}^{n} b_{ii}a_i\right)^2 - \frac{1}{2}\sum_{i,j=1}^{n} b_{ij}^2 (a_j - a_i)^2 \geq -\sum_{j=1}^{n} a_j^2 b.$$

Thus, we complete the proof of (1) of Lemma.

For the proof of (2), we consider the function

$$f(y) = \sum_{i=1}^{n} y_i^4 - \frac{B^2}{n}$$

with constraint conditions $\sum_{i=1}^{n} y_i = 0$ and $\sum_{i=1}^{n} y_i^2 = B$.

Since $\sum_{i=1}^{n} y_i^2 = B$, we know that at least one of the y_i^2's is not less than B/n. We assume the $y_n^2 \geq B/n$, without loss of generality. From $\sum_{i=1}^{n} y_i = 0$, we have

$$y_n^2 = \left(\sum_{i=1}^{n-1} y_i\right)^2 \leq (n-1)\sum_{i=1}^{n-1} y_i^2 = (n-1)(B - y_n^2),$$

$$y_n^2 - \frac{B}{2} = \sum_{1 \leq i < j \leq n-1} y_i y_j,$$

$$y_n^2 \leq \frac{(n-1)B}{n}.$$

Hence,

$$\begin{aligned} f(y) &= \sum_{i=1}^{n-1} y_i^4 + y_n^4 - \frac{B^2}{n} \\ &= \left(\sum_{i=1}^{n-1} y_i^2\right)^2 - 2\sum_{1 \leq i < j \leq n-1} y_i^2 y_j^2 + y_n^4 - \frac{B^2}{n} \\ &\leq (B - y_n^2)^2 - \frac{4}{(n-1)(n-2)}\left(\sum_{1 \leq i < j \leq n-1} y_i y_j\right)^2 + y_n^4 - \frac{B^2}{n} \\ &= \frac{2n(n-3)}{(n-1)(n-2)}(y_n^4 - By_n^2) + \left(\frac{n-1}{n} - \frac{1}{(n-1)(n-2)}\right)B^2. \end{aligned}$$

Since the maximum of the function $t^2 - Bt$ in the interval $[(1/n)B, ((n-1)/n)B]$ is $-((n-1)/n^2)B^2$, we obtain

$$f(y) \leq \frac{(n-2)^2}{n(n-1)}B^2.$$

This completes the proof of (2) of Lemma.

Making use of the Lagrangian multipliers, we calculate the minimum of the function $g(x) = \sum_{i=1}^{n} x_i b_i^2$ with constraint conditions $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = a$. If the function $g(x)$ attains its minimum g_0 at some point x, then we have, at x,

$$b_i^2 + \lambda + 2\mu x_i = 0, \quad \text{for } i = 1, \ldots, n,$$

where λ and μ are the Lagrangian multipliers. Hence, we have

$$g_0 = -2\mu a, \quad \lambda = -\frac{\sum_{i=1}^n b_i^2}{n},$$

$$\sum_{i=1}^n b_i^4 - \frac{(\sum_{i=1}^n b_i^2)^2}{n} + 2\mu g_0 = 0.$$

Thus, (3) of Lemma is true. □

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