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Capelli type identities on certain scalar generalized Verma modules, II

Dedicated to Professor Ryoshi Hotta on his sixtieth birthday

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Abstract. In the preceding paper we gave an analogue of the Capelli identity for relative invariants in Hermitian symmetric settings, and the analogue was constructed on scalar generalized Verma modules. In this paper we give an analogue of the Capelli identity of lower degrees. This analogue contains non-principal minors in contrast with the original Capelli identity.

Introduction.

In the nineteenth century, Capelli [**2**] discovered the following identities:

$$\det[x_{ij}]_{1 \le i,j \le n} \det\left[\frac{\partial}{\partial x_{ij}}\right]_{1 \le i,j \le n} = \det\left[\sum_{k=1}^{n} x_{ki}\frac{\partial}{\partial x_{kj}} + (n-j)\delta_{ij}\right]_{1 \le i,j \le n}, \tag{0.1}$$

$$\sum_{I,J} \det[x_{ij}]_{i \in I, j \in J} \det\left[\frac{\partial}{\partial x_{ij}}\right]_{i \in I, j \in J} = \sum_{I} \det\left[\sum_{k=1}^{n} x_{ki}\frac{\partial}{\partial x_{kj}} + (d-j)\delta_{ij}\right]_{i,j \in I}, \tag{0.2}$$

where n^2 variables x_{ij} are natural coordinate functions on the vector space $V = \mathrm{Mat}(n, \boldsymbol{C})$ of $n \times n$ matrices, and I and J run over all subsets of $\{1, \dots, n\}$ with cardinality d. These formulas, called the *Capelli identities*, play an important role in classical invariant theory.

Let us interpret the Capelli identities from two different points of view. A simple interpretation of the Capelli identities is to regard them as a non-commutative version of the formula of the determinant over a commutative ring: $\det {}^t\!A \det B = \det {}^t\!AB$. Another interpretation is more representation theoretical. Set $L = GL(n, \boldsymbol{C}) \times GL(n, \boldsymbol{C})$ and $\mathfrak{l} = \mathrm{Lie}(L)$. Let $U(\mathfrak{l})$ denote the enveloping algebra of $\mathfrak{l}$, and $Z(\mathfrak{l})$ its center. Let D_V be the ring of differential operators on V with polynomial coefficients. Since L acts on V by $(g,h).X = gXh^{-1}$, we have an algebra homomorphism φ from $Z(\mathfrak{l})$ to the subalgebra D_V^L of all the L-invariants of D_V. Then, the Capelli identities are regarded as the formulas representing an element in D_V^L (the left hand sides in (0.1) and (0.2)) as the image of an element in $Z(\mathfrak{l})$ through φ (the right hand sides there). Moreover, the differential operators in (0.1) and (0.2) give a generator of the algebra D_V^L.

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A century after, Howe and Umeda [**5**] investigated, from the representation theoretical view point, the Capelli identity and its generalization in the context of multiplicity-free action. Precisely, an action of a connected complex algebraic group L on a vector space V is said to be multiplicity-free, if the L-module $\boldsymbol{C}[V]$ of all the polynomials on V decomposes into irreducibles with multiplicity one. Kac [**6**] completely classified multiplicity-free actions (L, V) such that V is irreducible under L-action. In [**5**], the Capelli identities were studied for every irreducible multiplicity-free action (L, V).

A large number of irreducible multiplicity-free actions come from prehomogeneous vector spaces $(L, \mathfrak{n}^+)$ of commutative parabolic type, attached to complex simple Lie algebras $\mathfrak{g}$ of Hermitian type. Namely, $\mathfrak{n}^+$ is the nilradical of a maximal parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, and $\mathfrak{n}^+$ is assumed to be commutative. We write $\mathfrak{l}$ for a Levi subalgebra of $\mathfrak{p}$. Then, a connected algebraic group L with Lie algebra $\mathfrak{l}$ acts on $\mathfrak{n}^+$ prehomogeneously through the adjoint representation, and this action is irreducible and multiplicity-free. For such a pair $(\mathfrak{g}, \mathfrak{p})$, we construct the scalar generalized Verma modules $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \boldsymbol{C}_\lambda$ induced from one-dimensional representations $\boldsymbol{C}_\lambda$ of $\mathfrak{p}$. $M(\lambda)$ is realized on the polynomial ring $\boldsymbol{C}[\mathfrak{n}^+]$ in the canonical way, and we denote by Ψ_λ the corresponding $\mathfrak{g}$-action on $\boldsymbol{C}[\mathfrak{n}^+]$.

The purpose of this paper is to give a *Ψ_λ-analogue* of the Capelli identities of lower degrees (0.2) on scalar generalized Verma modules $M(\lambda)$ for classical Lie algebras $\mathfrak{g}$ of Hermitian type–$(\mathrm{A}_{p+q-1}, p), (\mathrm{B}_n, 1), (\mathrm{C}_n, n)$ and $(\mathrm{D}_n, 1)$ (for the notation, see §1). We also give the Ψ_λ-analogue of the Turnbull identities for the case (D_n, n). This is a continuation of our work [**12**], where we established such an analogue of the identity (0.1) for the relative invariants of regular prehomogeneous vector spaces $(L, \mathfrak{n}^+)$ of commutative parabolic type. In what follows, the original Capelli identities (due to Capelli, Howe and Umeda) will be called *classical*, in order to distinguish them from our Ψ_λ-analogue.

Taking (A_{2n-1}, n) for example (Theorem 2.2), we give further explanations of our result:

$$\Psi_\lambda\left(\sum_{IJ} f_{IJ}\,{}^t f_{IJ}\right) = (-1)^d \sum_{IJ} \Psi_{2\lambda+2\rho}(u_{IJ}) \Psi_0(u_{JI}), \tag{0.3}$$

$$f_{IJ} = \det[E_{n+j,i}]_{i\in I, j\in J}, \quad {}^t f_{IJ} = \det[E_{i,n+j}]_{i\in I, j\in J},$$

$$u_{IJ} = \det[-E_{ij} + (j-1)\delta_{ij}]_{i\in I, j\in J},$$

where I and J run over all subsets of $\{1, \ldots, n\}$ with cardinality d, and E_{ij} is the matrix unit of $\mathfrak{g} = \mathfrak{gl}(2n, \boldsymbol{C})$. The Ψ_λ-analogue (0.3) expresses an L-invariant operator (left hand side) as an image of $Z(\mathfrak{l})$ (right hand side). On the left hand side, the term $\sum_{IJ} f_{IJ}\,{}^t f_{IJ}$ is a generator of $(S(\mathfrak{n}^-)S(\mathfrak{n}^+))^L$ corresponding to the left hand side of (0.2) through the canonical isomorphism from $(S(\mathfrak{n}^-)S(\mathfrak{n}^+))^L$ to $D^L_{\mathfrak{n}^+}$. On the right hand side, the summands contain non-principal minors contrary to the classical Capelli identities. Both sides of (0.3) have order $2n$ as differential operators, which is a fundamental difference from the classical Capelli identity. To prove the Ψ_λ-analogue for $(\mathrm{A}_{p+q-1}, p), (\mathrm{C}_n, n)$ and (D_n, n), we use the idea of Noumi, Umeda and Wakayama [**7**].

The essence of the idea is to use an exterior algebra for expressing determinants. In the forthcoming paper, we will discuss the Ψ_λ-analogue for the case $(E_6, 1)$ and $(E_7, 7)$.

Let us explain a close connection between the Ψ_λ-analogue and the structure of generalized Verma modules. Every irreducible unitary highest weight module is expressed as a quotient of a generalized Verma module $M(\lambda)$ of $\mathfrak{g}$ of Hermitian type [**3**], and the b-function of the prehomogeneous vector space $(L, \mathfrak{n}^+)$ controls which λ gives the irreducible unitary quotient. In addition, the b-function controls the irreducibility of generalized Verma modules ([**8**], [**4**]). Using the classical Capelli identity we can compute the b-function, and, analogously, using the Ψ_λ-analogue we can compute an analogue of the b-function, which is twisted by the character λ. This analogue of the b-function intrinsically reveals why the b-function controls the structure of the generalized Verma modules [**11**]. The Ψ_λ-analogue thus has a close connection with the structure of the generalized Verma modules via the b-function.

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1. Scalar generalized Verma module.

In this section we will give a realization of scalar generalized Verma modules as a representation on a certain polynomial ring. We first fix the notation. Let $\mathfrak{g}, \mathfrak{h}, \Delta$ and Δ^+ be a simple Lie algebra, its Cartan subalgebra, the root system and a positive root system, respectively. We denote the simple roots by $\alpha_1, \dots, \alpha_n$ and the corresponding fundamental weights by $\varpi_1, \dots, \varpi_n$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ including $\mathfrak{h}$ and all the positive root spaces. Let $\mathfrak{l}$ be the Levi subalgebra of $\mathfrak{p}$ including $\mathfrak{h}$, and $\mathfrak{n}^+$ the nilpotent radical of $\mathfrak{p}$. Let Δ_L and Δ_N^+ be the sets of roots occurring in $\mathfrak{l}$ and $\mathfrak{n}^+$, respectively. Set $\mathfrak{n}^- = \sum_{\alpha \in \Delta_N^+} \mathfrak{g}^{-\alpha}$, where $\mathfrak{g}^\alpha$ denotes the α-root space of $\mathfrak{g}$. We fix an invariant bilinear form $\langle \, , \rangle$ on $\mathfrak{g}$.

Throughout this paper, a pair $(\mathfrak{g}, \mathfrak{p})$ is assumed to be of Hermitian symmetric type, that is, $\mathfrak{n}^+$ is nonzero and commutative. In this situation, $\mathfrak{p}$ has to be a maximal parabolic subalgebra, and therefore there exists the unique simple root α_{i_0} which does not belong to Δ_L. We often denote $(\mathfrak{g}, \mathfrak{p})$ by $(\mathfrak{g}, i_0)$ using Bourbaki's numbering of simple roots ([**1**]).

For a character $\lambda \in \mathrm{Hom}(\mathfrak{p}, \boldsymbol{C})$, we define

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \boldsymbol{C}_\lambda, \tag{1.1}$$

which is said to be the scalar generalized Verma module induced from λ, where $\boldsymbol{C}_\lambda$ denotes the representation space of λ. There exists a linear isomorphism $M(\lambda) \simeq U(\mathfrak{n}^-) = S(\mathfrak{n}^-) \simeq \boldsymbol{C}[\mathfrak{n}^+]$, since both $\mathfrak{n}^-$ and $\mathfrak{n}^+$ are commutative Lie algebras, in which formula the last isomorphism is due to the identification $\mathfrak{n}^- \simeq (\mathfrak{n}^+)^*$ via $\langle \, , \rangle$. This isomorphism yields a representation $(U(\mathfrak{g}), \Psi_\lambda, \boldsymbol{C}[\mathfrak{n}^+])$, and its explicit form is as follows:

LEMMA 1.1 (cf. [**11**]). *Let $\{F_k\}$ be a basis of $\mathfrak{n}^-$. Then we have*

(1) $\Psi_\lambda(X) = X \qquad (X \in \mathfrak{n}^-),$

(2) $\Psi_\lambda(X) = \mathrm{ad}(X) + \lambda(X)$

$$= \sum_k [X, F_k] \frac{\partial}{\partial F_k} + \lambda(X) \qquad (X \in \mathfrak{l}),$$

(3) $\Psi_\lambda(X) = \frac{1}{2}\sum_{k,l} [[X, F_k], F_l] \frac{\partial}{\partial F_k} \frac{\partial}{\partial F_l} + \sum_k \lambda([X, F_k]) \frac{\partial}{\partial F_k} \qquad (X \in \mathfrak{n}^+).$

The operator in (1) *means the multiplication operator.*

Next we fix a Chevalley basis $\{X_\alpha \in \mathfrak{g}^\alpha \mid \alpha \in \Delta\} \cup \{H_i \mid i = 1, \dots, n\}$ of $\mathfrak{g}$, where H_i denotes the coroot of α_i. In addition, for $P \in S(\mathfrak{n}^+)$, we define a constant coefficient differential operator $P(\partial)$ on $\mathfrak{n}^+$ by

$$P(\partial) \exp\langle x, y\rangle = P(y) \exp\langle x, y\rangle \quad (x \in \mathfrak{n}^+, y \in \mathfrak{n}^-).$$

We define three involutions; one is on $U(\mathfrak{g})$ and the rest are on $D_{\mathfrak{n}^+}$.

Definition 1.2. Let $D_{\mathfrak{n}^+}$ be the ring of polynomial coefficient differential operators on $\mathfrak{n}^+$. For a linear endomorphism ψ on some ring, we call it an anti-involution if $\psi \neq \mathrm{id}$, $\psi^2 = \mathrm{id}$, and $\psi(xy) = \psi(y)\psi(x)$ for all x, y. We have

(1) Define an anti-involution $x \mapsto {}^t x$ on $U(\mathfrak{g})$ by

$$ {}^t X_\alpha = X_{-\alpha} \quad (\alpha \in \Delta),$$
$$ {}^t H_i = H_i \quad (i \in \{1, \dots, n\}),$$

where X_α and H_i are members of the fixed Chevalley basis.

(2) Define an anti-involution σ on $D_{\mathfrak{n}^+}$ by

$$\sigma(F_j) = F_j,$$
$$\sigma\left(\frac{\partial}{\partial F_j}\right) = -\frac{\partial}{\partial F_j},$$

where $\{F_j\}$ is a basis of $\mathfrak{n}^-$, and this definition is clearly independent of the choice of a basis.

(3) Define an anti-involution τ on $D_{\mathfrak{n}^+}$ by

$$\tau(X_{-\alpha}) = X_\alpha(\partial),$$
$$\tau(X_\alpha(\partial)) = X_{-\alpha} \quad (\alpha \in \Delta_N^+),$$

where X_α is a member of the fixed Chevalley basis, and we note that

$$X_\alpha(\partial) = \langle X_\alpha, X_{-\alpha}\rangle \frac{\partial}{\partial X_{-\alpha}} = \frac{2}{(\alpha, \alpha)} \frac{\partial}{\partial X_{-\alpha}}.$$

Then we have the following lemma.

Lemma 1.3 (cf. [**12**]). (1) *The anti-involution $x \mapsto {}^t x$ on $U(\mathfrak{g})$ is the identity mapping on the center $Z(\mathfrak{l})$ of $U(\mathfrak{l})$.*

(2) *The anti-involution* σ *on* $D_{\mathfrak{n}^+}$ *satisfies*

$$\sigma(\Psi_\lambda(u)) = \Psi_{-\lambda-2\rho}(s(u)) \quad (u \in U(\mathfrak{g})),$$

where s *is the anti-involution on* $U(\mathfrak{g})$ *defined by*

$$s(X) = \begin{cases} -X & (X \in \mathfrak{l}), \\ X & (X \in \mathfrak{n}^+ + \mathfrak{n}^-), \end{cases}$$

and $\rho \in \mathrm{Hom}(\mathfrak{p}, \boldsymbol{C})$ *is the half sum of roots in* Δ_N^+.

(3) *The anti-involution* τ *on* $D_{\mathfrak{n}^+}$ *satisfies*

$$\tau(\Psi_\lambda(u)) = \Psi_\lambda({}^t u) \quad (u \in U(\mathfrak{l})).$$

(4) *The anti-involution* τ *is the identity mapping on* $\mathrm{Ad}(L)$*-invariant subspace* $D_{\mathfrak{n}^+}^L$ *of* $D_{\mathfrak{n}^+}$.

2. Main theorems.

In this section we will give a realization of $\mathfrak{g}$, and give the Ψ_λ-analogue on $M(\lambda)$ of the classical Capelli identity for each type of $(\mathrm{A}_{p+q-1}, p), (\mathrm{B}_n, 1), (\mathrm{C}_n, n), (\mathrm{D}_n, 1), (\mathrm{D}_n, n)$, or for each type of $GL_p \otimes GL_q$, $O_{2n} \otimes GL_1$, $S^2 GL_n$, $O_{2n-1} \otimes GL_1$, $\Lambda^2 GL_n$ using the notation of [**5**], and the point of the analogue in this paper is that there appear minors in contrast with [**12**].

For the pair (D_n, n) or $\Lambda^2 GL_n$, the classical Capelli identity is a formula using Pfaffians. We, however, give the Ψ_λ-analogue of the Turnbull identity instead, which is a formula using permanents. The Turnbull identity first appeared in [**9**], and in this paper we mean the Turnbull identity by the formula explicitly given in [**10**]. We note that the Turnbull identity, however, is not a complete substitute for the Capelli identity, since the Turnbull identity does not write all generators of $D_{\mathfrak{n}^+}^L$ as images of $Z(\mathfrak{l})$.

We make a note on the proofs of the Ψ_λ-analogues. For the cases where the prehomogeneous vector space $(L, \mathrm{Ad}, \mathfrak{n}^+)$ is regular (i.e. has a relative invariant), [**12**] gives a type-independent proof of Ψ_λ-analogues of the classical Capelli identities for the relative invariants. In this paper we generalize the result of [**12**], the idea of the proof in this paper is due to [**7**], and the proof depends on the types of pairs of $(\mathfrak{g}, i_0)$.

2.1. (A_{p+q-1}, p) or $GL_p \otimes GL_q$.

Set $\mathfrak{g} = \mathfrak{gl}(p+q, \boldsymbol{C})$. Let $\mathfrak{h}$ be the set of diagonal matrices of $\mathfrak{g}$, and E_{ij} the matrix unit, and we define $\varepsilon_i \in \mathfrak{h}^*$ $(i \in \{1, \dots, p+q\})$ by $\varepsilon_i(E_{jj}) = \delta_{ij}$. We summarize data such as the root system or a Chevalley basis (C. B.) in the following list, where Π and Π_L denote the sets of simple roots of $\mathfrak{g}$ and $\mathfrak{l}$, respectively.

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{p+q-1} - \varepsilon_{p+q}\},$$

$$\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le p+q\},$$

$$E_{ij} : (\varepsilon_i - \varepsilon_j)\text{-root vector for } i \neq j,$$

$$\Pi_L = \Pi \backslash \{\varepsilon_p - \varepsilon_{p+1}\},$$

$$\Delta_L^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le p\} \cup \{\varepsilon_i - \varepsilon_j \mid p+1 \le i < j \le p+q\},$$

$$\varpi_{i_0} = (q(\varepsilon_1 + \cdots + \varepsilon_p) - p(\varepsilon_{p+1} + \cdots + \varepsilon_{p+q}))/(p+q),$$
$$2\rho = (p+q)\varpi_{i_0},$$
$$\langle X, Y\rangle = \mathrm{Tr}(XY) \quad (X, Y \in \mathfrak{g}),$$
$$\text{C. B.} : \{E_{ij} \mid i \neq j\} \cup \{E_{ii} - E_{i+1,i+1} \mid 1 \leq i < p+q\}.$$

Subalgebras $\mathfrak{p}, \mathfrak{n}^+$ and $\mathfrak{l}$ are as follows:

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathfrak{g} \mid A \in \mathfrak{gl}(p, \boldsymbol{C}), B \in \mathrm{Mat}(p,q;\boldsymbol{C}), D \in \mathfrak{gl}(q, \boldsymbol{C}) \right\},$$
$$\mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \mid B \in \mathrm{Mat}(p,q;\boldsymbol{C}) \right\},$$
$$\mathfrak{l} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{g} \mid A \in \mathfrak{gl}(p, \boldsymbol{C}), D \in \mathfrak{gl}(q, \boldsymbol{C}) \right\}.$$

We have ${}^tE_{ij} = E_{ji}$ with respect to the involution $x \mapsto {}^tx$ of Definition 1.2. We obtain a linear coordinate system $\{x_{ij}\}$ on $\mathfrak{n}^+$ by defining $x_{ij} = E_{p+j,i}$, and we set $\partial_{ij} = \partial/\partial x_{ij}$. The lemma below follows from an easy calculation using Lemma 1.1.

Lemma 2.1. (1) $\Psi_\lambda(E_{ij}) = -\sum_{k=1}^{q} x_{jk}\partial_{ik} + \lambda(E_{ij})$ $(1 \leq i, j \leq p)$,

(2) $\Psi_\lambda(E_{p+i,p+j}) = \sum_{k=1}^{p} x_{ki}\partial_{kj} + \lambda(E_{p+i,p+j})$ $(1 \leq i, j \leq q)$,

(3) $\Psi_\lambda(E_{i,p+j}) = -\sum_{1 \leq k \leq q, 1 \leq l \leq p} x_{lk}\partial_{ik}\partial_{lj} + \lambda^0\partial_{ij}$ $(1 \leq i \leq p, 1 \leq j \leq q)$,

where λ^0 is the complex number such that $\lambda = \lambda^0\varpi_{i_0}$.

For $1 \leq d \leq \min(p,q)$, we set

$$f_{IJ} = \det[x_{I(s)J(t)}]_{1 \leq s,t \leq d},$$

where $I \subset \{1, \ldots, p\}$, $J \subset \{1, \ldots, q\}$ with $\#I = \#J = d$, and then we have

$${}^tf_{IJ}(\partial) = \det[\partial_{I(s)J(t)}]_{1 \leq s,t \leq d},$$

where $I(1), I(2), \ldots, I(d)$ are members of I in ascending order. In the special case where $p = q$,

$$f = \det[x_{ij}]_{1 \leq i,j \leq p},$$

is the relative invariant with weight $-2\varpi_{i_0}$.

Theorem 2.2. *For $1 \leq d \leq \min(p,q)$, $H, I \subset \{1, \ldots, p\}$ with $\#H = \#I = d$, we set*

$$u_{HI}^{L} = \det[-E_{H(s)I(t)} + (t-1)\delta_{H(s)I(t)}]_{1 \leq s,t \leq d},$$
$$u_{HI}^{LT} = \det[-E_{H(t)I(s)} + (d-t)\delta_{H(t)I(s)}]_{1 \leq s,t \leq d},$$
$$v_{HI}^{L} = \det[-E_{H(s)I(t)} + (q-d+t)\delta_{H(s)I(t)}]_{1 \leq s,t \leq d},$$
$$v_{HI}^{LT} = \det[-E_{H(t)I(s)} + (q-t+1)\delta_{H(t)I(s)}]_{1 \leq s,t \leq d},$$

and we first have

$$\sum_{IJ} f_{IJ}{}^t f_{IJ}(\partial) = \sum_I \mathrm{ad}(u_{II}^L), \tag{2.1}$$

$$\sum_{IJ} f_{IJ}{}^t f_{IJ}(\partial) = \sum_I \mathrm{ad}(u_{II}^{LT}), \tag{2.2}$$

$$\sum_{IJ} {}^t f_{IJ}(\partial) f_{IJ} = \sum_I \mathrm{ad}(v_{II}^L), \tag{2.3}$$

$$\sum_{IJ} {}^t f_{IJ}(\partial) f_{IJ} = \sum_I \mathrm{ad}(v_{II}^{LT}). \tag{2.4}$$

Second we have

$$\sum_{IJ} \Psi_\lambda(f_{IJ}{}^t f_{IJ}) = (-1)^d \sum_{HI} \Psi_{((p+q)/q)\lambda+(2p/q)\rho}(u_{HI}^L)\Psi_0(u_{IH}^L), \tag{2.5}$$

$$\sum_{IJ} \Psi_\lambda(f_{IJ}{}^t f_{IJ}) = (-1)^d \sum_{HI} \Psi_0(u_{HI}^{LT})\Psi_{((p+q)/q)\lambda+(2p/q)\rho}(u_{IH}^{LT}), \tag{2.6}$$

$$\sum_{IJ} \Psi_\lambda({}^t f_{IJ} f_{IJ}) = (-1)^d \sum_{HI} \Psi_0(v_{HI}^L)\Psi_{((p+q)/q)\lambda+2\rho}(v_{IH}^L), \tag{2.7}$$

$$\sum_{IJ} \Psi_\lambda({}^t f_{IJ} f_{IJ}) = (-1)^d \sum_{HI} \Psi_{((p+q)/q)\lambda+2\rho}(v_{HI}^{LT})\Psi_0(v_{IH}^{LT}). \tag{2.8}$$

In the summations above, H and I run over all subsets of $\{1,\dots,p\}$ with cardinality d, and J runs over all subsets of $\{1,\dots,q\}$ with cardinality d. We regard f_{IJ} as an element in $\boldsymbol{C}[\mathfrak{n}^+]$ from (2.1) *to* (2.4), *and as an element in $S(\mathfrak{n}^-)$ from* (2.5) *to* (2.8).

Remark 2.3. (1) The formula (2.5) does not hold if we interchange $\Psi_{((p+q)/q)\lambda+(2p/q)\rho}(u_{HI}^L)$ with $\Psi_0(u_{IH}^L)$, because u_{HI}^L does not belong to $Z(\mathfrak{l})$ in general. Similarly in the equalities (2.6), (2.7) and (2.8), we can not interchange the order of the multiplications in the right hand sides.

(2) There only appear principal minors in the right hand sides of the classical Capelli identities from (2.1) to (2.4), while there appear all $d \times d$ minors in the Ψ_λ-analogues from (2.5) to (2.8).

The rest of this subsection is devoted to proving Theorem 2.2. We first recall the way to express determinants in terms of an exterior algebra. We take the exterior algebra $\bigwedge(\boldsymbol{C}^n)$ of $\boldsymbol{C}^n$, and we define a ring structure on the tensor product $\bigwedge(\boldsymbol{C}^n) \otimes_{\boldsymbol{C}} D_{\mathfrak{n}^+}$ by $(x \otimes u)\cdot(y \otimes v) = xy \otimes uv$. We omit symbols $\wedge$ in writing elements of $\bigwedge(\boldsymbol{C}^n)$, like xy instead of $x \wedge y$. Let $\{a_1,\dots,a_n\}$ be a basis of $\boldsymbol{C}^n$. For a matrix $[A_{ij}] \in \mathrm{Mat}(n,m; D_{\mathfrak{n}^+})$, we define $\eta_j \in \bigwedge(\boldsymbol{C}^n) \otimes_{\boldsymbol{C}} D_{\mathfrak{n}^+}$ by

$$(\eta_1,\dots,\eta_m) = (a_1,\dots,a_n)[A_{ij}]_{1\le i\le n,\,1\le j\le m}.$$

Then we can compute determinants as follows, for $J \subset \{1,\dots,m\}$ with $\#J = d$:

$$\begin{aligned}\eta_{J(1)}\cdots\eta_{J(d)} &= \sum_{1\le i_1,\dots,i_d\le n} a_{i_1}A_{i_1J(1)}\cdots a_{i_d}A_{i_dJ(d)}\\ &= \sum_{1\le i_1,\dots,i_d\le n} a_{i_1}\cdots a_{i_d}A_{i_1J(1)}\cdots A_{i_dJ(d)}\\ &= \sum_{I\subset\{1,\dots,n\},\#I=d,\sigma\in\mathfrak{S}_d} a_{I(\sigma(1))}\cdots a_{I(\sigma(d))}A_{I(\sigma(1))J(1)}\cdots A_{I(\sigma(d))J(d)}\\ &= \sum_I a_{I(1)}\cdots a_{I(d)}\sum_\sigma \varepsilon(\sigma)A_{I(\sigma(1))J(1)}\cdots A_{I(\sigma(d))J(d)}\\ &= \sum_I a_{I(1)}\cdots a_{I(d)}\det[A_{I(s)J(t)}]_{1\le s,t\le d}. \end{aligned}\tag{2.9}$$

We prove two lemmas before proving the theorem.

Lemma 2.4. *For $H, I\subset\{1,\dots,p\}$ with $\#H=\#I=d$, we have*

(1) $$\sum_J f_{IJ}{}^tf_{HJ}(\partial) = \mathrm{ad}(\det[-E_{H(t)I(s)}+(d-t)\delta_{H(t)I(s)}]_{1\le s,t\le d}),$$

(2) $$\sum_J {}^tf_{HJ}(\partial)f_{IJ} = \mathrm{ad}(\det[-E_{H(t)I(s)}+(q-t+1)\delta_{H(t)I(s)}]_{1\le s,t\le d}).$$

In the summations above, J runs over all subsets of $\{1,\dots,q\}$ with cardinality d. We remark that we can obtain the classical Capelli identities, when we consider the special case of $H=I$ in each of the formulas above, and take a sum over I.

Proof. [proof of (1)]
First we set

$$(\eta_1,\dots,\eta_q) = (a_1,\dots,a_p)[x_{ij}]_{1\le i\le p,1\le j\le q},$$

and then it follows from (2.9) for $J\subset\{1,\dots,q\}$ with $\#J=d$ that

$$\eta_{J(1)}\cdots\eta_{J(d)} = \sum_{I\subset\{1,\dots,p\},\#I=d} a_{I(1)}\cdots a_{I(d)}f_{IJ}.$$

In addition, we have $\eta_i\eta_j=-\eta_j\eta_i$, since x_{ij}'s commute with each other. Second we set

$$(\zeta_1,\dots,\zeta_p) = (\eta_1,\dots,\eta_q)[\partial_{ij}]_{1\le j\le q,1\le i\le p},$$

and we then have the commutation relation $\zeta_h\eta_j=-\eta_j\zeta_h+\eta_ja_h$ using $[\partial_{hk},\eta_j]=\delta_{kj}a_h$. We can also show that $(\zeta_1,\dots,\zeta_p)=(a_1,\dots,a_n)[\mathrm{ad}(-E_{hi})]_{1\le i,h\le p}$. Next we set

$$(\zeta_1(u),\dots,\zeta_p(u)) = (a_1,\dots,a_p)[\mathrm{ad}(-E_{hi})-u\delta_{hi}]_{1\le i,h\le p},$$

and then we have $\zeta_h(u)=\zeta_h-ua_h$. Hence we have the commutation relation $\zeta_h(u)\eta_j = -\eta_j\zeta_h(u+1)$.

Let us calculate $\zeta_{H(1)}(-d+1)\cdots\zeta_{H(d-1)}(-1)\zeta_{H(d)}(0)$ for $H\subset\{1,\dots,p\}$ with $\#H=d$ in two different ways. First we calculate as follows:

$$\zeta_{H(1)}(-d+1)\cdots\zeta_{H(d-1)}(-1)\zeta_{H(d)}(0) = \zeta_{H(1)}(-d+1)\cdots\zeta_{H(d-1)}(-1)\sum_{j_d=1}^q \eta_{j_d}\partial_{H(d)j_d}.$$

Here we repeatedly use $\zeta_h(u)\eta_j = -\eta_j\zeta_h(u+1)$, and thereby we can calculate the right hand side of this formula as follows:

$$
\begin{aligned}
(-1)^{d-1} & \sum_{j_d} \eta_{j_d}\zeta_{H(1)}(-d+2)\cdots\zeta_{H(d-1)}(0)\partial_{H(d)j_d} \\
&= \\
&\vdots \\
&= ((-1)^{d-1})^d \sum_{1\le j_1,\dots,j_d\le q} \eta_{j_1}\cdots\eta_{j_d}\partial_{H(1)j_1}\cdots\partial_{H(d)j_d} \\
&= \sum_{J\subset\{1,\dots,q\},\#J=d,\sigma\in\mathfrak{S}_d} \varepsilon(\sigma)\eta_{J(1)}\cdots\eta_{J(d)}\partial_{H(1)J(\sigma(1))}\cdots\partial_{H(d)J(\sigma(d))} \\
&= \sum_{I\subset\{1,\dots,p\},\#I=d} a_{I(1)}\cdots a_{I(d)} \sum_{J\subset\{1,\dots,q\},\#J=d} f_{IJ}\,{}^tf_{HJ}(\partial).
\end{aligned}
$$

In the last equality above we used (2.9).

Second we calculate as follows:

$$
\begin{aligned}
&\zeta_{H(1)}(-d+1)\cdots\zeta_{H(d-1)}(-1)\zeta_{H(d)}(0) \\
&\quad= \left\{\sum_{i_1=1}^{p} a_{i_1}(\mathrm{ad}(-E_{H(1)i_1}) - (-d+1)\delta_{H(1)i_1})\right\}\cdots\left\{\sum_{i_d=1}^{p} a_{i_d}(\mathrm{ad}(-E_{H(d)i_d}) - 0\delta_{H(d)i_d})\right\} \\
&\quad= \sum_{1\le i_1,\dots,i_d\le p} a_{i_1}\cdots a_{i_d}\{\mathrm{ad}(-E_{H(1)i_1}) - (-d+1)\delta_{H(1)i_1}\}\cdots\{\mathrm{ad}(-E_{H(d)i_d}) - 0\delta_{H(d)i_d}\} \\
&\quad= \sum_{I\subset\{1,\dots,p\},\#I=d} a_{I(1)}\cdots a_{I(d)}\,\mathrm{ad}(\det[-E_{H(t)I(s)} - (-d+t)\delta_{H(t)I(s)}]_{1\le s,t\le d}).
\end{aligned}
$$

Here we have calculated in a way similar to (2.9). Taking a look at these two result of the calculations, we conclude (1) holds since a summand in the first result and that in the second result must coincide, when they correspond to the same I.

[proof of (2)]

First it follows from $[\partial_{hk},\eta_j] = \delta_{kj}a_h$ that $\sum_{j=1}^{q}\partial_{ij}\eta_j = \sum_j(\eta_j\partial_{ij} + \delta_{jj}a_i) = \zeta_i + qa_i = \zeta_j(-q)$. This time we calculate $\zeta_{H(1)}(-q)\zeta_{H(2)}(-q+1)\cdots\zeta_{H(d)}(-q+d-1)$ for $H\subset\{1,\dots,p\}$ with $\#H = d$ in two different ways like the proof of (1). Then we have the assertion. □

Lemma 2.5. *For $I\subset\{1,\dots,p\}$, $J\subset\{1,\dots,q\}$ with $\#I = \#J = d$, we have*

(1) $$\Psi_\lambda({}^tf_{IJ}) = \sum_H {}^tf_{HJ}(\partial)\,\mathrm{ad}(\det[E_{I(t)H(s)} + (\lambda^0 + p - d + t)\delta_{I(t)H(s)}]_{1\le s,t\le d}),$$

(2) $$\Psi_\lambda({}^tf_{IJ}) = \sum_H \mathrm{ad}(\det[E_{I(t)H(s)} + (\lambda^0 + t - 1)\delta_{I(t)H(s)}]_{1\le s,t\le d})\,{}^tf_{HJ}(\partial),$$

where H runs over all subsets of $\{1,\dots,p\}$ with cardinality d.

PROOF. [proof of (1)]

While we have discussed in $\bigwedge(\boldsymbol{C}^p)\otimes_{\boldsymbol{C}} D_{\mathfrak{n}^+}$ in the previous lemma, we will discuss in $\bigwedge(\boldsymbol{C}^q)\otimes_{\boldsymbol{C}} D_{\mathfrak{n}^+}$ in this lemma. We set $(\mu_1,\dots,\mu_p)=(a_1,\dots,a_q)[\partial_{ij}]_{1\le j\le q,1\le i\le p}$, and it follows from (2.9) for $I\subset\{1,\dots,p\}$ with $\#I=d$ that

$$\mu_{I(1)}\cdots\mu_{I(d)}=\sum_{J\subset\{1,\dots,q\},\#J=d} a_{J(1)}\cdots a_{J(d)}\,{}^t\!f_{IJ}(\partial).$$

Similarly we set $(\xi_1,\dots,\xi_p)=(a_1,\dots,a_q)[\Psi_\lambda(E_{i,p+j})]_{1\le j\le q,1\le i\le p}$, and we have

$$\xi_{I(1)}\cdots\xi_{I(d)}=\sum_{J\subset\{1,\dots,q\},\#J=d} a_{J(1)}\cdots a_{J(d)}\Psi_\lambda({}^t\!f_{IJ}), \tag{2.10}$$

for $I\subset\{1,\dots,p\}$ with $\#I=d$. Here we have two expressions of ξ_h $(1\le h\le p)$. First one is

$$\xi_h=\sum_{i=1}^p(\mathrm{ad}(E_{hi})+\lambda^0\delta_{hi})\mu_i,$$

which is proved easily. Using the commutation relation $[\mathrm{ad}(E_{hi}),\mu_g]=\delta_{ig}\mu_h$ for $1\le g$, $h,i\le p$, we can obtain the second expression:

$$\xi_h=\sum_{i=1}^p\mu_i\{\mathrm{ad}(E_{hi})+(\lambda^0+p)\delta_{hi}\}.$$

We have the relation $\xi_h\mu_g=-\mu_g\xi_h-\mu_h\mu_g$, thanks to the relation $[\mathrm{ad}(E_{hi}),\mu_g]=\delta_{ig}\mu_h$ used before.

Next we set $(\xi_1(u),\dots,\xi_p(u))=(\mu_1,\dots,\mu_p)[\mathrm{ad}(E_{hi})+(\lambda^0+p+u)\delta_{hi}]_{1\le i,h\le p}$. Then we have $\xi_h(u)=\xi_h+u\mu_h$, and hence we obtain the commutation relation between $\xi_h(u)$ and μ_i: $\xi_h(u)\mu_i=-\mu_i\xi_h(u-1)$.

Let us calculate $\xi_{I(1)}\cdots\xi_{I(d)}$ for $I\subset\{1,\dots,p\}$ with $\#I=d$ in a way different from (2.10).

$$\begin{aligned}\xi_{I(1)}\cdots\xi_{I(d)}&=\xi_{I(1)}(0)\cdots\xi_{I(d)}(0)\\&=\xi_{I(1)}(0)\cdots\xi_{I(d-1)}(0)\sum_{h=1}^p\mu_h(\mathrm{ad}(E_{I(d)h})+(\lambda^0+p)\delta_{I(d)h}).\end{aligned}$$

Here we repeatedly use $\xi_h(u)\mu_i=-\mu_i\xi_h(u-1)$, and thereby we can calculate the right hand side of the formula above:

$$\begin{aligned}&(-1)^{d-1}\sum_{h_d=1}^p\mu_{h_d}\xi_{I(1)}(-1)\cdots\xi_{I(d-1)}(-1)\{\mathrm{ad}(E_{I(d)h_d})+(\lambda^0+p)\delta_{I(d)h_d}\}\\&=\\&\ \vdots\\&=((-1)^{d-1})^d\sum_{1\le h_1,\dots,h_d\le p}\mu_{h_1}\cdots\mu_{h_d}\{\mathrm{ad}(E_{I(1)h_1})+(\lambda^0+p-(d-1))\delta_{I(1)h_1}\}\times\end{aligned}$$

$$\times \cdots \times \{\mathrm{ad}(E_{I(d)h_d}) + (\lambda^0 + p - 0)\delta_{I(d)h_d}\}$$

$$= \sum_{J \subset \{1,\dots,q\},\, \#J=d} a_{J(1)} \cdots a_{J(d)} \sum_{H \subset \{1,\dots,p\},\, \#H=d} {}^t f_{HJ}(\partial)$$

$$\times\, \mathrm{ad}(\det[E_{I(t)H(s)} + (\lambda^0 + p - d + t)\delta_{I(t)H(s)}]_{1 \le s,t \le d}).$$

Fixing J, we compare this formula with (2.10), and we thus have proved (1).

[proof of (2)]

We calculate $\xi_{I(1)} \cdots \xi_{I(d)}$ in another way. The strategy of the last calculation of the proof of (1) is to bring μ_h's to the front of the expression. Here we bring μ_h's to the end of the expression. We then obtain

$$\xi_{I(1)} \cdots \xi_{I(d)} = \sum_{J \subset \{1,\dots,q\},\, \#J=d} a_{J(1)} \cdots a_{J(d)} \sum_{H \subset \{1,\dots,p\},\, \#H=d} \mathrm{ad}(\det[E_{I(t)J(s)} + (\lambda^0 + t - 1)\delta_{I(t)J(s)}]_{1 \le s,t \le d})\, {}^t f_{HJ}(\partial).$$

Fixing J, we compare this formula with (2.10), and thereby we obtain (2). □

Proof of Theorem 2.2. First of all, we can prove (2.2) and (2.4) by setting $H = I$ and taking sums over I in Lemma 2.4 (1) and (2), respectively. Let us prove (2.6). We first note that $\mu(E_{ii}) = \mu^0 q/(p+q)$ for a character $\mu = \mu^0 \varpi_{i_0}$ of $\mathfrak{p}$ and $1 \le i \le p$, and we can rewrite Lemma 2.5 (1) as follows:

$$\Psi_\lambda({}^t f_{IJ}) = (-1)^d \sum_H {}^t f_{HJ}(\partial) \Psi_{((p+q)/q)\lambda + (2p/q)\rho}(u_{IH}^{LT}),$$

and Lemma 2.4 (1) as follows:

$$\sum_J f_{IJ}\, {}^t f_{HJ}(\partial) = \mathrm{ad}(u_{HI}^{LT}) = \Psi_0(u_{HI}^{LT}).$$

Using these expressions, we have

$$\sum_{\substack{I \subset \{1,\dots,p\},\, J \subset \{1,\dots,q\}, \\ \#I = \#J = d}} \Psi_\lambda(f_{IJ}\, {}^t f_{IJ}) = (-1)^d \sum_{IJ} f_{IJ} \sum_{H \subset \{1,\dots,p\},\, \#H=d} {}^t f_{HJ}(\partial) \Psi_{((p+q)/q)\lambda + (2p/q)\rho}(u_{IH}^{LT})$$

$$= (-1)^d \sum_{HI} \Psi_0(u_{HI}^{LT}) \Psi_{((p+q)/q)\lambda + (2p/q)\rho}(u_{IH}^{LT}),$$

and hence we obtain (2.6). We also obtain (2.8) similarly from Lemma 2.5 (2) and Lemma 2.4 (2).

Let us prove the rest of the formulas by using the anti-involutions σ, s and τ defined in Definition 1.2. A composite mapping $\sigma\tau$, as well as $s \circ {}^t\cdot$ is an algebra automorphism, and $s \circ {}^t\cdot$ acts on $\mathfrak{l}$ by $s({}^t X_\alpha) = -X_{-\alpha}$ ($\alpha \in \Delta_L$), and we therefore obtain for $H, I \subset \{1, \dots, p\}$ with $\#H = \#I = d$:

$$\begin{aligned}\sigma\tau(\Psi_\mu(u_{HI}^{LT})) &= \Psi_{-\mu-2\rho}(s({}^t(u_{HI}^{LT})))\\ &= \Psi_{-\mu-2\rho}(\det[E_{I(s)H(t)} + (d-t)\delta_{I(s)H(t)}]_{1\le s,t\le d})\\ &= \Psi_{-\mu}(\det[E_{I(s)H(t)} + (d-t-q)\delta_{I(s)H(t)}]_{1\le s,t\le d})\\ &= (-1)^d \Psi_{-\mu}(v_{IH}^L).\end{aligned}$$

Using this formula and the formula $\mathrm{ad}(u) = \Psi_0(u)$ for $u \in U(\mathfrak{l})$, we have

$$\begin{aligned}\sigma\tau((2.2)\ \mathrm{RHS}) &= \sigma\tau\left(\sum_I \mathrm{ad}(u_{II}^{LT})\right)\\ &= (-1)^d \sum_I \mathrm{ad}(v_{II}^L) = (-1)^d((2.3)\ \mathrm{RHS}).\end{aligned}$$

Next we note that the left hand side of (2.2) belongs to $D_{\mathfrak{n}^+}^L$, and hence it is τ-invariant. Then using the definition of σ, we have

$$\begin{aligned}\sigma\tau((2.2)\ \mathrm{LHS}) &= \sigma((2.2)\ \mathrm{LHS})\\ &= \sigma\left(\sum_{IJ} f_{IJ}\, {}^t f_{IJ}(\partial)\right)\\ &= (-1)^d \sum_{IJ} {}^t f_{IJ}(\partial) f_{IJ} = (-1)^d((2.3)\ \mathrm{LHS}).\end{aligned}$$

Thus we obtain (2.3) by comparing these two formulas. Similarly we have (2.1) by applying $\sigma\tau$ to (2.4).

Let us prove (2.7). We calculate in a way similar to the calculation above. On the one hand we have

$$\begin{aligned}\sigma\tau((2.6)\ \mathrm{RHS}) &= \sigma\tau\left((-1)^d \sum_{HI} \Psi_0(u_{HI}^{LT}) \Psi_{((p+q)/q)\lambda+(2p/q)\rho}(u_{IH}^{LT})\right)\\ &= (-1)^d \sum_{HI} (-1)^d \Psi_0(v_{IH}^L) \cdot (-1)^d \Psi_{-((p+q)/q)\lambda-(2p/q)\rho}(v_{HI}^L).\end{aligned}$$

On the other hand we have

$$\begin{aligned}\sigma\tau((2.6)\ \mathrm{LHS}) &= \sigma((2.6)\ \mathrm{LHS})\\ &= \sigma\left(\sum_{IJ} \Psi_\lambda(f_{IJ}\, {}^t f_{IJ})\right)\\ &= \sum_{IJ} \Psi_{-\lambda-2\rho}({}^t f_{IJ} f_{IJ}).\end{aligned}$$

We replace λ with $-\lambda - 2\rho$, compare these two formulas, and thus we have proved (2.7). We can prove (2.5) similarly. □

2.2. (C_n, n) **or** S^2GL_n.

Set

$$\mathfrak{g}=\mathfrak{sp}(n,\boldsymbol{C})=\left\{\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}\in\mathfrak{gl}(2n,\boldsymbol{C})\;\middle|\;\begin{array}{l} A\in\mathfrak{gl}(n,\boldsymbol{C}),\\ B,C\in\mathrm{Sym}(n,\boldsymbol{C}) \end{array}\right\}.$$

Let $\mathfrak{h}$ be the set consisting of diagonal matrices of $\mathfrak{g}$. For $i,j\in\{1,\dots,n\}$, we set

$$H_{ij}=E_{ij}-E_{n+j,n+i},$$

$$G_{ij}=E_{i,n+j}+E_{j,n+i},$$

$$F_{ij}=E_{n+i,j}+E_{n+j,i},$$

and then we have bracket relations

$$[H_{ij},H_{kl}]=\delta_{jk}H_{il}-\delta_{il}H_{kj},$$

$$[H_{ij},G_{kl}]=\delta_{jk}G_{il}+\delta_{jl}G_{ik},$$

$$[H_{ij},F_{kl}]=-\delta_{ik}F_{jl}-\delta_{il}F_{jk},$$

$$[G_{ij},F_{kl}]=\delta_{jk}H_{il}+\delta_{il}H_{jk}+\delta_{jl}H_{ik}+\delta_{ik}H_{jl}.$$

We define $\varepsilon_i\in\mathfrak{h}^*$ $(i\in\{1,\dots,n\})$ by $\varepsilon_i(H_{jj})=\delta_{ij}$. We summarize data such as the root system in the following list.

$$\Pi=\{\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{n-1}-\varepsilon_n,2\varepsilon_n\},$$

$$\Delta^+=\{\varepsilon_i\pm\varepsilon_j\,|\,1\le i<j\le n\}\cup\{2\varepsilon_i\},$$

$H_{ij}:(\varepsilon_i-\varepsilon_j)$-root vector for $i\neq j$,

$G_{ij}:(\varepsilon_i+\varepsilon_j)$-root vector,

$F_{ij}:-(\varepsilon_i+\varepsilon_j)$-root vector,

$$\Pi_L=\Pi\backslash\{2\varepsilon_n\},$$

$$\Delta_L^+=\{\varepsilon_i-\varepsilon_j\,|\,1\le i<j\le n\},$$

$$\varpi_{i_0}=\varepsilon_1+\cdots+\varepsilon_n,$$

$$2\rho=(n+1)\varpi_{i_0},$$

$$\langle X,Y\rangle=\mathrm{Tr}(XY)/2\quad(X,Y\in\mathfrak{g}),$$

C. B. : $\{H_{ij}\}\cup\{G_{ij}\,|\,i<j\}\cup\{(1/2)G_{ii}\}\cup\{F_{ij}\,|\,i<j\}\cup\{(1/2)F_{ii}\}$.

Subalgebras $\mathfrak{p},\mathfrak{n}^+$ and $\mathfrak{l}$ are as follows:

$$\mathfrak{p}=\left\{\begin{pmatrix} A & B \\ 0 & -{}^tA \end{pmatrix}\in\mathfrak{g}\,|\,A\in\mathfrak{gl}(n,\boldsymbol{C}),B\in\mathrm{Sym}(n,\boldsymbol{C})\right\},$$

$$\mathfrak{n}^+=\left\{\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\in\mathfrak{g}\,|\,B\in\mathrm{Sym}(n,\boldsymbol{C})\right\},$$

$$\mathfrak{l}=\left\{\begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix}\in\mathfrak{g}\,|\,A\in\mathfrak{gl}(n,\boldsymbol{C})\right\}.$$

With respect to the anti-involution $x \mapsto {}^t x$ of Definition 1.2, we have

$$ {}^t H_{ij} = H_{ji}, \quad {}^t G_{ij} = F_{ij}, \quad {}^t F_{ij} = G_{ij}. $$

For $i, j \in \{1, \dots, n\}$, we set $x_{ij} = F_{ij}$, and then $\{x_{ij} \mid i \leq j\}$ forms a linear coordinate system on $\mathfrak{n}^+$. In addition, we set $\partial_{ij} = \partial / \partial x_{ij}$. For $i \neq j$, we have ${}^t F_{ij}(\partial) = G_{ij}(\partial) = \langle G_{ij}, F_{ij} \rangle \partial_{ij} = \partial_{ij}$, while we have ${}^t F_{ii}(\partial) = \langle G_{ii}, F_{ii} \rangle \partial_{ii} = 2\partial_{ii}$. For this reason, we set $\tilde{\partial}_{ij} = (1 + \delta_{ij}) \partial_{ij}$. We use Lemma 1.1, and we easily have the following lemma.

LEMMA 2.6. *For $1 \leq i, j \leq n$, we have*
(1) $\Psi_\lambda(H_{ij}) = -\sum_{k=1}^n x_{jk} \tilde{\partial}_{ik} + \lambda^0 \delta_{ij}$,

(2) $\Psi_\lambda(G_{ij}) = -\sum_{k,l=1}^n x_{kl} \tilde{\partial}_{il} \tilde{\partial}_{jk} + 2\lambda^0 \tilde{\partial}_{ij}$.

For $1 \leq d \leq n$, we set

$$ f_{IJ} = \det[x_{I(s)J(t)}]_{1 \leq s, t \leq d} $$

where $I, J \subset \{1, \dots, n\}$ with $\# I = \# J = d$. In particular, when $I = J = \{1, \dots, n\}$, a function

$$ f = \det[x_{ij}]_{1 \leq s, t \leq n} $$

is the relative invariant with weight $-2\varpi_{i_0}$. We also have

$$ {}^t f_{IJ}(\partial) = \det[\partial_{I(s)J(t)}]_{1 \leq s, t \leq d}. $$

THEOREM 2.7. *For $1 \leq d \leq n$ and $J, K \subset \{1, \dots, n\}$ with $\# J = \# K = d$, we set*

$$ u_{KJ} = \det[-H_{K(s)J(t)} + (t-1)\delta_{K(s)J(t)}]_{1 \leq s, t \leq d}, $$

$$ u_{KJ}^T = \det[-H_{K(t)J(s)} + (d-t)\delta_{K(t)J(s)}]_{1 \leq s, t \leq d}, $$

$$ v_{KJ} = \det[-H_{K(s)J(t)} + (n+t-d+1)\delta_{K(s)J(t)}]_{1 \leq s, t \leq d}, $$

$$ v_{KJ}^T = \det[-H_{K(t)J(s)} + (n-t+2)\delta_{K(t)J(s)}]_{1 \leq s, t \leq d}, $$

and we then have

$$ \sum_{IJ} f_{IJ} {}^t f_{IJ}(\partial) = \sum_J \mathrm{ad}(u_{JJ}), \tag{2.11} $$

$$ \sum_{IJ} f_{IJ} {}^t f_{IJ}(\partial) = \sum_J \mathrm{ad}(u_{JJ}^T), \tag{2.12} $$

$$ \sum_{IJ} {}^t f_{IJ}(\partial) f_{IJ} = \sum_J \mathrm{ad}(v_{JJ}), \tag{2.13} $$

$$ \sum_{IJ} {}^t f_{IJ}(\partial) f_{IJ} = \sum_J \mathrm{ad}(v_{JJ}^T). \tag{2.14} $$

Moreover we have

$$ \sum_{IJ} \Psi_\lambda(f_{IJ} {}^t f_{IJ}) = (-1)^d \sum_{JK} \Psi_{2\lambda+2\rho}(u_{KJ}) \Psi_0(u_{JK}), \tag{2.15} $$

$$\sum_{IJ} \Psi_\lambda(f_{IJ}\,{}^t f_{IJ}) = (-1)^d \sum_{JK} \Psi_0(u_{KJ}^T)\Psi_{2\lambda+2\rho}(u_{JK}^T), \tag{2.16}$$

$$\sum_{IJ} \Psi_\lambda({}^t f_{IJ} f_{IJ}) = (-1)^d \sum_{JK} \Psi_0(v_{KJ})\Psi_{2\lambda+2\rho}(v_{JK}), \tag{2.17}$$

$$\sum_{IJ} \Psi_\lambda({}^t f_{IJ} f_{IJ}) = (-1)^d \sum_{JK} \Psi_{2\lambda+2\rho}(v_{KJ}^T)\Psi_0(v_{JK}^T). \tag{2.18}$$

In the summations above, I, J and K run over all subsets of $\{1,\dots,n\}$ with cardinality d.

Remark 2.8. In the last formula above, we can not interchange $\Psi_{2\lambda+2\rho}(v_{KJ}^T)$ with $\Psi_0(v_{JK}^T)$, since v_{KJ}^T does not belong to $Z(\mathfrak{l})$ in general. Similarly we can not interchange the order of multiplications on the right hand sides in the formulas from (2.15) to (2.18).

The rest of this subsection is devoted to proving Theorem 2.7. The essence of the proof is the same as in the case of (A_{p+q-1}, p), and we give only outlines of the proofs of the theorem and lemmas. The main differences of the proof are just differences of commutation relations.

Lemma 2.9. *For $J, K \in \{1,\dots,n\}$ with $\#J = \#K = d$, we have*

(1) $\sum_{I} f_{IJ}\,{}^t f_{IK}(\partial) = \mathrm{ad}(\det[-H_{K(t)J(s)} + (d-t)\delta_{K(t)J(s)}]_{1\le s,t\le d})$,

(2) $\sum_{I} {}^t f_{IK}(\partial) f_{IJ} = \mathrm{ad}(\det[-H_{K(t)J(s)} + (n-t+2)\delta_{K(t)J(s)}]_{1\le s,t\le d})$.

In the summations above, I runs over all subsets of $\{1,\dots,n\}$ with cardinality d. We remark that we can obtain the classical Capelli identities, when we consider the special case of $K = J$ in each of the formulas above, and take a sum over J.

Proof. We work in $\bigwedge(\boldsymbol{C}^n) \otimes_{\boldsymbol{C}} D_{\mathfrak{n}^+}$ as in the case of (A_{p+q-1}, p). Let $\{a_1,\dots,a_n\}$ be a basis of $\boldsymbol{C}^n$. Set $(\eta_1,\dots,\eta_n) = (a_1,\dots,a_n)[x_{ij}]_{1\le i,j\le n}$. Then we have $\eta_i\eta_j = -\eta_j\eta_i$ and

$$\eta_{I(1)}\cdots\eta_{I(d)} = \sum_{J\subset\{1,\dots,n\},\,\#J=d} a_{J(1)}\cdots a_{J(d)} f_{IJ}.$$

Next we set $(\zeta_1,\dots,\zeta_n) = (\eta_1,\dots,\eta_n)[\tilde{\partial}_{ij}]_{1\le i,j\le n}$, and $(\zeta_1(u),\dots,\zeta_n(u)) = (a_1,\dots,a_n)\cdot[\mathrm{ad}(-H_{ij}) - u\delta_{ij}]_{1\le i,j\le n}$. Then we have $\zeta_i(0) = \zeta_i$ and $\zeta_i(u)\eta_j = -\eta_j\zeta_i(u+1)$.

These relations are the same as in the proof of Lemma 2.4 (1). We thus have (1) of this lemma by computing $\zeta_{K(1)}(-d+1)\cdots\zeta_{K(d)}(0)$ in two different ways like Lemma 2.4 (1).

For proving (2), we can show that $[\tilde{\partial}_{ij}]_{1\le i,j\le n}{}^t(\eta_1,\dots,\eta_n) = {}^t(\zeta_1(-n-1),\dots,\zeta_n(-n-1))$, which is the difference from the case of (A_{p+q-1}, p). We then have (2) of this lemma by computing $\zeta_{K(1)}(-n-1)\cdots\zeta_{K(d)}(-n+d-2)$ in two different ways. □

Lemma 2.10. *For $I, J \subset \{1,\dots,n\}$ with $\#I = \#J = d$, we have*

(1) $\quad \Psi_\lambda({}^t f_{IJ}) = \sum_K {}^t f_{IK}(\partial)\,\mathrm{ad}(\det[H_{J(t)K(s)} + (2\lambda^0 + n + 1 + t - d)\delta_{J(t)K(s)}]_{1\le s,t\le d}),$

(2) $\quad \Psi_\lambda({}^t f_{IJ}) = \sum_K \mathrm{ad}(\det[H_{J(t)K(s)} + (2\lambda^0 + t - 1)\delta_{J(t)K(s)}]_{1\le s,t\le d})\,{}^t f_{IK}(\partial),$

where K runs over all subsets of $\{1,\dots,n\}$ with cardinality d.

PROOF. Set $(\mu_1,\dots,\mu_n) = (a_1,\dots,a_n)[\tilde{\partial}_{ij}]_{1\le i,j\le n}$. Then we have

$$\mu_{I(1)}\cdots\mu_{I(d)} = \sum_{J\subset\{1,\dots,n\},\,\#J=d} a_{J(1)}\cdots a_{J(d)}\,{}^t f_{IJ}(\partial).$$

Next we set $(\xi_1,\dots,\xi_n) = (a_1,\dots,a_n)[\Psi_\lambda(G_{ij})]_{1\le i,j\le n}$, and we have

$$\xi_{I(1)}\cdots\xi_{I(d)} = \sum_{J\subset\{1,\dots,n\},\,\#J=d} a_{J(1)}\cdots a_{J(d)}\,\Psi_\lambda({}^t f_{IJ}).$$

We can obtain two more expressions for $\xi_{I(1)}\cdots\xi_{I(d)}$ as follows, and each of them proves (1) or (2) of this lemma by comparing with the formula for $\xi_{I(1)}\cdots\xi_{I(d)}$ stated above.

We set $(\xi_1(u),\dots,\xi_n(u)) = (\mu_1,\dots,\mu_n)[\mathrm{ad}(H_{ji}) + (2\lambda^0 + n + 1 + u)\delta_{ji}]_{1\le i,j\le n}$, which is different from the case of (A_{p+q-1}, p) in the diagonal shift, and then we have $\xi_i(0) = \xi_i$ and $\xi_i(u)\mu_j = -\mu_j\xi_i(u-1)$. Here we can compute $\xi_{I(1)}\cdots\xi_{I(d)}$ in a way to bring μ_j's to the front of the expression as in the proof of Lemma 2.5 (1). Comparing it with the formula for $\xi_{I(1)}\cdots\xi_{I(d)}$ above, we have (1) of this lemma.

For proving (2), we can show that ${}^t(\xi_1(u),\dots,\xi_n(u)) = [\mathrm{ad}(H_{ji}) + (2\lambda^0 + u)\delta_{ji}]_{1\le i,j\le n}\,{}^t(\mu_1,\dots,\mu_n)$, and we can compute $\xi_{I(1)}\cdots\xi_{I(d)}$ in a way to bring μ_j's to the end of the expression. We thus have (2) of the lemma. □

PROOF OF THEOREM 2.7. (2.12) and (2.14) are proved by Lemma 2.9 (1) and (2), respectively. (2.16) is proved by Lemma 2.10 (1) and Lemma 2.9 (1). (2.18) is proved by Lemma 2.10 (2) and Lemma 2.9 (2). Similarly to the case of (A_{p+q-1}, p), we can show the rest of the formulas by applying the anti-automorphisms σ and τ to the proved formulas. □

2.3. (D_n, n) **or** $\Lambda^2 GL_n$**.**

Set

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^t A \end{pmatrix} \in \mathfrak{gl}(2n,\boldsymbol{C}) \;\middle|\; \begin{array}{l} A \in \mathfrak{gl}(n,\boldsymbol{C}), \\ B, C \in \mathrm{Alt}(n,\boldsymbol{C}) \end{array} \right\},$$

and let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. For $i,j \in \{1,\dots,n\}$, we set

$$H_{ij} = E_{ij} - E_{n+j,n+i},$$

$$G_{ij} = E_{i,n+j} - E_{j,n+i},$$

$$F_{ij} = E_{n+j,i} - E_{n+i,j},$$

and we have the following relations:

$$[H_{ij}, H_{kl}] = \delta_{jk} H_{il} - \delta_{il} H_{kj},$$

$$[H_{ij}, G_{kl}] = \delta_{jk} G_{il} + \delta_{jl} G_{ki},$$

$$[H_{ij}, F_{kl}] = \delta_{ik} F_{lj} + \delta_{il} F_{jk},$$

$$[G_{ij}, F_{kl}] = \delta_{jl} H_{ik} + \delta_{ik} H_{jl} - \delta_{jk} H_{il} - \delta_{il} H_{jk}.$$

Define $\varepsilon_i \in \mathfrak{h}^*$ $(i \in \{1, \dots, n\})$ by $\varepsilon_i(H_{jj}) = \delta_{ij}$, and we list the data such as the root system in the following list.

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\},$$

$$\Delta^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\},$$

$$H_{ij} : (\varepsilon_i - \varepsilon_j)\text{-root vector for } i \ne j,$$

$$G_{ij} : (\varepsilon_i + \varepsilon_j)\text{-root vector for } i \ne j,$$

$$F_{ij} : -(\varepsilon_i + \varepsilon_j)\text{-root vector for } i \ne j,$$

$$\Pi_L = \Pi \backslash \{\varepsilon_{n-1} + \varepsilon_n\},$$

$$\Delta_L^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\},$$

$$\varpi_{i_0} = (\varepsilon_1 + \cdots + \varepsilon_n)/2,$$

$$2\rho = 2(n-1)\varpi_{i_0},$$

$$\langle X, Y \rangle = \mathrm{Tr}(XY)/2 \quad (X, Y \in \mathfrak{g}),$$

$$\text{C. B.} : \{H_{ij}\} \cup \{G_{ij} \mid i < j\} \cup \{F_{ij} \mid i < j\}.$$

Subalgebras $\mathfrak{p}, \mathfrak{n}^+$ and $\mathfrak{l}$ of $\mathfrak{g}$ are as follows:

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ 0 & -{}^t A \end{pmatrix} \in \mathfrak{g} \mid A \in \mathfrak{gl}(n, \boldsymbol{C}), B \in \mathrm{Alt}(n, \boldsymbol{C}) \right\},$$

$$\mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \mid B \in \mathrm{Alt}(n, \boldsymbol{C}) \right\},$$

$$\mathfrak{l} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^t A \end{pmatrix} \in \mathfrak{g} \mid A \in \mathfrak{gl}(n, \boldsymbol{C}) \right\}.$$

It follows from the definition of $x \mapsto {}^t x$ in Definition 1.2 that

$${}^t H_{ij} = H_{ji}, \quad {}^t G_{ij} = F_{ij}, \quad {}^t F_{ij} = G_{ij}.$$

We define $x_{ij} = F_{ij}$ for $i, j \in \{1, \dots, n\}$, and then $\{x_{ij} \mid i < j\}$ forms a liner coordinate system on $\mathfrak{n}^+$. In addition, we set $\partial_{ij} = \partial/\partial x_{ij}$. The next lemma follows easily from Lemma 1.1.

Lemma 2.11. *For* $1 \le i, j \le n$, *we have*

(1) $\Psi_\lambda(H_{ij}) = -\sum_{1 \le k \le n, k \ne i} x_{kj} \partial_{ki} + \dfrac{1}{2} \lambda^0 \delta_{ij}$,

(2) $\Psi_\lambda(G_{ij}) = -\sum_{k \ne j, l \ne i} x_{kl} \partial_{il} \partial_{kj} + \lambda^0 \partial_{ij}$.

In the case of (D_n, n), the classical Capelli identity has a complicated expression, and moreover we can not prove the Ψ_λ-analogue by a method similar to that of (A_{p+q-1}, p) or (C_n, n). We therefore give a Ψ_λ-analogue of the Turnbull identity. We first define the permanent of matrices in which entries do not necessarily commutes with each other, which are called column permanents. Set

$$\mathrm{Per}[A_{ij}]_{1 \le i, j \le d} = \sum_{\sigma \in \mathfrak{S}_d} A_{\sigma(1)1} \cdots A_{\sigma(d)d}.$$

The column permanent and the row permanent, which is defined similarly, coincide when the entries commute with each other. The multiplication formula of permanents is more complicated than that of determinants, even when the entries commute with each other. Let R be a commutative ring, $A, B, C \in \mathrm{Mat}(n, R)$, $C = AB$ and $1 \le d \le n$, and we then have the multiplication formula of permanents,

$$\mathrm{Per}(C_{IK}) = \sum_{\#J=d} \frac{1}{J!} \mathrm{Per}(A_{IJ}) \mathrm{Per}(B_{JK}),$$

where I, J, K are indices such that $1 \le I(1) \le \cdots \le I(d) \le n$ and so on, in which case we simply write $\#I = d$, and C_{IK} is the $d \times d$ matrix determined according to I and K, which is not a submatrix of C in general, and

$J! = ($the number of 1's appearing in $J)! \cdots ($the number of n's appearing in $J)!$.

For indices I, J with $\#I = \#J = d$ $(1 \le d \le n)$, we set

$$f_{IJ} = \mathrm{Per}[x_{I(s)J(t)}]_{1 \le s, t \le d},$$

and we then have

$${}^t f_{IJ} = \mathrm{Per}[G_{I(s)J(t)}]_{1 \le s, t \le d}, \quad {}^t f_{IJ}(\partial) = \mathrm{Per}[\partial_{I(s)J(t)}]_{1 \le s, t \le d}.$$

Theorem 2.12. *Let $1 \le d \le n$, and let I be an index satisfying $1 \le I(1) \le \cdots \le I(d) \le n$, in which case we write $\#I = d$, and let J be an index with $\#J = d$. We set*

$$u_{IJ} = \mathrm{Per}[-H_{I(s)J(t)} - (t-1)\delta_{I(s)J(t)}]_{1 \le s, t \le d},$$

$$u_{IJ}^T = \mathrm{Per}[-H_{I(t)J(s)} - (d-t)\delta_{I(t)J(s)}]_{1 \le s, t \le d},$$

$$v_{IJ} = \mathrm{Per}[-H_{I(s)J(t)} + (d+n-1-t)\delta_{I(s)J(t)}]_{1 \le s, t \le d},$$

$$v_{IJ}^T = \mathrm{Per}[-H_{I(t)J(s)} + (n-2+t)\delta_{I(t)J(s)}]_{1 \le s, t \le d},$$

and we then have

$$\sum_{\#I=\#J=d} \frac{f_{IJ}\,{}^t f_{IJ}(\partial)}{I!J!} = \sum_{\#I=d} \frac{\mathrm{ad}(u_{II})}{I!}, \tag{2.19}$$

$$\sum_{\#I=\#J=d} \frac{f_{IJ}\,{}^t f_{IJ}(\partial)}{I!J!} = \sum_{\#I=d} \frac{\mathrm{ad}(u_{II}^T)}{I!}, \tag{2.20}$$

$$\sum_{\#I=\#J=d} \frac{{}^t f_{IJ}(\partial) f_{IJ}}{I!J!} = \sum_{\#I=d} \frac{\mathrm{ad}(v_{II})}{I!}, \tag{2.21}$$

$$\sum_{\#I=\#J=d} \frac{{}^t f_{IJ}(\partial) f_{IJ}}{I!J!} = \sum_{\#I=d} \frac{\mathrm{ad}(v_{II}^T)}{I!}. \tag{2.22}$$

Moreover we have

$$\sum_{\#I=\#J=d} \Psi_\lambda\left(\frac{f_{IJ}\,{}^t f_{IJ}}{I!J!}\right) = (-1)^d \sum_{\#I=\#J=d} \frac{1}{I!J!} \Psi_{2\lambda+2\rho}(u_{IJ})\Psi_0(u_{JI}), \tag{2.23}$$

$$\sum_{\#I=\#J=d} \Psi_\lambda\left(\frac{f_{IJ}\,{}^t f_{IJ}}{I!J!}\right) = (-1)^d \sum_{\#I=\#J=d} \frac{1}{I!J!} \Psi_0(u_{IJ}^T)\Psi_{2\lambda+2\rho}(u_{JI}^T), \tag{2.24}$$

$$\sum_{\#I=\#J=d} \Psi_\lambda\left(\frac{{}^t f_{IJ} f_{IJ}}{I!J!}\right) = (-1)^d \sum_{\#I=\#J=d} \frac{1}{I!J!} \Psi_0(v_{IJ})\Psi_{2\lambda+2\rho}(v_{JI}), \tag{2.25}$$

$$\sum_{\#I=\#J=d} \Psi_\lambda\left(\frac{{}^t f_{IJ} f_{IJ}}{I!J!}\right) = (-1)^d \sum_{\#I=\#J=d} \frac{1}{I!J!} \Psi_{2\lambda+2\rho}(v_{IJ}^T)\Psi_0(v_{JI}^T). \tag{2.26}$$

Remark 2.13. (1) The formulas (2.19) and (2.20) in the theorem above are exactly the same as the formulas of [**10**, Theorem 3.1], where we use [**10**, expression (2.8)] and [**10**, expression (2.7)], respectively, for D_N. Thus we give the outline of the proof, in the proof of Lemma 2.14.

(2) We have $\sum_I u_{II}/I!$, $\sum_I u_{II}^T/I! \in Z(\mathfrak{l})$ due to [**10**, Theorem 2.3], and $\sum_I v_{II}/I!$, $\sum_I v_{II}^T/I!$ are also in $Z(\mathfrak{l})$.

We prove two lemmas before we prove the theorem. As we have discussed the cases of determinants using $\bigwedge(\boldsymbol{C}^n) \otimes_{\boldsymbol{C}} D_{\mathfrak{n}^+}$, we will discuss this case of permanents using $\boldsymbol{C}[b_1,\dots,b_n] \otimes_{\boldsymbol{C}} D_{\mathfrak{n}^+}$, where $\boldsymbol{C}[b_1,\dots,b_n]$ is the polynomial ring with indeterminates b_j. Set

$$(\eta_1,\dots,\eta_n) = (b_1,\dots,b_n)[A_{ij}]_{1\le i,j\le n} \quad (A_{ij} \in D_{\mathfrak{n}^+}),$$

and we then have the following formula for $\#I = d$,

$$\begin{aligned}
\eta_{I(1)}\cdots\eta_{I(d)} &= \sum_{1\le i_t\le n} b_{i_1}\cdots b_{i_d} A_{i_1 I(1)}\cdots A_{i_d I(d)} \\
&= \sum_{\#J=d,\,\sigma\in\mathfrak{S}_d} \frac{1}{J!} b_{J(\sigma(1))}\cdots b_{J(\sigma(d))} A_{J(\sigma(1))I(1)}\cdots A_{J(\sigma(d))I(d)} \\
&= \sum_{\#J=d} \frac{1}{J!} b_{J(1)}\cdots b_{J(d)} \sum_{\sigma\in\mathfrak{S}_d} A_{J(\sigma(1))I(1)}\cdots A_{J(\sigma(d))I(d)} \\
&= \sum_{\#J=d} \frac{1}{J!} b_{J(1)}\cdots b_{J(d)} \operatorname{Per} A_{JI}. \qquad (2.27)
\end{aligned}$$

Lemma 2.14. *For $\#I = \#J = d$, we have*

(1) $\displaystyle\sum_{\#K=d} \frac{1}{K!} f_{KJ}\,{}^t f_{KI}(\partial) = (-1)^d \operatorname{ad}(\operatorname{Per}[H_{I(t)J(s)} + (d-t)\delta_{I(t)J(s)}]_{1\le s,t\le d}),$

(2) $\displaystyle\sum_{\#K=d} \frac{1}{K!} {}^t f_{KI}(\partial) f_{KJ} = (-1)^d \operatorname{ad}(\operatorname{Per}[H_{I(t)J(s)} + (2-n-t)\delta_{I(t)J(s)}]_{1\le s,t\le d}).$

We remark that we can obtain the Turnbull identity, when we consider the special case of $I = J$ in each of the formulas above, divide it by $I!$, and take a sum over I.

Proof. [proof of (1)]

First we note that $\operatorname{Per} A = \operatorname{Per} {}^tA$ for a square matrix A in which all the entries commute with each other, and that $x_{ji} = -x_{ij}$ in the present setting. Set

$$(\eta_1, \dots, \eta_n) = (b_1, \dots, b_n)[x_{ij}]_{1 \le i,j \le n},$$

and, for the reason above, it follows for $\# I = d$ from (2.27) that

$$\begin{aligned}\eta_{I(1)} \cdots \eta_{I(d)} &= \sum_{\# J = d} \frac{1}{J!} b_{J(1)} \cdots b_{J(d)} f_{JI} \\ &= (-1)^d \sum_{\# J = d} \frac{1}{J!} b_{J(1)} \cdots b_{J(d)} f_{IJ}.\end{aligned}$$

We set $(\zeta_1, \dots, \zeta_n) = (\eta_1, \dots, \eta_n)[\partial_{ij}]_{1 \le i,j \le n}$, and then we have the commutation relation between ζ_j and η_i: $\zeta_j \eta_i = \eta_i \zeta_j - \eta_i b_j$. We also set

$$(\zeta_1(u), \dots, \zeta_n(u)) = (b_1, \dots, b_n)[\operatorname{ad}(H_{ji}) + u\delta_{ji}]_{1 \le i,j \le n},$$

and then it follows that $\zeta_j(u) = \zeta_j + u b_j$, and hence we have the commutation relation $\zeta_j(u)\eta_i = \eta_i \zeta_j(u-1)$.

Let us calculate $\zeta_{I(1)}(d-1)\zeta_{I(2)}(d-2) \cdots \zeta_{I(d)}(0)$ in two different ways for $\# I = d$. First we have

$$\zeta_{I(1)}(d-1)\zeta_{I(2)}(d-2) \cdots \zeta_{I(d)}(0) = (-1)^d \sum_{\# K = d} \frac{1}{K!} \sum_{\# J = d} \frac{1}{J!} b_{J(1)} \cdots b_{J(d)} f_{KJ} {}^t f_{KI}(\partial),$$

using the commutation relation between $\zeta_j(u)$ and η_i. Second we calculate as follows:

$$\zeta_{I(1)}(d-1) \cdots \zeta_{I(d)}(0) = \sum_{\# J = d} \frac{1}{J!} b_{J(1)} \cdots b_{J(d)} \operatorname{ad}(\operatorname{Per}[H_{I(t)J(s)} + (d-t)\delta_{I(t)J(s)}]_{1 \le s,t \le d}).$$

These two results of the calculations above must coincide summand by summand with respect to J, and we therefore obtain (1).

[proof of (2)]

We have $\sum_i \partial_{ij} \eta_i = \zeta_j(1-n)$. We calculate $\zeta_{I(1)}(1-n-0)\zeta_{I(2)}(1-n-1) \cdots \zeta_{I(d)}(1-n-(d-1))$ in two different ways as in the proof of (1), and we have the assertion. □

Lemma 2.15. *For $\# I = \# J = d$, we have*

(1) $$\Psi_\lambda({}^t f_{IJ}) = (-1)^d \sum_{\# K = d} \frac{1}{K!} {}^t f_{KI}(\partial) \operatorname{ad}(\operatorname{Per}[H_{J(t)K(s)} + (\lambda^0 + n - 1 + d - t)\delta_{J(t)K(s)}]_{st}),$$

(2) $$\Psi_\lambda({}^t f_{IJ}) = (-1)^d \sum_{\# K = d} \frac{1}{K!} \operatorname{ad}(\operatorname{Per}[H_{J(t)K(s)} + (\lambda^0 - t + 1)\delta_{J(t)K(s)}]_{st}) {}^t f_{KI}(\partial).$$

Proof. [proof of (1)]

First we set $(\mu_1, \dots, \mu_n) = (b_1, \dots, b_n)[\partial_{ij}]_{1 \le i,j \le n}$, and then it follows from (2.27) that

$$\mu_{I(1)} \cdots \mu_{I(d)} = (-1)^d \sum_{\# J = d} \frac{1}{J!} b_{J(1)} \cdots b_{J(d)} {}^t f_{IJ}(\partial).$$

Next we set $(\xi_1, \dots, \xi_n) = (b_1, \dots, b_n)[\Psi_\lambda(G_{ij})]_{1 \le i,j \le n}$, and we then have

$$\xi_{I(1)} \cdots \xi_{I(d)} = \sum_{\#J=d} \frac{1}{J!} b_{J(1)} \cdots b_{J(d)} \Psi_\lambda({}^t f_{JI}). \tag{2.28}$$

Here we give two different expressions of ξ_j. First one is $\xi_j = \sum_l (\mathrm{ad}(H_{jl}) + \lambda^0 \delta_{jl}) \mu_l$, which is proved easily. Here we can find the commutation relation between ξ_j and μ_i: $\xi_j \mu_i = \mu_i(\xi_j + \mu_j)$, using $[\mathrm{ad}(H_{jl}), \mu_i] = \delta_{li}\mu_j - b_l \partial_{ij}$. Applying these relations to the first expression of ξ_j, we have the second expression, $\xi_j = \sum_l \mu_l(\mathrm{ad}(H_{jl}) + (\lambda^0 + n - 1)\delta_{jl})$.

Next we set $(\xi_1(u), \dots, \xi_n(u)) = (\mu_1, \dots, \mu_n)[\mathrm{ad}(H_{ji}) + (\lambda^0 + n - 1 + u)\delta_{ji}]_{1 \le i,j \le n}$, and we then have $\xi_j(u) = \xi_j + u\mu_j$. Hence we obtain the commutation relation between $\xi_j(u)$ and μ_i: $\xi_j(u)\mu_i = \mu_i \xi_j(u+1)$.

Let us calculate $\xi_{I(1)} \cdots \xi_{I(d)}$ for $\#I = d$ in a way different from (2.28).

$$\begin{aligned} \xi_{I(1)} \cdots \xi_{I(d)} &= \xi_{I(1)}(0) \cdots \xi_{I(d)}(0) \\ &= \xi_{I(1)}(0) \cdots \xi_{I(d-1)}(0) \sum_i \mu_i (\mathrm{ad}(H_{I(d)i}) + (\lambda^0 + n - 1)\delta_{I(d)i}). \end{aligned}$$

Here we repeatedly use $\xi_j(u)\mu_i = \mu_i \xi_j(u+1)$, and we can calculate the expression above as follows:

$$\begin{aligned} &\sum_i \mu_i \xi_{I(1)}(1) \cdots \xi_{I(d-1)}(1)\{\mathrm{ad}(H_{I(d)i}) + (\lambda^0 + n - 1 + 0)\delta_{I(d)i}\} \\ &= \\ &\vdots \\ &= \sum_{1 \le i_1, \dots, i_d \le n} \mu_{i_1} \cdots \mu_{i_d} \{\mathrm{ad}(H_{I(1)i_1}) + (\lambda^0 + n - 1 + (d-1))\delta_{I(1)i_1}\} \\ &\quad \times \cdots \times \{\mathrm{ad}(H_{I(d)i_d}) + (\lambda^0 + n - 1 + 0)\delta_{I(d)i_d}\} \\ &= (-1)^d \sum_{\#K=d} \frac{1}{K!} \left(\sum_{\#J=d} \frac{1}{J!} b_{J(1)} \cdots b_{J(d)} {}^t f_{KJ}(\partial) \right) \\ &\quad \times \mathrm{ad}(\mathrm{Per}[H_{I(t)K(s)} + (\lambda^0 + n - 1 + d - t)\delta_{I(t)K(s)}]_{1 \le s,t \le d}). \end{aligned}$$

Fixing J, we compare this expression with (2.28), interchange I with J, and thereby obtain (1).

[proof of (2)]

The strategy of proof is the same as in the proof of Lemma 2.5 (2). We will bring μ_i's to the end of the expression, using the commutation relation between μ_i and ξ_j. We then obtain

$$\begin{aligned} \xi_{I(1)} \cdots \xi_{I(d)} = \sum_{\#K=d} \frac{1}{K!} &\mathrm{ad}(\mathrm{Per}[H_{I(t)K(s)} + (\lambda^0 - t + 1)\delta_{I(t)K(s)}]_{1 \le s,t \le d}) \\ &\times (-1)^d \sum_{\#J=d} \frac{1}{J!} b_{J(1)} \cdots b_{J(d)} {}^t f_{KJ}(\partial). \end{aligned}$$

Fixing J, we compare this expression with (2.28), interchange I with J, and thereby we obtain (2). □

Proof of Theorem 2.12. First we can obtain (2.20) from Lemma 2.14 (1) by setting $I = J$, dividing by $I!$ and taking a sum over I in the lemma. We can prove (2.22) similarly. Second we prove (2.24). It follows from Lemma 2.15 (1) and Lemma 2.14 (1) that

$$\begin{aligned}\sum_{\#I=\#J=d} \Psi_\lambda\left(\frac{f_{IJ}\,{}^t f_{IJ}}{I!J!}\right) &= \sum_{\#I=\#J=d} \frac{f_{IJ}}{I!J!} \cdot (-1)^d \sum_{\#K=d} \frac{1}{K!}\,{}^t f_{KI}(\partial) \cdot (-1)^d \Psi_{2\lambda+2\rho}(u_{JK}^T) \\ &= (-1)^d \sum_{\#J=\#K=d} \frac{1}{J!K!}\left\{\sum_{\#I=d} \frac{1}{I!} f_{IJ}\,{}^t f_{IK}(\partial)\right\} \Psi_{2\lambda+2\rho}(u_{JK}^T) \\ &= (-1)^d \sum_{\#J=\#K=d} \frac{1}{J!K!} \operatorname{ad}(u_{KJ}^T) \Psi_{2\lambda+2\rho}(u_{JK}^T),\end{aligned}$$

and this proves (2.24). We can prove (2.26) similarly from Lemma 2.15 (2) and Lemma 2.14 (2).

For proving the rest of the assertion, we use anti-involutions σ, s and τ defined in Definition 1.2. Let us prove (2.21). Since the left hand side of (2.20) belongs to $D_{\mathfrak{n}^+}^L$, it is τ-invariant, and hence we have

$$\begin{aligned}\sigma\tau((2.20)\ \text{LHS}) &= \sigma((2.20)\ \text{LHS}) \\ &= (-1)^d \sum_{IJ} \frac{{}^t f_{IJ}(\partial) f_{IJ}}{I!J!} = (-1)^d((2.21)\ \text{LHS}).\end{aligned}$$

Next we note that a composite mapping $\sigma\tau$, as well as $s \circ {}^t\cdot$, is an automorphism, and we have

$$\sigma\tau(\Psi_\mu(u_{IJ}^T)) = (-1)^d \Psi_{-\mu}(v_{JI}). \tag{2.29}$$

Using this equation and the fact that $\operatorname{ad}(u) = \Psi_0(u)$ for $u \in U(\mathfrak{l})$, we have

$$\begin{aligned}\sigma\tau((2.20)\ \text{RHS}) &= \sigma\tau\left(\sum_{\#I=d} \frac{\operatorname{ad}(u_{II}^T)}{I!}\right) \\ &= (-1)^d \sum_{\#I=d} \frac{\operatorname{ad}(v_{II})}{I!} = (-1)^d((2.21)\ \text{RHS}).\end{aligned}$$

These two formulas prove (2.21). Similarly (2.19) is proved by applying $\sigma\tau$ to (2.22).

Finally we apply the automorphism $\sigma\tau$ to (2.24) and (2.26) using (2.29), replace λ with $-\lambda - 2\rho$, and thereby we obtain (2.25) and (2.23), respectively. □

2.4. $(\mathrm{D}_n, 1)$ **or** $O_{2n-1} \otimes GL_1$.

Set

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^t A \end{pmatrix} \in \mathfrak{gl}(2n, \boldsymbol{C}) \;\middle|\; \begin{array}{l} A \in \mathfrak{gl}(n, \boldsymbol{C}), \\ B, C \in \operatorname{Alt}(n, \boldsymbol{C}) \end{array} \right\},$$

and let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. We take a basis of $\mathfrak{g}$ in a way different from that of §§2.3. Set

$$H_{ij} = E_{\bar{i}\bar{j}} - E_{\overline{n+j},\overline{n+i}} \quad (i, j \in \boldsymbol{Z}_{>0}),$$

where $\bar{i}$ is the integer satisfying $1 \le \bar{i} \le 2n$ and $i \equiv \bar{i} \pmod{2n}$. Thanks to this setting, some formulas become simpler. All H_{ij} belong to $\mathfrak{g}$ and they satisfy

$$H_{n+i,n+j} = -H_{ji},$$

$$H_{i,n+i} = 0,$$

$$[H_{ij}, H_{kl}] = \delta_{\bar{j}\bar{k}} H_{il} - \delta_{\bar{l}\bar{i}} H_{kj} - \delta_{\bar{j}\,\overline{n+l}} H_{i,n+k} + \delta_{\bar{i}\,\overline{n+k}} H_{n+l,j}.$$

We define $\varepsilon_i \in \mathfrak{h}^*$ $(i \in \{1, \dots, n\})$ by $\varepsilon_i(H_{jj}) = \delta_{ij}$ $(j \in \{1, \dots, n\})$, and summarize data such as the root system in the following list.

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\},$$

$$\Delta^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\},$$

$$H_{ij} : (\varepsilon_i - \varepsilon_j)\text{-root vector } (1 \le i, j \le n, i \ne j),$$

$$H_{i,n+j} : (\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \le i, j \le n, i \ne j),$$

$$H_{n+j,i} : -(\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \le i, j \le n, i \ne j),$$

$$\Pi_L = \Pi \backslash \{\varepsilon_1 - \varepsilon_2\},$$

$$\Delta_L^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 < i < j \le n\},$$

$$\varpi_{i_0} = \varepsilon_1,$$

$$\rho = (n-1)\varpi_{i_0},$$

$$\langle X, Y \rangle = \mathrm{Tr}(XY)/2 \quad (X, Y \in \mathfrak{g}),$$

C. B. : $\{H_{ij} \mid 1 \le i, j \le n\} \cup \{H_{i,n+j} \mid 1 \le i < j \le n\} \cup \{H_{n+j,i} \mid 1 \le i < j \le n\}$.

Set $M = \{1, \dots, 2n\} \backslash \{1, n+1\}$. Subalgebras $\mathfrak{p}, \mathfrak{n}^+$ and $\mathfrak{l}$ are as follows:

$$\mathfrak{l} = \mathrm{span}_{\boldsymbol{C}}\{H_{11}, H_{ij}(1 < i, j \le n), H_{i,n+j}(1 < i < j \le n), H_{n+j,i}(1 < i < j \le n)\},$$

$$\mathfrak{n}^+ = \mathrm{span}_{\boldsymbol{C}}\{H_{1j}(j \in M)\},$$

$$\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+.$$

With respect to $x \mapsto {}^t x$ in Definition 1.2, we have

$${}^t H_{ij} = H_{ji}.$$

We set $x_i = H_{i1}$ for $i \in M$, and $\{x_i\}$ forms a linear coordinate system on $\mathfrak{n}^+$. We denote $\partial/\partial x_i$ by ∂_i and $x_{\bar{i}}$ by x_i etc. The following lemma follows directly from Lemma 1.1.

Lemma 2.16. (1) $\Psi_\lambda(H_{11}) = -\sum_{k \in M} x_k \partial_k + \lambda^0$,
(2) $\Psi_\lambda(H_{ij}) = x_i \partial_j - x_{n+j} \partial_{n+i}$ $(i, j \in M)$,
(3) $\Psi_\lambda(H_{1j}) = -\sum_{k \in M} x_k \partial_k \partial_j + \frac{1}{2} \sum_{k \in M} x_{n+j} \partial_k \partial_{n+k} + \lambda^0 \partial_j$ $(j \in M)$.

The relative invariant $f \in \boldsymbol{C}[\mathfrak{n}^+]$ with weight $-2\varpi_{i_0}$ is given by

$$f = x_2 x_{n+2} + x_3 x_{n+3} + \cdots + x_n x_{2n},$$

and we have

$${}^t f(\partial) = \partial_2 \partial_{n+2} + \cdots + \partial_n \partial_{2n}.$$

Theorem 2.17. *We set*

$$\begin{aligned} u_1 &= -H_{11}, \\ v_1 &= -H_{11} + 2n - 2, \\ u_2 &= \frac{1}{4} H_{11}(H_{11} - 2n + 4) - \frac{1}{4} c, \\ v_2 &= \frac{1}{4}(H_{11} - 2)(H_{11} - 2n + 2) - \frac{1}{4} c, \end{aligned}$$

where $c \in U(\mathfrak{l})$ is the Casimir element of $[\mathfrak{l}, \mathfrak{l}]$ with respect to $\langle\, , \rangle$. Then we have

$$\begin{aligned} \sum_{j \in M} x_j \partial_j &= \operatorname{ad}(u_1), \\ \sum_{j \in M} \partial_j x_j &= \operatorname{ad}(v_1), \\ f\,{}^t f(\partial) &= \operatorname{ad}(u_2), \\ {}^t f(\partial) f &= \operatorname{ad}(v_2), \end{aligned}$$

and moreover we have

$$\begin{aligned} \sum_{j \in M} \Psi_\lambda(H_{j1} H_{1j}) &= -\frac{1}{2} \Psi_0(u_1) \Psi_{2\lambda + 2\rho}(u_1) - \frac{1}{2} \operatorname{ad}(c), \\ \sum_{j \in M} \Psi_\lambda(H_{1j} H_{j1}) &= -\frac{1}{2} \Psi_0(v_1) \Psi_{2\lambda + 2\rho}(v_1) - \frac{1}{2} \operatorname{ad}(c), \\ \Psi_\lambda(f\,{}^t f) &= \Psi_0(u_2) \Psi_{2\lambda + 2\rho}(u_2), \\ \Psi_\lambda({}^t f f) &= \Psi_0(v_2) \Psi_{2\lambda + 2\rho}(v_2). \end{aligned}$$

Proof. As for the relative invariant f, we have proved the assertion in [**12**]. For the Ψ_λ-analogue corresponding to the Euler operator, we can prove the assertion by direct calculations using Lemma 2.16. □

2.5. $(\mathrm{B}_n, 1)$ **or** $O_{2n} \otimes GL_1$**.**
Set

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & a & b \\ -{}^t b & A & B \\ -{}^t a & C & -{}^t A \end{pmatrix} \in \mathfrak{gl}(2n+1, \boldsymbol{C}) \;\middle|\; \begin{array}{l} A \in \mathfrak{gl}(n, \boldsymbol{C}), \\ B, C \in \operatorname{Alt}(n, \boldsymbol{C}), \\ a, b \in \boldsymbol{C}^n \end{array} \right\},$$

and let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. We take a basis of $\mathfrak{g}$ by extending the basis in §§2.4. Counting the row number and the column number from zero, we set

$$H_{ij} = E_{\bar{i}\bar{j}} - E_{\overline{n+j},\overline{n+i}} \quad (i, j \in \boldsymbol{Z}_{>0}),$$

$$g_i = E_{0\bar{i}} - E_{\overline{n+i}0} \quad (i \in \boldsymbol{Z}_{>0}),$$

where $\bar{i}$ is the same as in §§2.4. Then all H_{ij} and g_i belong to $\mathfrak{g}$, and they satisfy

$$\begin{aligned}
H_{n+i,n+j} &= -H_{ji},\\
H_{i,n+i} &= 0,\\
g_{n+i} &= -{}^t g_i,\\
[H_{ij}, H_{kl}] &= \delta_{\bar{j}\bar{k}} H_{il} - \delta_{\bar{l}\bar{i}} H_{kj} - \delta_{\bar{j}\overline{n+l}} H_{i,n+k} + \delta_{\bar{i}\overline{n+k}} H_{n+l,j},\\
[H_{ij}, g_k] &= -\delta_{\bar{i}\bar{k}} g_j + \delta_{\bar{k}\overline{n+j}} g_{n+i},\\
[g_i, g_j] &= H_{n+j,i}.
\end{aligned}$$

We define $\varepsilon_i \in \mathfrak{h}^*$ $(i \in \{1, \dots, n\})$ by $\varepsilon_i(H_{jj}) = \delta_{ij}$ $(j \in \{1, \dots, n\})$, and we list data such as the root system in the following list.

$$\begin{aligned}
\Pi &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\},\\
\Delta^+ &= \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{\varepsilon_i \mid 1 \le i \le n\},\\
H_{ij} &: (\varepsilon_i - \varepsilon_j)\text{-root vector } (1 \le i, j \le n, i \ne j),\\
H_{i,n+j} &: (\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \le i < j \le n),\\
H_{n+j,i} &: -(\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \le i < j \le n),\\
g_{n+i} &: \varepsilon_i\text{-root vector } (1 \le i \le n),\\
g_i &: -\varepsilon_i\text{-root vector } (1 \le i \le n),\\
\Pi_L &= \Pi \backslash \{\varepsilon_1 - \varepsilon_2\},\\
\Delta_L^+ &= \{\varepsilon_i \pm \varepsilon_j \mid 1 < i < j \le n\} \cup \{\varepsilon_i \mid 1 < i \le n\},\\
\varpi_{i_0} &= \varepsilon_1,\\
2\rho &= (2n-1)\varpi_{i_0},\\
\langle X, Y \rangle &= \mathrm{Tr}(XY)/2 \quad (X, Y \in \mathfrak{g}),\\
\text{C. B.} &: \{H_{ij} \mid 1 \le i, j \le n\} \cup \{H_{i,n+j} \mid 1 \le i < j \le n\}\\
&\quad \cup \{H_{n+j,i} \mid 1 \le i < j \le n\} \cup \{\sqrt{-2} g_i \mid 1 \le i \le 2n\}.
\end{aligned}$$

Subalgebras $\mathfrak{p}, \mathfrak{n}^+$ and $\mathfrak{l}$ are as follows:

$$\mathfrak{l} = \mathrm{span}_{\boldsymbol{C}}\{H_{11}, H_{ij}(1 < i, j \le n), H_{i,n+j}(1 < i < j \le n), H_{n+j,i}(1 < i < j \le n), g_i(i \in M)\},$$

$$\mathfrak{n}^+ = \mathrm{span}_{\boldsymbol{C}}\{H_{1j}(j \in M), g_{n+1}\},$$

$$\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+,$$

where $M = \{1, \dots, 2n\} \backslash \{1, n+1\}$, which is the same as in §§2.4. With respect to $x \mapsto {}^t x$ defined in Definition 1.2, we have

$${}^t H_{ij} = H_{ji}, \quad {}^t g_i = -g_{n+i}.$$

We set $x_i = H_{i1}$ and $x_0 = g_1$ for $i \in M$, and we then obtain a linear coordinate system $\{x_i \mid i \in M_0\}$ on $\mathfrak{n}^+$, where $M_0 = M \cup \{0\}$. In addition, we set $\partial_i = \partial/\partial x_i$ for $i \in M_0$. The following lemma follows easily from Lemma 1.1.

Lemma 2.18. (1) $\Psi_\lambda(H_{11}) = -\sum_{j \in M_0} x_j \partial_j + \lambda^0$,
(2) $\Psi_\lambda(H_{ij}) = x_i \partial_j - x_{n+j} \partial_{n+i}$ $(i, j \in M)$,
(3) $\Psi_\lambda(g_i) = x_0 \partial_i - x_i \partial_0$ $(i \in M)$,
(4) $\Psi_\lambda(H_{1j}) = -\sum_{k \in M_0} x_k \partial_k \partial_j + \dfrac{1}{2} x_{n+j} (\sum_{k \in M} \partial_k \partial_{n+k} + \partial_0 \partial_0) + \lambda^0 \partial_j$,
(5) $\Psi_\lambda(g_{n+1}) = \sum_{k \in M_0} x_k \partial_k \partial_0 - \dfrac{1}{2} x_0 (\sum_{k \in M} \partial_k \partial_{n+k} + \partial_0 \partial_0) - \lambda^0 \partial_0$.

The relative invariant $f \in \boldsymbol{C}[\mathfrak{n}^+]$ with weight $-2\varpi_{i_0}$ is given by

$$f = x_2 x_{n+2} + x_3 x_{n+3} + \cdots + x_n x_{2n} + \frac{1}{2} x_0^2,$$

and we have

$${}^t f(\partial) = \partial_2 \partial_{n+2} + \cdots + \partial_n \partial_{2n} + \frac{1}{2} \partial_0 \partial_0.$$

Theorem 2.19. *We set*

$$u_1 = -H_{11},$$

$$v_1 = -H_{11} + 2n - 1,$$

$$u_2 = \frac{1}{4} H_{11}(H_{11} - 2n + 3) - \frac{1}{4} c,$$

$$v_2 = \frac{1}{4}(H_{11} - 2)(H_{11} - 2n + 1) - \frac{1}{4} c,$$

where $c \in U(\mathfrak{l})$ *is the Casimir element of* $[\mathfrak{l}, \mathfrak{l}]$ *with respect to* $\langle\,,\rangle$. *Then we have*

$$\sum_{j \in M_0} x_j \partial_j = \mathrm{ad}(u_1),$$

$$\sum_{j \in M_0} \partial_j x_j = \mathrm{ad}(v_1),$$

$$f\,{}^t f(\partial) = \mathrm{ad}(u_2),$$

$${}^t f(\partial) f = \mathrm{ad}(v_2),$$

and moreover we have

$$\Psi_\lambda\left(\sum_{j\in M} H_{j1}H_{1j} - g_1 g_{n+1}\right) = -\frac{1}{2}\Psi_0(u_1)\Psi_{2\lambda+2\rho}(u_1) - \frac{1}{2}\operatorname{ad}(c),$$

$$\Psi_\lambda\left(\sum_{j\in M} H_{1j}H_{j1} - g_{n+1} g_1\right) = -\frac{1}{2}\Psi_0(v_1)\Psi_{2\lambda+2\rho}(v_1) - \frac{1}{2}\operatorname{ad}(c),$$

$$\Psi_\lambda(f\,{}^t\!f) = \Psi_0(u_2)\Psi_{2\lambda+2\rho}(u_2),$$

$$\Psi_\lambda({}^t\!ff) = \Psi_0(v_2)\Psi_{2\lambda+2\rho}(v_2).$$

Proof. As for the relative invariant f, we have proved the assertion in [**12**]. For the Ψ_λ-analogue corresponding to the Euler operator, we can prove the assertion by direct calculations using Lemma 2.18. □

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