J. Math. Soc. Japan
Vol. 56, No. 2, 2004

On extension of solutions of Kirchhoff equations

By Kunihiko Kajitani and Atsushi Satoh

(Received Mar. 14, 2002)
(Revised Oct. 16, 2002)

Abstract. This paper is devoted to the study of the extention of the existence time of solutions in Sobolev spaces to the Cauchy problem for Kirchhoff equations.

0. Introduction.

We shall investigate the Cauchy problem for the Kirchhoff equation:

$$u_{tt}(t,x) = M(\|\nabla_x u(t)\|_{L^2}^2)\triangle u(t,x), \tag{0.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{0.2}$$

for $0 \le t < T$, $x \in \boldsymbol{R}^n$. It is well known (see Bernstein [**1**] and Pohozaev [**3**]) that if the initial data u_0, u_1 are real analytic then we have a unique time global solution of the above Kirchhoff equation, and that for the initial data u_i in Sobolev space H^{s-i} $(s \ge 2, i = 0,1)$ there is $T > 0$ such that there exists a unique solution $u \in \bigcap_{i=0}^2 C^i([0,T); H^{s-i})$ of the Cauchy problem (0.1)–(0.2). On the other hand recently Colombini, Del Santo and Kinoshita in [**2**] have obtained the C^∞ well-posedness for the Cauchy problem to linear second order hyperbolic equations of which the first derivatives of coefficients have singularities of order one with respect to time variable. In this paper, noting that the first derivative of the coefficient $M(\|\nabla_x u(t)\|_{L^2}^2)$ of Kirchhoff equation (0.1)–(0.2) have also singularities of order one with respect to time variable, we can apply the result of Colombini, Del Santo and Kinoshita in [**2**] to this solution and we shall show that we can extend beyond $t = T$ the solution in Sobolev space of Kirchhoff equation (0.1)–(0.2).

We assume that the coefficient $M(\eta) \in C^1([0,\infty))$ satisfies

$$0 < \exists\lambda_0 \le M(\eta) \quad (0 \le \eta). \tag{0.3}$$

For $u \in \bigcap_{j=0}^2 C^j([0,T); H^{s-j})$ satisfying (0.1) we introduce the following two energies,

$$E(t) = \frac{1}{2}\left\{\|u_t(t)\|_{L^2}^2 + \int_0^{\|\nabla_x u(t)\|_{L^2}^2} M(\eta)\,d\eta\right\}, \tag{0.4}$$

and for a non negative number s

$$e_s(t) = \frac{1}{2}\{\|u_t(t)\|_s^2 + M(\|\nabla_x u(t)\|_{L^2}^2)\|\nabla_x u(t)\|_s^2\}, \tag{0.5}$$

2000 *Mathematics Subject Classification.* Primary 35L70.
Key Words and Phrases. hyperbolic equation, nonregular coefficients, life span.

where H^s stands for Sobolev space in $\boldsymbol{R}^n$ with the norm $\|\cdot\|_s$. Then we can show easily (see Lemma 2.1 and Propositon 2.3 in the Section 2)

$$E(t) = E(0), \quad t > 0,$$

and

$$\sqrt{e_1(t)} \leq \frac{\sqrt{2}\lambda_0}{K(T-t)}, \quad 0 \leq t < T,$$

where

$$T = \frac{\sqrt{2}\lambda_0}{K\sqrt{e_1(0)}}, \quad K = \max_{0 \leq \eta \leq 2E(0)/\lambda_0} |M'(\eta)| 2\sqrt{\eta}. \tag{0.6}$$

Applying the result of Colombini, Del Santo and Kinoshita [**2**] to Kirchhoff equation (0.1)–(0.2), we can prove the following theorem.

Theorem 0.1. *Suppose the coefficient* $M \in C^1([0,\infty))$ *satisfies* (0.3) *and* $u_0 \in H^4$, $u_1 \in H^3$. *Let* $T > 0$ *be defined by* (0.6). *Then* $e_3(t)$ $(0 \leq t < T)$ *is bounded and moreover* $\lim_{t\to T-0} u(t,x)$ *and* $\lim_{t\to T-0} u_t(t,x)$ *exist and belong to* H^4 *and to* H^3 *respectively.*

It follows from Theorem 0.1 that we can consider the Cauchy problem for Kirchhoff equation for the initial plane $t = T$ and for the initial data $\lim_{t\to T-0} \partial_t^i u(t,x)$ $(i = 0, 1)$. Therefore we can extend the existence time of solutions of (0.1) beyond T. Repeating the same augument step by step, we can prove the following result.

Theorem 0.2. *Suppose the coefficient* $M \in C^1([0,\infty))$ *satisfies* (0.3). *Let* $u_0 \in H^4$, $u_1 \in H^3$ *and define* $T_k = (\sqrt{2}\lambda_0 / K\sqrt{e_1(T_{k-1})}) + T_{k-1}$ $(k = 1, 2, \ldots)$, $T_0 = 0$ *and*

$$\bar{T} = \frac{\sqrt{2}\lambda_0}{K} \sum_{k=1}^{\infty} \frac{1}{\sqrt{e_1(T_{k-1})}},$$

where $K = \max_{0\leq\eta\leq 2E(0)/\lambda_0} |M'(\eta)| 2\sqrt{\eta}$. *Then* $e_3(T_k)$ *for every* $k \geq 1$ *is finite and there exists a unique solution* $u \in \bigcap_{j=0}^{2} C^j([0,\bar{T}); H^{2-j})$ *of the Cauchy problem* (0.1)–(0.2).

We organize this paper as follows. In the section 1 we shall derive an energy estimate for a linear hyperbolic equation with non Lipschitz continuous coefficients following the idea given in [**2**]. In the section 2 we shall obtain a local solution for the Cauchy problem of Kirchhoff equation of which coefficient satisfies the conditions of the theorem in [**2**] and applying the estimate derived in the section 1 we shall prove Theorem 0.1 and Theorem 0.2.

1. Linear equations.

Let $T > 0$. We consider the following Cauchy problem for the linear equation

$$u_{tt}(t,x) = a(t) \triangle u(t,x), \quad 0 \leq t < T, \ x \in \boldsymbol{R}^n, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \boldsymbol{R}^n. \tag{1.2}$$

We assume that the coefficient $a = a(t)$ satisfies the following hypothesis.

Assumption 1.1. *Let $a = a(t) \in C^1([0,T))$ be a function satisfying the following properties.*

$$\text{(i)} \quad 0 < \exists\lambda_0 \le a(t) \le \exists M_0 \quad (0 \le t \le T)$$

$$\text{(ii)} \quad |a'(t)| \le \frac{\exists A_0}{T-t} \quad (0 \le t < T),$$

here we denote by $a'(t) = da(t)/dt$.

Under the above assumption we can prove the following proposition:

Proposition 1.2 (Colombini-Del Santo-Kinoshita [**2**]). *Suppose that Assumption* 1.1 *holds. Let $s \ge 2$ and denote $\delta = A_0/\lambda_0$. Then for any $u_0 \in H^{s+\delta}$, $u_1 \in H^{s+\delta-1}$ there exists a unique solution $u \in \bigcap_{i=0}^2 C^i([0,T]; H^{s-j})$ of* (1.1)–(1.2) *satisfying*

$$\|\nabla_x u(t)\|_s^2 + \|u_t(t)\|_s^2 \le B\{\|\nabla_x u_0\|_{s+\delta}^2 + \|u_1\|_{s+\delta}^2\} \quad (0 \le t \le T), \tag{1.3}$$

where

$$\begin{cases} B = \max(d, C'), & d = \tilde{C}e^{M_1/\lambda_0}T^\delta, \\ C' = \tilde{C}e^{M_1/2\lambda_0}2^\delta, & M_1 = M_0(1+M_0), \\ \tilde{C} = \dfrac{\max(1, M_0)}{\min(1, \lambda_0)}. \end{cases} \tag{1.4}$$

Proof. Let $v = v(t,\xi)$ be a solution of the next Cauchy problem:

$$\begin{cases} v_{tt}(t,\xi) = -a(t)|\xi|^2 v(t,\xi), \\ v(0,\xi) = u_0(\xi), \quad v_t(0,\xi) = \hat{u}_1(\xi) \end{cases} \tag{1.5}$$

where $\hat{u} = \hat{u}(\xi)$ stands for the Fourier transform of $u = u(x)$. Let $\varepsilon > 0$. When $\varepsilon < T$, we define

$$a_\varepsilon(t) = \begin{cases} a(t), & (0 \le t \le T-\varepsilon) \\ a(T-\varepsilon), & (T-\varepsilon \le t < T). \end{cases} \tag{1.6}$$

When $\varepsilon > T$ define

$$a_\varepsilon(t) = \begin{cases} a(t), & (0 \le t \le T/2) \\ a(T/2), & (T/2 \le t < T). \end{cases} \tag{1.7}$$

We introduce an energy for v as follows:

$$E_\varepsilon(t) = |v'(t)|^2 + a_\varepsilon(t)|\xi|^2|v(t)|^2.$$

Then taking account that v satisfies the equation (1.5) we calculate

$$\begin{aligned} \frac{d}{dt}E_\varepsilon(t) &= 2\operatorname{Re}(v''(t)\overline{v'(t)}) + a_\varepsilon'(t)|\xi|^2|v(t)|^2 + 2a_\varepsilon(t)|\xi|^2\operatorname{Re}(v'(t)\overline{v(t)}) \\ &= -2a(t)|\xi|^2\operatorname{Re}(v(t)\overline{v'(t)}) + a_\varepsilon'(t)|\xi|^2|v(t)|^2 + 2a_\varepsilon(t)|\xi|^2\operatorname{Re}(v'(t)\overline{v(t)}) \\ &= 2\{a_\varepsilon(t) - a(t)\}|\xi|^2\operatorname{Re}(v'(t)\overline{v(t)}) + a_\varepsilon'(t)|\xi|^2|v(t)|^2 \end{aligned}$$

$$\le 2\{|a_\varepsilon(t)-a(t)|\,|\xi|^2|v'(t)|\,|v(t)| + |a_\varepsilon'(t)|\,|\xi|^2|v(t)|^2\}\frac{a_\varepsilon(t)}{a_\varepsilon(t)}$$

$$\le \frac{|a_\varepsilon(t)-a(t)|}{a_\varepsilon(t)}|\xi|\{a_\varepsilon(t)|\xi|^2|v(t)|^2 + a_\varepsilon(t)|v'(t)|^2\} + \frac{|a_\varepsilon'(t)|}{a_\varepsilon(t)}a_\varepsilon(t)|\xi|^2|v(t)|^2.$$

Hence we get from (i) of Assumption 1.1,

$$\frac{d}{dt}E_\varepsilon(t) \le \frac{|a_\varepsilon(t)-a(t)|}{a_\varepsilon(t)}|\xi|\{E_\varepsilon(t) + E_\varepsilon(t)M_0\} + \frac{|a_\varepsilon'(t)|}{a_\varepsilon(t)}E_\varepsilon(t)$$

$$\le \frac{|a_\varepsilon(t)-a(t)|}{\lambda_0}|\xi|(1+M_0)E_\varepsilon(t) + \frac{|a_\varepsilon'(t)|}{\lambda_0}E_\varepsilon(t).$$

By using Gronwall's lemma, we obtain

$$E_\varepsilon(t) \le E_\varepsilon(0)e^{\int_0^t \alpha(\tau)\,d\tau} \quad (0 \le t < T), \tag{1.8}$$

where

$$\alpha(t) = \frac{1}{\lambda_0}(|a_\varepsilon(t)-a(t)|\,|\xi|(1+M_0) + |a_\varepsilon'(t)|).$$

Now let us estimate the integral

$$\int_0^t \alpha(\tau)\,d\tau = \int_0^t \frac{1}{\lambda_0}\{|a_\varepsilon(\tau)-a(\tau)|\,|\xi|(1+M_0) + |a_\varepsilon'(\tau)|\}\,d\tau.$$

From the definition (1.6) of $a_\varepsilon(t)$ we have

$$\int_0^t |a_\varepsilon'(\tau)|\,d\tau = \int_0^{T-\varepsilon}|a_\varepsilon'(\tau)|\,d\tau + \int_{T-\varepsilon}^T |a_\varepsilon'(\tau)|\,d\tau$$

$$= A_0\log\frac{T}{\varepsilon}$$

and

$$\int_0^t |a_\varepsilon(\tau)-a(\tau)|\,d\tau = \int_{T-\varepsilon}^T |a(T-\varepsilon)-a(\tau)|\,d\tau$$

$$\le \int_{T-\varepsilon}^T M_0\,d\tau = M_0\varepsilon.$$

Hence, if we choose $\varepsilon = |\xi|^{-1}$, we obtain

$$\int_0^t \alpha(\tau)\,d\tau \le \frac{1}{\lambda_0}\left(A_0\log\frac{T}{\varepsilon} + |\xi|(1+M_0)M_0\varepsilon\right)$$

$$= \frac{A_0}{\lambda_0}\log\frac{T}{\varepsilon} + \frac{M_1}{\lambda_0}|\xi|\varepsilon$$

$$= \frac{A_0}{\lambda_0}\log(T|\xi|) + \frac{M_1}{\lambda_0}, \quad M_1 = (1+M_0)M_0.$$

Therefore we get from (1.8)

$$E_\varepsilon(t) \le E_\varepsilon(0)e^{(A_0/\lambda_0)\log(T|\xi|)+(M_1/\lambda_0)} = E_\varepsilon(0)e^{M_1/\lambda_0}T^{A_0/\lambda_0}|\xi|^{A_0/\lambda_0}. \tag{1.9}$$

Noting that

$$\begin{aligned} E_\varepsilon(t) &= |v'(t)|^2 + a_\varepsilon(t)|\xi|^2|v(t)|^2 \ge |v'(t)|^2 + \lambda_0|\xi|^2|v(t)|^2 \\ &\ge \min(1,\lambda_0)\{|v'(t)|^2 + |\xi|^2|v(t)|^2\} \end{aligned}$$

and

$$\begin{aligned} E_\varepsilon(0) &= |v'(0)|^2 + a_\varepsilon(0)|\xi|^2|v(0)|^2 \le |v'(0)|^2 + M_0|\xi|^2|v(0)|^2 \\ &\le \max(1,M_0)\{|v'(0)|^2 + |\xi|^2|v(0)|^2\}, \end{aligned}$$

we get from (1.9)

$$|v'(t)|^2 + |\xi|^2|v(t)|^2 \le \tilde{C}e^{M_1/\lambda_0}T^\delta\{|v'(0)|^2 + |\xi|^2|v(0)|^2\}|\xi|^\delta, \tag{1.10}$$

for $|\xi| \ge T^{-1}$, where $\tilde{C} = \max(1,M_0)/\min(1,\lambda_0)$, $\delta = A_0/\lambda_0$. Next we consider the case of $T|\xi| < 1$. From the definition (1.7) of $a_\varepsilon(t)$ we get

$$\begin{aligned} \int_0^t |a_\varepsilon'(\tau)|\,d\tau &= \int_0^{T/2} |a'(\tau)|\,d\tau + \int_{T/2}^T |a_\varepsilon'(\tau)|\,d\tau \\ &\le \int_0^{T/2} \frac{A_0}{T-\tau}d\tau = A_0\log 2 \end{aligned}$$

and

$$\begin{aligned} \int_0^t |a_\varepsilon(\tau) - a(\tau)|\,d\tau &= \int_{T/2}^T \left|a\left(\frac{T}{2}\right) - a(\tau)\right| d\tau \\ &\le \int_{T/2}^T M_0\,d\tau = M_0\frac{T}{2}. \end{aligned}$$

Thus we obtain for $T|\xi| < 1$

$$\begin{aligned} \int_0^t \alpha(\tau)\,d\tau &\le \frac{1}{\lambda_0}\left\{A_0\log 2 + |\xi|(1+M_0)M_0\frac{T}{2}\right\} \\ &\le \frac{A_0}{\lambda_0}\log 2 + \frac{M_1}{2\lambda_0}, \end{aligned}$$

and consequently

$$E_\varepsilon(t) \le E_\varepsilon(0)2^{A_0/\lambda_0}e^{((M_1/2\lambda_0)T|\xi|)},$$

which implies

$$|v'(t)|^2 + |\xi|^2|v(t)|^2 \le \tilde{C}e^{(M_1/2\lambda_0)}2^\delta\{|v'(0)|^2 + |\xi|^2|v(0)|^2\}. \tag{1.11}$$

Multiplying $|\xi|^2|\hat{u}(t,\xi)|^2 + |\hat{u}_t(t,\xi)|^2$ by $\langle\xi\rangle^{2s}$ and integrating over $\boldsymbol{R}^n_\xi$, we obtain (1.3) from (1.10) and (1.11). □

2. Kirchhoff equations.

Next we consider the following Cauchy problem for the Kirchhoff equation:

$$\begin{cases} u_{tt}(t,x) = M(\|\nabla_x u(t)\|^2_{L^2})\triangle u(t,x), & 0 \le t < T,\ x \in \boldsymbol{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \boldsymbol{R}^n. \end{cases} \tag{2.1}$$

We begin to state a fundamental property of Kirchhoff equation.

Lemma 2.1. *For* $u \in \bigcap_{j=0}^2 C^i([0,\bar{T}); H^{2-j})$ *satisfying* (2.1) *set*

$$E(t) = \frac{1}{2}(\|u_t(t)\|^2_{L^2} + F(\|\nabla_x u(t)\|^2_{L^2})), \quad F(\eta) = \int_0^\eta M(\lambda)\,d\lambda.$$

Then

$$E(t) = E(0), \quad \forall t \in [0,T]. \tag{2.2}$$

Proof. We can see that (2.2) holds because of $(d/dt)E(t) = 0$. □

We suppose that the coefficient M satisfies the following hypothesis.

Assumption 2.2. *$M = M(\eta)$ is a function satisfying the following properties.*

(i) $0 < \exists\lambda_0 \le M(\eta)$

(ii) $M(\eta) \in C^1([0,\infty))$.

We denote

$$a(t) := M(\|\nabla_x u(t)\|^2_{L^2}). \tag{2.3}$$

We can show the local existence theorem of Kirchhoff equation (2.1).

Proposition 2.3. *Suppose Assumption* 2.2 *holds and* $s \ge 2$ *a positive integer. Then for* $u_i \in H^{s-i}$ $(i = 0,1)$ *there is a unique solution* $u \in \bigcap_{j=0}^2 C^j([0,T); H^{s-j})$ *of the equation* (2.1) *and moreover* $a(t)$ *defined by* (2.3) *satisfies*

$$|a'(t)| \le \frac{2\lambda_0}{T-t}, \tag{2.4}$$

where $T = \sqrt{2}\lambda_0/K\sqrt{e_1(0)}$ *and* $K = \max_{0\le\eta\le 2E(0)/\lambda_0}|M'(\eta)|2\sqrt{\eta}$.

Proof. Differentiating $a(t) = M(\|\nabla_x u(t)\|^2_{L^2})$, we have

$$\begin{aligned} |a'(t)| &= |M'(\|\nabla_x u(t)\|^2_{L^2}) \cdot 2\,\mathrm{Re}(\nabla_x u_t, \nabla_x u)_{L^2}| \\ &\le |M'(\|\nabla_x u(t)\|^2_{L^2})| \cdot 2 \cdot \|\nabla_x u_t\|_{L^2}\|\nabla_x u\|_{L^2}. \end{aligned} \tag{2.5}$$

Using Assumption 2.2 and Lemma 2.1 and noting

$$\lambda_0\|\nabla_x u(t)\|^2_{L^2} \le F(\|\nabla_x u(t)\|^2_{L^2}) \le 2E(t) = 2E(0),$$

we obtain

$$|M'(\|\nabla_x u(t)\|_{L^2}^2)| \le \max_{0\le\eta\le 2E(0)/\lambda_0} |M'(\eta)| 2\sqrt{\eta} =: K.$$

Hence we can see from (2.5)

$$|a'(t)| \le K\|\nabla_x u_t\|_{L^2}. \tag{2.6}$$

Differentiating $e_1(t)$ given by (0.5) with respect to t, and using the equation (2.1) we have

$$\begin{aligned}
\frac{d}{dt}e_1(t) &= \operatorname{Re}(u_{tt}, u_t)_1 + \frac{1}{2}a'(t)\|\nabla_x u(t)\|_1^2 + a(t)\operatorname{Re}(\nabla_x u_t, \nabla_x u)_1 \\
&= \operatorname{Re}(a(t)\triangle u, u_t)_1 + a(t)\operatorname{Re}(\nabla_x u_t, \nabla_x u)_1 + \frac{1}{2}a'(t)\|\nabla_x u(t)\|_1^2 \\
&= -a(t)\operatorname{Re}(\nabla_x u, \nabla_x u_t)_1 + a(t)\operatorname{Re}(\nabla_x u_t, \nabla_x u)_1 + \frac{1}{2}a'(t)\|\nabla_x u(t)\|_1^2 \\
&\le \frac{1}{2}|a'(t)|\,\|\nabla_x u(t)\|_1^2.
\end{aligned} \tag{2.7}$$

Since from (2.6)

$$|a'(t)| \le K\sqrt{2e_1(t)} \tag{2.8}$$

holds, we deduce from (2.7)

$$\begin{aligned}
\frac{d}{dt}e_1(t) &\le \frac{1}{2}K\sqrt{2e_1(t)}\|\nabla_x u(t)\|_1^2 \frac{a(t)}{a(t)} \\
&\le \frac{K}{2\lambda_0}\sqrt{2e_1(t)}a(t)\|\nabla_x u(t)\|_1^2 \\
&\le \frac{\sqrt{2}K}{\lambda_0}e_1(t)^{3/2}.
\end{aligned} \tag{2.9}$$

Set $\beta(t) = \sqrt{e_1(t)}$. Then we have from (2.9)

$$2\beta'(t)\beta(t) \le \frac{\sqrt{2}K}{\lambda_0}\beta(t)^3.$$

Solving this inequality, we have

$$\beta(t) \le \frac{1}{K/\sqrt{2}\lambda_0}\left(\frac{\sqrt{2}\lambda_0}{K\beta(0)} - t\right)^{-1},$$

which yields together with (2.8),

$$|a'(t)| \le \frac{2\lambda_0}{(\sqrt{2}\lambda_0/K\beta(0)) - t}$$

and hence we obtain (2.4). Thus we have proved this proposition. □

Combining Proposition 1.2 and Proposition 2.3, we obtain the following Proposition.

Proposition 2.4. *For* $u_0 \in H^4$, $u_1 \in H^3$ *there exists a unique solution* $u \in \bigcap_{j=0}^{2} C^j([0,T); H^{2-j})$ *of the Cauchy problem* (2.1) *which satisfies*

$$\|\nabla_x u(t)\|_1^2 + \|u_t(t)\|_1^2 \le B(T)\{\|\nabla u_0\|_3^2 + \|u_1\|_3^2\} \quad (0 \le t < T), \tag{2.10}$$

where

$$\begin{cases} T = \dfrac{\sqrt{2}\lambda_0}{K\beta(0)}, \quad B(T) = \dfrac{\max(1, M_0)}{\min(1, \lambda_0)} e^{M_1/\lambda_0} \max(T^2, 2^2), \\ K = \max_{0 \le \eta \le 2E(0)/\lambda_0} |M'(\eta)| 2\sqrt{\eta}, \\ M_1 = M_0(1 + M_0), \quad M_0 = \max_{0 \le \eta \le 2E(0)/\lambda_0} |M(\eta)|. \end{cases} \tag{2.11}$$

Proof. It follows from Proposition 2.3 that we can take $T = \sqrt{2}\lambda_0/K\beta(0)$, $K = \max_{0 \le \eta \le 2E(0)/\lambda_0} |M'(\eta)| 2\sqrt{\eta}$, $M_0 = \max_{0 \le \eta \le 2E(0)/\lambda_0} M(\eta)$, $A_0 = 2\lambda_0$ and $\delta_0 = 2$ in Proposition 1.2 and, consequently, we get (2.10) from (1.3). □

Proposition 2.5. *Let* $1 \le s \le 3$ *and* $u_0 \in H^4$ *and* $u_1 \in H^3$. *Then it holds*

$$e_s(t) \le e_s(0) e^{C_1 t} \quad (0 \le t < T), \tag{2.12}$$

where $T = \sqrt{2}\lambda_0/K\sqrt{e_1(0)}$, $C_1 = K\sqrt{B(T)Qe_3(0)}/\lambda_0$, $Q = 2/\min(1, \lambda_0)$ *and* $B(T)$ *is given in Proposition* 2.4.

Proof. We get from (2.10)

$$\begin{aligned} \|\nabla u(t)\|_1^2 + \|u_t(t)\|_1^2 &\le B(T)(\|\nabla_x u_0\|_3^2 + \|u_1\|_3^2) \\ &\le B(T)Qe_3(0). \end{aligned}$$

Therefore we obtain from (2.8)

$$|a'(t)| \le K\sqrt{B(T)Qe_3(0)}.$$

Now differentiating $e_s(t)$ with respect to t, we have

$$\begin{aligned} \frac{d}{dt} e_s(t) &= \frac{1}{2} a'(t) \|\nabla_x u(t)\|_3^2 \\ &\le \frac{1}{2} K\sqrt{B(T)Qe_3(0)} \|\nabla_x u(t)\|_3^2 \frac{a(t)}{a(t)} \\ &\le \frac{1}{2} K\sqrt{B(T)Qe_3(0)} \frac{a(t)\|\nabla_x u(t)\|_s^2}{\lambda_0} \\ &\le \frac{K\sqrt{B(T)Qe_s(0)}}{\lambda_0} e_s(t) = C_1 e_s(t), \end{aligned}$$

where $C_1 = K\sqrt{B(T)Qe_3(0)}/\lambda_0$. Gronwall's inequality gives (2.12). □

Now we can prove Theorem 0.1. $e_3(t)$ is bounded in $[0,T)$. Therefore using the equation (2.1) we can see that $\|u_{tt}(t)\|_2$ and $\|u_t(t)\|_3$ are bounded in $[0,T)$ and so $\lim_{t\to T-0} u_t(t)$ and $\lim_{t\to T-0} u_t(t)$ exist in H^2 and in H^3 respectively. On the other hand a subsequence of $\{u_t(t)\}$ and of $\{u(t)\}$ converge weakly in H^3 and H^4 respectively. Therefore $\lim_{t\to T-0} u_t(t)$ and $\lim_{t\to T-0} u_t(t)$ belong to H^3 and H^4 respectively. Thus we have proved Theorem 0.1.

Next we shall extend the solution of Kirchhoff equation (2.1) beyond $T = T_1 = \sqrt{2}\lambda_0/K\beta(0)$. Since it follows from Proposition 2.5 that $u_1^i(x) = \lim_{t\to T_1-0}(\partial^i/\partial t^i)u(t,x)$ $(i = 0,1)$ exist in H^3 and in H^4 respectively. Hence we can consider the following Cauchy problem,

$$\begin{cases} u_{tt}(t,x) = M(\|\nabla_x u(t)\|_{L^2}^2)\triangle u(t,x), & (T_1 \le t < T_2) \\ u(T_1,x) = u_1^0(x), \quad u_t(T_1,x) = u_1^1(x). \end{cases} \tag{2.13}$$

Repeating the same argument as one in the proof of Proposition 2.5 we can show the next Proposition.

Proposition 2.6. *Let* $1 \le s \le 3$ *and* $u_0 \in H^4$ *and* $u_1 \in H^3$. *Then we get*

$$e_s(t) \le e_s(T_1)e^{C_2(t-T_1)} \quad (T_1 \le t \le T_2), \tag{2.14}$$

where $T_1 = \sqrt{2}\lambda_0/(K\sqrt{e_1(0)})$, $T_2 = \sqrt{2}\lambda_0/(K\sqrt{e_1(T_1)}) + T_1$, $C_2 = (K\sqrt{B(T_2-T_1)Q\cdot e_3(T_1)})/\lambda_0$, $Q = 2/\min(1,\lambda_0)$ *and* $B(T)$ *is given in Proposition* 2.4.

Proof. Denote $v(t,x) = u(T_1+t,x)$. Then v satisfies

$$\begin{cases} v_{tt}(t,x) = M(\|\nabla_x v(t)\|_{L^2}^2)\triangle v(t,x), & (0 \le t < T) \\ v(0,x) = u(T_1,x), \quad v_t(0,x) = u_t(T_1,x). \end{cases} \tag{2.15}$$

Put $a(t) = M(\|\nabla_x v(t)\|_{L^2}^2)$ and define an energy of v as follows,

$$e_s(v(t)) = \frac{1}{2}(\|v_t(t)\|_s + a(t)\|\nabla_x v(t)\|_s).$$

Then it follows from Propostion 2.3 that we have

$$e_s(v(t)) \le e^{C_2 t}e_s(v(0)) \quad (0 \le t < T), \tag{2.16}$$

where $T = \sqrt{2}\lambda_0/(K\sqrt{e_1(v(0))})$, $C_2 = K\sqrt{B(T)Qe_3(v(0))}/\lambda_0$. $e_s(v(0)) = e_s(T_1)$ gives $T = \sqrt{2}\lambda_0/(K\sqrt{e_1(T_1)})$. Denote $T_2 = T_1 + T = T_1 + \sqrt{2}\lambda_0/(K\sqrt{e_1(0)})$. Then $B(T) = B(T_2 - T_1)$. Hence we can see that (2.16) implies (2.14). □

Inductively repeating the same argument we can obtain the next result.

Proposition 2.7. *Let* $1 \le s \le 3$ *and* $u_0 \in H^4$, $u_1 \in H^3$ *and a positive integer* k. *Then we get*

$$e_s(t) \le e_s(T_{k-1})e^{C_k(t-T_{k-1})} \quad (T_{k-1} \le t \le T_k), \tag{2.17}$$

where $T_k = \sqrt{2}\lambda_0/(K\sqrt{e_1(T_{k-1})}) + T_{k-1}$, $T_0 = 0$ *and* $C_k = K\sqrt{B(T_k - T_{k-1})Qe_3(T_{k-1})}/\lambda_0$.

It follows from (2.17) that we get

$$e_s(t) \le e_s(0)\exp\{C_k(t-T_{k-1}) + C_{k-1}(T_{k-1}-T_{k-2}) + \cdots + C_2(T_2-T_1) + C_1(T_1-T_0)\}, \tag{2.18}$$

for $1 \le s \le 3$. Put $\hat{C}_k = \max_{1\le j\le k} C_j$ which defines an increasing sequence. Then we obtain with $T_0 = 0$ the estimate from (2.18) with $s = 1$

$$\sqrt{e_1(T_k)} \le \sqrt{e_1(T_0)e^{\hat{C}_k T_k}}, \tag{2.19}$$

for $k = 1, 2, \ldots.$ Moreover,

$$T_k = \frac{\sqrt{2}\lambda_0}{K\sqrt{e_1(T_{k-1})}} + T_{k-1}$$

implies

$$T_k = \frac{\sqrt{2}\lambda_0}{K}\sum_{l=1}^{k}\frac{1}{\sqrt{e_1(T_{l-1})}},$$

for $k = 1, 2, \ldots.$ Now we put

$$\bar{T} = \frac{\sqrt{2}\lambda_0}{K}\sum_{k=0}^{\infty}\frac{1}{\sqrt{e_1(T_k)}}.$$

Assume that $\bar{T}$ is finite. Then we obtain from (2.19)

$$\begin{aligned}\bar{T} &= \frac{\sqrt{2}\lambda_0}{K}\sum_{k=0}^{\infty}\frac{1}{\sqrt{e_1(T_k)}} \\ &\ge \frac{\sqrt{2}\lambda_0}{K}\left(\sum_{k=1}^{\infty}\frac{1}{\sqrt{e^{\hat{C}_k T_k}e_1(0)}} + \frac{1}{\sqrt{e_1(0)}}\right) = \frac{\sqrt{2}\lambda_0}{K\sqrt{e_1(0)}}\left(\sum_{k=1}^{\infty}\frac{1}{\sqrt{e^{\hat{C}_k T_k}}} + 1\right) \\ &\ge \frac{\sqrt{2}\lambda_0}{K\sqrt{e_1(0)}}\left(\sum_{k=1}^{\infty}\frac{1}{\sqrt{e^{\hat{C}_k \bar{T}}}} + 1\right).\end{aligned} \tag{2.20}$$

Noting that $B(T_k - T_{k-1}) \le B(\bar{T})$, we have from (2.18) with $s = 3$

$$\begin{aligned}C_k = \frac{K\sqrt{B(T_k - T_{k-1})Qe_3(T_{k-1})}}{\lambda_0} &\le \frac{K\sqrt{B(\bar{T})Qe_3(T_{k-1})}}{\lambda_0} \\ &\le \frac{K\sqrt{B(\bar{T})Qe_3(0)e^{\hat{C}_{k-1}\bar{T}}}}{\lambda_0}.\end{aligned}$$

Therefore we get

$$\hat{C}_k = \max_{l=1,\ldots,k} C_l \le \frac{K\sqrt{B(\bar{T})Qe_3(0)e^{\hat{C}_{k-1}\bar{T}}}}{\lambda_0} \tag{2.21}$$

for $k = 2, 3, \ldots.$ Define

$$f(T, \eta) = \frac{K\sqrt{B(T)Qe_3(0)e^{\eta T}}}{\lambda_0}.$$

Thus we can see from (2.21)

$$\begin{aligned}\hat{C}_k &\le f(\bar{T}, \hat{C}_{k-1}) \le f(\bar{T}, f(\bar{T}, \hat{C}_{k-2}))\\ &\le \cdots \le \underbrace{f(\bar{T}, f(\bar{T}, f(\bar{T}, f(\cdots f(\bar{T}, C_1))\cdots)))}_{k-1 \text{ times}} := G_k(\bar{T}).\end{aligned}$$

Here $C_1 = K\sqrt{B(T_1)Qe_3(0)}/\lambda_0 = K\sqrt{B(\sqrt{2}\lambda_0/K\sqrt{e_1(0)})Qe_3(0)}/\lambda_0$. Hence we get from (2.20)

$$\frac{\sqrt{2}\lambda_0}{K\sqrt{e_1(0)}}\sum_{k=1}^{\infty}\frac{1}{\sqrt{e^{\hat{C}_k T_k}}} \ge \frac{\sqrt{2}\lambda_0}{K\sqrt{e_1(0)}}\sum_{k=1}^{\infty}\frac{1}{e^{\bar{T}G_k(\bar{T})}}. \tag{2.22}$$

Define for $T \ge 0$

$$H(T) = \frac{\sqrt{2}\lambda_0}{K\sqrt{e_1(0)}}\left(\sum_{k=1}^{\infty}\frac{1}{e^{TG_k(T)}} + 1\right).$$

It follows from (2.20) and (2.22) that

$$\bar{T} \ge H(\bar{T}). \tag{2.23}$$

Let T^* be a fixed point of $H(\bar{T})$. Then we can find from (2.23)

$$\bar{T} \ge T^*. \tag{2.24}$$

Thus we have proved the following result.

THEOREM 2.8. *Assume* $u_0 \in H^4$, $u_1 \in H^3$ *and define*

$$\bar{T} = \frac{\sqrt{2}\lambda_0}{K}\sum_{k=1}^{\infty}\frac{1}{\sqrt{e_1(T_{k-1})}},$$

where $T_0 = 0$, $T_k = \sqrt{2}\lambda_0/(K\sqrt{e_1(T_{k-1})}) + T_{k-1}$ *and* K *is given by Proposition* 2.3.

PROOF. Define $u_k^i(x) = \lim_{t\to T_k}(\partial^i/\partial t^i)u_{k-1}(t,x)$ $(i = 0, 1)$ inductively. Let $u^k(t,x)$ be the solution of the Cauchy problem,

$$\begin{cases} u_{tt}(t,x) = M(\|\nabla_x u(t)\|_{L^2}^2)\triangle u(t,x), & (T_k \le t < T_{k+1})\\ u_k(T_k, x) = u_k^0(x), \quad u_t(T_k, x) = u_k^1(x). \end{cases}$$

Then it follows from Proposition 2.7 that we can see inductively that $u_k(t,x)$ belongs to $\bigcap_{j=0}^{2} C^j([T_k, T_{k+1}]; H^{2-j})$ and $u_{k+1}^i(x)$ are in H^{3+i}. We define $u(t,x) = u_k(t,x)$ for $t \in [T_k, T_{k+1}), k = 0, 1, \ldots.$ Then $u(t,x)$ is in $\bigcap_{j=0}^{2} C^j([0, \bar{T}); H^{2-j})$ and satisfies the equation (2.1) in $(0, \bar{T})$. □

We can see easily that Theorem 0.2 in the introduction follows from the above theorem.

References

[1] S. Bernstein, Sur une classe d'équations fonctionelles aux dérivées partielles, Izv. Akad. Nauk SSSR Ser. Mat., No. 4 (1940), 17–26.
[2] F. Colombini, D. Del Santo and T. Kinoshita, Well-posedness of the Cauchy problem for a hyperbolic equation with non-Lipschitz coefficients, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), No. I (2002), 327–358.
[3] S. I. Pohozaev, On a class of quasilinear hyperbolic equations, Math. USSR-Sb., **25** (1975), 145–158.

Kunihiko KAJITANI
Institute of Mathematics
University of Tsukuba
Tsukuba-shi, Ibaraki, 305-8571
Japan

Atsushi SATOH
Institute of Mathematics
University of Tsukuba
Tsukuba-shi, Ibaraki, 305-8571
Japan