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On the graded ring of Siegel modular forms of degree 2, level 3

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Abstract. In general, it is difficult to determine the dimension of the space of Siegel modular forms with low weight. In this paper, we consider the spaces of modular forms belonging to the principal congruence subgroup of level 3 as the representation spaces of the finite symplectic group to calculate their dimensions.

0. Introduction.

The aim of this paper is to give the dimension of the space of Siegel modular forms $M_k(\Gamma(3))$ of degree 2, level 3 and weight k for each k. Our main result is

THEOREM.

$$\dim M_k(\Gamma(3)) = \begin{cases} 5 & k = 1; \\ 15 & k = 2; \\ 40 & k = 3; \\ \dfrac{1}{2}(6k^3 - 27k^2 + 79k - 78) & k \geq 4. \end{cases}$$

In other words, we have the generating function:

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma(3)) t^k = \frac{1 + t + t^2 + 6t^3 + 6t^4 + t^5 + t^6 + t^7}{(1-t)^4}.$$

About the space of cusp forms, the dimension formula of weight $k \geq 4$ is shown by Morita ([**11**]) Christian ([**2**], [**3**]) and Yamazaki ([**17**]), by using the Selberg trace formula or Riemann-Roch theorem. Therefore for weight $k \geq 5$, the calculation of the dimension of $M_k(\Gamma(3))$ is not so hard using Eisenstein series and the Siegel Φ-operator. In the case of low weights, the fact that $M_k(\Gamma(3))$ are representation spaces of the finite group $Sp(2, \boldsymbol{F}_3)$ is crucial. Fortunately all the irreducible characters of $Sp(2, \boldsymbol{F}_3)$ are given by Srinivasan ([**16**]). By utilizing this table we can determine these representations, hence all the more the dimensions.

Now we review the contents of our paper in detail. Firstly we recall some basic facts on elliptic modular forms in §1. In §2 the dimension formula of the spaces of Siegel cusp forms of higher weights is reviewed, and we examine the boundaries of the Satake compactification of $\Gamma(3)\backslash \boldsymbol{H}_2$ in §4.

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We determine the dimensions in §5 and §6. The space of modular forms $M_k(\Gamma(3))$ is decomposed into the space of cusp forms $S_k(\Gamma(3))$, the space of Siegel Eisenstein series $E_k(\Gamma(3))$ and the space of Klingen Eisenstein series $K_k(\Gamma(3))$ in §§5.1 and 5.2. In §§5.3 we define some elements of $M_1(\Gamma(3))$ by using the theory of theta series of quadratic forms, and we calculate the dimensions of $M_1(\Gamma(3))$ and $M_2(\Gamma(3))$ in §§5.4. Finally in §6 the dimensions of $M_3(\Gamma(3))$ and $M_4(\Gamma(3))$ are determined exactly by using the theory of non-holomorphic Eisenstein series.

This paper is the author's master thesis at the University of Tokyo. After finishing this paper, we are informed that Freitag and Salvati Manni are also investigating the same problem for a slightly different method (cf. [**5**]). They give not only generating functions but also relations for our graded ring.

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Notations.

- $\boldsymbol{H}_g = \{Z \in M_g(\boldsymbol{C}) \mid Z = {}^tZ, \operatorname{Im} Z > 0\}$
- For a commutative ring R,

$$Sp(g,R) = \{M \in M_{2g}(R) \mid {}^tMJM = J, J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}\}$$

- $\Gamma^{(g)} = Sp(g,\boldsymbol{Z})$
- $\Gamma^{(g)}(N) = \{M \in \Gamma^{(g)} \mid M \equiv 1_{2g} \mod N\}$
  If $g = 2$ we write simply Γ, $\Gamma(N)$ respectively dropping superscript.
- For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\boldsymbol{R})$ and $Z \in \boldsymbol{H}_g$, we set

$$M\langle Z\rangle = (AZ+B)(CZ+D)^{-1} \in \boldsymbol{H}_g.$$

- For a holomorphic function f on $\boldsymbol{H}_g$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\boldsymbol{R})$, we put

$$f|_kM(Z) = \det(CZ+D)^{-k} f(M\langle Z\rangle).$$

  Let Γ' be a subgroup of $\Gamma^{(g)}$ which contains some $\Gamma^{(g)}(N)$.
- The space of Siegel modular forms of weight k is given by

$$M_k(\Gamma') = \{f \text{ a holomorphic function on } \boldsymbol{H}_g \mid$$
$$f|_kM = f \text{ for all } M \in \Gamma, f \text{ is holomorphic at each cusp if } g = 1\}.$$

  For $f \in M_k(\Gamma')$ and $M \in \Gamma^{(g)}$ we write $f|M$ instead of $f|_kM$. This function belongs to $M_k(M^{-1}\Gamma'M)$.
- For $f \in M_k(\Gamma')$ and $z \in \boldsymbol{H}_r$ $(0 \le r \le g-1)$, we define an operator Φ^r on $M_k(\Gamma')$ by

$$\Phi^r(f)(z) = \lim_{\lambda\to\infty} f\begin{pmatrix} z & 0 \\ 0 & i\lambda 1_{g-r}\end{pmatrix}.$$

The image $\Phi^r(f)$ is a holomorphic function on $\boldsymbol{H}_r$.

- We write $S_k(\Gamma') = \{f \in M_k(\Gamma') \mid \Phi^{g-1}(f|M) = 0 \text{ for any } M \in \Gamma^{(g)}\}$. This is called the space of cusp forms of weight k.

1. Elliptic modular forms.

In this section we review basic facts about elliptic modular forms i.e. modular forms of degree one. Among others, we give a basis of the space of elliptic modular forms of level 3 and of weight k for each k. For details, refer to [**14**, Chapters 1, 2] and [**12**, Chapter IV].

The dimension formula of elliptic modular forms is already well known ([**14**, Theorems 2.24 and 2.25]). For the case level 3, it gives the following.

Lemma 1.1.

$$(1)\quad \dim S_k(\Gamma^{(1)}(3)) = \begin{cases} k-3 & (k\geq 4)\\ 0 & (k\leq 3)\end{cases}$$

$$(2)\quad \dim M_k(\Gamma^{(1)}(3)) = k+1 \quad (k\geq 0)$$

It implies immediately the following.

Theorem 1.1. *The graded ring* $\bigoplus_{k=0}^{\infty} M_k(\Gamma^{(1)}(3))$ *is isomorphic to the polynomial ring of two variables* $\boldsymbol{C}[X,Y]$.

Proof. Take a basis f, g of $M_1(\Gamma^{(1)}(3))$. In view of the dimensions, it suffices to prove their algebraic independence.

Now assume that f and g satisfy some homogeneous relation of degree k:

$$a_0 f^k + a_1 f^{k-1} g + \cdots + a_{k-1} f g^{k-1} + a_k g^k = 0$$

Then $a_0(f/g)^k + a_1(f/g)^{k-1} + \cdots + a_k = 0$. Hence f/g is algebraic over $\boldsymbol{C}$, which means $f = \lambda g$ for $\lambda \in \boldsymbol{C}$, to get a contradiction. □

We want to have the generators X, Y in the above theorem, i.e. a basis of $M_1(\Gamma^{(1)}(3))$. For that purpose we use the theory of theta series of quadratic forms ([**8**, pp. 428–460] and also we explain it in this paper at §5).

Let $Q = \begin{pmatrix} 2 & 3 \\ 3 & 6\end{pmatrix}$. For $z \in \boldsymbol{H}_1$, we set

$$\vartheta(z) = \sum_{m\in \boldsymbol{Z}^2} \exp(\pi i\, {}^t m Q m z)$$

$$\chi(z) = \frac{1}{3} \sum_{\substack{m\in \boldsymbol{Z}^2 \\ m\equiv (0,1) \bmod 3}} \exp\left(\frac{1}{9}\pi i\, {}^t m Q m z\right).$$

Then $\vartheta, \chi \in M_1(\Gamma^{(1)}(3))$ and for $q = e^{2\pi i z}$, $p = e^{2\pi i z/3}$, the Fourier expansions are as follows.

$$\vartheta = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + \cdots$$

$$\chi = p + p^4 + 2p^7 + 2p^{13} + p^{16} + \cdots$$

And for $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, ϑ and χ satisfy

$$\vartheta|J = \frac{\sqrt{-3}}{3}\vartheta + 2\sqrt{3}\chi$$

$$\chi|J = \frac{\sqrt{-3}}{9}\vartheta - \frac{\sqrt{-3}}{3}\chi.$$

The $\Gamma^{(1)}(3)$-nonequivalent cusps of $\boldsymbol{H}_1 \cup \boldsymbol{Q} \cup \{\infty\}$ are ∞, 0, 1, and -1. For each cusp x, as $\gamma \in SL_2(\boldsymbol{Z})$ such that $\gamma\langle\infty\rangle = x$ we can take the element respectively,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Definition 1.1. For $f \in M_k(\Gamma^{(1)})$ and γ defined above, the value of the 0-th Fourier coefficient of $f|\gamma$ is called the value of f at the cusp of ∞, 0, 1, -1 respectively.

Remark. If k is even, the value at cusp x is independent on the choice of γ such that $\gamma\langle\infty\rangle = x$, but if k is odd the sign may depend on the choice.

Now we consider the space of modular forms of higher weights.

For $k \geq 3$, and ${}^t(c,d) \in \boldsymbol{Z}^2$, we put

$$e^k_{(c,d)}(z) = \sum_{(m,n)\equiv(c,d) \mod 3} (mz+n)^{-k}$$

$$e^{k*}_{(c,d)}(z) = \sum_{\substack{(m,n)\equiv(c,d) \mod 3 \\ (m,n)=1}} (mz+n)^{-k}.$$

Then $e^k_{(c,d)}, e^{k*}_{(c,d)} \in M_k(\Gamma(3))$ and for $M \in SL_2(\boldsymbol{Z})$, $(c,d)M = (c',d')$, $e^k_{(c,d)}|M = e^k_{(c',d')}$ and $e^{k*}_{(c,d)}|M = e^{k*}_{(c',d')}$.

Now we take $(0,1)$, $(1,0)$, $(1,1)$ and $(1,-1)$ as (c,d). The value of e^{k*}s at each cusp are given as follows.

	∞	0	1	-1
$e^{k*}_{(0,1)}$	1	0	0	0
$e^{k*}_{(1,0)}$	0	1	0	0
$e^{k*}_{(1,1)}$	0	0	0	1
$e^{k*}_{(1,-1)}$	0	0	$(-1)^k$	0

In particular when $k = 3$, the set $\{e^3_{(c,d)}\}$ are basis of this space since $\dim M_3(\Gamma(3)) = 4$.

To consider Fourier expansion of e^{k*}, first we recall the expansion of e^k ([**12**, Chapter IV, Proposition 17]):

$$e^k_{(c,d)}(z) = \sum_\lambda a_\lambda p^\lambda \quad (p = e^{2\pi i z/3}),$$

where

$$a_0 = \begin{cases} 0 & (c \not\equiv 0 \mod 3) \\ \sum_{n\equiv d \mod 3} n^{-k} & (c \equiv 0 \mod 3), \end{cases}$$

$$a_\lambda = \frac{(-2\pi i)^k}{3^k(k-1)!} \sum_{\substack{\nu, \\ m\nu=\lambda, \\ m\equiv c \mod 3}} (\operatorname{sgn} \nu)\nu^{k-1} e^{2\pi i \nu d/3}.$$

Since $e^{k*}_{(c,d)} = c_1 e^k_{(c,d)} + c_2 e^k_{(c,d)}$ with

$$c_1 = \sum_{a\equiv 1 \mod 3, a>0} \mu(a)a^{-3}, \quad c_2 = \sum_{a\equiv 2 \mod 3, a>0} \mu(a)a^{-3} \quad (\mu\text{: Möbius function}),$$

we can get the Fourier coefficients of $e^{3*}_{(c,d)}$ as follows.

	$e^{3*}_{(0,1)}$	$e^{3*}_{(1,0)}$	$e^{3*}_{(1,1)}$	$e^{3*}_{(1,-1)}$
a_0	$1 = (b_1 - b_2)(c_1 - c_2)$	0	0	0
a_1	0	$K(c_1 - c_2)$	$\omega K(c_1 - c_2)$	$\omega^2 K(c_1 - c_2)$
a_2	0	$3K(c_1 - c_2)$	$3\omega^2 K(c_1 - c_2)$	$3\omega K(c_1 - c_2)$
a_3	$\sqrt{-3}K(c_1 - c_2)$	$9K(c_1 - c_2)$	$9K(c_1 - c_2)$	$9K(c_1 - c_2)$

Here $K = 3^{-3}2^{-1}(-2\pi i)^3$, $b_1 = \sum_{n=0}^\infty (1+3n)^{-3}$, $b_2 = \sum_{n=0}^\infty (2+3n)^{-3}$, and $\omega = e^{2\pi i/3} = (-1+\sqrt{-3})/2$.

Now $\vartheta^3 \in M_3(\Gamma^{(1)}(3))$ and by comparison of Fourier coefficients, we get

$$\vartheta^3 = e^{3*}_{(0,1)} - \frac{\sqrt{-3}}{9}(e^{3*}_{(1,0)} + e^{3*}_{(1,1)} + e^{3*}_{(1,-1)}) \quad \text{and} \quad K(c_1 - c_2) = 3\sqrt{-3},$$

and the exact values of Fourier coefficients.

Similarly, we have

$$\vartheta^4 = e^{4*}_{(0,1)} + \frac{1}{9}(e^{4*}_{(1,0)} + e^{4*}_{(1,1)} + e^{4*}_{(1,-1)}).$$

In the table below (Table 1-A), there are some of the Fourier coefficients of $e^{3*}_{(c,d)}$ and $e^{4*}_{(c,d)}$.

In addition we get the values at each cusp of ϑ and χ as follows.

	∞	0	1	-1
ϑ	1	$\frac{\sqrt{-3}}{3}$	$-\frac{\sqrt{-3}}{3}$	$\frac{\sqrt{-3}}{3}$
χ	0	$\frac{\sqrt{-3}}{9}$	$-\frac{\sqrt{-3}}{9}\omega$	$\frac{\sqrt{-3}}{9}\omega^2$

Next we consider the element $h = \chi e^{3*}_{(0,1)} \in M_4(\Gamma^{(1)}(3))$. This function takes value 0 at each cusp, this means $h \in S_4(\Gamma^{(1)}(3))$. The Fourier expansion of h is

$$h = p - 8p^4 + 20p^7 - 70p^{13} + \cdots.$$

	$e^{3*}_{(0,1)}$	$e^{3*}_{(1,0)}$	$e^{3*}_{(1,1)}$	$e^{3*}_{(1,-1)}$	$e^{4*}_{(0,1)}$	$e^{4*}_{(1,0)}$	$e^{4*}_{(1,1)}$	$e^{4*}_{(1,-1)}$
a_0	1	0	0	0	1	0	0	0
a_1	0	$3\sqrt{-3}$	$3\sqrt{-3}\omega$	$3\sqrt{-3}\omega^2$	0	3	3ω	$3\omega^2$
a_2	0	$9\sqrt{-3}$	$9\sqrt{-3}\omega^2$	$9\sqrt{-3}\omega$	0	27	$27\omega^2$	27ω
a_3	-9	$27\sqrt{-3}$	$27\sqrt{-3}$	$27\sqrt{-3}$	-3	81	81	81
a_4	0	$39\sqrt{-3}$	$39\sqrt{-3}\omega$	$39\sqrt{-3}\omega^2$	0	219	219ω	$219\omega^2$
a_5	0	$72\sqrt{-3}$	$72\sqrt{-3}\omega^2$	$72\sqrt{-3}\omega$	0	378	$378\omega^2$	378ω
a_6	27	$81\sqrt{-3}$	$81\sqrt{-3}$	$81\sqrt{-3}$	-27	729	729	729
a_7	0	$150\sqrt{-3}$	$150\sqrt{-3}\omega$	$150\sqrt{-3}\omega^2$	0	1032	1032ω	$1032\omega^2$
a_8	0	$153\sqrt{-3}$	$153\sqrt{-3}\omega^2$	$153\sqrt{-3}\omega$	0	1755	$1755\omega^2$	1755ω
a_9	-9	$243\sqrt{-3}$	$243\sqrt{-3}$	$243\sqrt{-3}$	159	2187	2187	2187

Table 1-A.

For our later use we have to specify the representation of $SL_2(\boldsymbol{F}_3)$ on $M_k(\Gamma^{(1)}(3))$. The action of $SL_2(\boldsymbol{Z})$ on $M_k(\Gamma^{(1)}(3))$ given by $(\gamma, f) \mapsto f|\gamma^{-1}$ for $f \in M_k(\Gamma^{(1)}(3))$ and $\gamma \in SL_2(\boldsymbol{F}_3)$ induces a representation of $SL_2(\boldsymbol{F}_3) \cong SL_2(\boldsymbol{Z})/\Gamma^{(1)}(3)$.

THEOREM 1.2. (1) *If we denote the representation of* $SL_2(\boldsymbol{F}_3)$ *on* $M_k(\Gamma^{(1)}(3))$ *by* η_k, *then*

$$\eta_1 \cong A^+, \quad \eta_k \cong \operatorname{Sym}^k \eta_1 \cong \operatorname{Sym}^k A^+.$$

(2) *For* $k \geq 4$, *the representation on* $S_k(\Gamma^{(1)}(3))$ *is given by* $B^+ \otimes \operatorname{Sym}^{k-4} A^+$.
Here the character of all representation of $SL_2(\boldsymbol{F}_3)$ $(A^+, B^+$ *etc.*) *is given in the table below* (*Table 1-B*).

PROOF. It is easily proved by seeing the action of $SL_2(\boldsymbol{Z})$ to ϑ and χ, and Theorem 1.1 and the fact that the basis of $S_k(\Gamma^{(1)}(3))$ are given by $\{h\vartheta^a\chi^b\}_{a,b\geq 0}^{a+b=k-4}$. □

2. The dimension formula for Siegel cusp forms.

For the dimension of the Siegel cusp forms of degree two with level $l \geq 3$, there exists a closed formula for weight $k \geq 4$. There is a vanishing result for the space of cusps of weight 3. We recall these results to use in latter section.

First we quote the following fundamental result.

THEOREM 2.1 ([**17**, §4 THEOREM], [**2**, Satz] and [**11**, Main Theorem]). *For* $k \geq 4$ *and* $l \geq 3$,

$$\dim S_k(\Gamma(l)) = \{2^{-10}3^{-3}5^{-1}(2k-2)(2k-3)(2k-4)l^{10} - 2^{-6}3^{-2}(2k-3)l^8 + 2^{-5}3^{-1}l^7\}\prod_{p|l}(1-p^{-2})(1-p^{-4}).$$

conj. class	card.	1	N	Z	$A^{\pm}$	$B^{\pm}$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	1	3	2	2	1
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	1	1	3	-2	-2	1
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	4	1	0	-1	$1+\omega^{\pm 1}$	$\omega^{\pm 1}$
$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	4	1	0	-1	$1+\omega^{\mp 1}$	$\omega^{\mp 1}$
$\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$	4	1	0	1	$-(1+\omega^{\pm 1})$	$\omega^{\pm 1}$
$\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$	4	1	0	1	$-(1+\omega^{\mp 1})$	$\omega^{\mp 1}$
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	6	1	-1	0	0	1

Table 1-B.

Put $l = 3$ in this theorem, then we have following.

Lemma 2.1. *For* $k \geq 4$,

$$\dim S_k(\Gamma(3)) = \frac{1}{2}(6k^3 - 27k^2 - k + 82).$$

Now we have to consider the case of the low weight k.

Lemma 2.2. $\dim S_3(\Gamma(3)) = 0$.

Proof. Let X be a smooth compactification of $\Gamma(3)\backslash \boldsymbol{H}_2$. Then according to [**6**, Corollary 3.4] X is rational, hence we get $\dim \Gamma(X, \Omega_X^3) = 0$. On the other hand, there is an isomorphism:

$$S_3(\Gamma(3)) \to \Gamma(X, \Omega_X^3) \quad f \mapsto f\, dz_1 \wedge dz_2 \wedge dz_3$$

([**4**, Chapter III, Satz 2.6]), which proves the lemma. □

Lemma 2.3. $\dim S_1(\Gamma(3)) = 0$.

Proof. If $f \neq 0 \in S_1(\Gamma(3))$, $f^3 \neq 0 \in S_3(\Gamma(3))$. Therefore, the lemma is immediately induced by Lemma 2.2. □

Remark. Since we have $\dim M_1(\Gamma(3)) \neq 0$ in §5, a similar argument as the proof of Lemma 2.3 shows $\dim S_2(\Gamma(3)) = 0$ (Lemma 5.3).

3. The representation of $Sp(2, \boldsymbol{F}_3)$.

Since the fact that $M_k(\Gamma(3))$ and $S_k(\Gamma(3))$ are representations of the finite symplectic group $Sp(2, \boldsymbol{F}_3)$ is crucial in our investigation, we review the known result on the irreducible representations of $Sp(2, \boldsymbol{F}_3)$. In this section, extracting from [**16**] we give a list of part of irreducible characters of $G = Sp(2, \boldsymbol{F}_3)$ which is necessary in this paper.

Now as representative element of conjugation class of G, we take following elements.

$$A_1 = 1_4, \quad A_1' = -1_4, \quad A_{21} = \begin{pmatrix} 1_2 & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \\ 0 & 1_2 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 1_2 & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} \\ 0 & 1_2 \end{pmatrix}$$

$$A_{31} = \begin{pmatrix} 1_2 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \\ 0 & 1_2 \end{pmatrix} \quad A_{32} = \begin{pmatrix} 1_2 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \\ 0 & 1_2 \end{pmatrix} \quad A_{41} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \end{pmatrix}$$

$$A_{42} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 1 & 0 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \end{pmatrix} \quad B_1(1) = \underset{\text{element of order 10}}{\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}} \quad B_1(2) = \underset{\text{element of order 5}}{\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}$$

$$B_2(1) = \underset{\text{element of order 8}}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}} \quad B_6(1) = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \quad B_7(1) = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$$

$$C_1(1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad C_{21}(1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad C_{22}(1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix} \quad D_{21} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix} \quad D_{22} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix}$$

$$D_{23} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix} \quad D_{24} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix} \quad D_{31} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix}$$

$$D_{32} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix} \quad D_{33} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix} \quad D_{34} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix}$$

And we write X' for $-X$.

The following table is a part of character tables of $Sp(2,\boldsymbol{F}_3)$ determined by Srinivasan.

conj. class	card.	θ_3	θ_9	θ_{11}	Φ_2	Φ_4	Φ_9
A_1	1	5	24	15	10	30	30
A_1'	1	5	24	15	10	30	30
A_{21}	40	$2+3\omega$	6	-3	$-2+3\omega^2$	$3+9\omega^2$	3
A_{21}'	40	$2+3\omega$	6	-3	$-2+3\omega^2$	$3+9\omega^2$	3
A_{22}	40	$2+3\omega^2$	6	-3	$-2+3\omega$	$3+9\omega$	3
A_{22}'	40	$2+3\omega^2$	6	-3	$-2+3\omega$	$3+9\omega$	3
A_{31}	480	2	3	3	1	0	3
A_{31}'	480	2	3	3	1	0	3
A_{32}	240	-1	0	0	1	-3	3
A_{32}'	240	-1	0	0	1	-3	3
A_{41}	2880	$-\omega^2$	0	0	ω^2	0	0
A_{41}'	2880	$-\omega^2$	0	0	ω^2	0	0
A_{42}	2880	$-\omega$	0	0	ω	0	0
A_{42}'	2880	$-\omega$	0	0	ω	0	0
$B_1(1)$	5184	0	-1	0	0	0	0
$B_1(2)$	5184	0	-1	0	0	0	0
$B_2(1)$	6480	-1	0	1	0	0	0
$B_6(1)$	540	1	0	3	-2	2	2
$B_7(1)$	4320	1	0	0	1	-1	-1
$C_1(1)$	540	1	0	-1	2	2	-2
$C_1'(1)$	540	1	0	-1	2	2	-2
$C_{21}(1)$	2160	ω	0	-1	$-\omega^2$	$-\omega$	1
$C_{21}'(1)$	2160	ω	0	-1	$-\omega^2$	$-\omega$	1
$C_{22}(1)$	2160	ω^2	0	-1	$-\omega$	$-\omega^2$	1
$C_{22}'(1)$	2160	ω^2	0	-1	$-\omega$	$-\omega^2$	1
D_1	90	-3	8	7	2	6	-10
D_{21}	360	$-(2+\omega)$	2	1	$2+3\omega^2$	$-1+\omega^2$	-1
D_{22}	360	$-(2+\omega^2)$	2	1	$2+3\omega$	$-1+\omega$	-1
D_{23}	360	$-(2+\omega)$	2	1	$2+3\omega^2$	$-1+\omega^2$	-1
D_{24}	360	$-(2+\omega^2)$	2	1	$2+3\omega$	$-1+\omega$	-1
D_{31}	1440	$\omega^2-\omega$	2	-2	-1	$\omega^2-\omega$	-1
D_{32}	1440	0	-1	1	-1	0	-1
D_{33}	1440	0	-1	1	-1	0	-1
D_{34}	1440	$\omega-\omega^2$	2	-2	-1	$\omega-\omega^2$	-1

And θ_4 is the complex conjugate of θ_3.

In this paper we have to calculate the symmetric tensor product or induced representation of a given representation. We use the following formulae.

Let ρ be a representation of a finite group G and $\chi = \chi_\rho$ be its character. For $\gamma \in G$, we have

$$\chi_{\mathrm{Sym}^2(\rho)}(\gamma) = \frac{1}{2}(\chi(\gamma)^2 + \chi(\gamma^2)),$$
$$\chi_{\mathrm{Sym}^3(\rho)}(\gamma) = \frac{1}{6}(\chi(\gamma)^3 + \chi(\gamma^2)\chi(\gamma) + 2\chi(\gamma^3)),$$
$$\chi_{\mathrm{Sym}^4(\rho)}(\gamma) = \frac{1}{24}(\chi(\gamma)^4 + 6\chi(\gamma^2)\chi(\gamma)^2 + 3\chi(\gamma^2)^2 + 8\chi(\gamma^3)\chi(\gamma) + 6\chi(\gamma^4)).$$

Let H be a subgroup of G and $\{g_\lambda\}$ be a representative system of G/H. Then for a representation ρ of H, we have

$$\chi_{\mathrm{Ind}_H^G(\rho)}(\gamma) = \sum_{\substack{\gamma \\ g_\lambda^{-1}\gamma g_\lambda \in H}} \chi_\rho(g_\lambda^{-1}\gamma g_\lambda).$$

4. Boundaries of the Satake compactification.

In this section we examine the boundary components of Satake compactification of $\Gamma(3)\backslash \boldsymbol{H}_2$. About the definition of Satake compactification, we refer to [**13**, Exposé 12, 13].

DEFINITION 4.1. (1) For a standard parabolic subgroup

$$P_0 = \{\gamma \in \Gamma \mid \gamma = \begin{pmatrix} U & T \\ 0 & {}^tU^{-1} \end{pmatrix}\},$$

the map $u_0 : P_0 \to \{\pm 1\} \subset \boldsymbol{C}$ is given by $u_0(\gamma) = \det U$.

(2) For another standard parabolic subgroup

$$P_1 = \{\gamma \in \Gamma \mid \gamma = \begin{pmatrix} a_1 & 0 & b_1 & b_2 \\ a_2 & a_3 & b_3 & b_4 \\ c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{pmatrix}\},$$

the map $u_1 : P_1 \to \{\pm 1\} \subset \boldsymbol{C}$ is given by $u_1(\gamma) = d_4$.

Further we set $\pi(\gamma) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, for $\pi : P_1 \to SL_2(\boldsymbol{Z})$.

We fix an integer $N \geq 3$. Now we decompose Γ into double cosets by P_0 or P_1 and $\Gamma(N)$ as

$$\Gamma = \bigcup_\mu \Gamma(N) M_\mu^0 P_0$$

or

$$\Gamma = \bigcup_\lambda \Gamma(N) M_\lambda^1 P_1,$$

respectively.

Similarly we have the decomposition

$$SL_2(\boldsymbol{Z}) = \bigcup_\nu \Gamma^{(1)}(N) M_\nu P_0^{(1)}.$$

Here $P_0^{(1)} = \{\gamma \in SL_2(\boldsymbol{Z}) \mid \gamma = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\}$.

And we define $\iota : SL_2(\boldsymbol{Z}) \to Sp(2, \boldsymbol{Z})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now we consider the element $(f_\lambda)_\lambda$ of $\prod_\lambda M_k(\Gamma^{(1)}(N))$ which satisfies the following condition.

$(*)$ For $\lambda_1, \lambda_2, \nu_1, \nu_2$ and μ such that

$$M^1_{\lambda_1}\iota(M_{\nu_1}) = \alpha M^0_\mu \gamma_1 \quad \text{and} \quad M^1_{\lambda_2}\iota(M_{\nu_2}) = \beta M^0_\mu \gamma_2, \ \ \alpha, \beta \in \Gamma(N) \quad \gamma_1, \gamma_2 \in P_0,$$

$$\text{(the 0-th Fourier coefficient of } f_{\lambda_1}|M_{\nu_1}) \times u_0(\gamma_1)^k$$

$$= \text{(the 0-th Fourier coefficient of } f_{\lambda_2}|M_{\nu_2}) \times u_0(\gamma_2)^k.$$

Then the above value depends only on M^0_μ, and we call it the value of $(f_\lambda)_\lambda$ at 0-th cusp M^0_μ.

Definition 4.2. The subspace of $\prod_\lambda M_k(\Gamma^{(1)}(N))$ spanned by the elements which satisfy the above condition $(*)$ is called the space of the boundary values. We denote it by ∂M_k.

Next we consider the Siegel Φ-operator (refer to [**13**, Exposé 14]).

Definition 4.3. For $f \in M_k(\Gamma(N))$, we define the operator

$$\Phi^1_\lambda : M_k(\Gamma(N)) \to M_k(\Gamma^{(1)}(N)) \quad \text{by}$$

$$\Phi^1_\lambda(f)(z) = \Phi^1(f|M^1_\lambda)(z) = \lim_{x\to\infty} f|M^1_\lambda \begin{pmatrix} z & 0 \\ 0 & ix \end{pmatrix}.$$

Similarly we put $\Phi^0_\mu : M_k(\Gamma(N)) \to \boldsymbol{C}$,

$$\Phi^0_\mu(f) = \Phi^0(f|M^0_\mu) = \lim_{x\to\infty} f|M^0_\mu(ix1_2).$$

The Siegel operator has the following properties.

Lemma 4.1. *For* $f \in M_k(\Gamma(N))$,

$$\Phi^1(f|\gamma) = u_1(\gamma)^k \Phi^1(f)|\pi(\gamma) \quad \textit{if} \ \ \gamma \in P_1,$$

$$\Phi^0(f|\gamma) = u_0(\gamma)^k \Phi^0(f) \quad \textit{if} \ \ \gamma \in P_0.$$

This is a straightforward calculation, so we omit the proof.
From this lemma, the following proposition is induced immediately.

Proposition 4.1. (1) *The image of the morphism* $\prod_\lambda \Phi^1_\lambda : M_k(\Gamma(N)) \to \prod_\lambda M_k(\Gamma^{(1)}(N))$ *is contained in* ∂M_k, *and the value of* $(\Phi^1_\lambda(f))_\lambda$ *at the cusp* M^0_μ *equal to* $\Phi^0_\mu(f)$.
(2) $S_k(\Gamma(N)) = \ker(\prod_\lambda \Phi^1_\lambda)$.

Now we consider the case $N = 3$. We can take $\{M^0_\mu\}$, and $\{M^1_\lambda\}$ as follows.

$$M^0_\mu : \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \quad \begin{pmatrix} \begin{matrix} a_1 & a_2 \\ a_2 & a_4 \end{matrix} & -1_2 \\ 1_2 & 0 \end{pmatrix} \quad \begin{pmatrix} a_1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ a_3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ a_3 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ a_3 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$a_i \in \{0, 1, -1\} \quad \#\{M^0_\mu\} = 40$$

$$M^1_\lambda : \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \quad \begin{pmatrix} \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 0 \\ -1 & 1 \end{matrix} \end{pmatrix} \quad \begin{pmatrix} \begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} \end{pmatrix}$$

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix} \quad \begin{pmatrix} \begin{matrix} 0 & a_2 \\ a_2 & a_4 \end{matrix} & -1_2 \\ 1_2 & 0 \end{pmatrix} \quad \begin{pmatrix} \begin{matrix} 0 & a_2 \\ a_2 & a_4 \end{matrix} & \begin{matrix} -1 & 0 \\ 1 & -1 \end{matrix} \\ \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} & 0 \end{pmatrix}$$

$$\begin{pmatrix} \begin{matrix} 0 & a_2 \\ a_2 & a_4 \end{matrix} & \begin{matrix} -1 & 0 \\ -1 & -1 \end{matrix} \\ \begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix} & 0 \end{pmatrix} \quad \begin{pmatrix} \begin{matrix} -a_4 & a_2 \\ 0 & a_4 \end{matrix} & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \end{pmatrix}$$

$$a_i \in \{0, 1, -1\} \quad \#\{M^1_\lambda\} = 40$$

And we put M_ν as follows, which correspond to the cusp $\infty, 0, 1$ and -1 of $\Gamma^{(1)}(3)$ respectively.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now we want to describe the manner of intersections of boundaries precisely. Namely in the language of discrete subgroups, we have to calculate λ, μ, ν satisfying

$$M^1_\lambda \iota(M_\nu) \in \Gamma(3) M^0_\mu \gamma, \quad \gamma \in P_0$$

and the value of $u_0(\gamma)$ at the same time.

First, if M^1_λ is the form of $\begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix}$, the point cusps and the associate values of u_0 on the 1 dimensional boundary components corresponding to M^1_λ are given by following table (Table 4-A).

Next if M^1_λ is the form $\begin{pmatrix} A & -{}^tU^{-1} \\ U & 0 \end{pmatrix}$, the descriptions of boundaries are as follows. In this case, when $x = 0, 1, -1$, it corresponds to cusp $\infty, 1, -1$ respectively, and the rest element corresponds to cusp 0 (Table 4-B).

This fact shows the dimension of ∂M_k for each k as follows.

$M_\lambda^1\backslash$0-th cusp	∞	0	1	-1
$\begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}$	$\begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & * \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & * \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & * \end{pmatrix}$
value of u_0	1	-1	1	-1
$\begin{pmatrix} \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix} \end{pmatrix}$	$\begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix} & * \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix} & * \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix} & * \end{pmatrix}$
value of u_0	1	-1	1	-1
$\begin{pmatrix} \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \end{pmatrix}$	$\begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} & * \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} & * \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} & * \end{pmatrix}$
value of u_0	1	1	-1	1
$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$	$\begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} & * \\ \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} & * \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & * \\ \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} & * \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & * \\ \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} & * \end{pmatrix}$
value of u_0	1	1	-1	1

Table 4-A.

Lemma 4.2. *For* $k \geq 3$, $\dim \partial M_k = 40k - 80$.

Proof. We define the morphism $\Psi : \partial M_k \to \prod_\mu \boldsymbol{C}$ to send the element of ∂M_k to the value at each 0-th cusp M_μ^0. Then it is clear

$$\ker \Psi = \{(f_\lambda)_\lambda \in \partial M_k \mid f_\lambda \in S_k(\Gamma^{(1)}(3)) \text{ for all } \lambda\}.$$

M_λ^1		
$\begin{pmatrix} \begin{smallmatrix} 0 & a_2 \\ a_2 & a_4 \end{smallmatrix} & -1_2 \\ 1_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} -x & a_2 \\ a_2 & a_4 \end{smallmatrix} & -1_2 \\ 1_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ a_4 & 0 \end{smallmatrix} & * \\ \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} & * \end{pmatrix}$
value of u_0	1	-1
$\begin{pmatrix} \begin{smallmatrix} 0 & a_2 \\ a_2 & a_4 \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix} \\ \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} & 0 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} -x & -x+a_2 \\ -x+a_2 & -x-a_2+a_4 \end{smallmatrix} & -1_2 \\ 1_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ a_2+a_4 & -1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} & * \end{pmatrix}$
value of u_0	1	-1
$\begin{pmatrix} \begin{smallmatrix} 0 & a_2 \\ a_2 & a_4 \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix} \\ \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} & 0 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} -x & -x+a_2 \\ -x+a_2 & -x+a_2+a_4 \end{smallmatrix} & -1_2 \\ 1_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ a_2-a_4 & -1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix} & * \end{pmatrix}$
value of u_0	1	1
$\begin{pmatrix} \begin{smallmatrix} -a_4 & a_2 \\ 0 & a_4 \end{smallmatrix} & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & 0 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} a_2 & a_4 \\ a_4 & -x \end{smallmatrix} & -1_2 \\ 1_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \begin{smallmatrix} a_2 & 0 \\ 0 & 1 \end{smallmatrix} & * \\ \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & * \end{pmatrix}$
value of u_0	1	1

$x \in \{0, 1, -1\}$

Table 4-B.

Since $k \geq 3$, $e^{k*}_{(c,d)} \in M_k(\Gamma^{(1)}(3))$, and it shows Ψ is surjective. By Lemma 1.1 $\dim S_k(\Gamma^{(1)}(3)) = k-3$, so we get

$$\dim \partial M_k = 40(k-3) + 40 = 40k - 80.$$

This proves the lemma. □

Lemma 4.3. $\dim \partial M_2 \leq 15$.

Proof. By Theorem 1.1, we can take a basis $\{\vartheta^2, \vartheta\chi, \chi^2\}$ of $M_2(\Gamma^{(1)}(3))$. In view of the values of these elements at each cusp, we see that for every element of $M_2(\Gamma^{(1)}(3))$ the sum of values at each cusp is equal to 0. In particular if $f \in M_2(\Gamma^{(1)}(3))$ takes value 0 at three cusps, we get $f = 0$.

In view of the configuration of boundaries and using the above argument, we can show that the morphism which send the element of ∂M_2 to the value at 0-th cusp $M^0_\mu = \begin{pmatrix} A & -1_2 \\ 1_2 & 0 \end{pmatrix}$ is injective. Here A represents these 15 matrices.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

This proves the lemma. □

In the same way of the proof of this lemma, we have following.

Lemma 4.4. $\dim \partial M_1 \leq 6$.

5. The space of modular forms as $Sp(2, \boldsymbol{F}_3)$-modules.

§§5.1. Eisenstein series.

First we consider the Siegel Eisenstein series and the Klingen Eisenstein series. Let $N \geq 3$. We put

$$E^{*k}(Z) = \sum_{P_0 \cap \Gamma(N) \backslash \Gamma(N)} \det(CZ + D)^{-k}$$

and for $g \in S_k(\Gamma^{(1)}(N))$,

$$E^{*k}(g, Z) = \sum_{P_1 \cap \Gamma(N) \backslash \Gamma(N)} g(\gamma\langle Z \rangle^*) \det(CZ + D)^{-k}.$$

Here $\gamma\langle Z \rangle^*$ means the (1,1)-component of $\gamma\langle Z \rangle$.

Then the above infinite sums defining $E^{*k}(Z)$ and $E^{*k}(g, Z)$ converges absolutely and uniformly in $V(d) = \{Z \in \boldsymbol{H}_2 \mid \mathrm{Im}(Z) \geq d1_2, \mathrm{Tr}((\mathrm{Re}(Z))^2) \leq d^{-1}\}$ for any $d > 0$, if $k > 3$, $k > 4$ respectively (cf. [**10**, §5 Theorem 1]). Clearly E^{*k} $(k \geq 4)$ and $E^{*k}(g, Z)$ $(k \geq 5)$ belong to $M_k(\Gamma(N))$.

Next we recall the relation of Eisenstein series and the Siegel Φ-operator. First we consider $\Phi^1(E^{*k}(g,Z))$. Since the sum defining $E^{*k}(g,Z)$ converges uniformly in $V(d)$ for any $d>0$, if we use the fact

$$\text{if } \gamma \notin P_1, \quad \lim_{x\to\infty} g\left(\gamma\left\langle\begin{pmatrix} z & 0 \\ 0 & ix\end{pmatrix}\right\rangle^*\right) \det\left(C\begin{pmatrix} z & 0 \\ 0 & ix\end{pmatrix}+D\right)^{-k} = 0$$

(cf. [**10**, §5 Proposition 5]), we see $\Phi^1(E^{*k}(g,Z)) = g$.

Since for $M \in \Gamma$ we have

$$E^{*k}(g,Z) \mid M = \sum_{\Gamma(N)\cap P_1\backslash\Gamma(N)M} g(\gamma\langle Z\rangle^*)\det(CZ+D)^{-k},$$

we can get

$$\Phi_\lambda^1(E^{*k}(g,Z)) = \begin{cases} g & \text{if } M_\lambda^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \\ 0 & \text{otherwise}\end{cases}$$

because $\{M_\lambda^1\}$ is a representative system of $\Gamma(N)\backslash\Gamma/P_1$.

We put $E_\lambda^{*k}(g,Z) = E^{*k}(g,Z) \mid (M_\lambda^1)^{-1}$. Since $\{(M_{\lambda_0}^1)^{-1}M_\lambda^1\}_\lambda$ is also a representative system of $\Gamma(N)\backslash\Gamma/P_1$ for fixed λ_0, by Lemma 4.1 we get

$$\Phi_\lambda^1(E_{\lambda_0}^{*k}(g,Z)) = \begin{cases} g & \text{if } \lambda = \lambda_0 \\ 0 & \text{otherwise.}\end{cases}$$

In the same way, if we put $E_\mu^{*k} = E^{*k}|(M_\mu^0)^{-1}$ we can get

$$\Phi_\mu^0(E_{\mu_0}^{*k}(Z)) = \begin{cases} 1 & \text{if } \mu = \mu_0 \\ 0 & \text{otherwise.}\end{cases}$$

By the above mentioned, we can show the following lemma.

Lemma 5.1. *For $k \geq 5$, the morphism $\prod_\lambda \Phi_\lambda^1 : M_k(\Gamma(N)) \to \partial M_k$ is surjective. In particular for $k \geq 5$,*

$$\dim M_k(\Gamma(3)) = \frac{1}{2}(6k^3 - 27k^2 + 79k - 78).$$

Proof. The latter half is induced by Lemma 2.1 and Lemma 4.2.

We take $(f_\lambda) \in \partial M_k$ and let a_μ be the value of (f_λ) at each 0-th cusp M_μ^0. We set $f' = \sum_\mu a_\mu E_\mu^{*k}$. If we write $(g_\lambda) = (f_\lambda) - \prod_\lambda \Phi_\lambda^1(f') \in \partial M_k$, then $g_\lambda \in S_k(\Gamma^{(1)}(N))$ for all λ.

Hence if we put $f = f' + \sum_\lambda E_\lambda^{*k}(g_\lambda, Z)$, we get $\Phi_\lambda^1(f) = f_\lambda$ for each λ. □

As a conclusion of this subsection, we have the following.

Proposition 5.1. *For $k \geq 5$, we can get the decomposition of the $M_k(\Gamma(N))$,*

$$M_k(\Gamma(N)) = S_k(\Gamma(N)) \oplus K_k(\Gamma(N)) \oplus E_k(\Gamma(N)).$$

Here,

$$S_k(\Gamma(N))\text{: the space of cusp forms}$$
$$K_k(\Gamma(N))\text{: the space spanned by } E_\lambda^{*k}(g,Z)$$
$$E_k(\Gamma(N))\text{: the space spanned by } E_\mu^{*k}(Z),$$

and each space is closed the action of Γ.

§§5.2. The representations associated with Eisenstein series.

Now we want to decompose the space $M_k(\Gamma(N))$ for $k \leq 4$, similarly the case $k \geq 5$.

Definition 5.1. For $k \geq 1$, we assume that $M_k(\Gamma(N))$ is decomposed as

$$M_k(\Gamma(N)) = S_k(\Gamma(N)) \oplus K_k(\Gamma(N)) \oplus E_k(\Gamma(N))$$

$S_k(\Gamma(N))$: the space of cusp forms

$K_k(\Gamma(N))$: the subspace of the complement space of $S_k(\Gamma(N))$ such that

$$\Phi_\lambda^1(f) \in S_k(\Gamma^{(1)}(N)) \text{ for all } \lambda$$

$E_k(\Gamma(N))$: the complement space of $S_k(\Gamma(N)) \oplus K_k(\Gamma(N)) = \ker\left(\prod_\mu \Phi_\mu^0\right)$

and each space is closed under the action of Γ.

Then we call $E_k(\Gamma(N))$ and $K_k(\Gamma(N))$ the space of Siegel Eisenstein series and Klingen Eisenstein series respectively.

Remark. This decomposition always exists by the complete reducibility of the representation of finite groups, but it is not unique.

Proposition 5.2. *We put $G = Sp(2, \boldsymbol{Z}/N\boldsymbol{Z})$, and $\bar{P}_0, \bar{P}_1$ the image of P_0, P_1 respectively by the natural map $\Gamma \to G$.*

(1) *For $k \geq 1$ the representation of G on $E_k(\Gamma(N))$ is given by the subrepresentation of $\mathrm{Ind}_{\bar{P}_0}^G(u_0^k)$.*

(2) *For $k \geq 1$ the representation of G on $K_k(\Gamma(N))$ is given by the subrepresentation of $\mathrm{Ind}_{\bar{P}_1}^G(u_1^k \otimes (\rho_k \circ \pi))$.*

Here ρ_k means the representation of $SL_2(\boldsymbol{Z}/N\boldsymbol{Z})$ on $S_k(\Gamma^{(1)}(N))$. Moreover π, u_0, u_1 denote the map from $\bar{P}_0, \bar{P}_1$ to $SL_2(\boldsymbol{Z}/N\boldsymbol{Z})$ or $\boldsymbol{C}^*$, naturally induced from π, u_0, u_1 given in Definition 4.1.

In the case of $N = 3$, we have $\rho_k = B^+ \otimes \mathrm{Sym}^{k-4} A^+$ by Theorem 1.2 (2).

Proof. We only prove (2), since one can easily show (1) in the similar way.

Let take a basis $h^1, \dots, h^m$ of $S_k(\Gamma^{(1)}(N))$. First we assume that for each i $(1 \leq i \leq m)$, there exists an element $f^i \in K_k(\Gamma(N))$ such that

$$\Phi^1_\lambda(f^i) = \begin{cases} h^i & \text{if } M^1_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0 & \text{otherwise.} \end{cases}$$

Then for $f^i_\lambda = f^i|(M^1_\lambda)^{-1}$,

$$\Phi^1_\lambda(f^i_{\lambda_0}) = \begin{cases} h^i & \text{if } \lambda = \lambda_0 \\ 0 & \text{otherwise.} \end{cases}$$

Now since $\{M^1_\lambda\}$ is a representative system of $\Gamma(N)\backslash\Gamma/P_1$, the set $\{(M^1_\lambda)^{-1}\}$ is a representative system of $\Gamma(N)P_1\backslash\Gamma$. Hence if we fix a $\gamma \in \Gamma$, for each λ there exists λ' such that

$$(M^1_\lambda)^{-1}\gamma = xp_\lambda(M^1_{\lambda'})^{-1} \quad x \in \Gamma(N) \ \ p_\lambda \in P_1, \tag{1}$$

and when λ runs through the representative system, λ' also runs the system.

Further because $S_k(\Gamma(N)) \cap K_k(\Gamma(N)) = 0$, if we can write

$$u_1(p_\lambda)^k h^i \,|\, \pi(p_\lambda) = \sum_{j=1}^m a^\lambda_{ij} h^j \quad a^\lambda_{ij} \in \boldsymbol{C}, \tag{2}$$

we have

$$f^i_\lambda \,|\, \gamma = \sum_{j=1}^m a^\lambda_{ij} f^j_{\lambda'}. \tag{3}$$

In particular the space spanned by $\{f^i\}_{1 \le i \le m}$ is closed under the action of P_1, and the representation of $\bar{P}_1$ on this space is given by $u_1^k \otimes (\rho_k \circ \pi)$. Since $\{f^i_\lambda\}_{\lambda,i}$ is a basis of $K_k(\Gamma(N))$, the representation of G on $K_k(\Gamma(N))$ is given by $\mathrm{Ind}^G_{\bar{P}_1}(u_1^k \otimes (\rho_k \circ \pi))$ in this case.

In general, we consider the $\boldsymbol{C}$-vector space V which is spanned by free basis $\{f^i_\lambda\}$, and induce the action of Γ by (1), (2) and (3). Next we define the morphism $\alpha : K_k(\Gamma(N)) \to V$ by $\alpha(f) = \sum_{\lambda,i} c^i_\lambda f^i_\lambda$ for $f \in K_k(\Gamma(N))$ such that $\Phi^1_\lambda(f) = \sum_{i=1}^m c^i_\lambda h^i$. Then α is injective and by the construction, α is the homomorphism of G-modules. Hence the representation of G is given by the subrepresentation of $\mathrm{Ind}^G_{\bar{P}_1}(u_1^k \otimes (\rho_k \circ \pi))$. □

We put $N = 3$. Then we have the following corollary.

Corollary 5.1. (1) *When k is even, the representation of $G = Sp(2, \boldsymbol{F}_3)$ on $E_k(\Gamma(3))$ is given by for $k \ge 4$*

$$\mathrm{Ind}^G_{\bar{P}_0}(1_{\bar{P}_0}) \simeq 1_G \oplus \theta_{11} \oplus \theta_9,$$

and its subrepresentation if $k = 2$.

(2) *When k is odd, the representation of G on $E_k(\Gamma(3))$ is given by for $k \ge 5$*

$$\mathrm{Ind}^G_{\bar{P}_0}(u_1) \simeq \theta_3 \oplus \theta_4 \oplus \Phi_9,$$

and its subrepresentation if $k = 1$ or $k = 3$.

PROOF. In view of the proposition, it suffices to show the corresponding relations:

$$(1)\quad \mathrm{Ind}_{\bar{P}_0}^{G}(1_{\bar{P}_0}) \simeq 1_G \oplus \theta_{11} \oplus \theta_9$$

$$(2)\quad \mathrm{Ind}_{\bar{P}_0}^{G}(u_1) \simeq \theta_3 \oplus \theta_4 \oplus \Phi_9.$$

These follows by direct computation of the induced characters. One should be careful that a representative system of $G/\bar{P}_0$ is given by $\{M_\mu^0\}$. Here is the table (Table 5-A) of the values of these induced characters. By comparing the values of characters in §3, we can check the isomorphism. □

§§5.3. Theta series of quadratic forms.

As in §1, we use the theory of theta series of quadratic forms in order to get the element of modular forms of weight 1.

DEFINITION 5.2. Let $Q \in M_m(\boldsymbol{Z})$ be a symmetric positive definite matrix with even diagonal entries, and let q be the minimum positive integer such that $qQ^{-1} \in M_m(\boldsymbol{Z})$ has even diagonal entries.

We put $T^n(Q) = \{T \in M_{m,n}(\boldsymbol{Z}) \mid QT \equiv 0 \mod q\}$. Then we define for $Z \in \boldsymbol{H}_n$ and $T \in T^n(Q)$,

$$\theta^n(Z, Q|T) = \sum_{N \in M_{m,n}(\boldsymbol{Z})} \exp \pi i \,\mathrm{Tr}\left({}^t\left(N + \frac{1}{q}T\right) Q \left(N + \frac{1}{q}T\right) Z\right).$$

PROPOSITION 5.3 ([**1**, Proposition 1.3.14, Exercise 2.2.3]).

$$(1)\quad \theta^n\left(\begin{pmatrix} V & 0 \\ 0 & {}^tV^{-1} \end{pmatrix}\langle Z\rangle, Q|T\right) = \theta^n(Z, Q \mid TV) \quad V \in GL_n(\boldsymbol{Z}).$$

$$\theta^n\left(\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}\langle Z\rangle, Q|T\right) = \exp \pi i \,\mathrm{Tr}\left(\frac{1}{q^2}\,{}^tTQTS\right)\theta^n(Z, Q|T).$$

$$(2)\quad \theta^n\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\langle Z\rangle, Q|T\right)$$

$$= (\det Q)^{-n/2}(\det(-iZ))^{m/2} \sum_{\substack{T' \in T^n(Q) \\ \mathrm{mod}\, q}} \exp 2\pi i \,\mathrm{Tr}\left(\frac{1}{q^2}\,{}^tTQT'\right)\theta^n(Z, Q|T').$$

$$(3)\quad \theta^n(Z, Q|T) \in M_{m/2}(\Gamma^{(n)}(q)).$$

Now we put $m = n = 2$, $Q = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$. Then we get $q = 3$, and $\theta^2(Z, Q|T) \in M_1(\Gamma(3))$.

The set $\{T \in M_2(\boldsymbol{F}_3) \mid QT = 0\}$ consisting of 9 matrices is given as follows.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

conj. class	card.	$\mathrm{Ind}_{\bar{P}_0}^G(1_{\bar{P}_0})$	$\mathrm{Ind}_{\bar{P}_0}^G(u_1)$
A_1	1	40	40
A_1'	1	40	40
A_{21}	40	4	4
A_{21}'	40	4	4
A_{22}	40	4	4
A_{22}'	40	4	4
A_{31}	480	7	7
A_{31}'	480	7	7
A_{32}	240	1	1
A_{32}'	240	1	1
A_{41}	2880	1	1
A_{41}'	2880	1	1
A_{42}	2880	1	1
A_{42}'	2880	1	1
$B_1(1)$	5184	0	0
$B_1(2)$	5184	0	0
$B_2(1)$	6480	2	-2
$B_6(1)$	540	4	4
$B_7(1)$	4320	1	1
$C_1(1)$	540	0	0
$C_1'(1)$	540	0	0
$C_{21}(1)$	2160	0	0
$C_{21}'(1)$	2160	0	0
$C_{22}(1)$	2160	0	0
$C_{22}'(1)$	2160	0	0
D_1	90	16	-16
D_{21}	360	4	-4
D_{22}	360	4	-4
D_{23}	360	4	-4
D_{24}	360	4	-4
D_{31}	1440	1	-1
D_{32}	1440	1	-1
D_{33}	1440	1	-1
D_{34}	1440	1	-1

Table 5-A.

And clearly, $\theta^2(Z,Q|T) = \theta^2(Z,Q|-T)$. Now we put

$$T_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix},$$

and

$$t_i = \theta^2(Z, Q|T_i) = \sum_{\substack{\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \equiv T_i \\ \bmod 3}} \exp \pi i \operatorname{Tr}\left(\frac{1}{9}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} Z\right).$$

Let $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$ and $p_j = e^{2\pi i z_j/3}$. Then using these p_j, the Fourier expansions of t_j are written as follows.

$$t_1 = 1 + 6p_1^3 + 6p_3^3 + 12p_1^3 p_2^3 p_3^3 + 12p_1^3 p_2^{-3} p_3^3 + 6p_1^3 p_2^6 p_3^3 + \cdots,$$

$$t_2 = 3p_1 + 6p_1 p_3^3 + 6p_1 p_2^3 p_3^3 + 6p_1 p_2^{-3} p_3^3 + \cdots,$$

$$t_3 = 3p_3 + 6p_1^3 p_3 + 6p_1^3 p_2^3 p_3 + 6p_1^3 p_2^{-3} p_3 + \cdots,$$

$$t_4 = 6p_1 p_2^{-1} p_3 + 3p_1 p_2^2 p_3 + 6p_1^4 p_2^2 p_3 + 3p_1^4 p_2^{-4} p_3 + 6p_1 p_2^2 p_3^4 + 3p_1 p_2^{-4} p_3^4 + \cdots,$$

$$t_5 = 6p_1 p_2 p_3 + 3p_1 p_2^{-2} p_3 + 6p_1^4 p_2^{-2} p_3 + 3p_1^4 p_2^4 p_3 + 6p_1 p_2^{-2} p_3^4 + 3p_1 p_2^4 p_3^4 + \cdots.$$

For any element $f \in M_k(\Gamma(3))$ if we write the Fourier expansion of f as $\sum c_{l_1 l_2 l_3} p_1^{l_1} p_2^{l_2} p_3^{l_3}$, then we consider the set of indices (l_1, l_2, l_3) with non-vanishing coefficients:

$$S(f) = \{(l_1, l_2, l_3) \in \mathbf{Z}^3 \mid C_{l_1, l_2, l_3} \neq 0\}.$$

M_λ^1	$\Phi_\lambda^1(t_1)$	$\Phi_\lambda^1(t_2)$	$\Phi_\lambda^1(t_3)$	$\Phi_\lambda^1(t_4)$	$\Phi_\lambda^1(t_5)$
$\begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}$	ϑ	3χ	0	0	0
$\begin{pmatrix} \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix} \end{pmatrix}$	ϑ	0	0	0	3χ
$\begin{pmatrix} \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \end{pmatrix}$	ϑ	0	0	3χ	0
$\begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$	ϑ	0	3χ	0	0
$\begin{pmatrix} \begin{smallmatrix} 0 & a_2 \\ a_2 & a_4 \end{smallmatrix} & -1_2 \\ 1_2 & 0 \end{pmatrix}$	x	y	$\omega^{a_4} x$	$\omega^{a_4 - a_2} y$	$\omega^{a_2 + a_4} y$
$\begin{pmatrix} \begin{smallmatrix} 0 & a_2 \\ a_2 & a_4 \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix} \\ \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} & 0 \end{pmatrix}$	x	y	$\omega^{a_4 - a_2} y$	$\omega^{a_2 + a_4} x$	$\omega^{a_4} y$
$\begin{pmatrix} \begin{smallmatrix} 0 & a_2 \\ a_2 & a_4 \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix} \\ \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} & 0 \end{pmatrix}$	x	y	$\omega^{a_4 + a_2} y$	$\omega^{a_4} y$	$\omega^{a_4 - a_2} x$
$\begin{pmatrix} \begin{smallmatrix} -a_4 & a_2 \\ 0 & a_4 \end{smallmatrix} & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & 0 \end{pmatrix}$	x	$\omega^{a_2} x$	y	$\omega^{a_2 - a_4} y$	$\omega^{a_2 + a_4} y$

Table 5-B.

Here, $x = -(\vartheta + 6\chi)/3$, $y = -(\vartheta - 3\chi)/3$.

On the other hand, we write the residue classes

$$R(m_1, m_2, m_3) = (m_1, m_2, m_3) + (3\boldsymbol{Z})^3$$

in $\boldsymbol{Z}^3$ with respect to the subgroup $3\boldsymbol{Z}^3$. Then

$$S(t_1) \subset R(0,0,0), \quad S(t_2) \subset R(1,0,0), \quad S(t_3) \subset R(0,0,1),$$
$$S(t_4) \subset R(1,-1,1), \quad S(t_5) \subset R(1,1,1).$$

In particular, $t_1, \cdots, t_5$ are linearly independent.

Next we consider the Siegel Φ-operator. The action of Γ to t_i are given by Proposition 5.3, thus we can calculate easily the image of Φ_λ^1.

§§5.4. Determination of the dimensions in low weight cases.

Now we can determine the dimensions of $M_k(\Gamma(3))$ for $k \leq 4$.

Lemma 5.2. $\dim M_1(\Gamma(3)) = 5$ *and the representation of* $G = Sp(2, \boldsymbol{F}_3)$ *on* $M_1(\Gamma(3))$ *is given by* θ_4.

Proof. In view of Proposition 5.3, we can see the representation of G on the subspace of $M_1(\Gamma(3))$ spanned by $\{t_i\}$ is given by θ_4 (We consider the action of Γ as $(\gamma, f) \mapsto f|\gamma^{-1}$). This fact and Corollary 5.1 and Lemma 2.3, 4.4, immediately prove this lemma. □

Lemma 5.3. $\dim M_2(\Gamma(3)) = 15$, $\dim S_2(\Gamma(3)) = 0$ *and the representation of* G *on* $M_2(\Gamma(3))$ *is given by* θ_{11}.

Proof. The map $S_2(\Gamma(3)) \to S_3(\Gamma(3)) f \mapsto t_1 f$ is clearly injective, hence we get $\dim S_2(\Gamma(3)) = 0$ by Lemma 2.2.

By Lemma 4.3 and Corollary 5.1, the representation of G on $M_2(\Gamma(3))$ is the subrepresentation of $1_G \oplus \theta_{11} \oplus \theta_9$ with dimension ≤ 15. Since $M_2(\Gamma) = \{0\}$ ([**10**, §9, Theorem]), the dimension of $M_2(\Gamma(3))$ is 15 and the representation is θ_{11}. □

Lemma 5.4. *The representation of* G *on* $S_4(\Gamma(3))$ *is given by* θ_{11}.

Proof. From [**18**, Proposition 4.1], we have $\dim S_4(\Gamma_0(3)) = 1$, where $\Gamma_0(3) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \mid C \equiv 0 \mod 3\}$. The image of $\Gamma_0(3)$ by the natural map to G is equal to $\bar{P}_0$.

On the other hand, if we write ρ the representation of G to $S_4(\Gamma(3))$, Frobenius reciprocity law tells

$$(\rho|_{\bar{P}_0}, 1_{\bar{P}_0})_{\bar{P}_0} = (\rho, \mathrm{Ind}_{\bar{P}_0}^G(1_{\bar{P}_0}))_G$$
$$= (\rho, 1_G)_G + (\rho, \theta_{11})_G + (\rho, \theta_9)_G.$$

We know already that $(\rho|_{\bar{P}_0}, 1_{\bar{P}_0})_{\bar{P}_0} = \dim S_4(\Gamma_0(3)) = 1$. Since $\dim S_4(\Gamma) = 0$, ρ

does not contain the trivial representation of G. Finally $\dim\theta_9 = 24$, $\dim\theta_{11} = 15$, and $\dim\rho = 15$, we get $\rho = \theta_{11}$. □

Next we consider the case $k = 3$ or $k = 4$. Since there is a natural map from $\operatorname{Sym}^k M_1(\Gamma(3))$ to $M_k(\Gamma(3))$, we have to calculate the k-th symmetric tensor product of the representation on $M_1(\Gamma(3))$.

By direct computation we have

$$\operatorname{Sym}^2\theta_4 \cong \underset{15\,\mathrm{dim}}{\theta_{11}},$$

$$\operatorname{Sym}^3\theta_4 \cong \underset{5\,\mathrm{dim}}{\theta_3} \oplus \underset{30\,\mathrm{dim}}{\Phi_9},$$

$$\operatorname{Sym}^4\theta_4 \cong 1_G \oplus \underset{15\,\mathrm{dim}}{\theta_{11}} \oplus \underset{24\,\mathrm{dim}}{\theta_9} \oplus \underset{30\,\mathrm{dim}}{\Phi_4}.$$

On the other hand by Proposition 5.2 and Corollary 5.1, $M_3(\Gamma(3))$ is the subrepresentation of

$$\underset{5\,\mathrm{dim}}{\theta_3} \oplus \underset{5\,\mathrm{dim}}{\theta_4} \oplus \underset{30\,\mathrm{dim}}{\Phi_9},$$

and $M_4(\Gamma(3))$ is the subrepresentation of

$$\begin{aligned} &1_G \oplus \theta_{11} \oplus \theta_9 \oplus \operatorname{Ind}_{P_1}^G(\rho_4 \circ \pi) \oplus \theta_{11} \\ &\cong \underbrace{1_G \oplus \underset{15\,\mathrm{dim}}{\theta_{11}} \oplus \underset{24\,\mathrm{dim}}{\theta_9}}_{E_4(\Gamma(3))} \oplus \underbrace{\underset{10\,\mathrm{dim}}{\Phi_2} \oplus \underset{30\,\mathrm{dim}}{\Phi_4}}_{K_4(\Gamma(3))} \oplus \underbrace{\underset{15\,\mathrm{dim}}{\theta_{11}}}_{S_4(\Gamma(3))}. \end{aligned}$$

Lemma 5.5. *The natural map* $\operatorname{Sym}^4 M_1(\Gamma(3)) \to M_4(\Gamma(3))$ *is injective. That is* t_1, t_2, t_3, t_4 *and* t_5 *does not have homogeneous relations of degree* 4.

By this lemma and by enumerating of the representations of low degree, we can get immediately the following.

Lemma 5.6. *We get*

$$\dim M_3(\Gamma(3)) = 35 \text{ or } 40, \quad \dim M_4(\Gamma(3)) = 85 \text{ or } 95.$$

Proof of Lemma 5.5. We have to show that the 70 monomials of t_i are linearly independent. Since $S(t_i)$ is contained in some class $R(m_1, m_2, m_3)$, for any monomial x of t_i the support of Fourier coefficients $S(x)$ is contained in some $R(m_1, m_2, m_3)$. We have the following table (Table 5-C).

The elements which contain different classes are clearly independent, so we have only to show the linearly independence of the elements contained in the same classes. It can be shown by using the Siegel Φ-operator. For example, consider the 2 elements contained in the class $R(1,0,1)$. For $M_\lambda^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we see

$$\Phi_\lambda^1(t_1^2 t_2 t_3) = x^3 y \quad \Phi_\lambda^1(t_4^2 t_5^2) = y^4.$$

Since x and y are algebraic independent (cf. Theorem 1.1) $t_1^2t_2t_3$ and $t_4^2t_5^2$ are linearly independent.

In the same way we can show all elements are linearly independent. □

Now we consider the 6 elements that are contained in the class $R(0,0,0), t_1^4, t_1t_2^3, t_1t_3^3, t_1t_4^3, t_1t_5^3$, and $t_2t_3t_4t_5$.

If we put

$$g = t_1^4 - t_1(t_2^3 + t_3^3 + t_4^3 + t_5^3) + 3t_2t_3t_4t_5,$$

$\Phi_\lambda^1(g) = 0$ for $M_\lambda^1 = \begin{pmatrix} A & -{}^tU^{-1} \\ U & 0 \end{pmatrix}$.

$R(0,0,0)$	t_1^4	$t_1t_2^3$	$t_1t_3^3$	$t_1t_4^3$	$t_1t_5^3$	$t_2t_3t_4t_5$
$R(1,0,0)$	$t_1^3t_2$	t_2^4	$t_2t_3^3$	$t_2t_4^3$	$t_2t_5^3$	
$R(0,0,1)$	$t_1^3t_3$	$t_2^3t_3$	t_3^4	$t_3t_4^3$	$t_3t_5^3$	
$R(1,-1,1)$	$t_1^3t_4$	$t_2^3t_4$	$t_3^3t_4$	t_4^4	$t_4t_5^3$	
$R(1,1,1)$	$t_1^3t_5$	$t_2^3t_5$	$t_3^3t_5$	$t_4^3t_5$	t_5^4	
$R(-1,0,0)$	$t_1^2t_2^2$	$t_1t_3t_4t_5$				
$R(1,0,1)$	$t_1^2t_2t_3$	$t_4^2t_5^2$				
$R(-1,-1,1)$	$t_1^2t_2t_4$	$t_3^2t_5^2$				
$R(-1,1,1)$	$t_1^2t_2t_5$	$t_3^2t_4^2$				
$R(0,0,-1)$	$t_1^2t_3^2$	$t_1t_2t_4t_5$				
$R(1,-1,-1)$	$t_1^2t_3t_4$	$t_2^2t_5^2$				
$R(1,1,-1)$	$t_1^2t_3t_5$	$t_2^2t_4^2$				
$R(-1,1,-1)$	$t_1^2t_4^2$	$t_1t_2t_3t_5$				
$R(-1,0,-1)$	$t_1^2t_4t_5$	$t_2^2t_3^2$				
$R(-1,0,1)$	$t_1t_2^2t_3$	$t_3^2t_4t_5$				
$R(1,0,-1)$	$t_1t_2t_3^2$	$t_2^2t_4t_5$				
$R(-1,-1,-1)$	$t_1t_2t_3t_4$	$t_1^2t_5^2$				
$R(0,-1,1)$	$t_1t_2^2t_4$	$t_3t_4^2t_5$				
$R(0,1,-1)$	$t_1t_2t_4^2$	$t_2^2t_3t_5$				
$R(0,1,1)$	$t_1t_2^2t_5$	$t_3t_4t_5^2$				
$R(0,-1,-1)$	$t_1t_2t_5^2$	$t_2^2t_3t_4$				
$R(1,1,0)$	$t_1t_3^2t_5$	$t_2t_4t_5^2$				
$R(1,-1,0)$	$t_1t_3^2t_4$	$t_2t_4^2t_5$				
$R(-1,1,0)$	$t_1t_3t_4^2$	$t_2t_3^2t_5$				
$R(-1,-1,0)$	$t_1t_3t_5^2$	$t_2t_3^2t_4$				
$R(0,1,0)$	$t_1t_4t_5^2$	$t_2t_3t_4^2$				
$R(0,-1,0)$	$t_1t_4^2t_5$	$t_2t_3t_5^2$				

Table 5-C.

And for $M_\lambda^1 = \begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix}$, $\Phi_\lambda^1(g) = \vartheta^4 - 27\vartheta\chi^3 = e_{(0,1)}^{4*}$.
This means

$$\Phi_\mu^0(g) = \begin{cases} 1 & \text{if } M_\mu^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0 & \text{otherwise.} \end{cases}$$

Using this fact we can see the following.

LEMMA 5.7. *The image* W *of the natural map* $\mathrm{Sym}^4 M_1(\Gamma(3)) \to M_4(\Gamma(3))$ *satisfies* $W \cap S_4(\Gamma(3)) = \{0\}$ *and the image* $W/S_4(\Gamma(3))$ *in* $M_4(\Gamma(3))/S_4(\Gamma(3))$ *coincides with the image of the Siegel Eisenstein series* $E_4(\Gamma(3))$.

6. Main result.

In this section we will prove the main result:

THEOREM 6.1.

$$(1) \quad \dim M_k(\Gamma(3)) = \begin{cases} 5 & k = 1; \\ 15 & k = 2; \\ 40 & k = 3; \\ \dfrac{1}{2}(6k^3 - 27k^2 + 79k - 78) & k \geq 4. \end{cases}$$

$$(2) \quad \dim S_k(\Gamma(3)) = \begin{cases} 0 & k = 1, 2, 3; \\ \dfrac{1}{2}(6k^3 - 27k^2 - k + 82) & k \geq 4. \end{cases}$$

COROLLARY 6.1.

$$\sum_{k=0}^{\infty} (\dim M_k(\Gamma(3)))t^k = \frac{1 + t + t^2 + 6t^3 + 6t^4 + t^5 + t^6 + t^7}{(1 - t)^4}.$$

PROOF OF THE THEOREM. First the statement (2) is already proved by Lemma 2.1, 2.2, 2.3 and 5.3. Also the assertion (1), it is proved except for the cases $k = 3$, $k = 4$, (cf. Lemma 5.1, 5.2 and 5.3). Further because of Lemma 5.6,

$$\dim M_3(\Gamma(3)) = 35 \text{ or } 40, \quad \text{and} \quad \dim M_4(\Gamma(3)) = 85 \text{ or } 95.$$

So if we show that the latter cases are true, the proof is finished.

LEMMA 6.1. *If* $\dim M_3(\Gamma(3)) = 40$, *then* $\dim M_4(\Gamma(3)) = 95$.

PROOF. Since $\dim S_3(\Gamma(3)) = 0$ and $\dim S_3(\Gamma^{(1)}(3)) = 0$, the assumption $\dim M_3(\Gamma(3)) = 40$ means $\dim E_3(\Gamma(3)) = 40$. Hence there exists an element $f \in M_3(\Gamma(3))$ such that

$$\Phi_\mu^0(f) = \begin{cases} 1 & \text{if } M_\mu^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ 0 & \text{otherwise.} \end{cases}$$

This means that

$$\Phi_\lambda^1(f) = \begin{cases} e_{(0,1)}^{3*} & \text{if } M_\lambda^1 = \begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence if we consider $t_2 f \in M_4(\Gamma(3))$,

$$\Phi_\lambda^1(t_2 f) = \begin{cases} 3e_{(0,1)}^{3*}\chi = 3h & \text{if } M_\mu^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\dim K_4(\Gamma(3)) = 40$, and from the result of §§5.4, we get $\dim M_4(\Gamma(3)) = 95$. □

Now it suffices to show that $\dim M_3(\Gamma(3)) = 40$, or equivalently $\dim M_3(\Gamma(3)) > 35$ by Lemma 5.6. Namely we have to show there exists a element of $M_3(\Gamma(3))$ which is not contained in the image W of the natural map $\mathrm{Sym}^3 M_1(\Gamma(3)) \to M_3(\Gamma(3))$.

To show this, we use the theory of non-holomorphic Eisenstein series.

For $s \in \boldsymbol{C}$, $\mathrm{Re}(s) > 0$ and $Z \in \boldsymbol{H}_2$, let

$$E(s, Z) = \sum_{P_0 \cap \Gamma(3) \backslash \Gamma(3)} \det(CZ + D)^{-3} \, |\det(CZ + D)|^{-s}.$$

Since $P_0 \Gamma(3) = \Gamma_0(3)$, this series can also be written as

$$E(s, Z) = \sum_{P_0 \backslash \Gamma_0(3)} \psi(\det D) \det(CZ + D)^{-3} \, |\det(CZ + D)|^{-s}.$$

Here ψ is the primitive Dirichlet character modulo 3.

Proposition 6.1 ([**15**, Theorem 7.1]). *For each Z, $E(s, Z)$ extends to a meromorphic function on $\boldsymbol{C}$, and it is holomorphic at $s = 0$. Further $E(0, Z)$ is holomorphic in Z and $E(0, Z) \in M_3(\Gamma(3))$.*

Our aim is to prove that $E = E(0, Z)$ is not contained in W. First we show a property of E:

Lemma 6.2. *For $\gamma \in P_0$, E satisfies*

$$E|\gamma = u_0(\gamma) E.$$

Proof. For $\mathrm{Re}(s) > 0$, the infinite sum defining $E(s, Z)$ converges absolutely and uniformly on $V(d) = \{Z \in \boldsymbol{H}_2 \mid \mathrm{Im}(Z) \geq d 1_2, \mathrm{Tr}(\mathrm{Re}(Z))^2 \leq d^{-1}\}$ for any $d > 0$. Now from the definition

$$E(s, Z) \mid \gamma = \sum_{P_0 \cap \Gamma(3) \backslash \Gamma(3)\gamma} (\cdots).$$

Since $\Gamma(3)$ and $P_0 \cap \Gamma(3)$ are normal subgroups of Γ and P_0 respectively, we get

$$E(s, Z) \mid \gamma = \sum_{\gamma(P_0 \cap \Gamma(3) \backslash \Gamma(3))} (\cdots).$$

Hence for $\mathrm{Re}(s) > 0$, $E(s, Z) \mid \gamma = u_0(\gamma) E(s, Z)$. Since it is holomorphic at $s = 0$, $E|\gamma = u_0(\gamma) E$. □

Lemma 6.3. *E is not contained in the image of the natural map* $\mathrm{Sym}^3 M_1(\Gamma(3)) \to M_3(\Gamma(3))$. *Hence* $\dim M_3(\Gamma(3)) = 40$.

Proof. We assume that E is contained in the image. Because E is invariant under $\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \in P_0$, $S(E)$ is contained in $R(0,0,0)$. Hence E can be written by five elements $t_1^3, t_2^3, t_3^3, t_4^3$, and t_5^3. Further, since $\begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix}$ induces permutation of t_2, t_3, t_4, t_5, we get

$$E = at_1^3 + b(t_2^3 + t_3^3 + t_4^3 + t_5^3), \quad a, b \in \boldsymbol{C}. \tag{$*$}$$

Now from [**15**, expression (7.8a) (7.8b) (7.8c) and (7.13)], we can see some information of the Fourier coefficients of $E|M_\lambda^1$, for $M_\lambda^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In the Fourier expansion

$$E(s, Z) \mid M_\lambda^1 = \sum_{h \geq 0} a(h, \mathrm{Im}(Z), s) \exp(2\pi i \,\mathrm{Tr}(h \,\mathrm{Re}(Z))),$$

the coefficient function is written as

$$a(h, \mathrm{Im}(Z), s) = c a_\infty(h, \mathrm{Im}(Z), s) a_f(h, s)$$

with

$$a_\infty(h, \mathrm{Im}(Z), 0) = -\frac{1}{2}(2\pi)^6 \Gamma_2(3)^{-1} (\det h)^{3/2} \exp(-2\pi \,\mathrm{Tr}(h \,\mathrm{Im}(Z))).$$

In particular $a(h, \mathrm{Im}(Z), 0) = 0$ if $\det h = 0$. This means

$$\Phi_\lambda^1(E) = 0 \quad \text{for } M_\lambda^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence from ($*$) and Table 5-B, we have

$$ax^3 + b(x^3 + 3y^3) = 0.$$

Since x and y are algebraically independent, we have $a = b = 0$. It is a contradiction. □

Now we have finished the proof of Theorem 6.1.

7. Remarks for further investigation.

§§7.1. Representation of G.

Here is a collection of the results on the representations of G obtained in the paper.

Theorem 7.1. *For $k \leq 4$ the representations of G on $M_k(\Gamma(3))$ are given as follows.*

$$\begin{cases} \theta_4 & k = 1; \\ \theta_{11} & k = 2; \\ \theta_3 \oplus \theta_4 \oplus \Phi_9 & k = 3; \\ \underbrace{1_G \oplus \theta_{11} \oplus \theta_9}_{E_4(\Gamma(3))} \oplus \underbrace{\Phi_2 \oplus \Phi_4}_{K_4(\Gamma(3))} \oplus \underbrace{\theta_{11}}_{S_4(\Gamma(3))} & k = 4. \end{cases}$$

For $k \geq 5$ *the representation on* $E_k(\Gamma(3))$ *is given by*

$$\begin{cases} 1_G \oplus \theta_{11} \oplus \theta_9 & k\text{: even;} \\ \theta_3 \oplus \theta_4 \oplus \Phi_9 & k\text{: odd.} \end{cases}$$

The representation on $K_k(\Gamma(3))$ *is given by*

$$\mathrm{Ind}_{P_1}^{G}(u_1^k \otimes ((B^+ \otimes \mathrm{Sym}^{k-4} A^+) \circ \pi)).$$

§§7.2. Generators and relations for the ring $\bigoplus_{k=0}^{\infty} M_k(\Gamma(3))$.

Since $\dim M_1(\Gamma(3)) = 5$, it is better to rewrite the generating function of Corollary 6.1 as

$$\frac{1 + 5t^3 - 5t^5 - t^8}{(1-t)^5}.$$

The quotient $M_3(\Gamma(3))/\mathrm{Sym}^3 M_1(\Gamma(3))$ has dimension 5. Therefore we further rewrite this function as

$$\frac{1 - 5t^5 + \{\text{higher terms}\}}{(1-t)^5(1-t^3)^5}.$$

Thus it is easy to see that we need 5 generators of weight 1 and another 5 generators of weight 3 in the graded ring $\bigoplus_{k=0}^{\infty} M_k(\Gamma(3))$, which has 5 linearly independent nontrivial relations of weight 5.

However one should be careful, because the above generating function itself gives no answer for the question whether there are nontrivial relations of weight 4 or not. Fortunately we can work out this problem.

Proposition 7.1. *All the elements of weight* 4 *are generated by the elements of weight* 1 *and* 3. *Therefore there is no nontrivial relation of weight* 4 *between the generators of weight* 1 *and weight* 3.

Proof. We use some results of [**9**]. As we show in the proofs of Lemma 6.1 and 6.3, there exists a unique element $E^{*3} \in M_3(\Gamma(3))$ such that

$$\Phi_\mu^0(E^{*3}) = \begin{cases} 1 & \text{if } M_\mu^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ 0 & \text{otherwise.} \end{cases}$$

Then E^{*3} is contained in the space $M_3(\Gamma_0(3), (-3/(\det D)))$ by the property of Siegel Φ-operator. On the other hand [**9**, Theorem 4] shows the following (this lemma gives the another proof of Lemma 6.3).

Lemma 7.1. *Let us define* 3 *elements* E_4, E_4', Θ_4 *of* $M_4(\Gamma_0(3))$ *by*

$$E_4(Z) = \sum_{P_0 \backslash \Gamma} \det(CZ + D)^{-4}, \quad E_4'(Z) = E_4(3Z)$$

and

$$\Theta_4 = \sum_{N \in M_{4,2}(\boldsymbol{Z})} P(N) \exp \pi i \operatorname{Tr}({}^t NSNZ)$$

for $S = \begin{pmatrix} \begin{smallmatrix} 2 & 3 \\ 3 & 6 \end{smallmatrix} & 0_2 \\ 0_2 & \begin{smallmatrix} 2 & 3 \\ 3 & 6 \end{smallmatrix} \end{pmatrix}$, $P\begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \\ m_3 & n_3 \\ m_4 & n_4 \end{pmatrix} = (m_1 n_2 - n_1 m_2 + m_3 n_4 - n_3 m_4)^2 - (m_1 n_4 - n_1 m_4 + m_2 n_3 - n_2 m_3 + m_1 n_3 - n_1 m_3)^2$.

Then the three functions

$$t_1^3, \quad \frac{x_4}{t_1} = \frac{E_4 + 81E_4' - 82t_1^4 + 324\Theta_4}{16t_1}, \quad \text{and} \quad \frac{y_4}{t_1} = \frac{E_4 - 81E_4'}{10t_1}$$

make a basis of the space $M_3(\Gamma_0(3), ((-3)/(\det D)))$.

Proof of the lemma. We have to show the divisibility of x_4 and y_4 by t_1. This is shown by the relations $x_4^2 = \alpha_1^2 w_6$ and $y_4^2 = \alpha_1^2 v_6$ in Theorem 4 of [**9**]. Again by the same theorem t_1^3 (essentially equals to α_1^3), x_4/t_1 and y_4/t_1 are linearly independent. Meanwhile the fact that there are only 3 equivalence classes of 0 dimensional cusps of $\Gamma_0(3)$ implies $\dim M_3(\Gamma_0(3), (-3/(\det D))) \le 3$. This settles the proof of our lemma. □

Now let us return to the proof of Proposition 7.1. By comparison of the values at each 0 dimensional cusp of $\Gamma_0(3)$ we get

$$E^{*3} = \frac{1}{2} t_1^3 + \frac{x_4}{8t_1} - \frac{y_4}{16t_1}.$$

This implies a relation:

$$t_1 E^{*3} = -\frac{9}{64} t_1^4 + \frac{1}{640} E_4 + \frac{729}{640} E_4' + \frac{81}{32} \Theta_4.$$

As we saw in §§5.4, $t_1 E^{*3}$ and $g = t_1^4 - t_1(t_2^3 + t_3^3 + t_4^4 + t_5^3) + 3t_2 t_3 t_4 t_5$ have the same value at each 1 dimensional cusp. Hence we have $t_1 E^{*3} = g + a\Theta_4$, $a \in \boldsymbol{C}$. By comparing the Fourier coefficient of $e^{2\pi i(z_1 + z_3)}$ we have

$$t_1 E^{*3} = t_1^4 - t_1(t_2^3 + t_3^3 + t_4^3 + t_5^3) + 3t_2 t_3 t_4 t_5 + \frac{81}{8} \Theta_4.$$

Let V be a subspace of $M_4(\Gamma(3))$ generated by the elements of weight 1 and 3. The above relation shows that $S_4(\Gamma(3)) \subset V$ in view of the irreducibility of the representation of $Sp(2, \boldsymbol{F}_3)$ on $S_4(\Gamma(3))$. Furthermore since

$$\Phi_\mu^0(t_1 E^{*3}) = \begin{cases} 1 & \text{if } M_\mu^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Phi_\lambda^1(t_2E^{*3}) = \begin{cases} 3h \in S_4(\Gamma^{(1)}(3)) & \text{if } M_\lambda^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0 & \text{otherwise,} \end{cases}$$

we have $E_4(\Gamma(3)) \subset V$ and $K_4(\Gamma(3)) \subset V$, hence finally $V = M_4(\Gamma(3))$. This proves the proposition. □

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