J. Math. Soc. Japan
Vol. 56, No. 2, 2004

Hulls and kernels from topological dynamical systems and their applications to homeomorphism C^*-algebras

By Jun TOMIYAMA

(Received Apr. 10, 2001)
(Revised Sept. 27, 2002)

Abstract. For a topological dynamical system $\Sigma = (X, \sigma)$ where σ is a homeomorphism in an arbitrary compact Hausdorff space X, we consider the noncommutative hulls and kernels with respect to the action σ in the associated C^*-algebra $A(\Sigma)$. We show that several ideals important for the structure of $A(\Sigma)$ have the form of such kernels and give topological characterizations of their hulls from the behavior of orbits in the dynamical system.

1. Introduction.

Let X be a compact Hausdorff space and $C(X)$ be the algebra of all complex valued continuous functions on X. For a subset S of X the kernel $k(S)$ in $C(X)$ is the closed ideal of $C(X)$ defined as

$$k(S) = \{f \in C(X) \mid f|S = 0\}.$$

On the other hand, the hull of a closed ideal I of $C(X)$, $h(I)$ is defined as

$$h(I) = \{x \in X \mid f(x) = 0 \ \ \forall f \in I\},$$

which naturally turns out to be a closed subset of X. These notions have been playing basic roles in functional analysis.

Moreover, regarding $C(X)$ as the prototype of commutative unital C^*-algebras, we can replace $C(X)$ by a noncommutative unital C^*-algebra A with its dual $\hat{A}$ (with hull-kernel topology) instead of the space X, and look for the role for this generalization. We regard these things as hull-kernels without actions or as simple minded noncommutative versions of hull-kernels.

The purpose of this article is to discuss hulls and kernels with an action on the space X, that is, actual noncommutative versions of hulls and kernels (written hereafter as Hulls and Kernels) in the homeomorphism algebra $A(\Sigma)$. This algebra is the C^*-crossed product constructed from a topological dynamical system $\Sigma = (X, \sigma)$ where σ is a homeomorphism on a compact Hausdorff space X. These notions stem originally from the article [**18**] in the context of C^*-crossed products by amenable discrete groups. We however modify the arguments there in a more suitable way fitted to the algebra $A(\Sigma)$. Here contrary to the case of usual hull-kernels without actions, not every closed ideal of $A(\Sigma)$ is expressed as such a Kernel. We shall show most important ideals for

---

2000 *Mathematical Subject Classfication*. Primary 46L55, Secondary 37A55, 54H15.
Key Words and Phrases. Homeomorphism, noncommutative hull-kernel, crossed product, recurrent point, periodic point.

the structure of the C^*-algebra $A(\Sigma)$ can be expressed as such Kernels and give topological characterizations of their Hulls. Those ideals belong to rather elementary classes of ideals from the point of view of C^*-theory, but their Hulls play important roles in dynamical systems. Actually, our main results (Theorems 4.3 and 4.7) show how the behavior of orbits yields compact operators in the images of infinite dimensional irreducible representations.

2. Notations and preliminaries.

Throughout this paper we consider a topological dynamical system $\Sigma = (X, \sigma)$ on an arbitrary (not necessarily metrizable) compact Hausdorff space X with single homeomorphism σ. With the automorphism α on $C(X)$ defined by $\alpha(f)(x) = f(\sigma^{-1}x)$, we write the C^*-crossed product $C(X) \times_\alpha Z$ as $A(\Sigma)$ and call it a homeomorphism C^*-algebra. We denote the generating unitary element of $A(\Sigma)$ by δ and the canonical projection of norm one from $A(\Sigma)$ to $C(X)$ by E. Write usual n-th Fourier coefficient of an element a of $A(\Sigma)$ as $a(n)$, which is defined as $a(n) = E(a\delta^{*n})$.

A representation of $A(\Sigma)$ is written as $\tilde{\pi} = \pi \times u$, where π means a representation of $C(X)$ as the restriction of $\tilde{\pi}$ to $C(X)$ and $u = \tilde{\pi}(\delta)$ a unitary element on a representing Hilbert space such that

$$\pi(\alpha(a)) = u\pi(a)u* = Adu(\pi(a)).$$

Through our discussions, we shall often make use of the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ derived from a representation $\tilde{\pi} = \pi \times u$. As explained in [**13**, p. 26], we define this dynamical system as

$$X_\pi = h(\pi^{-1}(0)) \quad \text{and} \quad \sigma_\pi = \sigma | X_\pi,$$

where $h(\pi^{-1}(0))$ means the standard hull of the kernel ideal of π in $C(X)$ and X_π turns out to be a closed invariant subset of X. It is then easily seen that this system Σ_π is topologically conjugate to the dynamical system $\Sigma'_\pi = (X'_\pi, \sigma'_\pi)$ where X'_π is the spectrum of $\pi(C(X))$ and the map σ'_π is the homeomorphism of X'_π induced from the automorphism Adu on $\pi(C(X))$. Thus we identify these two dynamical systems. Note that in this identification the action of $\pi(f)$ on the space X'_π is regarded simply as the restriction of the function f to the set X_π.

Now suppose $\tilde{\pi} = \pi \times u$ is particularly an irreducible representation induced by a point x of X. This is the irreducible representation following GNS-construction by a pure state extension φ of μ_x to $A(\Sigma)$, where μ_x means the point evaluation on $C(X)$ at x. Then by [**13**, Proposition 4.3] we see that $\pi(f) = 0$ if and only if f vanishes on the orbit $O(x)$ of x. Hence the set X_π turns out to be $\overline{O(x)}$, closure of $O(x)$.

We write $Per(\sigma)$ the set of all periodic points and $Aper(\sigma)$ the set of all aperiodic points. We say that a system Σ is topologically free if $Aper(\sigma)$ is dense in X. This class is wide enough to cover almost all reasonable dynamical systems. Nevertheless the structure of this class is quite compatible with the theory of C^*-algebras and we find many significant results for this class as seen from [**13**]. We call a system Σ is topologically transitive if for any pair of open subets of $X, \{U, V\}$, there exists an integer n such that $\sigma^n(U) \cap V \neq \emptyset$. When X is metrizable this property is known to be equivalent to the fact that there exists a point with dense orbit.

The algebra $A(\Sigma)$ is known to be the closed linear span of the generalized polynomials $\{\sum_{-n}^{n} f_k\delta^k\}$ over $C(X)$ so that every element of $A(\Sigma)$ can be approximated by those polynomials. In our arguments however, we need approximating polynomials whose coefficients are specified enough to consist of linear modification of Fourier coefficients of the original element. Thus, for this purpose, the following noncommutative version of Fejér's theorem of Cèsaro mean is quite useful (cf. [**16**]).

Theorem A. *The n-th generalized Cèsaro mean of an element a,*

$$\sigma_n(a) = \sum_{-n}^{n}\left(1 - \frac{|j|}{n+1}\right)a(j)\delta^j$$

converges to a in norm.

We remark that for each summability kernel in Fourier analysis such as de la Vallée Poussin kernel and Jackson kernel etc. we obtain the corresponding approximation sequence converging to a in norm ([**16**]).

Henceforth we mean an ideal of $A(\Sigma)$ a closed ideal if no confusion arises. For the irreducible representations induced by the points of X, we notice that if x is an aperiodic point the pure state extension φ to $A(\Sigma)$ is unique, whereas if y is a periodic point the set of pure state extensions is parametrized by the torus as $\{\varphi_{y,\lambda}\}$. Hence we denote their associated irreducible representations as $\tilde{\pi}_x$ and $\tilde{\pi}_{y,\lambda}$ respectively. Furthermore, by [**13**, Theorem 4.4 and remark afterwards] their kernels are determined by the orbit only if x is aperiodic and by the orbit together with the parameter λ if y is a periodic point. Therefore we denote those kernels as $P(\bar{x})$ and $P(\bar{y},\lambda)$. Besides, we write by $Q(\bar{y})$ the intersection of those kernels $\{P(\bar{y},\lambda)\}$ with all parameters. For all these things we refer to [**12**], [**13**].

3. Basic properties of noncommutative hulls and kernels in the homeomorphism algebra $A(\Sigma)$.

Let S be a subset of X and define the closed set $\mathrm{Ker}(S)$ in $A(\Sigma)$ as

$$\mathrm{Ker}(S) = \{a \in A(\Sigma) \mid a(n)(x) = 0 \ \ \forall x \in S, n \in Z\}.$$

By definition, $\mathrm{Ker}(S)$ is a closed subspace of $A(\Sigma)$.

For a closed ideal I of $A(\Sigma)$, define the closed set, $\mathrm{Hull}(I)$, in X as

$$\begin{aligned}\mathrm{Hull}(I) &= \{x \in X \mid a(n)(x) = 0 \ \ a \in I, n \in Z\}\\ &= \{x \in X \mid E(a)(x) = 0 \ \ a \in I\}.\end{aligned}$$

We regard these sets as Hulls and Kernels with the action σ on X, or α on $C(X)$, and call them noncommutative Hulls and Kernels.

On the other hand, we write usual kernels and hulls for $C(X)$ as we mentioned before. Namely,

$$k(S) = \{f \in C(X) \mid f(x) = 0 \ \ \forall x \in S\}$$

and for a closed ideal J of $C(X)$

$$h(J) = \{x \in X \mid f(x) = 0 \ \forall f \in J\}.$$

Let $\bar{S}$ be the closure of S in X. We have then as is well known,

$$h(k(S)) = \bar{S}, \quad \text{and} \quad k(h(J)) = J.$$

Now comparing with the case of simple minded generalization, it is not so obvious to see the properties of $\mathrm{Ker}(S)$ and $\mathrm{Hull}(I)$. We however still have the following facts known before in [**18**].

Proposition 3.1. (1) *If S is an invariant subset of X, then* $\mathrm{Ker}(S)$ *becomes a closed ideal of $A(\Sigma)$. Moreover, it is a closed linear span of generalized polynomials over the subalgebra $k(S)$ of $C(X)$ written as $J(k(S))$.*

(2) $\mathrm{Hull}(I)$ *is a closed invariant subset of X.*

We give a part of the proof for completeness.

In order to show the property of ideal for $\mathrm{Ker}(S)$, take an element a of $\mathrm{Ker}(S)$ and an arbitrary element b of $A(\Sigma)$. Then the generalized Cesàro mean of $a, \sigma_n(a)$ (clearly contained in $\mathrm{Ker}(S)$), converges to a in norm, hence $b\sigma_n(a)$ and $\sigma_n(a)b$ converge to ba and ab, respectively. On the other hand, we see by definition that $E(\mathrm{Ker}(S)) = k(S)$, which is apparantly contained in $\mathrm{Ker}(S)$. Now since

$$b\sigma_n(a) = \sum_{-n}^{n} \left(1 - \frac{|j|}{n+1}\right) ba(j)\delta^j \quad \text{where } a(j) \in k(S),$$

we have for any integer k

$$(b\sigma_n(a))(k)(x) = E(b\sigma_n(a)\delta^{\star k})(x) = \sum_{-n}^{n} \left(1 - \frac{|j|}{n+1}\right) b(k-j)(x)a(j)(\sigma^{j-k}(x)).$$

Hence if S is invariant the element $b\sigma_n(a)$ belongs to $\mathrm{Ker}(S)$, and similarly $\sigma_n(a)b$, too. Thus, both ba and ab belong to $\mathrm{Ker}(S)$.

For the assertion (2), we just note that for a point x in $\mathrm{Hull}(I)$ we have

$$a(n)(\sigma^{-1}x) = \alpha(a)(n)(x) = 0,$$

because $\alpha(a)$ belongs to I as well as a.

Henceforth, throughout this paper, we always mean $\mathrm{Ker}(S)$ the closed ideal of $A(\Sigma)$ defined by an invariant subset S of X.

For our noncommutative Hulls and Kernels we have also by definition

$$\mathrm{Hull}(\mathrm{Ker}(S)) = \bar{S}.$$

On the contrary, the ideal $\mathrm{Ker}(\mathrm{Hull}(I))$ for a closed ideal I of $A(\Sigma)$ contains I by definition, but the inclusion may happen to be strict. In fact, sometimes we meet a much worse situaton such as

$$\mathrm{Ker}(\mathrm{Hull}(I)) = A(\Sigma).$$

In fact, if we take a primitive ideal $P(\bar{y}, \lambda)$, the kernel of the finite dimensional irreducible representation $\tilde{\pi}_{y,\lambda}$, it is known that

$$E(P(\bar{y}, \lambda)) = C(X), \quad \text{hence } \mathrm{Hull}(P(\bar{y}, \lambda)) = \varnothing.$$

Thus its Kernel becomes the whole algebra.

Let I be a closed ideal of $A(\Sigma)$. Then the module properties of the projection E implies that the image $E(I)$ becomes an ideal of $C(X)$ (not necessarily closed). In general we do not know whether this ideal is a proper ideal or not. The case where this ideal is included in I is an extremely good situation for I, and it happens to be the ideal having the form of noncommutative kernel. Since our coming discussions heavily depend on the structure of the ideal $\mathrm{Ker}(S)$, we summarize here the related results for readers convenience, which are a little modifications of [**16**, Theorem 2]. However, we have to confirm first the following elementary fact because it is closely related to the assertion (1) of the theorem.

Namely, let S be a closed invariant subset of X, and consider the restricted dynamical system $\Sigma_S = (S, \sigma_S)$ where σ_S means the restriction of σ to S. We have then the $*$-homomorphism ρ from $C(X)$ to $C(S)$ by restriction and the automorphism α_S defined naturally on $C(S)$. Therefore, denoting the generating unitary in the homeomorphism algbra $A(\Sigma_S)$ by δ_S we obtain a $*$-homomorphism $\tilde{\rho} = \rho \times \delta_S$ from $A(\Sigma)$ to $A(\Sigma_S)$ in such a way that

$$\rho \circ E(a) = E_S \circ \tilde{\rho}(a).$$

Here the kernel of this homomorphism $\tilde{\rho}$ coincides naturally with the ideal $\mathrm{Ker}(S)$.

Theorem 3.2. *The following assertions are equivalent for a closed ideal I of $A(\Sigma)$:*

(1) $E(I) \subset I$, *consequently the quotient algebra $A(\Sigma)/I$ becomes the crossed product $q(C(X)) \times_{\alpha_I} Z$ where q is the quotient homomorphism and α_I is the automorphism of $q(C(X))$ defined as $\alpha_I(q(a)) = q(\alpha(a))$.*

Hence the quotient algebra is naturally isomorphisc to the homeomorphism C^*-algebra $A(\Sigma_S)$ for $S = \mathrm{Hull}(I)$;

(2)

$$I = \mathrm{Ker}(\mathrm{Hull}(I)) = J(k(\mathrm{Hull}(I))).$$

Here the third term means the closed linear span of generalized polynomials over the ideal of $C(X)$, $k(\mathrm{Hull}(I))$;

(3) *I is invariant by the dual action $\hat{\alpha}_t$,*

(4) *I is expressed as the intersection of families of primitive ideals $\{P(\overline{x_\alpha})\}$ and $\{Q(\overline{y_\beta})\}$ for aperiodic points $\{\overline{x_\alpha}\}$ and periodic points $\{\overline{y_\beta}\}$.*

Here we recall that the dual action on $A(\Sigma)$ is the one-parameter automorphism group $\{\hat{\alpha}_t \mid t \in T\}$ induced by the covariant action,

$$\hat{\alpha}_t(f) = f, \quad \hat{\alpha}_t(\delta) = e^{2\pi i t}\delta.$$

We further remark that as consequences of the assertion (4) and [**13**, Corollary 5.1.B] all ideals in $A(\Sigma)$ have the form of Kernel if the dynamical system is free, that is, no periodic points.

We shall show one more basic fact.

Proposition 3.3. *Let $\tilde{\pi} = \pi \times u$ be a topologically free representation of $A(\Sigma)$, that is, the induce dynamical system Σ_π is topologically free. Then for a closed invariant set S of X we have,*

$$\tilde{\pi}(\mathrm{Ker}(S)) = \mathrm{Ker}(X_\pi \cap S) \quad in \ A(\Sigma_\pi).$$

Hence the image of Ker(S) *has the same form of Kernel in this case. Moreover, in this case we have that*

$$\tilde{\pi}^{-1}(0) = \mathrm{Ker}(X_\pi).$$

Proof. Note first by [**13**, Corollary 5.1A] we can identify $\tilde{\pi}(A(\Sigma))$ with $A(\Sigma_\pi)$, hence from the above theorem we have,

$$\tilde{\pi}(\mathrm{Ker}(S)) = \tilde{\pi}(J(k(S))) = J(\pi(k(S))).$$

Here the action of $\pi(f)$ on X_π is nothing but the restriction of $\pi(f)$ to X_π. Therefore,

$$\pi(k(S)) = k(S) \,|\, X_\pi = k(S \cap X_\pi),$$

and

$$\tilde{\pi}(\mathrm{Ker}(S)) = J(k(S \cap X_\pi)) = \mathrm{Ker}(S \cap X_\pi).$$

The last assertion immediately follows from [**13**, Proposition 5.2] and Theorem 3.2 (2). □

The following proposition asserts that we can apply the above result to an arbitrary infinite dimensional irreducible representation of $A(\Sigma)$. Henceforth we shall often make use of this result.

Proposition 3.4. *If $\tilde{\pi} = \pi \times u$ is an infinite dimensional irreducible representation of $A(\Sigma)$, then the dynamical system Σ_π becomes topologically free and we have the same results as in the above Proposition* 3.3.

Proof. Since $\tilde{\pi}$ is irreducible the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ becomes topologically transitive by [**13**, Proposition 4.4]. Moreover, the space X_π is an infinite set (cf. [**14**, Proposition 3.4]). Thus, what we need is to show the assertion:

A topologically transitive dynamical system on an infinite set is topologically free.

The proof of this fact is just a combination of the arguments for the assertion $(b) \Rightarrow (a)$ of [**13**, Proposition 2.2] and the ones in the proof of [**13**, Theorem 4.6], and the result has been mentioned as [**13**, Corollary 5.1.B]. We however provide here the whole proof for readers' convenience because we shall often use this result in our coming discussions. For the proof, we assume for notational convenience the system $\Sigma = (X, \sigma)$ itself is topologically transitive acting on an infinite set X.

Now suppose that $Aper(\sigma)$ is not dense in X. Since a compact Hausdorff space is regular we can find an open set U whose closure $\overline{U}$ is also disjoint from $\overline{Aper(\sigma)}$. We have then

$$\overline{U} = \bigcup_{n=1}^{\infty} (Per^n(\sigma) \cap \overline{U}),$$

where

$$Per^n(\sigma) = \{x \in X \mid \sigma^n x = x\}.$$

It follows by the category theorem there exists an integer n such that the set $Per^n(\sigma) \cap \overline{U}$ contains an interior point. Let m be the smallest integer among such integers, then we can say that the set of m-periodic points $Per_m(\sigma)$ contains an interior point in the space $\overline{U}$. Thus we finally see that the set $Per(\sigma)$ contains an interior point in X, and the interior $(Per_m(\sigma))^\circ$ becomes an non-empty invariant open subset of X. Therefore it is dense in the space. Next take a point x in that interior and choose the compact neighborhood V of x contained in $Per_m(\sigma)$ such that the set $\{\sigma^i(V)\}$ for $0 \le i \le m-1$ are mutually disjoint. It follows that the union $\bigcup_{i=0}^{m-1} \sigma^i(V)$ is an invariant closed subset of X. Since $Per_m(\sigma)$ is dense, we have that

$$X = \bigcup_{i=0}^{m-1} \sigma^i(V).$$

Furthermore, if V contains another point y, choosing a smaller compact neighborhood W of x that does not contain y, we reach the same conclusion,

$$X = \bigcup_{i=0}^{m-1} \sigma^i(W).$$

Therefore, X has to be $O(x)$ for a m-periodic point x, a contradiction. The proof is completed, and we can simply apply this result to the derived dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$. □

4. Topological characterizations of Hulls of structural Kernel ideals of $A(\Sigma)$.

In this main section we consider several ideals of $A(\Sigma)$, which play important role for the structure of C^*-algebra $A(\Sigma)$ with the form of Kernels and give topological characterizations of their Hulls in connection with orbit behavior of the dynamical system.

As our first applications we reformulate our previous results [**13**, Theorem 4.6 (2)] and [**16**, Proposition 4] in a more transparent way. Namely let I_F and I_∞ be the intersection of all kernels of finite dimensional irreducible representations and infinite dimentional irreducible representations, respectively. We have then,

Theorem 4.1. (1) $I_F = \mathrm{Ker}(Per(\sigma))$, *hence* $A(\Sigma)$ *is residually finite dimensional* (*sufficiently many finite dimensional irreducible representations*) *if and only if* $Per(\sigma)$ *is dense in* X,

(2) $I_\infty = \mathrm{Ker}(Aper(\sigma))$, *hence the dynamical system* Σ *is topologically free if and only if there exist sufficiently many infinite dimensional irreducible representations*, *that is*, *there exists a family of infinite dimensional irreducible representations which separates any pair of elements of* $A(\Sigma)$.

It is worth to notice here that since the set of infinite dimensional irreducible representations is usually quite bigger than the set of infinite dimensional irreducible representations arising from aperiodic points, the equality of the second assertion is highly nontrivial, whereas the first equality is more or less straightforward.

Proof of the assertion (2). Since I_∞ is clearly invariant by the dual action, by Theorem 3.2 we may write as $I_\infty = \mathrm{Ker}(S)$ for an invariant closed subset S of X. We assert that $S = \overline{Aper(\sigma)}$. Now as I_∞ is contained in $\mathrm{Ker}(Aper(\sigma))$ the set S contains $\overline{Aper(\sigma)}$. Suppose S contains strictly the latter, and take a continuous function f vanishing on $\overline{Aper(\sigma)}$ and not on S. There exists then an infinite dimensional irreducible representation $\tilde{\pi} = \pi \times u$ such that $\pi(f) \neq 0$. Now consider the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$, which turns out to be topologically free by Proposition 3.4. It follows that $\pi(f)$ does not vanish on the set $Aper(\sigma_\pi)$. Since $\pi(f) \mid X_\pi$ is identified with the restriction $f|X_\pi$, the function f does not vanish on $Aper(\sigma_\pi)$ and not on $Aper(\sigma)$. This is a contradiction. □

The advantage of these results lies at the point to show that the size of those sets, $Per(\sigma)$ and $Aper(\sigma)$ are invariant by isomorphisms between homeomorphism C^*-algebras. As of now, we are still very far from establishment of the general isomorphism theorem, which may tell us exact relations between two dynamical systems when their associated homeomorphism C^*-algebras are isomorphic each other.

We should notice here that the ideal $\mathrm{Ker}(S)$ of $A(\Sigma)$ for an invariant closed subset of X might be understood as the C^*-crossed product $C_0(S^c) \times_\alpha Z$. This will be seen if we consider the covariant representation of the system $\{C_0(S^c), \alpha\}$ into the algebra $A(\Sigma)$ which is compatible with their canonical projections of norm one to $C_0(S^c)$ and $C(X)$, respectively. This is rather a standard notation (instead of $\mathrm{Ker}(S)$) in most literature. When we meet those Kernel ideals corresponding with various invariant subsets, however, this kind of expression turns out to be somewhat inconvenient to discuss with, particularly when we want to compare their locations in the algebra $A(\Sigma)$. This is the main reason of our understanding and notations of $\mathrm{Ker}(\cdot)$ and $\mathrm{Hull}(\cdot)$.

Next recall the definition of a recurrent point. Here we define a recurrent point x if for every neighborhood U of x there exists a nonzero integer n such that $\sigma^n(x)$ is in U. When X is metrizable, this means that there exists a subsequence $\{n_i\}$ such that either $\sigma^{n_i}(x)$ converge to x (positively recurrent) or $\sigma^{-n_i}(x)$ to x (negatively recurrent). It is to be noticed here that two other definitions of recurrent points are used in literature (for instance cf. [**6**, p. 129]). Namely, call a point x recurrent if it is either positively recurrent or both positively and negatively recurrent. We however employ the above definition in [**1**, p. 674] as the one which is most compatible with C^*-theory.

Now denote the set of all recurrent points by $c(\sigma)$, whose closure is known to be as the Birkhoff center or simply called the center. When X is metrizable it is known that the set $c(\sigma)$ is always nonempty, but this is also true for a dynamical system on an arbitrary compact space.

We call a C^*-algebra of type 1 if every irreducible representation contains a nonzero compact operator (hence contains all compact operators by irreducibility). By Sakai's result [**8**] this definition is equivalent to that of postliminality given in [**3**] as well as the usual definition of a C^*-algebra of type 1 (every representation generates a von Neumann algebra of type 1).

Now there are many literatures to discuss when the algebra $A(\Sigma)$ becomes an algebra of type 1 in the broad context of transformation group C^*-algebras. Those results are however formulated towards the theory of operator algebras and not for dynamical systems themselves. Thus even for our simplest dynamical systems (single homeomorphism on a compact space) it is often hard to see whether or not a given dynamical system yields the algebra $A(\Sigma)$ of type 1.

Now recall that a C^*-algebra A always contains the largest ideal K of type 1 such that the quotient algebra A/K has no nonzero ideal of type 1. We denote by $K(\sigma)$ this ideal in $A(\Sigma)$. In [**2**] when X is metrizable, we have precisely determined the size of the ideal $K(\sigma)$ in $A(\Sigma)$, from which we can easily see when the algebra becomes of type one. We shall reinforth the arguments there in terms of our Hull-Kernels and clarify the point where metrizability comes in.

Let $\tilde{\pi} = \pi \times u$ be an irreducible representation of $A(\Sigma)$ on a Hilbert space H. Denote the algebra of all compact operators on H by $C(H)$. The following proposition is the refined version of the key result, [**2**, Proposition 1], without the restriction of metrizability.

Proposition 4.2. *The image $\tilde{\pi}(A(\Sigma))$ contains the algebra of compact operators, $C(H)$, if and only if there exists a point x not belonging to the set $c(\sigma)\backslash Per(\sigma)$ with dense orbit in the space X_π.*

This point x becomes necessarily an isolated point of X_π.

We give here the whole proof for completeness.

Proof. We first note that in case of an irreducible representation of $A(\Sigma)$ it is finite dimensional if and only if X_π is a finite set by [**14**, Proposition 3.4]. Thus for the proof we may assume that X_π is an infinite set.

($\Rightarrow$). From the above assumption, the dynamical system Σ_π is topologically free, hence by [**13**, Theorem 5.4 (2)] (with Proposition 3.4 and [**13**, Corollary 5.1.A]) the intersection of $\pi(C(X))$ and $C(H)$ contains a nonzero selfadjoint element $\pi(f)$. Take a spectral projection p of $\pi(f)$ which is naturally contained in $\tilde{\pi}(A(\Sigma))$. It follows by [**13**, Theorem 5.4 (3)] together with the property of spectral projections that p belongs to $\pi(C(X))$. Let S be the support of p in X_π, then it is an open and closed set. Moreover as p is finite dimensional S must consist of isolated points of finite number. Now take a point x in S, then the orbit $O(x)$ is an invariant open (infinite) set. Hence it has to be dense because Σ_π is topologically transitive ([**13**, Proposition 4.4]), and since x is an isolated point it can not be a recurrent point.

($\Leftarrow$). Let x be the point in X_π with dense orbit which is not a recurrent point. Then x has to be an isolated point of X_π. Let p be the characteristic function of $\{x\}$. We have then for any a in $\tilde{\pi}(A(\Sigma))$ and g in $\pi(C(X))$,

$$papg = g(x)pap = gpap.$$

Therefore, by [**13**, Theorem 5.4 (3)], pap belongs to $\pi(C(X))$ and

$$pap = \lambda p \quad \text{for some scalar } \lambda.$$

This shows p is a minimal projection of $\tilde{\pi}(A(\Sigma))$. Now since this image is strongly dense in $B(H)$ it becomes a minimal projection of $B(H)$, and it is one dimensional. It follows that the image of $A(\Sigma)$ contains $C(H)$ because of the irreducibility of $\tilde{\pi}$. This completes the proof. □

A prototype of a dynamical system on a non-metrizable comapct space satisfying the above condition is the topologically transitive free dynamical system $(\beta Z, \sigma)$ discussed in [**12**, Example 3.3.2], where βZ is the Čech compactification of the integer group Z and σ is the extension of the simple shift on Z to βZ.

The next theorem is a reformulation of our previous result [**2**, Thorem 1] in terms of Hulls.

Theorem 4.3. Hull$(K(\sigma))$ *contains the set* $c(\sigma)\backslash Per(\sigma)$.
When X *is metrizable we have the equality*,

$$\mathrm{Hull}(K(\sigma)) = \overline{c(\sigma)\backslash Per(\sigma)}.$$

Proof. Note first that $K(\sigma)$ is invariant by the dual action, hence by Theorem 3.2 we can write it as

$$K(\sigma) = \mathrm{Ker}(\mathrm{Hull}(K(\sigma))).$$

Put $S = \mathrm{Hull}(K(\sigma))$ in short and take a point x in $c(\sigma)\backslash Per(\sigma)$. Then by the above Proposition we see that the image of the irreducible representation $\tilde{\pi}_x$ induced by x can not contain any non-zero compact operator, and $\tilde{\pi}_x(K(\sigma)) = \{0\}$. It follows from Proposition 3.3 that

$$S \cap X_{\pi_x} = X_{\pi_x} = \overline{O(x)}.$$

Hence S contains X_{π_x}, and x. Therefore, S contains the set $c(\sigma)\backslash Per(\sigma)$. Namely,

$$K(\sigma) \subset \mathrm{Ker}(c(\sigma)\backslash Per(\sigma)) = \mathrm{Ker}(\overline{c(\sigma)\backslash Per(\sigma)}).$$

When X is metrizable we can show that the ideal $\mathrm{Ker}(c(\sigma)\backslash Per(\sigma))$ itself is of type 1 (cf. [**2**]), hence the equality holds. □

Corollary 4.4. (1) *If* $A(\Sigma)$ *is of type* 1, *we have that* $c(\sigma) = Per(\sigma)$. *The converse holds when* X *is metrizable.*
(2) *If* $c(\sigma)\backslash Per(\sigma)$ *is dense, then* $K(\sigma) = \{0\}$ *i.e.* $A(\Sigma)$ *is antiliminal. The converse holds when* X *is metrizable.*

The result exactly shows how $A(\Sigma)$ differs from being of type 1 in terms of dynamical systems, and we naturally see the meaning of the difference $c(\sigma)\backslash Per(\sigma)$ in C^*-algebra theory.

Although the preceeding proposition 4.2 holds without the countability assumption for the space X, we do not know whether or not the metrizability is a crucial obstruction in the theorem.

In our theory of the interplay between topological dynamics and theory of operator algebras in a series of papers starting from the article [**12**], this is the first result for which we need the countability restriction, whereas we meet abundance of examples of topological dynamical systems on hyperstonean spaces (highly non-metrizable) coming from non-singular measurable dynamical systems on finite measure spaces. Therefore, it is quite desirable not to put the countability assumption for relevant topological dynamical systems.

In the proof of the theorem, a main trouble in non-metrizable case is at the point that for an irreducible representation $\tilde{\pi}$ we can not assume apriori the existence of a point with dense orbit, whereas in case X being metrizable it is the consequence of the equivalency between topological transitivity and dense orbit property. As it is noticed in [**15**, Proposition 1.2], however, every (nontrivial) topologically transitive dynamical system arising from a non-singular ergodic transformation on the non-atomic Lebesgue space reveals a counter-example for this equivalency in non-metrizable case. Nevertheless, we have been still unable to find a counter-example or to obtain the equality in the theorem without the metrizability assumption.

Henceforth though we do not know the converse of the above corollary, we call a dynamical system $\Sigma = (X, \sigma)$ of type 1 if $c(\sigma) = Per(\sigma)$, that is, no proper recurrent point. We often meet important examples of dynamical systems of type 1. The so-called Morse-Smale dynamical systems on the circle are such examples of the systems of type 1. In fact in this case the set of non-wandering points, $\Omega(\sigma)$, is a finite set and naturally consists of only periodic points. Actually we can assert more:

Any homeomorphism on the circle admitting periodic points becomes of type 1, hence in particular every orientation reversing homeomorphism is of type 1.

We refer our joint work [**10**] for detailed investigation of dynamical systems of type 1.

A C^*-algebra is said to be liminal if the image of every irreducible representation consists of compact operators. Hence a unital liminal C^*-algebra means that all of its irreducible representation are finite dimensional.

Next, recall that in a C^*-algebra there always exists the largest liminal ideal. It is defined as the set of all elements whose images of every irreducible representation are compact operators. Write this ideal of $A(\Sigma)$ as $L(\sigma)$. By definition, this ideal is invariant by the dual action hence by Theorem 3.2 we can write it as

$$L(\sigma) = \mathrm{Ker}(\mathrm{Hull}(L(\sigma))).$$

We shall discuss a characterization of $\mathrm{Hull}(L(\sigma))$. At first, for an invariant closed set S we consider the following condition $(*)$;

$(*)$ For every point x in $X \backslash S$ the set $\partial O(x) = \overline{O(x)} \backslash O(x)$ is contained in S.

Henceforth for convenience sake we sometimes call the above set the boundary of the orbit $O(x)$. We notice that this set becomes non-empty for any aperiodic point x by the category theorem, and the condition means S absorbes such boundaries of those aperiodic points outside of S. On the contrary, some periodic points may spread outside of S.

The ideal $L(\sigma)$ sits at the starting position of the structure of the C^*-algebra $A(\Sigma)$. On the other hand, in connection with the condition $(*)$ we notice that the behavior where an orbit absorbes another orbit is the simplest case to count nonwandering points.

Namely, if the orbit of a point y trails to the orbit of the another point x all points in $O(x)$ become nonwandering points.

We have then the next proposition. Similar arguments are found in the proof of [**17**, Lemma 2], where the space X is assumed to be metrizable and the set S contains the center, $\overline{c(\sigma)}$.

Proposition 4.5. *If the ideal* $\mathrm{Ker}(S)$ *for an invariant closed subset* S *is liminal,* S *satisfies the condition* $(*)$. *The converse holds if* X *is metrizable.*

Proof. Take a point x in $X \backslash S$ and let $\tilde{\pi} = \pi \times u$ be the irreducible representation on a Hilbert space H induced by x. We may assume here that x is an aperiodic point. Since then $\tilde{\pi}(\mathrm{Ker}(S)) \neq \{0\}$, the image $\tilde{\pi}(A(\Sigma))$ contains the algebra $C(H)$. Hence by Proposition 4.2 all points of $O(x)$ are isolated points in the space $X_\pi = \overline{O(x)}$ and the characteristic function of each point becomes a one dimensional projection in H. Now suppose there exists a point y_0 in the boundary $\partial O(x)$ which would not belong to S. There exists then a compact neighborhood U of y_0 in $X \backslash S$ containing infinite points of $O(x)$. Take a positive continuous function f on X satisfying the conditions $f|U = 1$ and $f|S = 0$. By definition, f belongs to $\mathrm{Ker}(S)$ and if we consider the induced dynamical system (X_π, σ_π) the image $\pi(f)$ is regarded as the restriction of f to X_π. Furthermore, since $\tilde{\pi}$ is an infinite dimensional irreducible representation we can apply [**13**, Theorem 5.4 (3)] by Proposition 3.4 together with [**13**, Corollary 5.1.A] to see that every spectral projection of $\pi(f)$ belongs to $C(X_\pi)$. Therefore, $\pi(f)$ can not be a compact operator on H, a contradiction. Thus, S satisfies the condition $(*)$.

Next assume that X is metrizable and suppose S satisfies the condition $(*)$. Let $\tilde{\pi} = \pi \times u$ be an irreducible representation of $A(\Sigma)$, which is also considered as an irreducible representation of $\mathrm{Ker}(S)$. We asssert that the image $\tilde{\pi}(\mathrm{Ker}(S))$ consists of compact operators.

We may assume here that $\tilde{\pi}$ is infinite dimensional. We have then by Proposition 3.4 and 3.3,

$$\tilde{\pi}(\mathrm{Ker}(S)) = \mathrm{Ker}(S \cap X_\pi) = J[k(S \cap X_\pi)].$$

Hence it is enough to show that every function f in $k(S \cap X_\pi)$ becomes a compact operator. Now note that there exists a point x in X_π whose orbit $O(x)$ is dense in X_π. If we assume $\tilde{\pi}$ is nonzero on $\mathrm{Ker}(S)$, x does not belong to S and by the condition $(*)$ $\partial O(x)$ is contained in $S \cap X_\pi$. Therefore f vanishes on $\partial O(x)$ and for any positive ε the set

$$\{y \in \overline{O(x)} \mid |f(y)| \geq \varepsilon\}$$

is a closed subset of $O(x)$. Hence by category theorem it must contain an isolated point, and consequently all points of this set become isolated points. Thus the set has to be a finite set, say $\{y_1, y_2, \dots, y_n\}$. This is also an open set in X_π. Moreover, as in the proof of Proposition 4.2 each characteristic function p_i of $\{y_i\}$ is a one dimensional projection in H. Now put the function g as

$$g(y) = \sum_{i=1}^{n} f(y_i) p_i(y).$$

Obviously, the function g is a compact operator on H and by definition $\|f - g\| < \varepsilon$. Hence the operator f is a compact operator.

This completes all proofs. □

Combining this proposition with Proposition 3.3 we have the following generalization of [**17**, Lemma 2].

Corollary 4.6. *Let S_1 and S_2 be invariant closed subsets of X where S_1 contains S_2. Then if the quotient ideal* $\mathrm{Ker}(S_2)/\mathrm{Ker}(S_1)$ *in $A(\Sigma_{S_1})$ becomes a liminal ideal, S_2 satisfies the condition $(*)$ in the space S_1. Moreover, if S_1 is metrizable the converse is also valid.*

Here in connection with the property of the ideal $\mathrm{Ker}(\Omega(\sigma))$ it would be worth to reinforce the results in [**17**] in terms of Kernels, which may bring more transparent insight. Throughout this argument we assume that X is a compact metric space. Then $\mathrm{Ker}(\Omega(\sigma))$ becomes a liminal ideal.

The shrinking steps of the nonwandering sets from $\Omega(\sigma)$ down to the center $\overline{c(\sigma)}$ is described by the family $\{\Omega_\lambda\}_{0 \le \lambda \le \gamma}$ parametrized by countable ordinals $\{\lambda\}$ such that $\Omega_0 = X$, $\Omega_\gamma = \overline{c(\sigma)}$ and

$$\Omega_{\lambda+1} = \Omega(\sigma|\Omega_\lambda).$$

Moreover,

$$\Omega_\lambda = \bigcap_{\rho<\lambda} \Omega_\rho \quad \text{if } \lambda \text{ is a limit ordinal.}$$

Now consider the family of ideals, $\{\mathrm{Ker}(\Omega_\lambda)\}$. This family becomes a so-called composition series of the C^*-algebra $\mathrm{Ker}(\overline{c(\sigma)})$ denoted by $J(\sigma)$ in [**17**]. Actually this turns out to be an ideal of $K(\sigma)$, that is, $K(\sigma) \cap \mathrm{Ker}(Per(\sigma))$. We have given there a characterization of this composition series.

In this series if the gap ideals $\mathrm{Ker}(\Omega_{\lambda+1})/\mathrm{Ker}(\Omega_\lambda)$ could become a C^*-algebra with continuous trace the result would be most desirable from the point of view of C^*-theory. The actual fact is however not in such a case; the ideal becomes the one similar to a C^*-algebra of continuous trace but different from such C^*-algebras ([**17**, Proposition 1, Theorem 1]).

As for the Hull of $L(\sigma)$, we obtain the folloing partial characterization. Consider the set

$$S_0 = \overline{\bigcup_{x \notin c(\sigma)} \partial O(x)} \cup \overline{c(\sigma) \backslash Per(\sigma)}.$$

We have then

Theorem 4.7. $\mathrm{Hull}(L(\sigma))$ *contains the set S_0.*
When X is metrizable, the equality holds, that is,

$$\mathrm{Hull}(L(\sigma)) = \overline{\bigcup_{x \notin c(\sigma)} \partial O(x)} \cup \overline{c(\sigma) \backslash Per(\sigma)}.$$

PROOF. Write $L(\sigma) = \mathrm{Ker}(S)$ for a closed invariant subset S. Then, by Theorem 4.3 S contains the set $c(\sigma)\backslash Per(\sigma)$ and moreover by the above Proposition it absorbes every boundary set of a non-recurrent point in its complement. For those points inside S, this naturally holds. Hence S contains S_0.

When X is metrizable, $\mathrm{Ker}(S_0)$ itself becomes a liminal ideal by Proposition 4.5 because S_0 satisfies the condition $(*)$. Therefore, it has to be the largest liminal ideal in $A(\Sigma)$ and $S_0 = \mathrm{Hull}(L(\sigma))$. □

As in the case of the ideal $K(\sigma)$, we meet here the same difficulty of countability assumption, which is concerned with the equivalency between topological transitivity and the dense orbit property in metrizable case.

We notice here that $\mathrm{Hull}(L(\sigma))$ may contain some periodic points besides the set of proper recurrent points. On the other hand, we see that the nonwandering set $\Omega(\sigma)$ satisfies the condition $(*)$ and $\mathrm{Ker}(\Omega(\sigma))$ becomes a liminal ideal provided that X is metrizable. Hence $\mathrm{Hull}(L(\sigma))$ is contained in $\Omega(\sigma)$. The difference between $\Omega(\sigma)$ and $\mathrm{Hull}(L(\sigma))$ becomes more clear if we consider the extreme case where X only consists of periodic points (such as the case of rational rotations). In fact, in this case $\Omega(\sigma) = X$ whereas $\mathrm{Hull}(L(\sigma))$ becomes empty. This last assertion holds however even if we drop the merizability assumption for X (cf. [**13**, Theorem 4.6 (1)]).

Now when the set $\overline{c(\sigma)\backslash Per(\sigma)}$ absorbes all boundary sets of non-recurrent points (in a metrizable sapce X), we have the situation,

$$S_0 = \overline{c(\sigma)\backslash Per(\sigma)}$$

and

$$L(\sigma) = K(\sigma) = \mathrm{Ker}(\overline{c(\sigma)\backslash Per(\sigma)}).$$

This holds, for instance, in the case of Denjoy homeomorphisms on the circle. Recall that a Denjoy homeomorphism is a homeomorphism having an irrational rotation number but not minimal. It is then well known that σ has a unique minimal invariant set S which absorbes the boundary of every orbit of a point in the complement of S. This means that S satisfies the condition $(*)$ in the above theorem, and $\mathrm{Ker}(S)$ becomes a liminal ideal. On the other hand, since σ does not have periodic points and S consists of proper recurrent points, we have that

$$L(\sigma) = K(\sigma) = \mathrm{Ker}(S).$$

This is a maximal ideal of $A(\Sigma)$ because its quotient algebra corresponds naturally to the simple homeomorphism algebra $A(\Sigma_S)$ with respect to the restricted dynamical system $\Sigma_S = (S, \sigma|S)$. We can however assert more. Namely, $\mathrm{Ker}(S)$ is the largest ideal among those ideals in $A(\Sigma)$. In fact, let P be the kernel of an irreducible representation $\tilde{\pi}$. Since here all irreducible representations are infinite dimensional, we can write it as $P = \mathrm{Ker}(\overline{O(x)})$ for a point x of T. As we have mentined above, the set $\overline{O(x)}$ should have nonempty intersection with S. It follows that $\overline{O(x)}$ contains the minimal set S and $P \subseteq \mathrm{Ker}(S)$.

Actually the readers may find the precise structure of this ideal in the article [**7**].

On the other hand, if σ is an orientation preserving homeomorphism with periodic points there is no proper recurrent points and the set, $Per(\sigma)$ $(= c(\sigma))$, becomes closed. Moreover, because of the orbit behavior of non-recurrent points we can see that $\mathrm{Hull}(L(\sigma))$ coincides really with the boundary set of $Per(\sigma)$. Namely we have that

$$\mathrm{Hull}(L(\sigma)) = Per(\sigma)\backslash Per(\sigma)^{\circ}.$$

Thus in this case,

$$A(\Sigma) = K(\sigma) \supsetneqq L(\sigma).$$

Next consider a trace τ on $A(\Sigma)$ and its left kernel,

$$L_{\tau} = \{a \in A(\Sigma) \mid \tau(a^*a) = 0\}.$$

Let $T(\sigma)$ be the intersection of all left kernels of traces on $A(\Sigma)$. This ideal is apparantly invariant by the dual action and we can write it as $T(\sigma) = \mathrm{Ker}(\mathrm{Hull}(T(\sigma)))$. We have then the following

Proposition 4.8. *The set* $\mathrm{Hull}(T(\sigma))$ *is the closure of the union of all supports of invariant probability measures in* X.
When X *is metrizable it is contained in the center* $\overline{c(\sigma)}$.

Proof. If μ is an invariant probability measure on X, $\mu \circ E$ is a trace on $A(\Sigma)$. Hence, if a belongs to $T(\sigma)$

$$\mu(E(a)^* E(a)) \leq \mu \circ E(a^*a) = 0.$$

As $E(a)$ is a continuous function, this implies that $E(a)$ vanishes on the support of μ. Hence $\mathrm{Hull}(T(\sigma))$ contains the union of all supports of invariant probability measures. On the other hand, a trace τ on $A(\Sigma)$ induces an invariant probability measure on X by the restriction of τ to $C(X)$. It follows that the set $\mathrm{Hull}(T(\sigma))$ coincides with the closure of the union of all supports of invariant probability measures.

The second assertion follows from Aoki's observation, [**1**, Remark 4.27], that in a compact metric space the set $c(\sigma)$ becomes a Borel set and $\mu(c(\sigma)) = 1$ for every invariant probability measure μ. □

Next let I_C be the commutater ideal of $A(\Sigma)$.

Proposition 4.9. $I_C = \mathrm{Ker}(Per_1(\sigma))$, *and the quotient algebra of* $A(\Sigma)$ *by* I_C *is canonically isomorphic to the tensor product* $C(Per_1(\sigma)) \otimes C(T)$.

Proof. As the ideal I_C is clearly invariant by the dual action, we can write it by Theorem 3.2 that $I_C = \mathrm{Ker}(S)$ for an invariant closed subet S. Moreover, S contains the set of fixed points $Per_1(\sigma)$ because I_C is contained in $\mathrm{Ker}(Per_1(\sigma))$. On the other hand, also by Theorem 3.2 (1) the quotient commutative algebra $A(\Sigma)/I_C$ is canonically isomorphic to the homeomorphism algebra $A(\Sigma_S)$. Hence the action of σ on S has to be trivial. Therefore, $S = Per_1(\sigma)$.

Once we have this identity the rest is easily seen from the fact that the quotient algebra of $A(\Sigma)$ by $\mathrm{Ker}(Per_1(\sigma))$ is regarded as the C^*-crossed product with respect to the trivial action. □

Acknowledgement. The author is deeply indebted to the referee whose advices with careful reading have brought considerable improvement of the understandability of the author's first draft.

References

[1] N. Aoki, Topological dynamics, Topics in General topology, (eds. K. Morita and J. Nagata), North-Holland, 1989, p. 625–740.

[2] N. Aoki and J. Tomiyama, Characterizations of topological dynamical systems whose transformation group C^*-algebras are antiliminal and of type 1, Ergodic Theory Dynam. Systems, **13** (1993), 1–5.

[3] J. Dixmier, C^*-algebras, North-Holland, 1977.

[4] E. G. Effros, Transformation groups and C^*-algebras, Ann. of Math., **81** (1965), 38–55.

[5] E. Gootman, The type of some C^*-algebras and W^*-algebras associated with transformation groups, Pacific J. Math., **48** (1973), 93–106.

[6] A. Katok and B. Husselblatt, Modern theory of dynamical systems, Cambridge Univ. Press, 1998.

[7] I. Putnam, K. Schmidt and C. Skau, C^*-algebaras associated with Denjoy homeomorphisms of the circle, J. Operator Theory, **16** (1986), 99–126.

[8] S. Sakai, On type 1 C^*-algebras, Proc. Amer. Math. Soc., **18** (1967), 861–863.

[9] S. Silvestrov, Representations of commutation relations, Ph.D thesis, Umear Univ., 1995.

[10] S. Silvestrov and J. Tomiyama, Topological dynamical systems of type 1, Expo. Math., **20** (2002), 117–142.

[11] M. Takesaki, Theory of operator algebras I, Springer, 1979.

[12] J. Tomiyama, Invitation to C^*-algebras and topological dynamical systems, No. 3 in series in Dynam. Sys. World Sci. Ltd., Singapore, 1987.

[13] J. Tomiyama, The interplay between topological dynamics and theory of C^*-algebras, Lecture note No. 2, Res. Inst. Math. Seoul, 1992.

[14] J. Tomiyama, C^*-algebras and topological dynamical systems, Rev. Math. Phys., **8** (1996), 741–760.

[15] J. Tomiyama, Representations of topological dynamical systems and C^*-algebras, Proc. Intern. Conf. on Operator Algebras and Operator Theory, Shanghai, 1997, Contemp. Math., **228** (1998), 351–364.

[16] J. Tomiyama, Structure of ideals and isomorphism problems of C^*-crossed products by single homeomorphisms, Tokyo J. Math., **23** (2000), 1–13.

[17] J. Tomiyama, Nonwandering sets of topological dynamical systems and C^*-algebras, Ergodic Theory Dynam. Systems, **23** (2003), 1611–1621.

[18] G. Zeller-Meier, Produit croisè d'une C^*-algèbre par un groupe d'automorphismes, J. Math. Pures Appl., **47** (1968), 102–239.

Jun Tomiyama

201,11-10 Nakane 1-chome
Meguro-ku, Tokyo 152-0031
Japan
E-mail: jtomiyama@fc.jwu.ac.jp