J. Math. Soc. Japan
Vol. 56, No. 2, 2004

On harmonic Hardy spaces and area integrals

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(Received Apr. 1, 2002)
(Revised Sept. 9, 2002)

Abstract. In this paper we prove several necessary and sufficient conditions for a harmonic function in the unit ball belong to $\mathscr{H}^p(B)$-Hardy harmonic space.

1. Introduction.

Throughout this paper $B(a,r) = \{x \in \boldsymbol{R}^n \mid |x-a| < r\}$ denotes the open ball centered at a of radius r, where $|x|$ denotes the norm of $x \in \boldsymbol{R}^n$ and B is the open unit ball in $\boldsymbol{R}^n$. $S = \partial B = \{x \in \boldsymbol{R}^n \mid |x| = 1\}$ is the boundary of B. Let dV denote the Lebesgue measure on $\boldsymbol{R}^n$, $d\sigma$ the surface measure on S, σ_n the surface area of S, dV_N the normalized Lebesgue measure on B, $d\sigma_N$ the normalized surface measure on S.

For $f \in C^1(B)$ we define the area integral by

$$\mathscr{A}(r,f) = \int_{rB} |\nabla f(x)|^2 \, dV_N(x), \quad r \in [0,1),$$

where $|\nabla f(x)| = (\sum_1^n |\partial f(x)/\partial x_i|^2)^{1/2}$, while

$$I_p(r) = \int_S |f(r\zeta)|^p \, d\sigma_N(\zeta).$$

Let $\mathscr{H}(B)$ denote the set of harmonic functions on B, $\mathscr{H}^p(B)$ denote the set of harmonic functions on B such that:

$$\|u\|_{\mathscr{H}^p(B)} = \sup_{0<r<1} \left(\int_S |u(r\zeta)|^p \, d\sigma_N(\zeta) \right)^{1/p} < +\infty.$$

Elements of $\mathscr{H}^p(B)$ theory can be found in [**1**, Chapter VI]. For elements of complex H^p theory see, for example, [**2**].

A function $f \in C^1(B)$ is said to be a Bloch function if

$$\|f\|_{\mathscr{B}} = \sup_{x \in B} (1-|x|) |\nabla f(x)| < +\infty.$$

The space of Bloch functions is denoted by $\mathscr{B}(B)$.

Let $p > 0$. A Borel function f, locally integrable on B, is said to be a $BMO_p(B)$ function if

$$\|f\|_{BMO_p} = \sup_{B(a,r) \subset B} \left(\frac{1}{V(B(a,r))} \int_{B(a,r)} |f(x) - f_{B(a,r)}|^p \, dV(x) \right)^{1/p} < +\infty$$

2000 *Mathematics Subject Classification*. Primary 31B05.

Key Words and Phrases. Hardy space, Lusin property, polar cap, area integral.

where the supremum is taken over all balls $B(a,r)$ in B, and $f_{B(a,r)}$ is the mean value of f over $B(a,r)$.

In [8] for $p \geq 1$, Muramoto proved that $\mathscr{B}(B) \cap \mathscr{H}(B)$ is isomorphic to $BMO_p(B) \cap \mathscr{H}(B)$ as Banach spaces. That paper inspired us to calculate exactly BMO_p norm for harmonic functions, which is theme of the [11].

In the proof of the main result in [11], we essentially proved a generalization of Hardy-Stein identity, see, for example, [5, p. 42]. This identity is included in the following lemma.

Lemma 1. *Let* $1 < p < +\infty$, $u \in \mathscr{H}(B)$, *then*

$$\int_S |u(r\zeta)|^p \, d\sigma_N(\zeta) = |u(0)|^p + \frac{p(p-1)}{n(n-2)} \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2 (|x|^{2-n} - r^{2-n}) \, dV_N(x), \quad n \geq 3. \tag{1}$$

We used this lemma in our investigations in [12]. In this note we continue investigate harmonic Hardy spaces $\mathscr{H}^p(B)$ using this lemma. In fact, we use the following corollary which is identity of Hardy-Stein type.

Corollary 1. *Let* $1 < p < +\infty$, $u \in \mathscr{H}(B)$, $r \in (0,1)$, $n \geq 3$, *then*

$$\frac{d}{dr} \int_S |u(r\zeta)|^p \, d\sigma_N(\zeta) = \frac{p(p-1)}{n} r^{1-n} \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2 \, dV_N(x). \tag{2}$$

In the case of holomorphic functions in $\boldsymbol{C}^n$, similar identity was proved in [13]. Another consequence of Lemma 1 is the following corollary (see [12, Theorem 1]).

Corollary 2. *Let* $1 < p < +\infty$ *and* $n \geq 3$. *A function* $u \in \mathscr{H}(B)$ *belongs to* $\mathscr{H}^p(B)$ *if and only if*

$$\int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|^2) \, dV_N(x) < +\infty.$$

In the sequel we keep our attention to the case $n \geq 3$. Analogous results hold in the case $n = 2$. Formulations and proofs of these results we leave to the reader.

We say that

$$con(\varsigma, \alpha) = \left\{ x \,|\, \cos\alpha \leq \frac{\langle \varsigma, \varsigma - x \rangle}{|\varsigma - x|} \leq 1 \right\}$$

is the cone with vertex at $\varsigma \in S$, axis coincident with the vector ς, and half-angle α. Let $con_0(z, \alpha)$ denote the cone with vertex at 0, axis coincident with the vector z, and half-angle α; that is

$$con_0(z, \alpha) = \left\{ y \,|\, \cos\alpha \leq \frac{\langle y, z \rangle}{|y|\,|z|} \leq 1 \right\}.$$

The cone $con_0(z, \alpha)$ determines a closed polar cap $cap(z, \alpha) = con_0(z, \alpha) \cap S$ having center z and spherical angle α.

Let $\mathscr{S}_\alpha(x)$ denote the Stoltz domain i.e.

$$\mathscr{S}_\alpha(x) = \left(con\left(\frac{x}{|x|}, \alpha\right) \cap con_0\left(x, \frac{\pi}{2} - \alpha\right) \right) \cup B(0, \sin\alpha).$$

Let G be a subdomain of the unit disk U in the complex plane such that the boundary of G has the only one point 1 in common with the unit circle. Assume that there exists $r_0 \in (0,1)$, depending on G, such that the intersection of G with each circle $\{|z| = r\}$, $r_0 < r < 1$, is of linear measure $r\phi(r)$, where

$$\liminf_{r \to 1} \frac{\phi(r)}{1-r} > 0$$

and

$$\limsup_{r \to 1} \frac{\phi(r)}{1-r} < \infty.$$

Let $\mathscr{G}$ be the family of all domains G of the type described above. A typical example of G is a triangular domain in U with one vertex at 1 which we call, for short, a triangular domain at 1. Denoting

$$G(\theta) = \{z \in U \mid e^{-i\theta} z \in G\}, \quad \theta \in [0, 2\pi],$$

we say that a holomorphic function f in U satisfies the p-Lusin property with respect to $G \in \mathscr{G}$ if

$$L_p(f, G, \theta) = \frac{p^2}{4} \iint_{G(\theta)} |f(z)|^{p-2} |f'(z)|^2 \, dxdy$$

is summable with respect to θ on $[0, 2\pi]$ $(0 < p < \infty)$. In [**14**] Yamashita proved the following theorem.

Theorem A. *Let f be a function holomorphic in the unit disc and let $0 < p < +\infty$. If $f \in H^p$, then f has the p-Lusin property with respect to a certain triangular domain of class $\mathscr{G}$. Conversely, if f has the p-Lusin property with respect to a certain triangular domain of class $\mathscr{G}$, then $f \in H^p$.*

Case $p = 2$ was previously considered by N. Lusin [**7**], and G. Piranian and W. Rudin [**10**, Theorem 1], by the coefficients of Taylor expansion of f about 0. Yamashita's proof is different from Lusin's, and Piranian-Rudin's, and it is based on Hardy-Stein identity.

First, we prove theorem of this type in the case of harmonic functions which are defined on B. Before we formulate this result we need a definition.

Definition 1. We say that u satisfies p-Lusin property with respect to Stoltz domain $S_\alpha(\zeta_0)$, $\zeta_0 \in S$, if

$$L_p(u, S_\alpha(\zeta)) = \int_{S_\alpha(\zeta)} (1 - |x|)^{2-n} |u(x)|^{p-2} |\nabla u(x)|^2 \, dV_N(x), \quad n \geq 3$$

is summable with respect $\zeta \in S$, $p \in (0, \infty)$.

Thus, it follows that $L_p(u, S_\alpha(\zeta)) < +\infty$ for almost every $\zeta \in S$.

The following theorem is a generalization of Theorem A.

Theorem 1. *Let $u \in \mathscr{H}(B)$, $p > 1$. If $u \in \mathscr{H}^p(B)$, then u has the p-Lusin property* *with respect to a certain Stoltz domain. Conversely, if u has the p-Lusin property with respect to a certain Stoltz domain, then $u \in \mathscr{H}^p(B)$.*

In [**6**] the authors proved the following theorem: Let f be holomorphic in U. Then, if $0 < p \leq 2$,

$$f \in H^p(U) \Rightarrow \int_0^1 A^{p/2}(r, f)\, dr < \infty,$$

while if $p \geq 2$,

$$\int_0^1 A^{p/2}(r, f)\, dr < \infty \Rightarrow f \in H^p(U),$$

where $A(r, f) = \int_{|z| \leq r} |f'(z)|^2\, dxdy$.

The main result in this paper is the following analogous, but slightly less perfect result for harmonic functions in the unit ball.

Theorem 2. *Let $u \in \mathscr{H}(B)$, and $\varepsilon > 0$. Then, if $p \in (1, 2]$,*

$$u \in \mathscr{H}^p(B) \Rightarrow \int_0^1 (1 - r)^{(n-2+\varepsilon)(2-p)/2} \mathscr{A}^{p/2}(r, u)\, dr < \infty,$$

while if $p > 2$,

$$\int_0^1 (1 - r)^{(n-2+\varepsilon)(2-p)/2} \mathscr{A}^{p/2}(r, u)\, dr < \infty \Rightarrow u \in \mathscr{H}^p(B).$$

Since complex analytic methods (the factorisation of zeros) are not available to establish sharp lower bounds for the integral average $I_p(r)$ in terms of the maximal function $M(r)$ there is an ε loss at one point.

2. Proof of Theorem 1.

In order to prove Theorem 1 we need an auxiliary result which is incorporated in the following lemma.

Lemma 2. *Let $S_\alpha(\zeta_0)$, $\zeta_0 \in S$ be a Stoltz domain in B, $\chi(x, \zeta)$ the characteristic function of $S_\alpha(\zeta)$, $\zeta \in S$, that is $\chi(x, \zeta) = 1$ if $x \in S_\alpha(\zeta)$ and $\chi(x, \zeta) = 0$ otherwise, and*

$$\phi(x) = \int_S \chi(x, \zeta)\, d\sigma(\zeta), \quad x \in B.$$

Then

$$\phi(x) \asymp c_\alpha (1 - |x|)^{n-1}.$$

Proof. Let $x \in B$ be fixed. Then $\phi(x)$ is the surface measure of a polar cap. Let β denote its half angle. By well known formula we have

$$\phi(x) = \sigma\left(cap\left(\frac{x}{|x|}, \beta\right)\right) = \sigma_{n-1} \int_0^\beta \sin^{n-2}\theta\, d\theta.$$

From this, for $\beta \in [0, \pi/2]$, we obtain

$$\frac{\sigma_{n-1}}{n-1}\left(\frac{2}{\pi}\right)^{n-2}\beta^{n-1} \le \sigma_{n-1}\int_0^\beta \sin^{n-2}\theta\,d\theta \le \frac{\sigma_{n-1}}{n-1}\beta^{n-1}.$$

Let c denote the side of the triangle in which other two sides have lengths 1 and $r = |x|$, and where the angle between c and the side which has length 1 is α. Then β is the angle between the sides with length 1 and r. Let us show that for x such that $|x| > \sin\alpha$, the inequalities

$$c_1\beta \le 1 - |x| \le c_2\beta, \tag{3}$$

hold, for some $c_1, c_2 > 0$.

By the cosine theorem we obtain

$$|c| = \frac{1 - r^2}{\cos\alpha + \sqrt{r^2 - \sin^2\alpha}}.$$

On the other hand by the sine theorem we have

$$\frac{\sin\beta}{(1 - r^2)/(\cos\alpha + \sqrt{r^2 - \sin^2\alpha})} = \frac{\sin\alpha}{r}.$$

Hence

$$\frac{1}{2}\tan\alpha \le \frac{\sin\beta}{1 - |x|} \le \frac{2}{\cos\alpha},$$

from which (3) follows in this case. For $x \in \overline{B(0, \sin\alpha)}$ the inequalities (3) are trivial. From which the result follows. □

Proof of Theorem 1. Let $S_\alpha(\zeta_0)$, $\zeta_0 \in S$ be a Stoltz domain in B and $\chi(x, \zeta)$ the characteristic function of $S_\alpha(\zeta)$, $\zeta \in S$. It is clear that

$$\phi(x) = \int_S \chi(x, \zeta)\,d\sigma(\zeta), \quad x \in B$$

is the surface measure of the set $\{\zeta \mid x \in S_\alpha(\zeta)\}$ and that $\phi(x)$ is a radial function i.e. $\phi(x) = \phi(|x|)$.

We have

$$\begin{aligned} L_p &= \int_S L_p(u, S_\alpha(\zeta))\,d\sigma(\zeta) \\ &= \int_B \left[\int_S \chi(x, \zeta)\,d\sigma(\zeta)\right](1 - |x|)^{2-n}|u(x)|^{p-2}|\nabla u(x)|^2\,dV_N(x). \end{aligned}$$

By Lemma 2 we have

$$\int_S \chi(x, \zeta)\,d\sigma(\zeta) \asymp c_\alpha(1 - |x|)^{n-1}.$$

Thus, integral L_p is equiconvergent to the integral

$$\int_B (1-|x|)|u(x)|^{p-2}|\nabla u(x)|^2\,dV_N(x).$$

By Corollary 2 we obtain our result. □

3. Proof of the main result.

We divide the proof of Theorem 2 into several steps. The following lemma is an inequality of Riesz-Fejér type, see [**3**], [**4**] and [**9**].

LEMMA 3. *Let* $u \in \mathscr{H}(B)$ *and* $\varepsilon > 0$. *Then*

$$\int_0^r (r-\rho)^{n-2+\varepsilon} M(\rho)^p\,d\rho \le c_{p,n,\varepsilon} r^{n-1+\varepsilon} I_p(r), \quad p > 1,$$

for some $c_{p,n,\varepsilon} > 0$, *which depends only on* p, n *and* ε, *and all* $r \in (0,1)$, *where*

$$M(r) = M(r,u) = \sup\{|u(x)| \mid |x| = r\}.$$

PROOF. We may suppose that $r = 1$ and $u(0) = 0$. By Poisson integral formula we have

$$u(x) = \int_{\partial B} \frac{1-|x|^2}{|x-\zeta|^n} u(\zeta)\,d\sigma_N(\zeta), \quad x \in B.$$

By Jensen's and Harnack's inequalities we obtain

$$|u(x)|^p \le \int_{\partial B} \frac{1-|x|^2}{|x-\zeta|^n}\,|u(\zeta)|^p\,d\sigma_N(\zeta) \le \frac{2\|u\|^p_{\mathscr{H}^p(B)}}{(1-|x|)^{n-1}},$$

i.e.

$$|u(x)|^p(1-|x|)^{n-1} \le 2\|u\|^p_{\mathscr{H}^p(B)}.$$

From this we obtain

$$M^p(\rho)(1-\rho)^{n-1} \le 2\|u\|^p_{\mathscr{H}^p(B)}, \quad \text{for } \rho \in (0,1).$$

Multiplying the last formula by $(1-\rho)^{-1+\varepsilon}$ and then integrating from 0 to 1 we obtain the desired inequality. □

LEMMA 4. *Let* $u \in \mathscr{H}(B)$, *and* $\varepsilon > 0$. *If* $p \in (1,2]$, *and* $u(0) = 0$,

$$I_p(r) \ge c_{p,n,\varepsilon} \int_0^r \left(\frac{\mathscr{A}(\rho,u)}{\rho^{n-1}}\right)^{p/2} (r-\rho)^{(n-2+\varepsilon)(2-p)/2}\,d\rho,$$

while if $p > 2$,

$$I_p(r) \le 2^{(p-2)/2}|u(0)|^{(p^2-p+2)/2} + c_{p,n,\varepsilon} \int_0^r \left(\frac{\mathscr{A}(\rho,u)}{\rho^{n-1}}\right)^{p/2} (r-\rho)^{(n-2+\varepsilon)(2-p)/2}\,d\rho,$$

for some $c_{p,n,\varepsilon} > 0$ *which depends only of* p, n *and* ε.

PROOF. If $p \in (1,2]$, then

$$\mathscr{A}(r,u) = \int_{rB} |\nabla u(x)|^2\, dV(x) \le M(r)^{2-p} \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2\, dV_N(x).$$

From this and by Corollary 1 we obtain

$$\frac{p(p-1)}{n} \mathscr{A}(r,u) \le M(r)^{2-p} r^{n-1} I_p'(r),$$

where $I_p'(r)$ is derivative of $I_p(r)$.

Multiplying by $(r-\rho)^{(n-2+\varepsilon)(2-p)/p}$ and applying again Hölder's inequality and Lemma 3 we obtain

$$\begin{aligned}
&\left(\frac{p(p-1)}{n}\right)^{p/2} \int_0^r \left(\frac{\mathscr{A}(\rho,u)}{\rho^{n-1}} (r-\rho)^{(n-2+\varepsilon)(2-p)/p}\right)^{p/2} d\rho \\
&\quad \le \int_0^r (r-\rho)^{(n-2+\varepsilon)(2-p)/2} M(\rho)^{(2-p)p/2} (I_p'(\rho))^{p/2}\, d\rho \\
&\quad \le \left(\int_0^r (r-\rho)^{n-2+\varepsilon} M(\rho)^p\, d\rho\right)^{(2-p)/2} (I_p(r) - I_p(0))^{p/2} \\
&\quad \le c_{p,n,\varepsilon}^{(2-p)/2} I_p(r),
\end{aligned}$$

as desired.

For $p > 2$, by Corollary 1 we have

$$\begin{aligned}
\frac{d}{dr} \int_S |u(r\zeta)|^p\, d\sigma_N(\zeta) &= \frac{p(p-1)}{n} r^{1-n} \int_{rB} |u(x)|^{p-2} |\nabla u(x)|^2\, dV_N(x) \\
&\le \frac{p(p-1)}{n} r^{1-n} M(r)^{p-2} \int_{rB} |\nabla u(x)|^2\, dV_N(x) \\
&= \frac{p(p-1)}{n} r^{1-n} M(r)^{p-2} \mathscr{A}(r,u).
\end{aligned}$$

By integration we obtain

$$I_p(r) \le |u(0)|^p + \frac{p(p-1)}{n} \int_0^r \rho^{1-n} M(\rho)^{p-2} \mathscr{A}(\rho,u)\, d\rho.$$

By Hölder's inequality and Lemma 3 we get

$$\begin{aligned}
I_p(r) &\le |u(0)|^p + \frac{p(p-1)}{n} \left(\int_0^r (r-\rho)^{n-2+\varepsilon} M(\rho)^p\, d\rho\right)^{(p-2)/p} \\
&\quad \times \left(\int_0^r \left(\frac{\mathscr{A}(\rho,u)}{\rho^{n-1}}\right)^{p/2} (r-\rho)^{(n-2+\varepsilon)(2-p)/2}\, d\rho\right)^{2/p} \\
&\le |u(0)|^p + \frac{p(p-1)}{n} c_{p,n,\varepsilon}^{(p-2)/p} I_p(r)^{(p-2)/p} \\
&\quad \times \left(\int_0^r \left(\frac{\mathscr{A}(\rho,u)}{\rho^{n-1}}\right)^{p/2} (r-\rho)^{(n-2+\varepsilon)(2-p)/2}\, d\rho\right)^{2/p}.
\end{aligned}$$

Hence

$$I_p(r)^{2/p} \le |u(0)|^{(p^2-p+2)/p} + \frac{p(p-1)}{n} c_{p,n,\varepsilon}^{(p-2)/p} \left(\int_0^r \left(\frac{\mathscr{A}(\rho,u)}{\rho^{n-1}} \right)^{p/2} (r-\rho)^{(n-2+\varepsilon)(2-p)/2} \, d\rho \right)^{2/p}.$$

From this the result follows in this case. □

Proof of Theorem 2. It is an easy consequence of Lemma 4. □

Corollary 3. *Let* $u \in \mathscr{H}^p(B)$ *and* $\varepsilon > 0$. *Then for* $p \in (1,2]$

$$\lim_{r\to 1} (1-r)^{((n-2+\varepsilon)(2-p)+2)/p} \mathscr{A}(r,u) = 0.$$

Proof. For $u \in \mathscr{H}^p(B)$, $p \in (1,2]$, it follows from Theorem 2 that

$$\int_0^1 (1-r)^{(n-2+\varepsilon)(2-p)/2} \mathscr{A}^{p/2}(r,u) \, dr < \infty.$$

Since $\mathscr{A}(r,u)$ is nondecreasing we have

$$\mathscr{A}(r,u)^{p/2} \int_r^1 (1-\rho)^{(n-2+\varepsilon)(2-p)/2} \, d\rho \le \int_r^1 (1-\rho)^{(n-2+\varepsilon)(2-p)/2} \mathscr{A}^{p/2}(\rho,u) \, d\rho$$

i.e.

$$(1-r)^{((n-2+\varepsilon)(2-p)+2)/2} \mathscr{A}(r,u)^{p/2} \le c_{p,n,\varepsilon} \int_r^1 (1-\rho)^{(n-2+\varepsilon)(2-p)/2} \mathscr{A}^{p/2}(\rho,u) \, d\rho \to 0,$$

as $r \to 1$, from which the result follows. □

A more precise result holds in the case of functions holomorphic in the unit disk, namely if $f \in H^p(U)$, $0 < p \le 2$, then

$$\lim_{r\to 1} (1-r)^{2/p} A(r,f) = 0;$$

see Theorem 2 in [**15**].

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