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# A role of Bargmann-Segal spaces in characterization and expansion of operators on Fock space

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**Abstract.** A rigged Hilbert space formalism is introduced to study Fock space operators. The symbols of continuous operators on a rigged Fock space are characterized in terms of Bargmann-Segal spaces and complex Gaussian integrals. In particular, characterizations of bounded operators and of operators of Hilbert-Schmidt class on the middle Fock space are obtained. As an application we establish an operator version of chaotic expansion (Wiener-Itô expansion) and describe a relation to the Fock expansion in terms of the Wick exponential of the number operator. As another application we discuss regularity property of a solution to a normal-ordered white noise differential equation generalizing a quantum stochastic differential equation.

## 1. Introduction.

A rigged Hilbert space approach [**3**], [**13**], widely accepted in various fields of mathematics and mathematical physics, is a powerful tool also in the study of operators on a (Boson) Fock space, e.g., [**4**], [**5**], [**15**], [**24**], [**25**], [**39**], [**40**]. In particular, a nuclear rigging (or also called a Gelfand triple) of a special type:

$$\mathscr{W} \subset \Gamma(\mathscr{H}_C) \subset \mathscr{W}^*$$

has been studied under the name of white noise theory or Hida calculus [**17**], [**19**], [**26**], [**27**], and much attention has been attracted also to the white noise operators $\mathscr{L}(\mathscr{W}, \mathscr{W}^*)$ in connection with quantum stochastic calculus and infinite dimensional harmonic analysis, see e.g., [**34**].

An important contribution of white noise theory is found in a series of characterization theorems of the $S$-transform:

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \quad \Phi \in \mathscr{W}^*,$$

and of the operator symbol:

$$\hat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \quad \Xi \in \mathscr{L}(\mathscr{W}, \mathscr{W}^*),$$

where $\phi_\xi$ is an exponential vector (or also called a coherent vector) and is defined by

$$\phi_\xi = \left(1, \frac{\xi}{1!}, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right).$$

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In fact, many variants of characterization theorems for the $S$-transforms and for the operator symbols have been obtained with a common feature: they are characterized as entire holomorphic functions on an infinite dimensional vector space having certain growth rates, see e.g., [**1**] and references cited therein for the $S$-transform; [**10**], [**33**] for the operator symbol; and [**22**] for a unification. However, since these characterizations depend heavily on the nuclearity of $\mathscr{W}$, elements in the Fock space $\Gamma(\mathscr{H}_{\boldsymbol{C}})$ itself or bounded operators on $\Gamma(\mathscr{H}_{\boldsymbol{C}})$ have not been characterized in terms of growth rates. A similar characterization for operator symbols is obtained in [**40**], where another type of rigged Fock space is constructed from entire vectors but is still a nuclear rigging.

Meanwhile, the complex Gaussian analysis has received some new understanding in connection with coherent state representations [**37**], [**38**] and the Bargmann-Segal space [**29**], [**43**], see also [**14**] for a historical survey and recent topics on loop spaces. In connection with the characterization theorems, a relation between the $S$-transform and the Segal-Bargmann transform was first pointed out in [**23**] and has been studied to some extent, see also [**28**]. Among others, the work of Grothaus, Kondratiev and Streit [**16**] should be drawn considerable attention: the $S$-transform of different classes of vectors are characterized by means of the Bargmann-Segal space and the complex Gaussian integrals. Their discussion is based on a more general rigging. Given a certain self-adjoint operator $K$ acting in a Hilbert space $\mathscr{H}_{\boldsymbol{C}}$, we first construct a rigged Hilbert space:

$$\begin{aligned}\mathscr{D}_\infty = \operatorname*{proj\,lim}_{p\to\infty} \mathscr{D}_p \subset \cdots \subset \mathscr{D}_p \subset \cdots \\ \subset \mathscr{D}_0 = \mathscr{H}_{\boldsymbol{C}} \subset \cdots \subset \mathscr{D}_{-p} \subset \cdots \subset \operatorname*{ind\,lim}_{p\to\infty} \mathscr{D}_{-p} = \mathscr{D}_\infty^*. \end{aligned} \tag{1.1}$$

Then, taking the Boson Fock space over the above Hilbert spaces, we obtain a rigged Fock space:

$$\begin{aligned}\mathscr{G}_\infty = \operatorname*{proj\,lim}_{p\to\infty} \mathscr{G}_p \subset \cdots \subset \mathscr{G}_p = \Gamma(\mathscr{D}_p) \subset \cdots \\ \subset \mathscr{G}_0 = \Gamma(\mathscr{H}_{\boldsymbol{C}}) \subset \cdots \subset \mathscr{G}_{-p} = \Gamma(\mathscr{D}_{-p}) \subset \cdots \subset \operatorname*{ind\,lim}_{p\to\infty} \mathscr{G}_{-p} = \mathscr{G}_\infty^*. \end{aligned} \tag{1.2}$$

These riggings are not necessarily nuclear so that their result [**16**] contains a characterization for the $S$-transform of $\Gamma(\mathscr{H}_{\boldsymbol{C}})$. Interesting examples of a non-nuclear rigging have been also discussed in [**21**], [**31**], [**42**].

It is therefore very natural to extend the idea of Grothaus, Kondratiev and Streit [**16**] and to characterize the symbols of several different classes of operators in the Fock space. We first note that the symbol $\Theta = \hat{\Xi}$ of $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ is uniquely extended to an entire function on $\mathscr{D}_p \times \mathscr{D}_{-q}$, where $p, q \in \boldsymbol{R}$. Thus, the question is to find additional condition for an entire function $\Theta : \mathscr{D}_p \times \mathscr{D}_{-q} \to \boldsymbol{C}$ to be the symbol of an operator. In fact, we describe such condition in terms of the complex Gaussian space $(\mathscr{N}_{\boldsymbol{C}}^*, v)$ and the Bargmann-Segal space $E^2(v)$. The main theorem (Theorem 5.2) claims that such an entire function $\Theta$ is the symbol of some operator $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ if and only if there exists a constant $C \geq 0$ such that

$$\left\| \sum_{i=1}^k a_i \Theta(\xi_i, K^q \cdot) \right\|_{E^2(v)} \leq C \left\| \sum_{i=1}^k a_i \phi_{\xi_i} \right\|_p \tag{1.3}$$

for any $k \geq 1$ and any choice of $\xi_i \in \mathscr{D}_p$ and $a_i \in \boldsymbol{C}$, $i = 1, \dots, k$. Usefulness of our main theorem is illustrated with two applications, where condition (1.3) is verified with no difficulty. Moreover, the symbol of an operator of Hilbert-Schmidt class $\Xi \in \mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)$ is also characterized in this line (Theorem 6.2).

As a first application of our main theorem we prove that each operator $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ admits an (operator version of) *chaotic expansion* or *Wiener-Itô expansion*:

$$\Xi = \sum_{l,m=0}^{\infty} I_{l,m}(K_{l,m}), \tag{1.4}$$

where $I_{l,m}(K_{l,m})$ is determined by

$$I_{l,m}(K_{l,m})\widehat{\ }(\xi, \eta) = \langle K_{l,m}\xi^{\otimes m}, \eta^{\otimes l} \rangle.$$

On the other hand, as is well known [**34**], the *Fock expansion* of $\Xi$ is given by

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(L_{l,m}), \tag{1.5}$$

where $\Xi_{l,m}(L_{l,m})$ is determined by

$$\Xi_{l,m}(L_{l,m})\widehat{\ }(\xi, \eta) = \langle L_{l,m}\xi^{\otimes m}, \eta^{\otimes l} \rangle e^{\langle \xi, \eta \rangle}.$$

It is noteworthy that the above two expansions (1.4) and (1.5) are related through the Wick (normal-ordered) exponential of the number operator $N$ in such a way that

$$I_{l,m}(K_{l,m}) = \text{wexp}(-N) \diamond \Xi_{l,m}(K_{l,m}).$$

Moreover, with this formula the chaotic expansion (1.4) becomes

$$\Xi = \sum_{l,m,n=0}^{\infty} \frac{(-1)^n}{n!} \Xi_{n+l,n+m}(I^{\otimes n} \otimes K_{l,m}), \quad \Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q). \tag{1.6}$$

The precise statements are given in Theorem 8.3 and its corollaries. The expansion (1.6) is a generalization of Attal [**2**], where only operators of Hilbert-Schmidt class are discussed as a quantum analogue of a multiple Wiener-Itô integral, see also [**30**].

The second application of our main theorem (Theorem 5.2) is found in a study of quantum stochastic differential equations. Along with our approach it is more natural to consider a normal-ordered white noise differential equation:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi, \quad \Xi(0) = I, \tag{1.7}$$

where $L_t$ is a given quantum stochastic process and $\diamond$ stands for the Wick product (or normal-ordered product). The solution $\Xi_t$ should be found in a space of white noise operators. If

$$L_t = L_1 a_t + L_2 a_t^* + L_3 a_t^* a_t + L_4 I,$$

where $a_t$ and $a_t^*$ are the annihilation and creation operators at a time point $t$, equation (1.7) becomes equivalent to a quantum stochastic differential equation:

$$d\Xi = (L_1\, dA_t + L_2\, dA_t^* + L_3\, d\Lambda_t + L_4)\Xi, \quad \Xi(0) = I,$$

for generalities see e.g., [**32**], [**41**]. A normal-ordered white noise differential equation (1.7), allowing to include more singular quantum noises such as higher powers of quantum white noises, is thus regarded as a natural generalization of a quantum stochastic differential equation. In a series of papers [**8**], [**9**], [**35**], [**36**] we proved several unique existence results, and in the recent paper [**10**] we sharpened the characterization theorem for operator symbols and found a weighted Fock space in which the solution acts. Now our characterization theorem (Theorem 5.2) offers a new method of examining regularity properties of a solution. For a particular coefficient $L_t$ we verify condition (1.3) and see that the solution lies in $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_{-q})$ for some $q \ge 0$. In other words, the solution is grasped as (unbounded) operators acting in the Boson Fock space $\mathscr{G}_{-q}$, which is different from the original $\Gamma(\mathscr{H}_{\boldsymbol{C}}) = \mathscr{G}_0$ unless $q = 0$. This result (Theorem 9.4) is also interesting for being free from the nuclearity of $\mathscr{W}$ that is required in the former works.

This paper is organized as follows: In Section 2 we review the basic construction of riggings of Fock space after [**3**], [**13**]. In Section 3 we mention the definitions of an exponential vector, $S$-transform and operator symbol. In Section 4 we introduce the Bargmann-Segal space after [**16**] and recall relationship between the $S$-transform and the duality transform (or Segal-Bargmann map). In Section 5 we prove characterization of a continuous operator in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$, $p, q \in \boldsymbol{R}$, in terms of the operator symbol and the Bargmann-Segal space. In Section 6 we investigate characterization of an operator of Hilbert-Schmidt class $\mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)$, $p, q \in \boldsymbol{R}$. In Section 7 an operator-version of chaotic expansion (Wiener-Itô expansion) is established with detailed argument of convergence. In Section 8 we mention relationship between the chaotic expansion and the Fock expansion. In Section 9 we study a normal-ordered white noise differential equation as an application of our main result.

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## 2. Construction of riggings of Fock space.

Let $\mathscr{H}$ be a real separable Hilbert space with norm $|\cdot|_0$ and inner product $\langle \cdot, \cdot \rangle$. The complexification is denoted by $\mathscr{H}_{\boldsymbol{C}} = \mathscr{H} + i\mathscr{H}$ whose norm is denoted by the same symbol. According to our convention, the inner product $\langle \cdot, \cdot \rangle$ is extended to a $\boldsymbol{C}$-bilinear form so that $|\xi|_0^2 = \langle \bar{\xi}, \xi \rangle$ for $\xi \in \mathscr{H}_{\boldsymbol{C}}$. Throughtout, to avoid confusion we do not use a specific symbol for the hermitian inner product of a complex Hilbert space. The Fock space over $\mathscr{H}_{\boldsymbol{C}}$, denoted by $\Gamma(\mathscr{H}_{\boldsymbol{C}})$, is by definition the space of all sequences $\phi = (f_n)_{n=0}^\infty$ where $f_n$ is a member of the $n$-fold symmetric tensor power $\mathscr{H}_{\boldsymbol{C}}^{\hat{\otimes} n}$ and

$$\|\phi\|_0^2 = \sum_{n=0}^\infty n! |f_n|_0^2 < \infty.$$

We denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the canonical $\boldsymbol{C}$-bilinear form on $\Gamma(\mathscr{H}_{\boldsymbol{C}})$ defined through $\langle \cdot, \cdot \rangle$.

By means of the standard method (see e.g., [**3**], [**13**]) we shall construct riggings of $\mathscr{H}_{\boldsymbol{C}}$ and of $\Gamma(\mathscr{H}_{\boldsymbol{C}})$. Let $K$ be a selfadjoint operator in $\mathscr{H}_{\boldsymbol{C}}$ with domain $\mathrm{Dom}(K)$ satisfying

(K1) $\inf \mathrm{Spec}(K) \geq 1$.

For each $p \geq 0$ the dense subspace $\mathrm{Dom}(K^p) \subset \mathscr{H}_{\boldsymbol{C}}$ becomes a Hilbert space equipped with the norm

$$|\xi|_p = |K^p \xi|_0, \quad \xi \in \mathrm{Dom}(K^p).$$

This Hilbert space is denoted by $\mathscr{D}_p = \mathscr{D}_p(K)$. From $\inf \mathrm{Spec}(K) \geq 1$ we see that $|\xi|_p \leq |\xi|_q$ for $0 \leq p \leq q$, and hence $\mathscr{D}_q \subset \mathscr{D}_p \subset \mathscr{D}_0 = \mathscr{H}_{\boldsymbol{C}}$. We then define a countable Hilbert space by

$$\mathscr{D}_\infty = \mathscr{D}_\infty(K) = \operatorname*{proj\,lim}_{p\to\infty} \mathscr{D}_p(K) = \bigcap_{p\geq 0} \mathscr{D}_p(K). \tag{2.1}$$

In other words, $\mathscr{D}_\infty(K)$ is the space of $C^\infty$-vectors for $K$ topologized in a natural way.

For $p \geq 0$ let $\mathscr{D}_{-p}$ be the completion of $\mathscr{H}_{\boldsymbol{C}}$ with respect to the norm $|\xi|_{-p} = |K^{-p}\xi|_0$. Then, we have $\mathscr{H}_{\boldsymbol{C}} = \mathscr{D}_0 \subset \mathscr{D}_{-p} \subset \mathscr{D}_{-q}$ for $0 \leq p \leq q$, and set

$$\mathscr{D}_{-\infty} = \mathscr{D}_{-\infty}(K) = \operatorname*{ind\,lim}_{p\to\infty} \mathscr{D}_{-p}(K) = \bigcup_{p\geq 0} \mathscr{D}_{-p}(K). \tag{2.2}$$

By the Riesz theorem the dual space of $\mathscr{H}_{\boldsymbol{C}}$ is identified with itself. More precisely, for $f \in \mathscr{H}_{\boldsymbol{C}}^*$ there exists a unique $\eta_f \in \mathscr{H}_{\boldsymbol{C}}$ such that $f(\xi) = \langle \eta_f, \xi \rangle$ for $\xi \in \mathscr{H}_{\boldsymbol{C}}$, and the map $f \mapsto \eta_f$ gives rise to an isometric isomorphism from $\mathscr{H}_{\boldsymbol{C}}^*$ onto $\mathscr{H}_{\boldsymbol{C}}$. By extending this isomorphism we identify the dual space of $\mathscr{D}_p$ with $\mathscr{D}_{-p}$. The canonical $\boldsymbol{C}$-bilinear form on $\mathscr{D}_{-p} \times \mathscr{D}_p$ is denoted by $\langle \cdot, \cdot \rangle$ for the compatibility. Moreover, it is known that the strong dual space of $\mathscr{D}_\infty$, denoted by $\mathscr{D}_\infty^*$, is identified with $\mathscr{D}_{-\infty}$ together with their topologies. Thus we come to a rigged Hilbert space:

$$\mathscr{D}_\infty = \operatorname*{proj\,lim}_{p\to\infty} \mathscr{D}_p(K) \subset \mathscr{D}_p(K) \subset \mathscr{H}_{\boldsymbol{C}} \subset \mathscr{D}_{-p}(K) \subset \operatorname*{ind\,lim}_{p\to\infty} \mathscr{D}_{-p} = \mathscr{D}_\infty^*. \tag{2.3}$$

We also note that for any $p, q \in \boldsymbol{R}$ the operator $K^{p-q}$ is naturally considered as an isometry from $\mathscr{D}_p(K)$ onto $\mathscr{D}_q(K)$.

We next construct a chain of Fock spaces over the rigged Hilbert space (2.3). For simplicity we set

$$\mathscr{G}_p = \Gamma(\mathscr{D}_p) = \Gamma(\mathscr{D}_p(K)), \quad p \in \boldsymbol{R}.$$

By definition, $\mathscr{G}_p$ is the space of sequences $\phi = (f_n)$ where $f_n \in \mathscr{D}_p^{\hat{\otimes} n}$ ($n$-fold symmetric tensor power of the Hilbert space $\mathscr{D}_p$) such that

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n! |f_n|_p^2 = \sum_{n=0}^{\infty} n! |(K^{\otimes n})^p f_n|_0^2 < \infty. \tag{2.4}$$

Then $\mathscr{G}_p$ becomes a Hilbert space with the norm defined in (2.4). Viewing an obvious inclusion relation: $\mathscr{G}_q \subset \mathscr{G}_p \subset \mathscr{G}_0 = \Gamma(\mathscr{H}_{\boldsymbol{C}}) \subset \mathscr{G}_{-p} \subset \mathscr{G}_{-q}$ for $0 \leq p \leq q$, we set

$$\mathscr{G}_\infty = \operatorname*{proj\,lim}_{p\to\infty} \mathscr{G}_p = \bigcap_{p\geq 0} \mathscr{G}_p, \quad \mathscr{G}_{-\infty} = \operatorname*{ind\,lim}_{p\to\infty} \mathscr{G}_{-p} = \bigcup_{p\geq 0} \mathscr{G}_{-p}. \tag{2.5}$$

Clearly, $\mathscr{G}_\infty$ becomes a countable Hilbert space equipped with the Hilbertian norms defined in (2.4), and $\mathscr{G}_\infty^*$, the strong dual space of $\mathscr{G}_\infty$, is identified with $\mathscr{G}_{-\infty}$. Thus we come to a rigged Fock space:

$$\mathscr{G}_\infty \subset \Gamma(\mathscr{H}_{\boldsymbol{C}}) \subset \mathscr{G}_\infty^*. \tag{2.6}$$

We use the same symbol $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ for the canonical $\boldsymbol{C}$-bilinear form on $\mathscr{G}_\infty^* \times \mathscr{G}_\infty$. Then,

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^\infty n! \langle F_n, f_n \rangle, \quad \Phi = (F_n) \in \mathscr{G}_\infty^*,\ \phi = (f_n) \in \mathscr{G}_\infty,$$

and the Schwartz inequality leads to:

$$|\langle\!\langle \Phi, \phi \rangle\!\rangle| \le \|\Phi\|_{-p} \|\phi\|_p.$$

We note that the norm of $\mathscr{G}_0 = \Gamma(\mathscr{H}_{\boldsymbol{C}})$ is given by $\|\phi\|_0^2 = \langle\!\langle \bar{\phi}, \phi \rangle\!\rangle$.

Proposition 2.1. *The countable Hilbert space $\mathscr{D}_\infty$ defined in* (2.1) *is nuclear if and only if $K^{-r}$ is of Hilbert-Schmidt class for some $r \ge 0$. Only in that case $\mathscr{G}_\infty$ defined in* (2.5) *is a nuclear space.*

Remark 2.2. If the selfadjoint operator $K$ satisfies the conditions: (i) $\inf \operatorname{Spec}(K) > 1$; and (ii) $K^{-r}$ is of Hilbert-Schmidt class for some $r \ge 0$, then the rigged Fock space (2.6) is called the *Hida-Kubo-Takenaka space* and provides the most prototype in the white noise distribution theory, see [**26**]. In this paper we do not assume these conditions, therefore, nuclearity of $\mathscr{G}_\infty$ is not assumed.

## 3. Exponential vectors, $S$-transform and operator symbol.

In general, a (formal) vector of the form:

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right)$$

is called an *exponential vector* or a *coherent vector*. From an obvious identity:

$$\|\phi_\xi\|_p^2 = \sum_{n=0}^\infty \frac{1}{n!} |\xi|_p^{2n} = e^{|\xi|_p^2}, \quad p \in \boldsymbol{R},$$

we obtain the following

Lemma 3.1. *Let $p \in \boldsymbol{R}$. An exponential vector $\phi_\xi$ belongs to $\mathscr{G}_p$ if and only if $\xi$ belongs to $\mathscr{D}_p$. In particular, $\phi_\xi$ belongs to $\Gamma(\mathscr{H}_{\boldsymbol{C}})$ if and only if $\xi$ belongs to $\mathscr{H}_{\boldsymbol{C}}$. Moreover, $\phi_\xi$ belongs to $\mathscr{G}_\infty$ (resp. $\mathscr{G}_\infty^*$) if and only if $\xi$ belongs to $\mathscr{D}_\infty$ (resp. $\mathscr{D}_\infty^*$).*

Moreover, by a standard argument we have

Lemma 3.2. *The exponential vectors $\{\phi_\xi; \xi \in \mathscr{D}_\infty\}$ span a dense subspace of $\mathscr{G}_\infty$, hence of $\mathscr{G}_p$ for all $p \in \boldsymbol{R}$ and of $\mathscr{G}_\infty^*$.*

Thus each $\Phi \in \mathscr{G}_\infty^*$ is uniquely specified by its values for the exponential vectors. We set

$$S\Phi(\xi) = \langle\!\langle \Phi, \phi_\xi \rangle\!\rangle, \quad \xi \in \mathscr{D}_\infty, \tag{3.1}$$

which is called the *S-transform* of $\Phi$ [**26**]. For $\Phi = (F_n) \in \mathscr{G}_\infty^*$ we have

$$S\Phi(\xi) = \sum_{n=0}^{\infty} n! \left\langle F_n, \frac{\xi^{\otimes n}}{n!} \right\rangle = \sum_{n=0}^{\infty} \langle F_n, \xi^{\otimes n} \rangle, \quad \xi \in \mathscr{D}_\infty. \tag{3.2}$$

Similarly, a continuous linear operator $\Xi \in \mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty^*)$ is uniquely specified by its matrix elements with respect to the exponential vectors. We set

$$\hat{\Xi}(\xi, \eta) = \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle, \quad \xi, \eta \in \mathscr{D}_\infty. \tag{3.3}$$

This $\boldsymbol{C}$-valued function $\hat{\Xi}$ defined on $\mathscr{D}_\infty \times \mathscr{D}_\infty$ is called the *symbol* of $\Xi$ [**33**]. The symbol is related to the $S$-transform in an obvious manner:

$$\hat{\Xi}(\xi, \eta) = S(\Xi\phi_\xi)(\eta) = S(\Xi^*\phi_\eta)(\xi), \quad \xi, \eta \in \mathscr{D}_\infty.$$

Now we note the following result, which is immediate from Lemma 3.1 and basic properties of entire functions on a Hilbert space.

PROPOSITION 3.3. *Let* $p, q \in \boldsymbol{R}$.

(1) *The S-transform of* $\Phi \in \mathscr{G}_p$ *is uniquely extended to an entire function on* $\mathscr{D}_{-p}$.

(2) *The symbol of* $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ *is uniquely extended to an entire function on* $\mathscr{D}_p \times \mathscr{D}_{-q}$.

## 4. Bargmann-Segal space.

We shall introduce the Bargmann-Segal space after Grothaus, Kondratiev and Streit [**16**]. In addition to (K1) we assume that the selfadjoint operator $K$ satisfies:

(K2) $K$ is a real operator, i.e., $K(\mathrm{Dom}(K) \cap \mathscr{H}) \subset \mathscr{H}$;

(K3) there is a real nuclear space $\mathscr{N}$ which is densely and continuously imbedded in $\mathscr{D}_\infty \cap \mathscr{H}$ and is kept invariant under $K$.

Thus we have

$$\mathscr{N} \subset \mathscr{D}_\infty \subset \mathscr{H} \subset \mathscr{D}_\infty^* \subset \mathscr{N}^*, \tag{4.1}$$

where the bilinear form on $\mathscr{N}^* \times \mathscr{N}$ is denoted by $\langle \cdot, \cdot \rangle$ again. Let $\mu_{1/2}$ be the Gaussian measure on $\mathscr{N}^*$ whose characteristic function is given by

$$\exp\left\{-\frac{1}{4}\langle \xi, \xi \rangle\right\} = \int_{\mathscr{N}^*} e^{i\langle x, \xi \rangle} \mu_{1/2}(dx), \quad \xi \in \mathscr{N}.$$

Define a probability measure $\nu$ on $\mathscr{N}_{\boldsymbol{C}}^* = \mathscr{N}^* + i\mathscr{N}^*$ in such a way that

$$\nu(dz) = \mu_{1/2}(dx) \times \mu_{1/2}(dy), \quad z = x + iy,\ x, y \in \mathscr{N}^*.$$

Following Hida [**18**] the probability space $(\mathscr{N}_{\boldsymbol{C}}^*, \nu)$ is called the (standard) *complex Gaussian space* associated with (4.1).

The *Bargmann-Segal space*, denoted by $E^2(\nu)$, is by definition the space of entire functions $g : \mathscr{H}_{\boldsymbol{C}} \to \boldsymbol{C}$ such that

$$\|g\|_{E^2(\nu)}^2 \equiv \sup_{P \in \mathscr{P}} \int_{\mathscr{N}_{\boldsymbol{C}}^*} |g(Pz)|^2 \nu(dz) < \infty, \tag{4.2}$$

where $\mathscr{P}$ is the set of all finite rank projections on $\mathscr{H}$ with ranges contained in $\mathscr{N}$. Note that $P \in \mathscr{P}$ is naturally extended to a continuous operator from $\mathscr{N}_{\boldsymbol{C}}^*$ into $\mathscr{H}_{\boldsymbol{C}}$ (in fact into $\mathscr{N}_{\boldsymbol{C}}$), which is denoted by the same symbol. The Bargmann-Segal space $E^2(\nu)$ is a Hilbert space with norm $\|\cdot\|_{E^2(\nu)}$, see also [**28**]. For $\phi = (f_n)_{n=0}^{\infty} \in \Gamma(\mathscr{H}_{\boldsymbol{C}})$ define

$$J\phi(\xi) = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, f_n \rangle, \quad \xi \in \mathscr{H}_{\boldsymbol{C}}, \tag{4.3}$$

where the right hand side converges uniformly on each bounded subset of $\mathscr{H}_{\boldsymbol{C}}$. Hence $J\phi$ becomes an entire function on $\mathscr{H}_{\boldsymbol{C}}$. Moreover, it is known (e.g., [**14**], [**16**], [**28**]) that $J$ becomes a unitary isomorphism from $\Gamma(\mathscr{H}_{\boldsymbol{C}})$ onto $E^2(\nu)$. Here we check only that $J$ is isometric. By definition

$$J\phi(Pz) = \sum_{n=0}^{\infty} \langle (Pz)^{\otimes n}, f_n \rangle = \sum_{n=0}^{\infty} \langle z^{\otimes n}, P^{\otimes n} f_n \rangle, \quad z \in \mathscr{N}_{\boldsymbol{C}}^*.$$

Then, using the orthogonal relation [**18**, Chapter 6], we easily obtain

$$\int_{\mathscr{N}_{\boldsymbol{C}}^*} |J\phi(Pz)|^2 \nu(dz) = \sum_{n=0}^{\infty} n! |P^{\otimes n} f_n|_0^2 = \|\Gamma(P)\phi\|_0^2, \tag{4.4}$$

from which we see that $J : \Gamma(\mathscr{H}_{\boldsymbol{C}}) \to E^2(\nu)$ is isometric; in fact,

$$\|J\phi\|_{E^2(\nu)}^2 = \sup_{p \in \mathscr{P}} \|\Gamma(P)\phi\|_0^2 = \|\phi\|_0^2, \quad \phi \in \Gamma(\mathscr{H}_{\boldsymbol{C}}).$$

Here we record a useful formula which follows from (4.4) and a simple identity $\langle\!\langle \phi, \phi_{Pz} \rangle\!\rangle = J\phi(Pz)$ for $\phi \in \Gamma(\mathscr{H}_{\boldsymbol{C}})$ and $z \in \mathscr{N}_{\boldsymbol{C}}^*$.

Lemma 4.1. *For $\phi \in \Gamma(\mathscr{H}_{\boldsymbol{C}})$ and $P \in \mathscr{P}$ it holds that*

$$\int_{\mathscr{N}_{\boldsymbol{C}}^*} |\langle\!\langle \phi, \phi_{Pz} \rangle\!\rangle|^2 \nu(dz) = \|\Gamma(P)\phi\|_0^2.$$

*Moreover,*

$$\sup_{P \in \mathscr{P}} \int_{\mathscr{N}_{\boldsymbol{C}}^*} |\langle\!\langle \phi, \phi_{Pz} \rangle\!\rangle|^2 \nu(dz) = \|\phi\|_0^2.$$

The map $J$ defined in (4.3) is called the *duality transform* and is related with the $S$-transform (3.1) in an obvious manner:

$$J\phi|_{\mathscr{D}_\infty} = S\phi, \quad \phi \in \Gamma(\mathscr{H}_{\boldsymbol{C}}),$$

which follows from (3.2) and (4.3).

Theorem 4.2 ([**16**]). *Let $p \in \boldsymbol{R}$. Then a $\boldsymbol{C}$-valued function $g$ on $\mathscr{D}_\infty$ is the $S$-transform of some $\Phi \in \mathscr{G}_p$ if and only if $g$ can be extended to a continuous function on $\mathscr{D}_{-p}$ and $g \circ K^p \in E^2(\nu)$.*

PROOF. Suppose we are given a $\boldsymbol{C}$-valued continuous function $g$ on $\mathscr{D}_{-p}$ such that $g \circ K^p \in E^2(\nu)$. In fact, $g$ is entire on $\mathscr{D}_{-p}$ since $K^p$ is an isometry from $\mathscr{H}_{\boldsymbol{C}}$ onto $\mathscr{D}_{-p}$. By the duality transform there exists $(f_n) \in \Gamma(\mathscr{H}_{\boldsymbol{C}})$ such that

$$g \circ K^p(\xi) = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, f_n \rangle, \quad \xi \in \mathscr{H}_{\boldsymbol{C}}.$$

Then, changing variables, we have

$$g(\xi) = \sum_{n=0}^{\infty} \langle (K^{-p})^{\otimes n} f_n, \xi^{\otimes n} \rangle, \quad \xi \in \mathscr{D}_{-p}.$$

Define $\Phi = ((K^{-p})^{\otimes n} f_n)$. Then by definition $\Phi \in \mathscr{G}_p$ and $S\Phi(\xi) = g(\xi)$ for $\xi \in \mathscr{D}_\infty$, see (3.2). Namely, $g|_{\mathscr{D}_\infty}$ is the $S$-transform of $\Phi \in \mathscr{G}_p$. The converse assertion is readily clear. □

During the above proof we have already established the following

PROPOSITION 4.3. *Let $p \in \boldsymbol{R}$ and $\Phi \in \mathscr{G}_p$. Then $S\Phi$ admits a continuous extension to $\mathscr{D}_{-p}$ and $S\Phi \circ K^p \in E^2(\nu)$. Moreover,*

$$\|\Phi\|_p = \|S\Phi \circ K^p\|_{E^2(\nu)}.$$

## 5. Characterization of bounded operators.

The symbols of continuous (equivalently, bounded) operators from $\mathscr{G}_p$ into $\mathscr{G}_q$ are characterized by means of the Bargmann-Segal space.

LEMMA 5.1. *Let $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ and put $\Theta = \hat{\Xi}$. Then, for any $k \ge 1$ and any choice of $\xi_i \in \mathscr{D}_p$ and $a_i \in \boldsymbol{C}$, $i = 1, 2, \dots, k$, it holds that*

$$\left\| \sum_{i=1}^{k} a_i \Theta(\xi_i, K^q \cdot) \right\|_{E^2(\nu)} = \left\| \sum_{i=1}^{k} a_i \Xi \phi_{\xi_i} \right\|_q. \tag{5.1}$$

PROOF. We first observe

$$\begin{aligned} \left\| \sum_{i=1}^{k} a_i \Theta(\xi_i, K^q \cdot) \right\|_{E^2(\nu)}^2 &= \sup_{P \in \mathscr{P}} \int_{\mathscr{N}_C^*} \left| \sum_{i=1}^{k} a_i \Theta(\xi_i, K^q P z) \right|^2 \nu(dz) \\ &= \sup_{P \in \mathscr{P}} \int_{\mathscr{N}_C^*} \left| \sum_{i=1}^{k} a_i \langle\langle \Xi \phi_{\xi_i}, \phi_{K^q P z} \rangle\rangle \right|^2 \nu(dz), \end{aligned} \tag{5.2}$$

which follws from the definitions (3.3) and (4.2). For simplicity we put $\psi = \sum_{i=1}^{k} a_i \phi_{\xi_i}$. Then (5.2) becomes

$$\begin{aligned} \sup_{P \in \mathscr{P}} \int_{\mathscr{N}_C^*} |\langle\langle \Xi\psi, \phi_{K^q P z} \rangle\rangle|^2 \nu(dz) &= \sup_{P \in \mathscr{P}} \int_{\mathscr{N}_C^*} |\langle\langle \Gamma(K^q) \Xi\psi, \phi_{P z} \rangle\rangle|^2 \nu(dz) \\ &= \|\Gamma(K^q)\Xi\psi\|_0^2 = \|\Xi\psi\|_q^2, \end{aligned} \tag{5.3}$$

where Lemma 4.1 is used. Then (5.1) follows from (5.2) and (5.3). □

Theorem 5.2. *Let $p, q \in \boldsymbol{R}$. A $\boldsymbol{C}$-valued function $\Theta$ on $\mathscr{D}_\infty \times \mathscr{D}_\infty$ is the symbol of some $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ if and only if*

(i) *$\Theta$ can be extended to an entire function on $\mathscr{D}_p \times \mathscr{D}_{-q}$;*

(ii) *there exists a constant $C \ge 0$ such that*

$$\left\| \sum_{i=1}^k a_i \Theta(\xi_i, K^q \cdot) \right\|_{E^2(\nu)} \le C \left\| \sum_{i=1}^k a_i \phi_{\xi_i} \right\|_p \tag{5.4}$$

*for any $k \ge 1$ and any choice of $\xi_i \in \mathscr{D}_p$ and $a_i \in \boldsymbol{C}$, $i = 1, \dots, k$.*

*In that case $\|\Xi\|_{\mathrm{OP}} \le C$.*

Proof. Suppose that $\Theta = \hat{\Xi}$ with $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$. As was mentioned in Proposition 3.3, condition (i) is obviously satisfied since $\Theta(\xi, \eta) = \langle\langle \Xi \phi_\xi, \phi_\eta \rangle\rangle$ is well-defined for $\xi \in \mathscr{D}_p$ and $\eta \in \mathscr{D}_{-q}$. As for (ii), we take $C \ge 0$ such that $\|\Xi \phi\|_q \le C \|\phi\|_p$. This choice is possible by assumption of $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$. Then (5.4) follows by Lemma 5.1.

We next prove the converse. Given $\xi \in \mathscr{D}_p$ we define a function $F_\xi : \mathscr{D}_{-q} \to \boldsymbol{C}$ by

$$F_\xi(\eta) = \Theta(\xi, \eta), \quad \eta \in \mathscr{D}_{-q}.$$

Then by condition (i) the function $F_\xi$ is entire on $\mathscr{D}_{-q}$. Moreover, $F_\xi \circ K^q \in E^2(\nu)$; in fact, by (ii) we have

$$\|F_\xi \circ K^q\|_{E^2(\nu)} = \|\Theta(\xi, K^q \cdot)\|_{E^2(\nu)} \le C \|\phi_\xi\|_p < \infty.$$

Then by Theorem 4.2, there exists a unique $\Phi_\xi \in \mathscr{G}_q$ such that $F_\xi = S\Phi_\xi$, i.e.,

$$S\Phi_\xi(\eta) = F_\xi(\eta) = \Theta(\xi, \eta). \tag{5.5}$$

Since the exponential vectors are linearly independent, a linear operator $\Xi$ is uniquely specified by

$$\Xi \phi_\xi = \Phi_\xi, \quad \xi \in \mathscr{D}_p. \tag{5.6}$$

Then we see from (5.5) that $\Theta = \hat{\Xi}$. Hence, to our goal, we need to show that $\Xi$ is extended to a continuous operator in $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$. Let $k \ge 1$ and take $\xi_i \in \mathscr{D}_p$ and $a_i \in \boldsymbol{C}$, $i = 1, 2, \dots, k$. By Lemma 4.1 we have

$$\begin{aligned}
\left\| \Xi \left( \sum_{i=1}^k a_i \phi_{\xi_i} \right) \right\|_q^2 &= \left\| \Gamma(K^q) \left( \sum_{i=1}^k a_i \Phi_{\xi_i} \right) \right\|_0^2 \\
&= \sup_{P \in \mathscr{P}} \int_{\mathscr{N}_C^*} \left| \left\langle\!\left\langle \Gamma(K^q) \left( \sum_{i=1}^k a_i \Phi_{\xi_i} \right), \phi_{Pz} \right\rangle\!\right\rangle \right|^2 \nu(dz) \\
&= \sup_{P \in \mathscr{P}} \int_{\mathscr{N}_C^*} \left| \left\langle\!\left\langle \sum_{i=1}^k a_i \Phi_{\xi_i}, \phi_{K^q P z} \right\rangle\!\right\rangle \right|^2 \nu(dz) \\
&= \sup_{P \in \mathscr{P}} \int_{\mathscr{N}_C^*} \left| \sum_{i=1}^k a_i \Theta(\xi_i, K^q P z) \right|^2 \nu(dz) \\
&= \left\| \sum_{i=1}^k a_i \Theta(\xi_i, K^q \cdot) \right\|_{E^2(\nu)}^2 .
\end{aligned}$$

Consequently, by (5.4) we have

$$\left\| \Xi\left(\sum_{i=1}^{k} a_i \phi_{\xi_i}\right)\right\|_q \le C\left\|\sum_{i=1}^{k} a_i \phi_{\xi_i}\right\|_p,$$

which proves that there exists $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ characterized by (5.6) since the exponential vectors $\{\phi_\xi; \xi \in \mathscr{D}_p\}$ span a dense subspace of $\mathscr{G}_p$. □

THEOREM 5.3. *A $\boldsymbol{C}$-valued function $\Theta$ defined on $\mathscr{D}_\infty \times \mathscr{D}_\infty$ is the symbol of an operator $\Xi \in \mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty^*)$ if and only if there exists some $p \in \boldsymbol{R}$ such that*
(i) *$\Theta$ can be extended to an entire function on $\mathscr{D}_p \times \mathscr{D}_p$;*
(ii) *there exists a constant $C \ge 0$ such that*

$$\left\|\sum_{i=1}^{k} a_i \Theta(\xi_i, K^{-p}\cdot)\right\|_{E^2(\nu)} \le C\left\|\sum_{i=1}^{k} a_i \phi_{\xi_i}\right\|_p$$

*for any $k \ge 1$ and any choice of $\xi_i \in \mathscr{D}_p$ and $a_i \in \boldsymbol{C}$, $i = 1, \dots, k$.*

PROOF. We see from general theory of countable Hilbert spaces that

$$\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty^*) = \bigcup_{p \ge 0} \mathscr{L}(\mathscr{G}_p, \mathscr{G}_{-p}).$$

Then the assertion is immediate from Theorem 5.2. □

THEOREM 5.4. *A $\boldsymbol{C}$-valued function $\Theta$ on $\mathscr{D}_\infty \times \mathscr{D}_\infty$ is the symbol of a bounded operator on $\Gamma(\mathscr{H}_{\boldsymbol{C}})$ if and only if*
(i) *$\Theta$ can be extended to an entire function on $\mathscr{H}_{\boldsymbol{C}} \times \mathscr{H}_{\boldsymbol{C}}$;*
(ii) *there exists a constant $C \ge 0$ such that*

$$\left\|\sum_{i=1}^{k} a_i \Theta(\xi_i, \cdot)\right\|_{E^2(\nu)} \le C\left\|\sum_{i=1}^{k} a_i \phi_{\xi_i}\right\|_0$$

*for any $k \ge 1$ and any choice of $\xi_i \in \mathscr{H}_{\boldsymbol{C}}$ and $a_i \in \boldsymbol{C}$, $i = 1, \dots, k$.*

PROOF. Immediate by specializing parameters in Theorem 5.2 as $p = q = 0$. □

Similarly the spaces $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_q)$ and $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty)$ are characterized as follows.

THEOREM 5.5. *A $\boldsymbol{C}$-valued function $\Theta$ on $\mathscr{D}_\infty \times \mathscr{D}_\infty$ is the symbol of an operator $\Xi \in \mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_q)$ with $q \in \boldsymbol{R}$ if and only if there exists $p \in \boldsymbol{R}$ satisfying conditions* (i) *and* (ii) *in Theorem 5.2. In particular, $\Theta$ is the symbol of an operator $\Xi \in \mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty)$ if and only if for any $q \ge 0$ there exists $p \in \boldsymbol{R}$ satisfying conditions* (i) *and* (ii) *in Theorem 5.2.*

## 6. Characterization of operators of Hilbert-Schmidt class.

Denote by $\mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q) \subset \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ the space of operators of Hilbert-Schmidt class. The Hilbert-Schmidt norm is denoted by $\|\cdot\|_{\mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)}$ or by $\|\cdot\|_{\mathrm{HS}}$ when there is no danger of confusion.

Lemma 6.1. *Let* $p \in \boldsymbol{R}$ *and let* $\Xi \in \mathscr{L}(\mathscr{G}_p, \Gamma(\mathscr{H}_{\boldsymbol{C}}))$. *If* $\sup_{P \in \mathscr{P}} \|\Gamma(P)\Xi\|_{\mathrm{HS}} < \infty$, *then* $\Xi$ *belongs to* $\mathscr{L}_2(\mathscr{G}_p, \Gamma(\mathscr{H}_{\boldsymbol{C}}))$ *and*

$$\|\Xi\|_{\mathrm{HS}} = \sup_{P \in \mathscr{P}} \|\Gamma(P)\Xi\|_{\mathrm{HS}}. \tag{6.1}$$

Proof. Let $\{P_m\}_{m=1}^{\infty} \subset \mathscr{P}$ be an increasing sequence of orthogonal projections converging strongly to the identity operator on $\mathscr{H}_{\boldsymbol{C}}$. Obviously, $\{\|\Gamma(P_m)\Xi\|_{\mathrm{HS}}\}$ is an increasing sequence, and by assumption $\lim_{m\to\infty} \|\Gamma(P_m)\Xi\|_{\mathrm{HS}} < \infty$. Let $\{\omega_n\}$ be a complete orthonormal basis of $\mathscr{G}_p$. Then, by the monotone convergence theorem we have

$$\lim_{m\to\infty} \|\Gamma(P_m)\Xi\|_{\mathrm{HS}}^2 = \lim_{m\to\infty} \sum_{n=1}^{\infty} \|\Gamma(P_m)\Xi\omega_n\|_0^2 = \sum_{n=1}^{\infty} \|\Xi\omega_n\|_0^2 = \|\Xi\|_{\mathrm{HS}}^2. \tag{6.2}$$

Hence $\Xi \in \mathscr{L}_2(\mathscr{G}_p, \Gamma(\mathscr{H}_{\boldsymbol{C}}))$. Furthermore, since $\|\Gamma(P)\Xi\|_{\mathrm{HS}} \le \|\Gamma(P)\|_{\mathrm{OP}} \|\Xi\|_{\mathrm{HS}} \le \|\Xi\|_{\mathrm{HS}}$, (6.1) follows from (6.2). □

Theorem 6.2. *Let* $p, q \in \boldsymbol{R}$. *A* $\boldsymbol{C}$*-valued function* $\Theta$ *on* $\mathscr{D}_\infty \times \mathscr{D}_\infty$ *is the symbol of an operator* $\Xi \in \mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)$ *if and only if*

(i) $\Theta$ *can be extended to an entire function on* $\mathscr{D}_p \times \mathscr{D}_{-q}$;

(ii) *there exists a non-negative, locally bounded function* $g$ *on* $\mathscr{D}_p$ *satisfying*

$$M^2 \equiv \sup_{P \in \mathscr{P}} \int_{\mathscr{N}_{\boldsymbol{C}}^*} g(K^{-p}Pz)^2 \nu(dz) < \infty \tag{6.3}$$

*and*

$$\|\Theta(\xi, K^q \cdot)\|_{E^2(\nu)} \le g(\xi), \quad \xi \in \mathscr{D}_p. \tag{6.4}$$

Proof. Suppose that $\Theta = \hat{\Xi}$ for some $\Xi \in \mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)$. Then condition (i) is obvious, see also Theorem 5.2. On the other hand, from Lemma 5.1 we see that

$$\|\Theta(\xi, K^q \cdot)\|_{E^2(\nu)} = \|\Xi\phi_\xi\|_q, \quad \xi \in \mathscr{D}_p.$$

It is sufficient to show that $g(\xi) = \|\Xi\phi_\xi\|_q \ge 0$ has the desired property. Since $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$, there exists $C \ge 0$ such that

$$g(\xi) = \|\Xi\phi_\xi\|_q \le C\|\phi_\xi\|_p = C\exp\left(\frac{1}{2}|\xi|_p^2\right).$$

Hence $g$ is bounded on every bounded subset of $\mathscr{D}_p$. We thus need only to show (6.3). Let $P \in \mathscr{P}$ and we take a complete orthonormal basis of $\mathscr{G}_{-q}$, say $\{\omega_n\}_{n=1}^{\infty}$. Then,

$$\begin{aligned} g(K^{-p}Pz)^2 &= \|\Xi\phi_{K^{-p}Pz}\|_q^2 = \sum_{n=1}^{\infty} |\langle\langle \Xi\phi_{K^{-p}Pz}, \omega_n \rangle\rangle|^2 \\ &= \sum_{n=1}^{\infty} |\langle\langle \Gamma(K^{-p})\Xi^*\omega_n, \phi_{Pz} \rangle\rangle|^2, \quad z \in \mathscr{N}_{\boldsymbol{C}}^*. \end{aligned}$$

Integrating over $\mathscr{N}_{\boldsymbol{C}}^*$ and applying Lemma 4.1, we come to

$$\sup_{P\in\mathscr{P}}\int_{\mathscr{N}_C^*} g(K^{-p}Pz)^2\nu(dz) = \sum_{n=1}^{\infty} \|\Gamma(K^{-p})\Xi^*\omega_n\|_0^2 = \sum_{n=1}^{\infty} \|\Xi^*\omega_n\|_{-p}^2 = \|\Xi^*\|_{\mathrm{HS}}^2, \quad (6.5)$$

which is finite since $\Xi^* \in \mathscr{L}_2(\mathscr{G}_{-q}, \mathscr{G}_{-p})$.

We next prove the converse. By a similar argument as in Theorem 5.2, for each $\xi \in \mathscr{D}_p$ there exists a unique $\Phi_\xi \in \mathscr{G}_q$ such that $F_\xi = S\Phi_\xi$, i.e.,

$$\langle\!\langle \Phi_\xi, \phi_\eta \rangle\!\rangle = S\Phi_\xi(\eta) = \Theta(\xi, \eta), \quad \xi \in \mathscr{D}_p,\ \eta \in \mathscr{D}_{-q}.$$

Moreover, from Proposition 4.3 and (6.4) we see that

$$\|\Phi_\xi\|_q = \|S\Phi_\xi \circ K^q\|_{E^2(\nu)} \le g(\xi), \quad \xi \in \mathscr{D}_p.$$

Now, we fix $\phi \in \mathscr{G}_{-q}$ and define a $\boldsymbol{C}$-valued function $G_\phi$ on $\mathscr{D}_p$ by

$$G_\phi(\xi) = \langle\!\langle \Phi_\xi, \phi \rangle\!\rangle, \quad \xi \in \mathscr{D}_p.$$

We shall show that $G_\phi \circ K^{-p} \in E^2(\nu)$. In view of

$$\begin{aligned} |G_\phi \circ K^{-p}(\xi)| &= |\langle\!\langle \Phi_{K^{-p}\xi}, \phi \rangle\!\rangle| \\ &\le \|\Phi_{K^{-p}\xi}\|_q \|\phi\|_{-q} \\ &\le g(K^{-p}\xi)\|\phi\|_{-q}, \quad \xi \in \mathscr{H}_{\boldsymbol{C}}, \end{aligned} \quad (6.6)$$

we see by assumption (6.3) that

$$\sup_{P\in\mathscr{P}}\int_{\mathscr{N}_C^*} |G_\phi \circ K^{-p}(Pz)|^2\nu(dz) \le M^2\|\phi\|_{-q}^2 < \infty. \quad (6.7)$$

In order to prove that $\xi \mapsto G_\phi \circ K^{-p}(\xi)$, $\xi \in \mathscr{H}_{\boldsymbol{C}}$, is entire it is sufficient to verify that $\lambda \mapsto G_\phi(\lambda\xi + \xi')$ is holomorphic on $\boldsymbol{C}$ for any $\xi, \xi' \in \mathscr{D}_p$ since $G_\phi \circ K^{-p}$ is locally bounded by (6.6), see e.g., [**12**]. Let $V$ be a space spanned by the exponential vectors $\{\phi_\eta; \eta \in \mathscr{D}_{-q}\}$. Then, obviously, $\lambda \mapsto G_\phi(\lambda\xi + \xi')$ is holomorphic for any choice of $\phi \in V$, $\xi, \xi' \in \mathscr{D}_p$. For an arbitrary $\phi \in \mathscr{G}_{-q}$ choose an approximating sequence $\{\phi_k\} \subset V$. Since $g$ is bounded on every bounded subset of $\mathscr{D}_q$, we can easily see that the functions $G_{\phi_k}(\lambda\xi + \xi')$ of $\lambda \in \boldsymbol{C}$ converge to $G_\phi(\lambda\xi + \xi')$ uniformly on every compact subset of $\boldsymbol{C}$. Therefore $\lambda \mapsto G_\phi(\lambda\xi + \xi')$ is holomorphic on $\boldsymbol{C}$, and consequently, $G_\phi \circ K^{-p} \in E^2(\nu)$.

By Theorem 4.2 there exists a unique $\Psi_\phi \in \mathscr{G}_{-p}$ such that $G_\phi = S\Psi_\phi$, i.e.,

$$\langle\!\langle \Psi_\phi, \phi_\xi \rangle\!\rangle = S\Psi_\phi(\xi) = G_\phi(\xi) = \langle\!\langle \Phi_\xi, \phi \rangle\!\rangle, \quad \xi \in \mathscr{D}_p. \quad (6.8)$$

Moreover, by (6.7) we have

$$\|\Psi_\phi\|_{-p} = \|G_\phi \circ K^{-p}\|_{E^2(\nu)} \le M\|\phi\|_{-q}.$$

In other words, $\phi \mapsto \Psi_\phi$ is a continuous linear operator from $\mathscr{G}_{-q}$ into $\mathscr{G}_{-p}$. Its adjoint operator is denoted by $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$. Then it follows from (6.8) that $\Xi\phi_\xi = \Phi_\xi$ and $\Theta = \hat{\Xi}$. We shall verify that $\Xi \in \mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)$, or equivalently $\Xi^* \in \mathscr{L}_2(\mathscr{G}_{-q}, \mathscr{G}_{-p})$. Let $\{\omega_n\}$ be a complete orthonormal basis of $\mathscr{G}_{-q}$ and $P \in \mathscr{P}$. Then,

$$\sum_{n=1}^{\infty}\int_{\mathcal{N}_C^*}|\langle\!\langle \Gamma(K^{-p})\Xi^*\omega_n, \phi_{Pz}\rangle\!\rangle|^2\nu(dz) = \sum_{n=1}^{\infty}\int_{\mathcal{N}_C^*}|\langle\!\langle \omega_n, \Xi\phi_{K^{-p}Pz}\rangle\!\rangle|^2\nu(dz)$$
$$= \int_{\mathcal{N}_C^*}\|\Xi\phi_{K^{-p}Pz}\|_q^2\nu(dz) = \int_{\mathcal{N}_C^*}\|\Phi_{K^{-p}Pz}\|_q^2\nu(dz)$$
$$\le \int_{\mathcal{N}_C^*} g(K^{-p}Pz)^2\nu(dz) \le M^2,$$

and, with the help of Lemma 4.1 we obtain

$$\|\Gamma(P)\Gamma(K^{-p})\Xi^*\|^2_{\mathcal{L}_2(\mathcal{G}_{-q},\Gamma(\mathcal{H}_C))} = \sum_{n=1}^{\infty}\|\Gamma(P)\Gamma(K^{-p})\Xi^*\omega_n\|_0^2 \le M^2.$$

Therefore, by Lemma 6.1, $\Gamma(K^{-p})\Xi^* \in \mathcal{L}_2(\mathcal{G}_{-q},\Gamma(\mathcal{H}_C))$ and hence $\Xi^* \in \mathcal{L}_2(\mathcal{G}_{-q},\mathcal{G}_{-p})$. □

Proposition 6.3. *If* $\Xi \in \mathcal{L}_2(\mathcal{G}_p,\mathcal{G}_q)$, *we have*

$$\|\Xi\|_{\mathrm{HS}}^2 = \sup_{P\in\mathcal{P}}\int_{\mathcal{N}_C^*}\|\Xi\phi_{K^{-p}Pz}\|_q^2\nu(dz)$$
$$= \sup_{P\in\mathcal{P}}\sup_{Q\in\mathcal{P}}\int_{\mathcal{N}_C^*}\int_{\mathcal{N}_C^*}|\hat{\Xi}(K^{-p}Pz, K^qQw)|^2\nu(dw)\nu(dz).$$

Proof. The first equality is immediate from (6.5). Let $\{Q_m\}\subset\mathcal{P}$ be an increasing sequence of orthogonal projections converging strongly to the identity operator on $\mathcal{H}_C$. Then we can easily prove that

$$\int_{\mathcal{N}_C^*}\|\Xi\phi_{K^{-p}Pz}\|_q^2\nu(dz) \ge \sup_{Q\in\mathcal{P}}\int_{\mathcal{N}_C^*}\|\Gamma(Q)\Gamma(K^q)\Xi\phi_{K^{-p}Pz}\|_0^2\nu(dz)$$
$$\ge \lim_{m\to\infty}\int_{\mathcal{N}_C^*}\|\Gamma(Q_m)\Gamma(K^q)\Xi\phi_{K^{-p}Pz}\|_0^2\nu(dz)$$
$$= \int_{\mathcal{N}_C^*}\|\Xi\phi_{K^{-p}Pz}\|_q^2\nu(dz),$$

where we used the monotone convergence theorem for the last equality. It follows that

$$\int_{\mathcal{N}_C^*}\|\Xi\phi_{K^{-p}Pz}\|_q^2\nu(dz) = \sup_{Q\in\mathcal{P}}\int_{\mathcal{N}_C^*}\|\Gamma(Q)\Gamma(K^q)\Xi\phi_{K^{-p}Pz}\|_0^2\nu(dz).$$

Hence in view of Lemma 4.1 the second equality holds. □

## 7. Chaotic expansion of operators.

Let $m\ge 0$ and $p\in\mathbf{R}$. Let $S_m$ denote the projection (symmetrizing operator) from $\mathcal{D}_p^{\otimes m}$ onto $\mathcal{D}_p^{\hat{\otimes} m}$. Furthermore, we define an injection $I_m \in \mathcal{L}(\mathcal{D}_p^{\hat{\otimes} m},\mathcal{G}_p)$ by

$$I_mF_m = (0,\dots,0,F_m,0,\dots), \quad F_m\in\mathcal{D}_p^{\hat{\otimes} m}, \tag{7.1}$$

where $F_m$ stays at the $m$-th position. The symbols $S_m$ and $I_m$ are used commonly for all $p \in \boldsymbol{R}$. We denote by $I_m^*$ the operator in $\mathscr{L}(\mathscr{G}_p, \mathscr{D}_p^{\hat{\otimes} m})$ defined by

$$I_m^*(F_0, \dots, F_{m-1}, F_m, \dots) = m! F_m. \tag{7.2}$$

Obviously, each $\Phi = (F_m) \in \mathscr{G}_p$ admits an expression:

$$\Phi = \sum_{m=0}^{\infty} I_m F_m,$$

which is referred to as the *chaotic expansion* or the *Wiener-Itô decomposition* of $\Phi$. In this section we study its operator version.

We start with the following

Proposition 7.1. *Let $p, q \in \boldsymbol{R}$. For each $K_{l,m} \in \mathscr{L}(\mathscr{D}_p^{\hat{\otimes} m}, \mathscr{D}_q^{\hat{\otimes} l})$ there exists a unique operator $I_{l,m}(K_{l,m}) \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ such that*

$$I_{l,m}(K_{l,m})\widehat{\ }(\xi, \eta) = \langle K_{l,m} \xi^{\otimes m}, \eta^{\otimes l} \rangle, \quad \xi, \eta \in \mathscr{D}_\infty. \tag{7.3}$$

*In this case, $\| I_{l,m}(K_{l,m}) \|_{\mathrm{OP}} \le \sqrt{l! m!} \| K_{l,m} \|_{\mathrm{OP}}$.*

Proof. For simplicity we denote the right hand side of (7.3) by $\Theta(\xi, \eta)$. Obviously, $\Theta$ is naturally extended to an entire function on $\mathscr{D}_p \times \mathscr{D}_{-q}$. To show condition (ii) in Theorem 5.2 take $\xi_i \in \mathscr{D}_p$, $a_i \in \boldsymbol{C}$, $i = 1, \dots, k$. By definition we have

$$\left\| \sum_{i=1}^k a_i \Theta(\xi_i, K^q \cdot) \right\|_{E^2(\nu)}^2 = \sup_{P \in \mathscr{P}} \int_{\mathscr{N}_C^*} \left| \sum_{i=1}^k a_i \Theta(\xi_i, K^q P z) \right|^2 \nu(dz). \tag{7.4}$$

For simplicity we put $\psi = \sum_{i=1}^k a_i \phi_{\xi_i}$. Using the identity:

$$\begin{aligned} \left| \sum_{i=1}^k a_i \Theta(\xi_i, K^q P z) \right|^2 &= |\langle\!\langle I_l K_{l,m} I_m^* \psi, \phi_{K^q P z} \rangle\!\rangle|^2 \\ &= |\langle\!\langle \Gamma(K^q) I_l K_{l,m} I_m^* \psi, \phi_{P z} \rangle\!\rangle|^2, \end{aligned}$$

which is verified by direct computation, we see that (7.4) becomes

$$\left\| \sum_{i=1}^k a_i \Theta(\xi_i, K^q \cdot) \right\|_{E^2(\nu)}^2 = \| \Gamma(K^q) I_l K_{l,m} I_m^* \psi \|_0^2 = \| I_l K_{l,m} I_m^* \psi \|_q^2.$$

In view of (7.1) and (7.2) we have

$$\| I_l K_{l,m} I_m^* \psi \|_q^2 = l! | K_{l,m} I_m^* \psi |_q^2 \le l! \| K_{l,m} \|_{\mathrm{OP}}^2 | I_m^* \psi |_p^2 \le l! m! \| K_{l,m} \|_{\mathrm{OP}}^2 \| \psi \|_p^2.$$

Consequently,

$$\left\| \sum_{i=1}^k a_i \Theta(\xi_i, K^q \cdot) \right\|_{E^2(\nu)} \le \sqrt{l! m!} \| K_{l,m} \|_{\mathrm{OP}} \| \psi \|_p.$$

Then the assertion is immediate by application of Theorem 5.2. □

Lemma 7.2. *If* $K_{l,m} \in \mathscr{L}(\mathscr{D}_p^{\hat{\otimes} m}, \mathscr{D}_q^{\hat{\otimes} l})$, *it holds that*

$$K_{l,m} = \frac{1}{l!m!} I_l^* I_{l,m}(K_{l,m}) I_m.$$

Proof. It is sufficient to show that

$$\langle K_{l,m}\xi^{\otimes m}, \eta^{\otimes l}\rangle = \frac{1}{l!m!}\langle I_l^* I_{l,m}(K_{l,m}) I_m \xi^{\otimes m}, \eta^{\otimes l}\rangle, \quad \xi, \eta \in \mathscr{D}_\infty,$$

which is immediate from (7.3). □

For a general $K_{l,m} \in \mathscr{L}(\mathscr{D}_p^{\otimes m}, \mathscr{D}_q^{\otimes l})$ we may define $I_{l,m}(K_{l,m})$ by the same formula (7.3). However, in that case the uniqueness of $K_{l,m}$ is not guaranteed; in fact, we have $I_{l,m}(K_{l,m}) = I_{l,m}(S_l K_{l,m} S_m)$.

For each $m \ge 0$ define a map by

$$\Pi_m = \frac{1}{m!} I_m I_m^* : (F_0, F_1, \dots, F_m, \dots) \mapsto (0, 0, \dots, 0, F_m, 0, \dots).$$

Obviously, $\Pi_m$ is an orthogonal projection on $\mathscr{G}_p$ for all $p \in \boldsymbol{R}$. Note also that $I_m^* I_m = m!$. Given $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$, we define

$$K_{l,m} = \frac{1}{l!m!} I_l^* \Xi I_m, \quad l, m \ge 0. \tag{7.5}$$

Then $K_{l,m} \in \mathscr{L}(\mathscr{D}_p^{\hat{\otimes} m}, \mathscr{D}_q^{\hat{\otimes} l})$ and by symbol calculus we have

$$I_{l,m}(K_{l,m}) = \Pi_l \Xi \Pi_m. \tag{7.6}$$

Since $\sum_{m=0}^\infty \Pi_m = I$ converges with respect to the strong operator topology of $\mathscr{L}(\mathscr{G}_p, \mathscr{G}_p)$, we can deduce the *chaotic expansion* of $\Xi$ given as in (7.7) below.

Theorem 7.3. *For any* $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ *there exists a unique family of operators* $K_{l,m} \in \mathscr{L}(\mathscr{D}_p^{\hat{\otimes} m}, \mathscr{D}_q^{\hat{\otimes} l})$, $l, m \ge 0$, *such that*

$$\Xi = \sum_{l,m=0}^\infty I_{l,m}(K_{l,m}), \tag{7.7}$$

*where the series converges weakly in the sense that*

$$\langle\!\langle \Xi\phi, \psi\rangle\!\rangle = \sum_{l,m=0}^\infty \langle\!\langle I_{l,m}(K_{l,m})\phi, \psi\rangle\!\rangle, \quad \phi \in \mathscr{G}_p, \ \psi \in \mathscr{G}_{-q}.$$

Since $\mathscr{D}_\infty$ is not necessarily a nuclear space (it is so if and only if $K^{-p}$ is of Hilbert-Schmidt class for some $p > 0$),

$$\mathscr{D}_\infty^{\hat{\otimes} m} = \bigcap_{p \ge 0} \mathscr{D}_p^{\hat{\otimes} m}$$

is taken as definition. With these notation we have

Corollary 7.4. *Let* $\Xi \in \mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty^*)$ *with the chaotic expansion as in* (7.7). *Then,* $K_{l,m} \in \mathscr{L}(\mathscr{D}_\infty^{\hat{\otimes} m}, (\mathscr{D}_\infty^{\hat{\otimes} l})^*)$. *If* $\Xi$ *belongs to* $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty)$ *or to* $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_p)$ *for some* $p \in \boldsymbol{R}$, *then* $K_{l,m}$ *belongs to* $\mathscr{L}(\mathscr{D}_\infty^{\hat{\otimes} m}, \mathscr{D}_\infty^{\hat{\otimes} l})$ *or to* $\mathscr{L}(\mathscr{D}_\infty^{\hat{\otimes} m}, \mathscr{D}_p^{\hat{\otimes} l})$, *respectively.*

We next consider operators of Hilbert-Schmidt class.

Lemma 7.5. *Let* $K_{l,m} \in \mathscr{L}(\mathscr{D}_p^{\hat{\otimes} m}, \mathscr{D}_q^{\hat{\otimes} l})$ *for some* $p, q \in \boldsymbol{R}$. *Then* $K_{l,m} \in \mathscr{L}_2(\mathscr{D}_p^{\hat{\otimes} m}, \mathscr{D}_q^{\hat{\otimes} l})$ *if and only if* $I_{l,m}(K_{l,m}) \in \mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)$. *In that case* $\|I_{l,m}(K_{l,m})\|_{\mathrm{HS}} = \sqrt{l!m!}\|K_{l,m}\|_{\mathrm{HS}}$.

Proof. Let $\{\omega_k\}$ and $\{\zeta_{k'}\}$ be complete orthonormal bases of $\mathscr{D}_p^{\hat{\otimes} m}$ and $\mathscr{D}_{-q}^{\hat{\otimes} l}$, respectively. Then $\{(m!)^{-1/2} I_m \omega_k\}$ and $\{(l!)^{-1/2} I_l \zeta_{k'}\}$ become complete orthonormal bases of $\Pi_m \mathscr{G}_p$ and $\Pi_l \mathscr{G}_{-q}$, respectively. By definition

$$\|I_{l,m}(K_{l,m})\|_{\mathrm{HS}}^2 = \sum_{k,k'} \frac{1}{l!m!} |\langle\!\langle I_{l,m}(K_{l,m}) I_m \omega_k, I_l \zeta_{k'} \rangle\!\rangle|^2.$$

On the other hand, by Lemma 7.2 we have

$$\|K_{l,m}\|_{\mathrm{HS}}^2 = \sum_{k,k'} |\langle K_{l,m}\omega_k, I_l \zeta_{k'} \rangle|^2 = \sum_{k,k'} \frac{1}{(l!m!)^2} |\langle I_l^* I_{l,m}(K_{l,m}) I_m \omega_k, I_l \zeta_{k'} \rangle|^2.$$

Therefore, $\|I_{l,m}(K_{l,m})\|_{\mathrm{HS}} = \sqrt{l!m!}\|K_{l,m}\|_{\mathrm{HS}} < \infty$. □

Theorem 7.6. *Let* $p, q \in \boldsymbol{R}$. *Given* $\Xi \in \mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)$ *let*

$$\Xi = \sum_{l,m=0}^{\infty} I_{l,m}(K_{l,m}), \tag{7.8}$$

*be the chaotic expansion. Then* $K_{l,m} \in \mathscr{L}_2(\mathscr{D}_p^{\hat{\otimes} m}, \mathscr{D}_q^{\hat{\otimes} l})$ *and the right hand side of* (7.8) *converges in* $\mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)$.

Proof. Since $\Xi$ is of Hilbert-Schmidt class, so is $I_{l,m}(K_{l,m})$ by (7.6). Then it follows by Lemma 7.5 that $K_{l,m} \in \mathscr{L}_2(\mathscr{D}_p^{\hat{\otimes} m}, \mathscr{D}_q^{\hat{\otimes} l})$. Since $\{\Pi_m\}$ is a resolution of the identity, in view of (7.6) we have

$$\|\Xi\|_{\mathrm{HS}}^2 = \sum_{l,m=0}^{\infty} \|\Pi_l \Xi \Pi_m\|_{\mathrm{HS}}^2 = \sum_{l,m=0}^{\infty} \|I_{l,m}(K_{l,m})\|_{\mathrm{HS}}^2.$$

This shows that the chaotic expansion (7.8) converges in $\mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)$. □

## 8. Relation with Fock expansion.

In this section, in stead of (K1) we assume a stronger condition:

(K4) $\inf \mathrm{Spec}(K) > 1$, i.e., $\rho \equiv \|K^{-1}\|_{\mathrm{OP}} < 1$.

We first modify the definition of an integral kernel operator [**34**] according to our present framework. Let $p, q \in \boldsymbol{R}$ and suppose we are given $K_{l,m} \in \mathscr{L}(\mathscr{D}_p^{\hat{\otimes} m}, \mathscr{D}_q^{\hat{\otimes} l})$. For $\phi = (f_n)$ define $\Xi_{l,m}(K_{l,m})\phi = (g_n)$ by

$$g_n = 0, \quad 0 \le n < l; \qquad g_{l+n} = \frac{(n+m)!}{n!} S_{l+n}(K_{l,m} \otimes I^{\otimes n}) f_{n+m}, \quad n \ge 0,$$

where $S_{l+m}$ stands for the symmetrizing operator. By virtue of direct norm estimates similar to [**34**, Section 4.3] we obtain

$$\|\Xi_{l,m}(K_{l,m})\phi\|_q \le \|K_{l,m}\|_{\mathrm{OP}}\rho^{r(m-1/2)}(l^l m^m)^{1/2}\left(\frac{\rho^{-r/2}}{-re\log\rho}\right)^{(l+m)/2}\|\phi\|_{(p\vee q)+r}, \tag{8.1}$$

where $p \vee q = \max\{p, q\}$, $r > 0$ is arbitrary. Therefore, $\Xi_{l,m}(K_{l,m}) \in \mathscr{L}(\mathscr{G}_{(p\vee q)+r}, \mathscr{G}_q)$. Such an operator $\Xi_{l,m}(K_{l,m})$ is called an *integral kernel operator*. The symbol is given by

$$\Xi_{l,m}(K_{l,m})\widehat{\ }(\xi,\eta) = \langle K_{l,m}\xi^{\otimes m}, \eta^{\otimes l}\rangle e^{\langle\xi,\eta\rangle}.$$

For $\Xi \in \mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty^*)$ an expression of the form

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(K_{l,m})$$

is called the *Fock expansion* and has been studied in the context of white noise theory [**6**], [**7**], [**34**], where nuclearity is important. Here we do not go into this direction.

The famous number operator $N$ is uniquely specified by the relation $NI_m = mI_m$ for $m \ge 0$, or equivalently by the action

$$N : (F_0, F_1, F_2, \dots, F_m, \dots) \mapsto (0, F_1, 2F_2, \dots, mF_m, \dots),$$

and admits a simple form of an integral kernel operator:

$$N = \Xi_{1,1}(I), \quad I\text{: identity operator on } \mathscr{D}_p.$$

The number operator $N$ is not a bounded operator on any $\mathscr{G}_p$ but a continuous operator from $\mathscr{G}_{p+q}$ into $\mathscr{G}_p$ for any $p \in \boldsymbol{R}$ and $q > 0$.

We next consider the Wick exponential of $-N$. In general, for two integral kernel operators $\Xi_{l_1,m_1}(K)$ and $\Xi_{l_2,m_2}(L)$ their *Wick product* or *normal-ordered product* is defined by

$$\Xi_{l_1,m_1}(K) \diamond \Xi_{l_2,m_2}(L) = \Xi_{l_1+l_2,m_1+m_2}(K \otimes L).$$

With these notation the Wick exponential of $-N$ is introduced:

$$\mathrm{wexp}(-N) = \sum_{n=0}^{\infty}\frac{1}{n!}(-N)^{\diamond n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Xi_{n,n}(I^{\otimes n}). \tag{8.2}$$

Lemma 8.1. *For any* $p \in \boldsymbol{R}$ *and* $q > 0$ *with* $\rho^{q/2}/(-q\log\rho) < 1$ *the series* (8.2) *converges absolutely in* $\mathscr{L}(\mathscr{G}_{p+q}, \mathscr{G}_p)$. *In particular*, $\mathrm{wexp}(-N) \in \mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty)$. *Moreover, for all* $p, q \in \boldsymbol{R}$, *the Wick exponential* $\mathrm{wexp}(-N)$ *is extended to an operator in* $\mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q)$.

Proof. Note first that $N^{\diamond n} = \Xi_{n,n}(I^{\otimes n})$ and observe from (8.1) that for any $p \in \boldsymbol{R}$ and $q > 0$,

$$\|N^{\diamond n}\phi\|_p \le n^n\rho^{-q/2}\left(\frac{\rho^{q/2}}{-qe\log\rho}\right)^n\|\phi\|_{p+q}.$$

Then, using $n^n \le e^n n!$ we easily check the convergence of (8.2). We next note that

$$\widehat{\mathrm{wexp}(-N)}(\xi,\eta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle\langle \Xi_{n,n}(I^{\otimes n})\phi_\xi, \phi_\eta \rangle\rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle \xi^{\otimes n}, \eta^{\otimes n} \rangle e^{\langle \xi,\eta \rangle} = 1.$$

Since a constant function obviously fulfills the conditions in Theorem 6.2, the last assertion follows immediately. □

In other words, there exists an operator $\Xi \in \mathscr{L}(\mathscr{G}_\infty^*, \mathscr{G}_\infty)$ such that

$$\hat{\Xi}(\xi,\eta) \equiv 1, \quad \Xi|_{\mathscr{G}_p} \in \mathscr{L}_2(\mathscr{G}_p, \mathscr{G}_q), \quad p, q \in \boldsymbol{R},$$

and $\Xi$ admits the Fock expansion as in (8.2) converging in $\mathscr{L}(\mathscr{G}_{p+q}, \mathscr{G}_p)$. Remark that

$$N^{\diamond n} = \Xi_{n,n}(I^{\otimes n}) = N(N-1)(N-2)\cdots(N-n+1).$$

Note also that $N^{\diamond n}$, $n \ge 1$, is not a bounded operator on any $\mathscr{G}_p$, $p \in \boldsymbol{R}$.

Proposition 8.2. *Let $p, q \in \boldsymbol{R}$ and $K_{l,m} \in \mathscr{L}(\mathscr{D}_p^{\hat{\otimes} m}, \mathscr{D}_q^{\hat{\otimes} l})$. Then*

$$I_{l,m}(K_{l,m}) = \mathrm{wexp}(-N) \diamond \Xi_{l,m}(K_{l,m}) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Xi_{n+l,n+m}(I^{\otimes n} \otimes K_{l,m}), \tag{8.3}$$

*where the series converges in $\mathscr{L}(\mathscr{G}_{(p\vee q)+r}, \mathscr{G}_q)$ for any $r > 0$ with $(2\rho^{r/2})/(-r\log\rho) < 1$. Hence,*

$$\Xi_{l,m}(K_{l,m}) = \mathrm{wexp}(N) \diamond I_{l,m}(K_{l,m}). \tag{8.4}$$

Proof. As was mentioned at the beginning of this section, $\Xi_{l,m}(K_{l,m}) \in \mathscr{L}(\mathscr{G}_{(p\vee q)+r}, \mathscr{G}_q)$ for any $r > 0$ and $I^{\otimes n} \otimes K_{l,m} \in \mathscr{L}(\mathscr{D}_{p\vee q}^{\otimes(n+m)}, \mathscr{D}_q^{\otimes(n+l)})$. Then in view of inequalities $n^n \le e^n n!$ and $(n+m)! \le 2^{n+m} n! m!$ we see that for any $\phi \in \mathscr{G}_{(p\vee q)+r}$,

$$\|\Xi_{n+l,n+m}(I^{\otimes n} \otimes K_{l,m})\phi\|_q \le C_{l,m} n! \left(\frac{2\rho^{r/2}}{-r\log\rho}\right)^n \|\phi\|_{(p\vee q)+r},$$

where

$$C_{l,m} = \sqrt{l!m!}\|K_{l,m}\|_{\mathrm{OP}} \rho^{rm-r/2} \left(\frac{2\rho^{-r/2}}{-r\log\rho}\right)^{(l+m)/2}.$$

Therefore, for any $r > 0$ with $(2\rho^{r/2})/(-r\log\rho) < 1$ we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|\Xi_{n+l,n+m}(I^{\otimes n} \otimes K_{l,m})\phi\|_q \le C_{l,m} \|\phi\|_{(p\vee q)+r} \sum_{n=0}^{\infty} \left(\frac{2\rho^{r/2}}{-r\log\rho}\right)^n < \infty,$$

from which the assertion follows. For (8.4) we need only to note that $\mathrm{wexp}(N) \diamond \mathrm{wexp}(-N) = \mathrm{wexp}(0) = I$ and $I$ is also the identity for the Wick product. □

Inserting (8.3) into the chaotic expansion (7.7), we obtain

$$\Xi = \sum_{l,m,n=0}^{\infty} \frac{(-1)^n}{n!} \Xi_{n+l,n+m}(I^{\otimes n} \otimes K_{l,m}). \tag{8.5}$$

As for the convergence, we mention the following

Theorem 8.3. *Let* $\Xi \in \mathscr{L}(\mathscr{G}_p, \mathscr{G}_q)$ *with some* $p, q \in \boldsymbol{R}$, *and let* $\Xi = \sum_{l,m=0}^{\infty} I_{l,m}(K_{l,m})$ *be the chaotic expansion. Then the expansion* (8.5) *converges in* $\mathscr{L}(\mathscr{G}_{(p\vee q)+r}, \mathscr{G}_{q-s})$ *for any* $r > 0$ *and* $s > 0$ *satisfying* $\rho^r/(-r\log\rho) < 1$ *and* $\rho^s/(-s\log\rho) < 1$.

Proof. From (7.5) we see that

$$\|K_{l,m}\|_{\mathrm{OP}} \le \frac{1}{\sqrt{l!m!}}\|\Xi\|_{\mathrm{OP}}, \quad l, m \ge 0. \tag{8.6}$$

On the other hand, by a standard way similar to [**34**, Section 4.3] we see that for any $r, s > 0$ and $\phi \in \mathscr{G}_{(p\vee q)+r}$

$$\|\Xi_{l+n,m+n}(I^{\otimes n} \otimes K_{l,m})\phi\|_{q-s} \le \|K_{l,m}\|_{\mathrm{OP}}\rho^{r(m+n)+s(l+n)-(r+s)/2} \times \left(\frac{(n+l)\rho^{-s}}{-2se\log\rho}\right)^{(n+l)/2}\left(\frac{(n+m)\rho^{-r}}{-2re\log\rho}\right)^{(n+m)/2}\|\phi\|_{(p\vee q)+r}.$$

Therefore, by (8.6) we have

$$\|\Xi_{n+l,n+m}(I^{\otimes n} \otimes K_{l,m})\phi\|_{q-s} \le \|\Xi\|_{\mathrm{OP}}\rho^{-(r+s)/2}n!\left(\frac{\rho^s}{-s\log\rho}\right)^{(n+l)/2}\left(\frac{\rho^r}{-r\log\rho}\right)^{(m+n)/2}\|\phi\|_{(p\vee q)+r}.$$

Hence

$$\sum_{l,m,n=0}^{\infty}\frac{1}{n!}\|\Xi_{n+l,n+m}(I^{\otimes n} \otimes K_{l,m})\phi\|_{q-s} \le \|\Xi\|_{\mathrm{OP}}\|\phi\|_{(p\vee q)+r}\rho^{-(r+s)/2} \times \sum_{l,m,n=0}^{\infty}\left(\frac{\rho^s}{-s\log\rho}\right)^{l/2}\left(\frac{\rho^r}{-r\log\rho}\right)^{m/2}\left\{\left(\frac{\rho^s}{-s\log\rho}\right)\left(\frac{\rho^r}{-r\log\rho}\right)\right\}^{n/2},$$

where the last series converges by assumption. Thus, the right hand side of (8.5) converges in $\mathscr{L}(\mathscr{G}_{(p\vee q)+r}, \mathscr{G}_{q-s})$ for any $r, s > 0$ satisfying $\rho^r/(-r\log\rho) < 1$ and $\rho^s/(-s\log\rho) < 1$. □

Corollary 8.4. *If* $\Xi$ *belongs to* $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_p)$ *for some* $p \in \boldsymbol{R}$, *then the expansion* (8.5) *converges in* $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_{p-r})$ *for any* $r > 0$ *satisfying* $\rho^r/(-r\log\rho) < 1$.

Corollary 8.5. *If* $\Xi$ *belongs to* $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty^*)$ *or to* $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_\infty)$, *then the expansion* (8.5) *converges in the respective spaces.*

An expansion of the form (8.5) is a generalization of the quantum analogue of multiple Wiener-Itô integrals due to Attal [**2**], where only operators of Hilbert-Schmidt class are considered by means of Maassen-Meyer kernel calculus [**32**]. Moreover, in quantum stochastic calculus we start with a particular Hilbert space, for example, $\mathscr{H}_{\boldsymbol{C}} = L^2(\boldsymbol{R}, dt)$ where $\boldsymbol{R}$ stands for a time axis. Then, operators discussed above admit more descriptive expressions and comparison with the multiple Wiener-Itô integrals [**2**] is more straightforward.

Take $\mathscr{H}_{\boldsymbol{C}} = L^2(\boldsymbol{R}, dt)$ and $K = A = 1 + t^2 - d^2/dt^2$. Then $\mathscr{D}_\infty(A) = \mathscr{S}(\boldsymbol{R})$, the Schwartz space of rapidly decreasing functions. In this case we usually write $(E) = \mathscr{G}_\infty$. For each $t \in \boldsymbol{R}$ the annihilation operator $a_t$ is uniquely specified by $a_t\phi_\xi = \xi(t)\phi_\xi$, $\xi \in \mathscr{D}_\infty(A)$, and becomes a continuous operator in $\mathscr{L}((E),(E))$. The creation operator $a_t^* \in \mathscr{L}((E)^*,(E)^*)$ is its adjoint. Then an integral kernel operator is expressed in a formal integral:

$$\Xi_{l,m}(K_{l,m}) = \int_{\boldsymbol{R}^{l+m}} \kappa_{l,m}(s_1,\dots,s_l,t_1,\dots,t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m}\, ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

where $\kappa_{l,m}$ is the kernel of the operator $K_{l,m}$, for more details see e.g., [**34**]. Similarly, we may write

$$\begin{aligned}&\Xi_{n+l,n+m}(I^{\otimes n} \otimes K_{l,m})\\ &\qquad = \int_{\boldsymbol{R}^{l+m}} \kappa_{l,m}(s_1,\dots,s_l,t_1,\dots,t_m) a_{s_1}^* \cdots a_{s_l}^* N^{\diamond n} a_{t_1} \cdots a_{t_m}\, ds_1 \cdots ds_l dt_1 \cdots dt_m\end{aligned}$$

and

$$N^{\diamond n} = \int_{\boldsymbol{R}^n} a_{s_1}^* \cdots a_{s_n}^* a_{s_1} \cdots a_{s_n}\, ds_1 \cdots ds_n.$$

On the other hand,

$$I_{l,m}(K_{l,m}) = \int_{\boldsymbol{R}^{l+m}} \kappa_{l,m}(s_1,\dots,s_l,t_1,\dots,t_m) a_{s_1}^* \cdots a_{s_l}^* \Pi_0 a_{t_1} \cdots a_{t_m}\, ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

where $\Pi_0$ is the vacuum projection. A similar expression appears also in [**20**].

## 9. Normal-ordered white noise differential equations.

In this section we take $\mathscr{H}_{\boldsymbol{C}} = L^2(\boldsymbol{R}, dt)$, $\mathscr{N} = \mathscr{S}(\boldsymbol{R})$ and a selfadjoint operator $K$ satisfying conditions (K2)–(K4). In general, a normal-ordered white noise differential equation takes a form:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi, \quad \Xi(0) = I, \tag{9.1}$$

where $\{L_t\}$ is a quantum stochastic process defined over a time interval $T$ containing 0, i.e., $L_t$ is an operator acting in the Boson Fock space $\Gamma(\mathscr{H}_{\boldsymbol{C}})$. To give a definite meaning to (9.1) we need a particular rigged Fock space, for example, a CKS-space [**11**] is convenient:

$$\mathscr{W}_\alpha \subset \Gamma(\mathscr{H}_{\boldsymbol{C}}) \subset \mathscr{W}_\alpha^*,$$

where $\alpha$ is a certain weight sequence. A quantum stochastic process is by definition a continuous map $t \mapsto L_t \in \mathscr{L}(\mathscr{W}_\alpha, \mathscr{W}_\alpha^*)$. Moreover, it is known that the Wick product introduced in the previous section is extended to a separately continuous bilinear map from $\mathscr{L}(\mathscr{W}_\alpha, \mathscr{W}_\alpha^*) \times \mathscr{L}(\mathscr{W}_\alpha, \mathscr{W}_\alpha^*)$ into $\mathscr{L}(\mathscr{W}_\alpha, \mathscr{W}_\alpha^*)$. Thus equation (9.1) is given a definite meaning.

Since the Wick product is commutative and (9.1) is a simple linear equation, a formal solution is given by the Wick exponential:

$$\Xi_t = \mathrm{wexp}\left(\int_0^t L_s\, ds\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int_0^t L_s\, ds\right)^{\diamond n}. \tag{9.2}$$

It is known that there is a unique solution to (9.1) in $\mathscr{L}(\mathscr{W}_\beta, \mathscr{W}_\beta^*)$ with a suitable choice of another weight sequence $\beta$. It occurs generally that $\mathscr{W}_\beta \subset \mathscr{W}_\alpha$ and $\mathscr{L}(\mathscr{W}_\alpha, \mathscr{W}_\alpha^*) \subset \mathscr{L}(\mathscr{W}_\beta, \mathscr{W}_\beta^*)$. Roughly speaking, unique existence of a solution is guaranteed always in the sense of "distributions." In fact, we have obtained further detailed properties of a solution, see [**10**] and references cited therein. The methods so far employed are, however, rather limited: by direct norm estimates of (9.2) or by the characterization theorem for operator symbols in terms of growth rates and its refinements.

We now illustrate how our characterization theorem (Theorem 5.2) gives a third method of finding a space in which the solution acts as an operator; in other words, regularity property of a solution (9.2) which makes already sense as a "distribution." First note that the symbol $\Theta_t = \hat{\Xi}_t$ of (9.2) is given by

$$\Theta_t(\xi,\eta) = e^{\langle \xi,\eta\rangle} \exp\{e^{-\langle \xi,\eta\rangle} \hat{M}_t(\xi,\eta)\}, \quad \xi,\eta \in \mathscr{D}_\infty, \tag{9.3}$$

where

$$M_t = \int_0^t L_s\, ds, \quad t \in T.$$

Then, applying Theorem 5.5, we obtain the following

Proposition 9.1. *A solution $\Xi_t$ to* (9.1) *lies in $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_q)$ with some $q \in \boldsymbol{R}$, if $\Theta_t$ defined in* (9.3) *satisfies the following conditions*:

(i) *there exists $p \in \boldsymbol{R}$ such that $\Theta_t$ can be extended to an entire function on $\mathscr{D}_p \times \mathscr{D}_{-q}$*;

(ii) *there exists a constant $C \ge 0$ such that*

$$\left\| \sum_{i=1}^k a_i \Theta_t(\xi_i, K^q \cdot)\right\|_{E^2(\nu)} \le C \left\| \sum_{i=1}^k a_i \phi_{\xi_i}\right\|_p \tag{9.4}$$

*for any $k \ge 1$ and any choice of $\xi_i \in \mathscr{D}_p$ and $a_i \in \boldsymbol{C}$, $i = 1, \dots, k$.*

To discuss a more concrete example we need some preliminaries. For each $\kappa \in \mathscr{D}_\infty^*$ there is $p \ge 0$ such that $\kappa \in \mathscr{D}_{-p}$ and the map $K : \xi \mapsto \langle \kappa, \xi\rangle$ becomes a continuous operator from $\mathscr{D}_p$ into $\boldsymbol{C}$, that is, $K \in \mathscr{L}(\mathscr{D}_p, \boldsymbol{C})$. Then, according to the definition in Section 8 we define an integral kernel operator $\Xi_{0,1}(K)$, denoted also by $\Xi_{0,1}(\kappa)$, which is characterized by

$$\Xi_{0,1}(\kappa)\phi_\xi = \langle \kappa, \xi\rangle \phi_\xi = (K\xi)\phi_\xi, \quad \xi \in \mathscr{D}_p.$$

This integral kernel operator $\Xi_{0,1}(K) = \Xi_{0,1}(\kappa)$ is called an annihilation operator. Now consider

$$e^{\Xi_{0,1}(\kappa)}\phi \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\Xi_{0,1}(\kappa)^n\phi = \sum_{n=0}^{\infty}\frac{1}{n!}\Xi_{0,n}(K^{\otimes n})\phi, \quad \phi \in \mathscr{G}_{\infty}, \tag{9.5}$$

where $K^{\otimes n}$ is the tensor power of $K \in \mathscr{L}(\mathscr{D}_p, \boldsymbol{C})$. Applying the norm estimate (8.1) and $\|K^{\otimes n}\|_{\mathrm{OP}} = \|K\|_{\mathrm{OP}}^n$, we see that (9.5) converges in any norm $\|\cdot\|_q$ and bounded by $\|\phi\|_{(p\vee q)+r}$ with arbitrary $r > 0$. Thus, the map $\phi \mapsto e^{\Xi_{0,1}(\kappa)}\phi$ is a continuous operator from $\mathscr{G}_{\infty}$ into itself. In short,

Lemma 9.2. *For $\kappa \in \mathscr{D}_{\infty}^*$ the exponential $e^{\Xi_{0,1}(\kappa)}$ defined in (9.5) belongs to $\mathscr{L}(\mathscr{G}_{\infty}, \mathscr{G}_{\infty})$.*

We next need multiplication operator $M_{\psi}$ associated with $\psi = (g_n) \in \Gamma(\mathscr{H}_{\boldsymbol{C}})$. For $\phi = (f_n)$ we define $M_{\psi}\phi = (h_l)$ by

$$h_l = \sum_{m+n=l}\sum_{k=0}^{\infty} k!\binom{m+k}{k}\binom{n+k}{k} g_{m+k}\,\hat{\otimes}_k\, f_{n+k}, \tag{9.6}$$

where $\hat{\otimes}_k$ is the right contraction, for details see [**34**]. This $M_{\psi}$ is called the *multiplication operator* by $\psi$. In fact, employing a Gaussian realization through the Wiener-Itô-Segal isomorphism, $M_{\psi}$ is nothing but a multiplication operator by $\psi$ which corresponds to a $\boldsymbol{C}$-valued function on a Gaussian space.

Lemma 9.3. *For $\zeta \in \mathscr{H}_{\boldsymbol{C}}$ and $T \in \mathscr{L}(\mathscr{D}_{\infty}, \mathscr{H}_{\boldsymbol{C}})$, $M_{\phi_\zeta}\Gamma(T)$ is extended to a continuous operator from $\mathscr{G}_{\infty}$ into $\Gamma(\mathscr{H}_{\boldsymbol{C}})$.*

Proof. We use similar argument as in the proof of [**34**, Theorem 3.5.6]. Let $\phi = (f_n) \in \mathscr{G}_{\infty}$ and set $M_{\phi_\zeta}\Gamma(T)\phi = (h_l)$. Then, by definition (9.6) we have

$$\begin{aligned} h_l &= \sum_{m+n=l}\sum_{k=0}^{\infty} k!\binom{m+k}{k}\binom{n+k}{k}\frac{\zeta^{\otimes(m+k)}}{(m+k)!}\,\hat{\otimes}_k\,(T^{\otimes(n+k)}f_{n+k}) \\ &= \sum_{m+n=l}\sum_{k=0}^{\infty}\frac{(n+k)!}{m!n!k!}\zeta^{\otimes(m+k)}\,\hat{\otimes}_k\,(T^{\otimes(n+k)}f_{n+k}). \end{aligned} \tag{9.7}$$

Fix $p \geq 0$ satisfying $T \in \mathscr{L}(\mathscr{D}_p, \mathscr{H}_{\boldsymbol{C}})$ and let $\|\!|T|\!\|$ be the operator norm in $\mathscr{L}(\mathscr{D}_p, \mathscr{H}_{\boldsymbol{C}})$. Choose $\alpha \geq 0$ such that $\max\{|\zeta|_0\|\!|T|\!\|, \|\!|T|\!\|\} \leq \rho^{-\alpha}$. (Recall condition (K4).) Then we have

$$\begin{aligned} |\zeta^{\otimes(m+k)}\,\hat{\otimes}_k\,(T^{\otimes(n+k)}f_{n+k})|_0 &\leq |\zeta|_0^{m+k}|T^{\otimes(n+k)}f_{n+k}|_0 \\ &\leq |\zeta|_0^{m+k}\|\!|T|\!\|^{n+k}|f_{n+k}|_p \\ &\leq |\zeta|_0^{m+k}\|\!|T|\!\|^{n+k}\rho^{\alpha(n+k)}|f_{n+k}|_{p+\alpha} \\ &\leq |\zeta|_0^{m}|f_{n+k}|_{p+\alpha}, \end{aligned}$$

and hence for any $\beta \geq 0$,

$$|\zeta^{\otimes(m+k)}\,\hat{\otimes}_k\,(T^{\otimes(n+k)}f_{n+k})|_0 \leq |\zeta|_0^{m}\rho^{\beta(n+k)}|f_{n+k}|_{p+\alpha+\beta}. \tag{9.8}$$

We are now in a position to estimate the norm. Combining (9.7) and (9.8), we obtain

$$\|M_{\phi_\zeta}\Gamma(T)\phi\|_0^2 = \sum_{l=0}^{\infty} l!|h_l|_0^2$$

$$\leq \sum_{l=0}^{\infty} l!\left(\sum_{m+n=l}\frac{1}{m!n!}|\zeta|_0^m\rho^{\beta n}\sum_{k=0}^{\infty}\frac{(n+k)!}{k!}\rho^{\beta k}|f_{n+k}|_{p+\alpha+\beta}\right)^2. \tag{9.9}$$

By the Schwartz inequality and an obvious inequality $(n+k)! \leq 2^{n+k}n!k!$ for $n, k \geq 0$, we have

$$\sum_{k=0}^{\infty}\frac{(n+k)!}{k!}\rho^{\beta k}|f_{n+k}|_{p+\alpha+\beta} \leq \|\phi\|_{p+\alpha+\beta}\left(\sum_{k=0}^{\infty}\frac{(n+k)!}{k!^2}\rho^{2\beta k}\right)^{1/2}$$

$$\leq \sqrt{n!}2^{n/2}e^{\rho^{2\beta}}\|\phi\|_{p+\alpha+\beta}.$$

With this (9.9) becomes

$$\|M_{\phi_\zeta}\Gamma(T)\phi\|_0^2 \leq \sum_{l=0}^{\infty} l!\left(\sum_{m+n=l}\frac{1}{m!n!}|\zeta|_0^m\rho^{\beta n}\sqrt{n!}2^{n/2}e^{\rho^{2\beta}}\|\phi\|_{p+\alpha+\beta}\right)^2$$

$$\leq e^{2\rho^{2\beta}}\|\phi\|_{p+\alpha+\beta}^2\sum_{l=0}^{\infty} l!\left(\sum_{m+n=l}\frac{1}{m!\sqrt{n!}}|\zeta|_0^m(\sqrt{2}\rho^\beta)^n\right)^2. \tag{9.10}$$

Here we note two obvious inequalities:

$$\frac{1}{\sqrt{m!}}(\varepsilon^{-1}|\zeta|_0)^m \leq e^{\varepsilon^{-2}|\zeta_0|^2/2}, \qquad \sqrt{\frac{(m+n)!}{m!n!}} \leq \frac{(m+n)!}{m!n!},$$

where $\varepsilon > 0$ and $m, n \geq 0$. Then (9.10) becomes

$$\|M_{\phi_\zeta}\Gamma(T)\phi\|_0^2 \leq e^{2\rho^{2\beta}}e^{\varepsilon^{-2}|\zeta_0|^2}\|\phi\|_{p+\alpha+\beta}^2\sum_{l=0}^{\infty}(\varepsilon+\sqrt{2}\rho^\beta)^{2l}.$$

The last sum becomes finite when $\beta \geq 0$ and $\varepsilon > 0$ are chosen in such a way that $\varepsilon + \sqrt{2}\rho^\beta < 1$, and we obtain the desired estimate. □

Theorem 9.4. *Consider a quantum stochastic process*

$$L_t = \Xi_{0,1}(I_{1,t}) + \Xi_{1,0}(I_{2,t}) + \Xi_{1,1}(I_{3,t}) \tag{9.11}$$

*defined on a time interval containing* 0, *where* $t \mapsto I_{1,t} \in \mathscr{N}_{\boldsymbol{C}}^*$, $t \mapsto I_{2,t} \in \mathscr{N}_{\boldsymbol{C}}^*$ *and* $t \mapsto I_{3,t} \in \mathscr{L}(\mathscr{N}_{\boldsymbol{C}}, \mathscr{N}_{\boldsymbol{C}}^*)$ *are continuous. Put*

$$J_{i,t} = \int_0^t I_{i,s}\,ds, \quad t \geq 0,\ i = 1,2,3.$$

*If there exists* $q \geq 0$ *such that* $J_{1,t} \in \mathscr{D}_\infty^*$, $J_{2,t} \in \mathscr{D}_{-q}$ *and* $J_{3,t} \in \mathscr{L}(\mathscr{D}_\infty, \mathscr{D}_{-q})$, *then the solution to the equation* (9.1) *with coefficient* $\{L_t\}$ *given in* (9.11) *lies in* $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_{-q})$.

PROOF. We apply Proposition 9.1 with $q$ being replaced with $-q$. Put

$$M_t = \int_0^t L_s\, ds = \Xi_{0,1}(J_{1,t}) + \Xi_{1,0}(J_{2,t}) + \Xi_{1,1}(J_{3,t})$$

and consider

$$\begin{aligned}\Theta_t(\xi,\eta) &= e^{\langle \xi,\eta\rangle} \exp\{e^{-\langle \xi,\eta\rangle} \hat{M}_t(\xi,\eta)\} \\ &= \exp\{\langle \xi,\eta\rangle + \langle J_{1,t},\xi\rangle + \langle J_{2,t},\eta\rangle + \langle J_{3,t}\xi,\eta\rangle\} \\ &= e^{\langle J_{1,t},\xi\rangle} \langle\!\langle \phi_{J_{2,t}+(J_{3,t}+I)\xi}, \phi_\eta \rangle\!\rangle.\end{aligned}$$

Condition (i) in Proposition 9.1 is obviously fulfilled. We shall check condition (ii) therein. By definition, we have

$$\begin{aligned}\left\| \sum_{i=1}^k a_i \Theta_t(\xi_i, K^{-q}\cdot) \right\|_{E^2(\nu)}^2 &= \sup_{P\in\mathscr{P}} \int_{\mathscr{N}_C^*} \left| \sum_{i=1}^k a_i \Theta_t(\xi_i, K^{-q}Pz) \right|^2 \nu(dz) \\ &= \left\| \sum_{i=1}^k a_i e^{\langle J_{1,t},\xi_i\rangle} \phi_{J_{2,t}+(J_{3,t}+I)\xi_i} \right\|_{-q}^2 \\ &= \left\| \sum_{i=1}^k a_i e^{\langle J_{1,t},\xi_i\rangle} \phi_{K^{-q}(J_{2,t}+(J_{3,t}+I)\xi_i)} \right\|_0^2. \qquad (9.12)\end{aligned}$$

Using the formula $\phi_{\xi+\eta} = e^{-\langle \xi,\eta\rangle}\phi_\xi\phi_\eta$, we see that

$$\phi_{K^{-q}(J_{2,t}+(J_{3,t}+I))\xi_i} = e^{-\langle K^{-q}J_{2,t}, K^{-q}(J_{3,t}+I)\xi_i\rangle} \phi_{K^{-q}J_{2,t}} \phi_{K^{-q}(J_{3,t}+I)\xi_i}.$$

For simplicity we set

$$\kappa = J_{1,t} - (J_{3,t}+I)^* K^{-2q} J_{2,t}, \quad \psi = \phi_{K^{-q}J_{2,t}}, \quad \phi = \sum_{i=1}^k a_i \phi_{\xi_i}.$$

Then, noting that

$$e^{\Xi_{0,1}(\kappa)}\phi_\xi = e^{\langle \kappa,\xi\rangle}\phi_\xi, \quad \phi_{K^{-q}(J_{3,t}+I)\xi} = \Gamma(K^{-q}(J_{3,t}+I))\phi_\xi,$$

we obtain

$$\sum_{i=1}^k a_i e^{\langle J_{1,t},\xi_i\rangle} \phi_{K^{-q}(J_{2,t}+(J_{3,t}+I))\xi_i} = M_\psi \Gamma(K^{-q}(J_{3,t}+I)) e^{\Xi_{0,1}(\kappa)}\phi.$$

It follows from Lemmas 9.2 and 9.3 that there exist $p \ge 0$ and $C \ge 0$ such that

$$\left\| \sum_{i=1}^k a_i e^{\langle J_{1,t},\xi_i\rangle} \phi_{K^{-q}(J_{2,t}+(J_{3,t}+I))\xi_i} \right\|_0 \le C\|\phi\|_p.$$

In other words, (9.12) is estimated as follows:

$$\left\| \sum_{i=1}^{k} a_i \Theta_t(\xi_i, K^{-q}\cdot) \right\|_{E^2(v)} \le C \left\| \sum_{i=1}^{k} a_i \phi_{\xi_i} \right\|_p .$$

This is condition (ii) that should be verified. Consequently, the unique solution $\Xi_t$ to (9.1) given by (9.2) lies in $\mathscr{L}(\mathscr{G}_\infty, \mathscr{G}_{-q})$. □

REMARK 9.5. It is worthwhile to note that nuclearity is not required during the above argument. If $K^{-r}$ is of Hilbert-Schmidt class for some $r > 0$, or equivalently, if $\mathscr{D}_\infty(K) = \mathscr{D}_\infty$ is nuclear, then $\mathscr{L}(\mathscr{D}_\infty, \mathscr{D}_{-q}) \cong \mathscr{D}_{-q} \otimes \mathscr{D}_\infty^*$ by the kernel theorem. In this case, Theorem 9.4 coincides with the statement of [**10**, Theorem 4.1] with $0 \le n \le 1$.

REMARK 9.6. If we choose

$$I_{1,t} = f_1(t)\delta_t, \quad I_{2,t} = f_2(t)\delta_t, \quad I_{3,t} = f_3(t)I,$$

where $f_i(t)$ is a continuous function, then the normal-ordered white noise differential equation (9.1) is equivalent to the quantum stochastic differential equation:

$$d\Xi_t = (f_1(t)\, dA_t + f_2(t)\, dA^*(t) + f_3(t)\, d\Lambda_t)\Xi_t, \quad \Xi(0) = I.$$

In this sense Theorem 9.4 extends the traditional scheme and gives a method of investigating solutions to quantum stochastic differential equations. Moreover, in a normal-ordered white noise differential equation the coefficients are not necessarily assumed to be adapted. For example, let $\gamma > 0$ and $\omega \in \boldsymbol{R}$ be constants and consider

$$I_t(s) = \frac{e^{-ist} - e^{-(i\omega+\gamma)t}}{i(\omega - s) + \gamma}.$$

Then, $I_t \in \mathscr{N}_{\boldsymbol{C}}^*$ for any $t \in \boldsymbol{R}$ and an equation involving the integral kernel operator $\Xi_{0,1}(I_t)$, which appears in a study of dissipative quantum systems, stays within our framework.

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