J. Math. Soc. Japan
Vol. 56, No. 3, 2004

Construction of K-stable foliations for two-dimensional dispersing billiards without eclipse

By Takehiko MORITA

(Received Aug. 19, 2002)
(Revised Mar. 6, 2003)

Abstract. Let T be the billiard map for a two-dimensional dispersing billiard without eclipse. We show that the nonwandering set Ω^+ for T has a hyperbolic structure quite similar to that of the horseshoe. We construct a sort of stable foliation for (Ω^+, T) each leaf of which is a K-decreasing curve. We call the foliation a K-stable foliation for (Ω^+, T). Moreover, we prove that the foliation is Lipschitz continuous with respect to the Euclidean distance in the so called (r, φ)-coordinates. It is well-known that we can not always expect the existence of such a Lipschitz continuous invariant foliation for a dynamical system even if the dynamical system itself is smooth. Therefore, we keep our construction as elementary and self-contained as possible so that one can see the concrete structure of the set Ω^+ and why the K-stable foliation turns out to be Lipschitz continuous.

1. Introduction.

Let $Q_1, Q_2, \dots, Q_J$ $(J \geq 3)$ be a finite number of bounded domains in $\boldsymbol{R}^2$ with boundaries $\partial Q_1, \partial Q_2, \dots, \partial Q_J$. Each of them is called a scatterer. Throughout the paper, we assume that these scatterers are located so that they can satisfy the Ikawa conditions (H.1) and (H.2) in [**3**] (see Figure 1.1).

(H.1) (dispersing) For each j, the boundary ∂Q_j of the domain Q_j is a strictly convex simply closed curve of class C^3.

(H.2) (no eclipse) For any triplet of distinct indices (j_1, j_2, j_3), we have

$$\mathrm{conv}(\overline{Q_{j_1}} \cup \overline{Q_{j_2}}) \cap \overline{Q_{j_3}} = \varnothing,$$

where $\mathrm{conv}(A)$ denotes the convex hull of the set A.

Consider the exterior domain $Q = \boldsymbol{R}^2 \setminus \bigcup_{j=1}^{J} \overline{Q_j}$ of the scatterers. Clearly, $\partial Q = \bigcup_{j=1}^{J} \partial Q_j$. For $q \in \partial Q$, $n(q)$ denotes the unit normal of ∂Q at q which is directed towards the inside of the domain Q. The billiard flow S^t is, in short, the Euclidean geodesic flow on the manifold $\bar{Q}$ obeying the law of reflections at the boundary.

Let $S\boldsymbol{R}^2 = \boldsymbol{R}^2 \times S^1$ denote the unit tangent bundle over $\boldsymbol{R}^2$ and $\pi : S\boldsymbol{R}^2 \to \boldsymbol{R}^2$; $(q, v) \mapsto q$ the natural projection. The state space M of the billiard flow S^t is given by

$$M = \pi^{-1}(Q) \cup (\pi^{-1}(\partial Q)/\sim),$$

where the equivalence relation $\sim$ on $\pi^{-1}(\partial Q)$ means that $(q, v) \sim (p, w)$ if and only if $q = p$ and $w = v - 2\langle v, n(q)\rangle n(q)$. Namely, the state of incidence and the state of

---

2000 *Mathematics Subject Classification.* Primary 37D50; Secondary 37D05, 37D10.
Key Words and Phrases. billiard map, Lipschitz continuous invariant foliation.

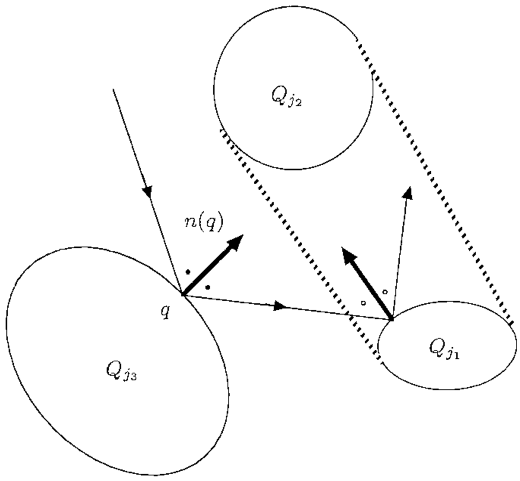

Figure 1.1.

reflection are identified. Therefore, selecting the states of reflection as representatives, we can regard $\pi^{-1}(\partial Q)/\sim$ as

$$M^+ = \{x = (q, v) : q \in \partial Q, \langle v, n(q) \rangle \geq 0\}.$$

Our main concern is the billiard flow restricted to its nonwandering set Ω. For the sake of simplicity we also denote the restriction by S^t.

We define the first collision time and the last collision time for the billiard flow at the point $x \in M^+$ as follows.

$$\begin{cases} t^+(x) = \inf\{t > 0 : S^t x \in M^+\} \\ t^-(x) = \sup\{t < 0 : S^t x \in M^+\}, \end{cases}$$

where $t^+(x)$ (resp. $t^-(x)$) is regarded as $+\infty$ (resp. $-\infty$) if the set in the above definition is empty. Set

$$\mathscr{D}_1 = \{x \in M^+ : t^+(x) < \infty\}, \quad \mathscr{D}_{-1} = \{x \in M^+ : t^-(x) > -\infty\}.$$

We define the local maps $T : \mathscr{D}_1 \to M^+$ and $T^{-1} : \mathscr{D}_{-1} \to M^+$ by

$$\begin{cases} Tx = S^{t^+(x)}x & (\text{if } x \in \mathscr{D}_1), \\ T^{-1}x = S^{t^-(x)}x & (\text{if } x \in \mathscr{D}_{-1}). \end{cases}$$

$\mathscr{D}_n$ and $\mathscr{D}_{-n}$ are defined to be the sets where one has the n-th iterations T^n and $(T^{-1})^n$, respectively. Note that the notation T^{-1} is compatible with the definition of the inverse map of T. Thus it is natural to denote $(T^{-1})^n$ by T^{-n}. The first collision map of T for the flow S^t is usually called the billiard map. Later we verify that for each positive integer n, the sets $\mathscr{D}_n$ and $\mathscr{D}_{-n}$ are identified with unions of mutually disjoint $J(J-1)^{n-1}$ quadrilarerals with respect to the so called (r, φ)-coordinates. In addition we can easily see that T^n and T^{-n} are diffeomorphisms of class C^2 from $\operatorname{int} \mathscr{D}_n$ onto $\operatorname{int} \mathscr{D}_{-n}$ and from $\operatorname{int} \mathscr{D}_{-n}$ onto $\operatorname{int} \mathscr{D}_n$, respectively.

Clearly, the nonwandering set of the billiard flow S^t can be expressed as

$$\Omega = \{x \in M : \pi(S^t x) \in \partial Q \text{ holds for both infinitely many } t > 0 \text{ and infinitely many } t < 0\}.$$

If we put $\Omega^+ = \Omega \cap M^+$, Ω^+ is expressed as $\Omega^+ = \bigcap_{n \in \boldsymbol{Z}} \mathscr{D}_n$ and T is clearly invertible on Ω^+, where $\mathscr{D}_0$ is defined to be $\mathscr{D}_1 \cup \mathscr{D}_{-1}$ for convenience. Note that the set Ω^+ and the map T play the roles of the Poincaré section and the Poincaré map for the flow S^t.

Let k, l be in $\boldsymbol{Z} \cup \{-\infty, \infty\}$ with $k < l$. To each element x in $\bigcap_{n=k}^{l} \mathscr{D}_n$, we assign a sequence $(\xi_n(x))_{n=k}^{l}$ so that $T^n x \in \partial Q_{\xi_n(x)}$. The sequence is called the itinerary of x from time k to time l. In [**7**] we show that for each $x \in \Omega^+$, the local stable curve $\gamma^s(x)$ and the local unstable curve $\gamma^u(x)$ can be written as

$$\gamma^s(x) = \left\{ y \in \bigcap_{n=0}^{\infty} \mathscr{D}_n : \xi_n(y) = \xi_n(x) \text{ for any } n \geq 0 \right\} \quad \text{and}$$

$$\gamma^u(x) = \left\{ y \in \bigcap_{n=-\infty}^{0} \mathscr{D}_n : \xi_n(y) = \xi_n(x) \text{ for any } n \leq 0 \right\}.$$

Moreover, it is shown that if the boundary ∂Q is of class C^n for $n \geq 2$, the local stable curve and the local unstable curve are K-monotone curves of class C^{n-1} whose lengths are bounded from below by a positive constant independent of x (see also [**5**]). The definition of K-monotonicity will be given in Section 2.

The main purpose of the present paper is to show the following theorem that asserts the existence of a kind of stable foliation. We call it a K-stable foliation for (Ω^+, T).

Theorem 1.1 (c.f. Theorem 3.1 in [**8**]). *Assume that the conditions* (H.1) *and* (H.2) *are satisfied. Then we can construct a foliation* $\mathscr{F}$ *supported on the set* $\mathscr{D}_1$ *with the following properties.*

($\mathscr{F}$.1) *Each leaf of* $\mathscr{F}$ *is a K-decreasing curve.*

($\mathscr{F}$.2) *For any* $x \in \Omega^+$, *the leaf* $\mathscr{F}(x)$ *containing* x *coincides with the local stable curve* $\gamma^s(x)$.

($\mathscr{F}$.3) *For any point* $x \in \mathscr{D}_2$, $T\mathscr{F}(x) \subset \mathscr{F}(Tx)$ *holds.*

($\mathscr{F}$.4) $\mathscr{F}$ *is a Lipschitz continuous foliation on* $\mathscr{D}_1$ *with respect to the Euclidean distance in the* (r, φ)*-coordinates.*

We note that there are three unpublished papers [**8**], [**9**], and [**10**] concerned with the present one. One may find that Theorem 1.1 is proved in the first half of [**8**]. We need to explain the position of Theorem 1.1 among these works. To this end we recall the results in [**6**]. Consider the set Σ defined by

$$\Sigma = \{\xi = (\xi_n)_{n \in \boldsymbol{Z}} \in \{1, 2, \ldots, J\}^{\boldsymbol{Z}} : \xi_n \neq \xi_{n+1} \text{ for any } n \in \boldsymbol{Z}\}.$$

Using a positive number $\theta \in (0, 1)$, we define a function $d_\theta : \Sigma \times \Sigma \to \boldsymbol{R}$ by

$$d_\theta(\xi, \eta) = \theta^n \quad \text{for } \xi = (\xi_n)_{n \in \boldsymbol{Z}} \text{ and } \eta = (\eta_n)_{n \in \boldsymbol{Z}},$$

where $n = \min\{j \geq 0 : \xi_j \neq \eta_j \text{ or } \xi_{-j} \neq \eta_{-j} \text{ hold}\}$. Then it is easy to see that d_θ becomes a metric on Σ which introduces the same topology as the product topology of the finite set $\{1, 2, \ldots, J\}$ with discrete topology. The shift map $\sigma : \Sigma \to \Sigma$ is defined so that

$$(\sigma\xi)_n = \xi_{n+1} \quad \text{for any } n \in \boldsymbol{Z}$$

holds for each $\xi \in \Sigma$.

One of the main results in [**6**] asserts that the map $\xi(\cdot) : \Omega^+ \to \Sigma$ gives a conjugacy between the dynamical systems (Ω^+, T) and (Σ, σ). Moreover, it is shown that for appropriately chosen θ, the inverse map $x(\cdot)$ of $\xi(\cdot)$ is Lipschitz continuous with respect to d_θ and if we define $f : \Sigma \to \boldsymbol{R}$ by $f(\xi) = t^+(x(\xi))$, f turns out to be d_θ-Lipschitz continuous. Consequently, we obtain the conjugacy between the billiard flow (Ω, S^t) and the suspension flow over (Σ, σ) with ceiling function f in such a way that the corresponding periodic orbits have the same periods. Therefore, we can apply the result in Parry and Pollicott [**12**] to the zeta function of the billiard flow defined by the following formal Euler product

$$\zeta(s) = \prod_\tau (1 - \exp(-sl(\tau)))^{-1},$$

where the product $\prod_\tau$ is taken over all the prime periodic orbits τ of S^t and $l(\tau)$ denotes the period of τ. Hence we see that there exists $h > 0$ such that the infinite pruduct above is absolutely convergent in the half-plane $\{s \in \boldsymbol{C} : \operatorname{Re} s > h\}$ and has a meromorphic extension to some half-plane containing $\{s \in \boldsymbol{C} : \operatorname{Re} s \geq h\}$ in which it has no zeros and $s = h$ is a unique pole and simple. Consequently we obtain the following analogue of the prime number theorem.

$$\#\{\tau : l(\tau) \leq u\} \frac{hu}{\exp(hu)} \to 1 \quad (u \to \infty).$$

Now it is natural to ask how wide the domain of meromorphy of $\zeta(\cdot)$ is and what kind of information we can obtain from the analytic properties of $\zeta(\cdot)$. In virtue of the results in Ruelle [**13**], $\zeta(\cdot)$ could be meromorphic on the entire complex plane if our billiard map T would be smooth enough and the totality of local stable curves would make up themselves into smooth invariant foliation supported on some neighborhood of Ω^+. Obviously we are not in such a good situaton because T has singularities and in addition the definition domain $\mathscr{D}_n$ of T^n shrinks as n becomes large. Theorem 1.1 above, however, enable us to show the following. First, by combining the results in the second half of [**8**] with [**9**], it is possible to find a positive number β such that $\zeta(\cdot)$ can be extended to a meromorphic function in the half-plane $\{s \in \boldsymbol{C} : \operatorname{Re} s > -\beta\}$ without zero. Secondly, we can show that we can calculate the special value of $\zeta(\cdot)$ as $\zeta(0) = -1/((J-2)2^{J-1})$ (see [**10**]). Hence we can conclude that if we obtain the information of the length spectrum of Q beforehand, we see how many scatterers there are.

This paper is organized as follows: In Section 2, we give some basic definitions and fundamental results for billiard maps. In Section 3 alternative proofs of the results in [**6**] and [**7**] are given by using the idea of I. Kubo. This enable us to keep our construction of K-stable foliation as elementary as possible. Section 4 is devoted to the construction of K-stable foliation. Finally, we show the validity ($\mathscr{F}$.4) of Lipschitz continuity in Section 5.

Acknowledgement. The author would like to thank I. Kubo for his comments on the preceding works of authors. He would also like to thank N. Aoki and N. Chernov for valuable comments on the Lipschitz continuity.

2. Preliminaries.

In this section, we recall some notions and fundamental properties of the billiard map. First of all, we introduce convenient coordinates to $\pi^{-1}\partial Q$. Choose a base point $q(j)$ for each $j = 1, 2, \ldots, J$ and define the following quantities for $x = (q, v) \in \pi^{-1}\partial Q$ (Figure 2.1).

$$\begin{cases} \xi_0(x) = j \quad \text{if } q \in \partial Q_j, \\ r(x) = \text{the arclength from } q(\xi_0(x)) \text{ to } q \\ \qquad \text{measured counterclockwise along the curve } \partial Q_j, \\ \varphi(x) = \text{the angle between the vector } v \text{ and the unit normal} \\ \qquad n(q) \text{ which is directed towards the inside of the domain } Q, \\ \qquad \text{measured counterclockwise from } n(q) \text{ to } v. \end{cases}$$

Such coordinates will be called the (r, φ)-coordinates of x. The quantities $\xi_0(x)$ and $\varphi(x)$ do not depend on the choice of the base point $q(j)$ but $r(x)$ does. Clearly, the change of the base point causes just the translation of the r-coordinate.

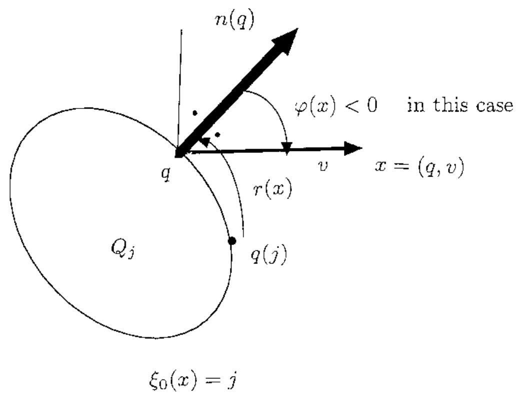

Figure 2.1.

Note that

$$\{(j, r, \varphi) : 0 \le r < \text{the perimeter of } \partial Q_j, -\pi < \varphi \le \pi\}$$

gives a global parametrization for the set $\pi^{-1}\partial Q_j$ but it is not a global parametrization for the C^3 manifold $\pi^{-1}\partial Q_j$. We often use this coordinates without specifying which points are chosen as base points. This ambiguity causes us no trouble because most of our investigations below are carried out locally. Therefore we often abuse the notation $x = (j, r, \varphi)$ for $x = (q, v)$ and (j, r) for q, respectively. Further, we drop the first coordinate j if there is no possibility of confusion. The totality of reflection states M^+ is expressed as

$$M^+ = \left\{ x \in \pi^{-1}\partial Q : -\frac{\pi}{2} \le \varphi(x) \le \frac{\pi}{2} \right\}.$$

For $j = 1, 2, \ldots, J$, set

$$M_j^+ = \{x \in M^+ : \xi_0(x) = j\}.$$

Each M_j^+ will be called a connected component of M^+. As we mentioned in Introduction, the state space M of the billiard flow can be regarded as

$$M = \pi^{-1}Q \cup M^+.$$

We define the first collision time and the last collision time for the billiard flow at the point $x \in M^+$ as follows.

$$\begin{cases} t^+(x) = \inf\{t > 0 : S^t x \in M^+\} \\ t^-(x) = \sup\{t < 0 : S^t x \in M^+\}, \end{cases}$$

where $t^+(x)$ (resp. $t^-(x)$) is regarded as $+\infty$ (resp. $-\infty$) if the set in the definition above is empty. Set

$$\mathscr{D}_1 = \{x \in M^+ : t^+(x) < \infty\}, \quad \mathscr{D}_{-1} = \{x \in M^+ : t^-(x) > -\infty\}.$$

We define the local maps $T : \mathscr{D}_1 \to M^+$ and $T^{-1} : \mathscr{D}_{-1} \to M^+$ by

$$\begin{cases} Tx = S^{t^+(x)}x & (\text{if } x \in \mathscr{D}_1), \\ T^{-1}x = S^{t^-(x)}x & (\text{if } x \in \mathscr{D}_{-1}). \end{cases}$$

For each positive integer n, we define the sets $\mathscr{D}_n$ and $\mathscr{D}_{-n}$ and the maps T^n and $(T^{-1})^n$ inductively as follows.

$$\begin{cases} \mathscr{D}_{n+1} = \{x \in \mathscr{D}_n : t^+(T^n x) < \infty\} \\ \mathscr{D}_{-(n+1)} = \{x \in \mathscr{D}_{-n} : t^-(T^{-n}x) > -\infty\} \end{cases}$$

and

$$\begin{cases} T^{n+1}x = T(T^n x), & \text{for } x \in \mathscr{D}_{n+1} \\ T^{-(n+1)}x = T^{-1}(T^{-n}x), & \text{for } x \in \mathscr{D}_{-(n+1)}. \end{cases}$$

Note that the notation T^{-1} is compatible with the definition of the inverse map of T and T^n (resp. T^{-n}) coincides with n-fold iteration of T (resp. T^{-1}). Usually the first collision map T is called the billiard map for the flow S^t.

It is not hard to see that the nonwandering set Ω of S^t is the totality of all initial states such that $\pi(S^t x) \in \partial Q$ holds for both infinitely many $t > 0$ and infinitely many $t < 0$. If we set $\Omega^+ = M^+ \cap \Omega$, Ω^+ can be expressed as $\Omega^+ = \bigcap_{n \in \mathbf{Z}} \mathscr{D}_n$ and T is clearly invertible on Ω^+, where $\mathscr{D}_0$ is defined to be $\mathscr{D}_1 \cup \mathscr{D}_{-1}$ for convenience.

For $x \in M^+$ we put

$$\xi_i(x) = \xi_0(T^i x), \quad \text{if } T^i \text{ is defined.}$$

For k and l with $-\infty \le k < l \le \infty$, a sequence $\{\xi_i\}_{i=k}^l$ is called the itinerary of $x \in M^+$ from time k to time l if $\xi_i = \xi_i(x)$ holds for each $i \in \mathbf{Z} \cap [k, l]$. The number $l - k + 1$, possibly ∞, is called the length of the itinerary. On the other hand, any sequence $\{\xi_i\}_{i=k}^l$ which can be the itinerary of some point from time k to time l is called an admissible word and the number $l - k + 1$ is called the length of the admissible word. $\mathscr{W}_n$ denotes the totality of admissible words of length n for $n \ge 2$ ($\mathscr{W}_1 = \{1, 2, \ldots, J\}$ for convenience). Under the conditions (H.1) and (H.2), the admissibility is equivalent to the condition that $\xi_i \in \{1, 2, \ldots, J\}$ for each $i \in \mathbf{Z} \cap [k, l]$ and $\xi_i \ne \xi_{i+1}$ for each $i \in \mathbf{Z} \cap [k, l-1]$.

We employ the following slightly abusive notations for our convenience. For $x = (q, v) = (j, r, \varphi)$, $k_i, r_i, \varphi_i, c_i, t_i^+$, and t_i^- denote $k(T^i x), r(T^i x), \varphi(T^i x), c(T^i x), t^+(T^i x)$, and $t^-(T^i x)$, respectively, where $k(x) = k(q) = k(j, r)$ denotes the curvature of ∂Q_j at $q = (j, r)$ and $c(x) = \cos\varphi(x)$.

By the help of the Implicit Function Theorem, the Jacobi matrix $D(T)$ for T and $D(T^{-1})$ for T^{-1} can be calculated as follows in terms of the (r, φ)-coordinates at the point where they are defined (see [**1**], [**2**], [**6**] and [**14**]).

$$\begin{cases} \dfrac{\partial r_1}{\partial r} = -\left(1 + \dfrac{t^+ k}{c}\right)\dfrac{c}{c_1}, \\ \dfrac{\partial r_1}{\partial \varphi} = -\dfrac{t^+}{c_1}, \\ \dfrac{\partial \varphi_1}{\partial r} = -k_1\left(1 + \dfrac{t^+ k}{c}\right)\dfrac{c}{c_1} - k, \\ \dfrac{\partial \varphi_1}{\partial \varphi} = -\left(1 + \dfrac{t^+ k_1}{c_1}\right), \end{cases} \qquad \begin{cases} \dfrac{\partial r_{-1}}{\partial r} = -\left(1 - \dfrac{t^- k}{c}\right)\dfrac{c}{c_{-1}}, \\ \dfrac{\partial r_{-1}}{\partial \varphi} = -\dfrac{t^-}{c_{-1}}, \\ \dfrac{\partial \varphi_{-1}}{\partial r} = k_{-1}\left(1 - \dfrac{t^- k}{c}\right)\dfrac{c}{c_{-1}} + k, \\ \dfrac{\partial \varphi_{-1}}{\partial \varphi} = -\left(1 - \dfrac{t^- k_{-1}}{c_{-1}}\right). \end{cases} \tag{2.1}$$

In addition, the partial derivatives of t^+ and t^- are given by

$$\begin{cases} \dfrac{\partial t^+}{\partial r} = \sin\varphi_1 \dfrac{\partial r_1}{\partial r} - \sin\varphi, \\ \dfrac{\partial t^+}{\partial \varphi} = -t^+ \tan\varphi_1, \end{cases} \qquad \begin{cases} \dfrac{\partial t^-}{\partial r} = \sin\varphi_{-1} \dfrac{\partial r_{-1}}{\partial r} - \sin\varphi, \\ \dfrac{\partial t^-}{\partial \varphi} = -t^- \tan\varphi_{-1}. \end{cases} \tag{2.2}$$

Then we obtain the following.

Lemma 2.1 (see [**4**], [**6**], [**8**], and [**14**]). *Let γ be a curve of class C^1 which is expressed as $\{(j, r, \varphi) : \varphi = \varphi(r), a \le r \le b\}$ in the (r, φ)-coordinates, where $\varphi(\cdot)$ is a C^1 function in r. Assume that T and T^{-1} are defined on γ. If the images $\gamma_1 = T\gamma$ and $\gamma_{-1} = T^{-1}\gamma$ are expressed as $\{(j_1, r_1, \varphi_1) : \varphi_1 = \varphi_1(r_1), a_1 \le r_1 \le b_1\}$ and $\{(j_{-1}, r_{-1}, \varphi_{-1}) : \varphi_{-1} = \varphi_{-1}(r_{-1}), a_{-1} \le r_{-1} \le b_{-1}\}$, where $\varphi_1(\cdot)$ and $\varphi_{-1}(\cdot)$ are C^1 functions in r_1 and r_{-1}, respectively, then we have the formulas*:

$$\frac{d\varphi_1}{dr_1} = k_1 + \frac{c_1}{c}\frac{1}{\dfrac{t^+}{c} + \dfrac{1}{\dfrac{d\varphi}{dr} + k}}, \qquad \frac{d\varphi_{-1}}{dr_{-1}} = -k_{-1} + \frac{c_{-1}}{c}\frac{1}{\dfrac{t^-}{c} + \dfrac{1}{\dfrac{d\varphi}{dr} - k}},$$

$$\frac{dr_1}{dr} = -\frac{c}{c_1}\left(1 + \frac{t^+\left(\dfrac{d\varphi}{dr} + k\right)}{c}\right), \qquad \frac{dr_{-1}}{dr} = -\frac{c}{c_{-1}}\left(1 + \frac{t^-\left(\dfrac{d\varphi}{dr} - k\right)}{c}\right),$$

$$\frac{d\varphi_1}{d\varphi} = -k_1\frac{c}{c_1}\frac{dr}{d\varphi} - \left(1 + \frac{t^+ k}{c_1}\right)\left(1 + k\frac{dr}{d\varphi}\right), \quad \frac{d\varphi_{-1}}{d\varphi} = k_{-1}\frac{c}{c_{-1}}\frac{dr}{d\varphi} - \left(1 - \frac{t^- k}{c_{-1}}\right)\left(1 - k\frac{dr}{d\varphi}\right),$$

$$\frac{dt^+}{dr} = \sin\varphi_1\frac{dr_1}{dr} - \sin\varphi, \qquad \frac{dt^-}{dr} = \sin\varphi_{-1}\frac{dr_{-1}}{dr} - \sin\varphi.$$

These formulas make sense even when $d\varphi/dr = 0$ and we obtain similar formulas if the role of the r-coordinate and that of the φ-coordinate are exchanged in the representaitions of curves γ, γ_1, and γ_{-1}.

Next we introduce the notions of increasing curves and decreasing curves. A curve in M^+ which is expressed as $\varphi = \varphi(r)$, $a \le r \le b$ in the (r, φ)-coordinates is said to be increasing (resp. dcreasing) if $\varphi(\cdot)$ is increasing (resp. decreasing) as a function of r. In the case when a curve is expressed as $r = r(\varphi)$, $\alpha \le \varphi \le \beta$, we also say it to be increasing or decreasing according as $r(\cdot)$ is increasing or decreasing.

Put

$$k_{\max} = \max\{k(q) : q \in \partial Q\}, \quad k_{\min} = \min\{k(q) : q \in \partial Q\},$$

$$t_{\min} = \min\{\operatorname{dist}(\overline{Q_{j_1}}, \overline{Q_{j_2}}) : j_1 \ne j_2\},$$

$$K_{\max} = k_{\max} + \frac{1}{t_{\min}}, \quad \theta = \frac{1}{1 + t_{\min} k_{\min}}.$$

An increasing (resp. decreasing) curve as above is called a K-increasing (resp. K-decreasing) curve if

$$k_{\min} \le \frac{\varphi(r_2) - \varphi(r_1)}{r_2 - r_1} \le K_{\max} \quad \left(\text{resp. } -K_{\max} \le \frac{\varphi(r_2) - \varphi(r_1)}{r_2 - r_1} \le -k_{\min}\right)$$

holds for any r_1 and r_2 with $a \le r_1 < r_2 \le b$. (K-)increasing curves and (K-)decreasing curves are often called (K-)monotone curves.

From Lemma 2.1, we can easily show:

Lemma 2.2. *Let γ be a C^1 curve in M^+ which is expressed as $\{(r, \varphi) : \varphi = \varphi(r), a < r < b\}$. Assume that γ is increasing (resp. decreasing) and T (resp. T^{-1}) is defined on γ. Then $T\gamma$ (resp. $T^{-1}\gamma$) turns out to be a C^1 curve which is expressed as $\{(r_1, \varphi_1) : \varphi_1 = \varphi_1(r_1), a_1 < r_1 < b_1\}$ (resp. $\{(r_{-1}, \varphi_{-1}) : \varphi_{-1} = \varphi_{-1}(r_{-1}), a_{-1} < r_{-1} < b_{-1}\}$) satisfying*

$$k_{\min} \le \frac{d\varphi_1}{dr_1} \le K_{\max} \quad \left(\textit{resp. } -K_{\max} \le \frac{d\varphi_{-1}}{dr_{-1}} \le -k_{\min}\right).$$

In addition, we have

$$\Theta(T\gamma) \ge \theta^{-1}\Theta(\gamma) \quad (\textit{resp. } \Theta(T^{-1}\gamma) \ge \theta^{-1}\Theta(\gamma)),$$

where $\Theta(\gamma)$ denotes the variation of the φ-coordinate along γ.

It follows immediately from Lemma 2.2 that if γ is an increasing curve (resp. a decreasing curve) of class C^1, then $T\gamma$ (resp. $T^{-1}\gamma$) is a K-increasing curve (resp. K-decreasing curve) of class C^1. In fact, we can see that the T-image (resp. T^{-1}-image) of any increasing curve (resp. decreasing curve) is always a K-increasing curve (K-decreasing curve) by a routine approximation argument.

3. Structure of Ω^+.

Next, we observe the sturcture of the definition domain $\mathscr{D}_n$ (resp. $\mathscr{D}_{-n}$) of T^n (resp. T^{-n}). Consider any admissible word $j_{-n} j_{-(n-1)} \cdots j_0 \cdots j_{n-1} j_n$ of length $2n+1$. We set

$$\mathscr{D}_1(j_0 j_1) = (\mathscr{D}_1 \cap M_{j_0}^+) \cap T^{-1}(\mathscr{D}_{-1} \cap M_{j_1}^+),$$

$$\mathscr{D}_{-1}(j_{-1} j_0) = (\mathscr{D}_{-1} \cap M_{j_0}^+) \cap T(\mathscr{D}_1 \cap M_{j_{-1}}^+),$$

$$\mathscr{D}_n(j_0 j_1 \cdots j_n) = T^{-1}(\mathscr{D}_{n-1}(j_1 j_2 \cdots j_n) \cap \mathscr{D}_{-1}(j_0 j_1)) \quad (n \geq 2),$$

$$\mathscr{D}_{-n}(j_{-n} j_{-(n-1)} \cdots j_0) = T(\mathscr{D}_{-(n-1)}(j_{-n} \cdots j_{-2} j_{-1}) \cap \mathscr{D}_1(j_{-1} j_0)) \quad (n \geq 2).$$

Then it is easy to see that $\mathscr{D}_n(j_0 j_1 \cdots j_n)$ (resp. $\mathscr{D}_{-n}(j_{-n} j_{-(n-1)} \cdots j_0)$) is a connected component of $\mathscr{D}_n$ (resp. $\mathscr{D}_{-n}$). Therefore we have

$$\mathscr{D}_n = \bigcup_{j_0 j_1 \cdots j_n \in \mathscr{W}_{n+1}} \mathscr{D}_n(j_0 j_1 \cdots j_n)$$

$$\mathscr{D}_{-n} = \bigcup_{j_{-n} j_{-(n-1)} \cdots j_0 \in \mathscr{W}_{n+1}} \mathscr{D}_{-n}(j_{-n} j_{-(n-1)} \cdots j_0).$$

For $j = 1, 2, \ldots, J$, put

$$S_j^+ = \left\{ x \in M^+ : \xi(x) = j, \varphi(x) = \frac{\pi}{2} \right\}, \quad S_j^- = \left\{ x \in M^+ : \xi(x) = j, \varphi(x) = -\frac{\pi}{2} \right\}$$

and put

$$S^+ = \bigcup_{j=1}^{J} S_j^+, \quad S^- = \bigcup_{j=1}^{J} S_j^-, \quad S = S^- \cup S^+.$$

Each S_j^+ (resp. S_j^-) is identified with the line segment in $\varphi = \pi/2$ (resp. $\varphi = -\pi/2$) in the (r, φ)-plane. If ij is admissible, we can show that $\mathscr{D}_1(ij)$ is a closed domain in M_i^+ enclosed by the four curves $T^{-1}S_j^+$, $\varphi = -\pi/2$, $T^{-1}S_j^-$, and $\varphi = \pi/2$. Similarly, $\mathscr{D}_{-1}(ji)$ is a closed domain in M_i^+ enclosed by the four curves TS_j^-, $\varphi = -\pi/2$, TS_j^+, and $\varphi = \pi/2$. Since $\cos\varphi_1 = 0$ (resp. $\cos\varphi_{-1} = 0$) holds on $T^{-1}S$ (resp. TS), $T^{-1}S_j^+$ and $T^{-1}S_j^-$ (resp. TS_j^- and TS_j^+) are K-decreasing curves (resp. K-increasing curves) expressed by the equation of the form

$$\frac{d\varphi}{dr} = -k - \frac{\cos\varphi}{t^+} \quad \left(\text{resp. } \frac{d\varphi}{dr} = k - \frac{\cos\varphi}{t^-} \right).$$

Combining these facts and Lemma 2.2, we can conclude inductively that if n is a positive integer and $j_0 j_1 \cdots j_n$ is admissible, then the set $\mathscr{D}_n(j_0 j_1 \cdots j_n)$ is a closed domain enclosed by a pair of K-decreasing curves and $\varphi = \pm\pi/2$ and the set $\mathscr{D}_{-n}(j_0 j_1 \cdots j_n)$ is a closed domain enclosed by a pair of K-increasing curves and $\varphi_n = \pm\pi/2$. We call such a closed domain enclosed by four curves a quadrilateral.

Following the idea of Izumi Kubo, we prove some estimates which are necessary for further investigations. Note that estimates which play similar roles to them are obtained in [**6**]. But in the present case, the following seem to be more useful than those in [**6**].

Lemma 3.1. *Assume that the conditions* (H.1) *and* (H.2) *are satisfied. Let* $j_{-n} \cdots j_0 \cdots j_n$ *be an admissible word of length* $2n + 1$. *Then we have the following.*

(1) *The Euclidean length of the boundary curve of the quadrilateral* $\mathscr{D}_n(j_0 \cdots j_n)$ (*resp.* $\mathscr{D}_{-n}(j_{-n} \cdots j_0)$) *contained in* $\varphi = \pm\pi/2$ *is not greater than* $C_1\theta^n$, *where* C_1 *is a positive constant depending only on the domain* Q.

(2) *The Hausdorff distance between K-decreasing curves* (*resp. K-increasing curves*) *in the boudary of* $\mathscr{D}_n(j_0 \cdots j_n)$ (*resp.* $\mathscr{D}_{-n}(j_{-n} \cdots j_0)$) *is not greater than* $C_1\theta^n$, *where* C_1 *is the same constant as in the assertion* (1).

(3) *There is a positive constant* $C_2 > 0$ *depending only on the domain* Q *such that the diameter of the set* $\mathscr{D}_{-n}(j_{-n} \cdots j_0) \cap \mathscr{D}_n(j_0 \cdots j_n)$ *is not greater than* $C_2\theta^n$.

In the above, the Euclidean length, the Hausdorff distance, and the diameter are measured in terms of the (r, φ)*-coordinates.*

Before we prove Lemma 3.1, we observe the structure of $\mathscr{D}_n(j_0 \cdots j_n)$ and $\mathscr{D}_{-n}(j_{-n} \cdots j_0)$. We can choose the base point $q(j_0)$ so that the corresponding (r, φ)-coordinates represents $\mathscr{D}_1(j_0)$ and $\mathscr{D}_{-1}(j_0)$ as quadrilaterals in the (r, φ)-plane. This fact enable us to identify $\mathscr{D}_1(j_0)$ and $\mathscr{D}_{-1}(j_0)$ with quadrilaterals in the (r, φ)-plane. We label the sides of quadrilateral $\mathscr{D}_n(j_0 \cdots j_n)$ (resp. $\mathscr{D}_{-n}(j_{-n} \cdots j_0)$) as $\gamma_t = \gamma_t(j_0 \cdots j_n)$, $\gamma_l = \gamma_l(j_0 \cdots j_n)$, $\gamma_b = \gamma_b(j_0 \cdots j_n)$, and $\gamma_r = \gamma_r(j_0 \cdots j_n)$ (resp. $\delta_t = \delta_t(j_{-n} \cdots j_0)$, $\delta_l = \delta_l(j_{-n} \cdots j_0)$, $\delta_b = \delta_b(j_{-n} \cdots j_0)$ and $\delta_r = \delta_r(j_{-n} \cdots j_0)$) in counterclockwise order so that γ_t (resp. δ_t) is contained in $\varphi = \pi/2$, γ_b (resp. δ_b) is contained in $\varphi = -\pi/2$, γ_l and γ_r are K-decreasing curves (resp. δ_l and δ_r are K-increasing curves) (see Figure 3.1). It is not hard to see the following facts:

(i) $T^k\gamma_t$ and $T^k\gamma_b$ are K-increasing curves (resp. $T^{-k}\delta_t$ and $T^{-k}\delta_b$ are K-decreasing curves) for each k with $1 \le k \le n$.

(ii) $T^k\gamma_l$ and $T^k\gamma_r$ are K-decreasing curves (resp. $T^{-k}\delta_l$ and $T^{-k}\delta_r$ are K-increasing curves) for each k with $0 \le k \le n-1$.

(iii) $T^n\mathscr{D}_n(j_0 \cdots j_n) = \mathscr{D}_{-n}(j_0 \cdots j_n)$ (resp. $T^{-n}\mathscr{D}_{-n}(j_{-n} \cdots j_0) = \mathscr{D}_n(j_{-n} \cdots j_0)$).

(iv) $T^k\gamma_t, T^k\gamma_l, T^k\gamma_b$, and $T^k\gamma_r$ (resp. $T^{-k}\delta_t, T^{-k}\delta_l, T^{-k}\delta_b$ and $T^{-k}\delta_r$) are the sides of the quadrilateral $T^k\mathscr{D}_n(j_0 \cdots j_n)$ (resp. $T^{-k}\mathscr{D}_{-n}(j_{-n} \cdots j_0)$). In particular, $T^n\gamma_r$ (resp. $T^{-n}\delta_l$) is contained in $\varphi_n = \pi/2$ or $\varphi_n = -\pi/2$ (resp. $\varphi_{-n} = \pi/2$ or $\varphi_{-n} = -\pi/2$) according as n is even or odd.

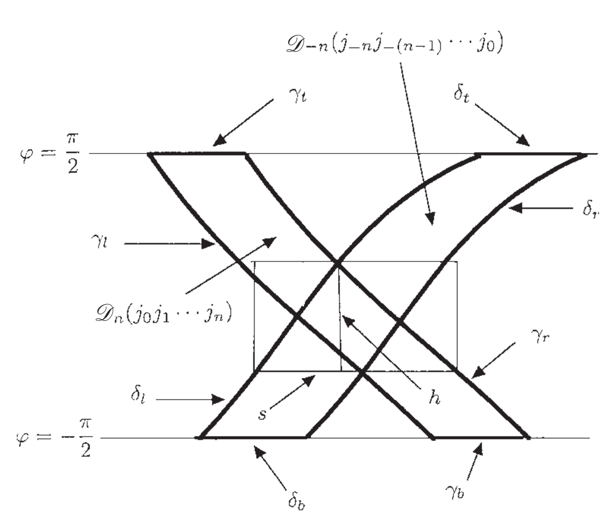

Figure 3.1.

Proof of Lemma 3.1. To establish the assertion (1), we estimate the length of γ_t and γ_b. Let γ_t is given by $\varphi = \pi/2$, $a \le r \le b$. In virtue of Lemma 2.1, the equation of the curve $T\gamma_t$ is given by

$$\frac{d\varphi_1}{dr_1} = k_1 + \frac{c_1}{t^+}$$

and $dr_1/dr = -t^+k/c_1$ for $a < r < b$. Therefore we have

$$\pi = \int_a^b \left|\frac{d\varphi_n}{dr}\right| dr = \int_a^b \left|\frac{d\varphi_n}{d\varphi_1}\frac{d\varphi_1}{dr_1}\frac{dr_1}{dr}\right| dr \ge \theta^{-n}k_{\min}(b-a).$$

We just note that the last inequality can be easily obtained from Lemma 2.1. For the curve γ_b, we get the same estimate. Now the first assertion of Lemma 3.1 follows if we put $C_1 = \pi(1/k_{\min})$.

The assertion (2) is proved as follows. Consider any line segment γ in $\mathscr{D}_n(j_0 \cdots j_n)$ parallel to the r-axis. Then we can apply the same argument as in the proof of the assertion (1) to γ instead of γ_t. Thus we conclude that the Euclidean length of γ is less than $C_1\theta^n$. Hence γ_r is contained in $C_1\theta^n$ neighborhood of γ_l. This implies that the assertion (2) is valid.

The assertion (3) can be shown as follows. Consider the line segment s passing through the vertex of $\mathscr{D}_{-n}(j_{-n} \cdots j_0) \cap \mathscr{D}_n(j_0 \cdots j_n)$ at the bottom, joining the curves δ_l and γ_r and parallel to the r-axis. The length of s is not greater than $2C_1\theta^n$ by the assertion (2). Next we consider the line segment h parallel to the φ-axis and joinning the vertex of $\mathscr{D}_{-n}(j_{-n} \cdots j_0) \cap \mathscr{D}_n(j_0 \cdots j_n)$ at the top and the line segment s. Since δ_l is K-increasing and γ_r is K-decreasing, it is easy to see that the length of h is not greater than $K_{\max}C_1\theta^n$. Clearly, the quadrilateral $\mathscr{D}_{-n}(j_{-n} \cdots j_0) \cap \mathscr{D}_n(j_0 \cdots j_n)$ is contained in a rectangle with base side s and with height not greater than $K_{\max}C_1\theta^n$. Hence C_2 is chosen to be $C_1\sqrt{K_{\max}^2 + 4} = (\pi/k_{\min})\sqrt{K_{\max}^2 + 4}$. □

Next we recall the itinerary problem studied in [**6**] (see [**15**] for heigher dimensional cases). The itinerary problem means the problem finding a point $x \in \Omega^+$ which satisfies the equation

$$\xi(x) = \xi$$

for a sequence ξ in Σ given beforehand. If the itinerary problem has a unique solution, we denote it by $x(\xi)$. For $\xi, \eta \in \Sigma$, put $d_\theta(\xi, \eta) = \theta^n$, where $n = \min\{i \ge 0 : \xi_{-i} \ne \eta_{-i}$ or $\xi_i \ne \eta_i\}$. Then d_θ is a metric on Σ which introduces the same topology that is induced by the product topology of $\{1, 2, \ldots, J\}^{\boldsymbol{Z}}$. In virtue of Lemma 3.1, we can show the Lipschitz well-posedness of the itinerary problem as follows.

Theorem 3.1. *Assume that the conditions* (H.1) *and* (H.2) *are satisfied. Then for any sequence* $\xi \in \Sigma$, *there exists a unique* $x \in \Omega^+$ *such that* $\xi(x) = \xi$. *Moreover, there exists a positive constant* C_3 *depending only on the domain* Q *such that*

$$|r(x(\xi)) - r(x(\eta))| \le C_3 d_\theta(\xi, \eta), \quad \textit{and} \quad |\varphi(x(\xi)) - \varphi(x(\eta))| \le C_3 d_\theta(\xi, \eta)$$

hold for any $\xi, \eta \in \Sigma$.

Proof. Let ξ be any element in Σ. Then $\{\mathscr{D}_n(\xi_0\cdots\xi_n)\cap\mathscr{D}_{-n}(\xi_{-n}\cdots\xi_0)\}_{n=1}^{\infty}$ is a decreasing sequence of compact subsets of $M_{\xi_0}^+$ such that the diameter of $\mathscr{D}_n(\xi_0\cdots\xi_n)\cap\mathscr{D}_{-n}(\xi_{-n}\cdots\xi_0)$ is not greater than $C_2\theta^n$ for each $n\geq 1$ in virtue of Lemma 3.1. Thus we see the validity of the existence and the uniqueness of the itinerary problem.

Let η be an element in Σ with $d_\theta(\xi,\eta)=\theta^n$ for some nonnegative integer n. If $n=0$, then we have $|r(x(\xi))-r(x(\eta))|\leq L$ and $|\varphi(x(\xi))-\varphi(x(\eta))|\leq\pi$, where L is the maximum of perimeters of ∂Q_j's. Assume $n\geq 1$. Then we see by definition that $\xi_i=\eta_i$ holds for each i with $|i|\leq n-1$. Therefore $x(\xi)$ and $x(\eta)$ are both contained in $\mathscr{D}_{n-1}(\xi_0\cdots\xi_{n-1})\cap\mathscr{D}_{-(n-1)}(\xi_{-(n-1)}\cdots\xi_0)$. Thus by Lemma 3.1 we obtain

$$|r(x(\xi))-r(x(\eta))|\leq C_2\theta^{-1}d_\theta(\xi,\eta),\quad\text{and}\quad|\varphi(x(\xi))-\varphi(x(\eta))|\leq C_2\theta^{-1}d_\theta(\xi,\eta).$$

Hence if we set $C_3=\max\{L,\pi,C_2\theta^{-1}\}$, we reach the desired inequalities. □

Next we summarize the facts on the structure of the local stable curve and the local unstable curve for $x\in\Omega^+$.

Theorem 3.2 ([7]). *The local stable curve $\gamma^s(x)$ (resp. the local unstable curve $\gamma^u(x)$) for $x\in\Omega^+$ coincides with the set*

$$\bigcap_{n=1}^{\infty}\mathscr{D}_n(\xi_0(x),\dots,\xi_n(x))\quad\left(\text{resp. }\bigcap_{n=1}^{\infty}\mathscr{D}_{-n}(\xi_{-n}(x),\dots,\xi_0(x))\right).$$

and it turns out to be a K-decreasing curve (resp. K-increasing curve) of class C^2 except for its end points. Moreover, there exists a function X^s (resp. X^u) in $C([-\pi/2,\pi/2])\cap C^1((-\pi/2,\pi/2))$ such that $\gamma^s(x)$ (resp. $\gamma^u(x)$) is expressed by the differential equation

$$\frac{dr}{d\varphi}(\varphi)=X^s(\varphi),\quad\left(\text{resp. }\frac{dr}{d\varphi}(\varphi)=X^u(\varphi)\right)\quad\text{for each }\varphi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

with respect to the appropriately chosen (r,φ)-coordinates.

Proof. Set

$$\begin{cases}X_1(y,s)=\dfrac{(kt^++c)s+t^+}{(k_1kt^++k_1c+c_1k)s+k_1t^++c_1} & \text{if } y\in\mathscr{D}_1 \text{ and } s\geq 0,\\[2ex] X_{-1}(y,s)=\dfrac{(-kt^-+c)s+t^-}{(k_{-1}kt^--k_{-1}c-c_{-1}k)s-k_{-1}t^-+c_{-1}} & \text{if } y\in\mathscr{D}_{-1} \text{ and } s\leq 0.\end{cases}\tag{3.1}$$

Inductively we can define for $n\geq 1$

$$\begin{cases}X_{n+1}(y,s)=X_1(T^ny,X_n(y,s)) & \text{if } y\in\mathscr{D}_n \text{ and } s\geq 0,\\ X_{-(n+1)}(y,s)=X_{-1}(T^{-n}y,X_{-n}(y,s)) & \text{if } y\in\mathscr{D}_{-n} \text{ and } s\leq 0.\end{cases}\tag{3.2}$$

We notice that if $\gamma^u(x)$ (resp. $\gamma^s(x)$) is expressed by the equation $r=r(\varphi)$, $\varphi\in[-\pi/2,\pi/2]$, we must have

$$\frac{dr}{d\varphi}(\varphi)=X_n\left(T^{-n}(r,\varphi),\frac{dr_{-n}}{d\varphi_{-n}}\right)\quad\left(\text{resp. }\frac{dr}{d\varphi}(\varphi)=X_{-n}\left(T^n(r,\varphi),\frac{dr_n}{d\varphi_n}\right)\right)$$

in $(-\pi/2,\pi/2)$.

Here we consider the case of $\gamma^u(x)$. $\gamma^s(x)$ is treated in the same way. By definition $\gamma^u(x)$ is a curve consisting of all points y such that the Euclidean distance between $T^{-n}x$ and $T^{-n}y$ goes to 0 as $n \to \infty$. By Lemma 3.1 (2), we can verify that $\gamma^u(x)$ is a K-increasing curve and

$$\gamma^u(x) = \bigcap_{n=1}^{\infty} \mathscr{D}_{-n}(\xi_{-n}(x)\cdots\xi_0(x)).$$

Let $\gamma^u(x)$ be expressed as $r = r(\varphi)$, $-\pi/2 \le \varphi \le \pi/2$. In this stage, we do not know whether $r(\varphi)$ is of class C^2 in $-\pi/2 < \varphi < \pi/2$ or not, but we know that it is absolutely continuous on $[-\pi/2, \pi/2]$. Thus using the approximation by C^1 function, we can show that $T^{-n}\gamma^u(x)$ is K-increasing for each $n \ge 1$ and

$$\frac{dr}{d\varphi}(\varphi) = X_n\left(r_{-n}, \varphi_{-n}, \frac{dr_{-n}}{d\varphi_{-n}}(\varphi_n)\right)$$

holds for alomost every φ, where we denote $T^{-n}(r,\varphi)$ by (r_{-n}, φ_{-n}). Therefore if we can show that $X_n(r_{-n}, \varphi_n, a_{-n})$ converges to some function uniformly on any compact set in $(-\pi/2, \pi/2)$, and the limit function $X(\varphi)$ is independent of the choice of the sequence $\{a_{-n}\}$ with $1/K_{\max} \le a_{-n} \le 1/k_{\min}$, then we can easily see that the function $r(\varphi)$ is of class C^1.

Differentiating X_1 by s, we hve

$$\begin{aligned}\frac{\partial X_1}{\partial s} &= \frac{cc_1}{(c_1 + k_1 t^+ + (kk_1 t^+ + k_1 c + c_1 k)s)^2} \\ &= \frac{c}{c_1} \frac{1}{\left(1 + \frac{k_1 t^+}{c_1} + \frac{(kk_1 t^+ + k_1 c + c_1 k)s}{c_1}\right)^2}.\end{aligned}$$

It follows that

$$0 \le \frac{\partial X_1}{\partial s} \le \frac{1}{(k_1 t^+)^2} \quad \text{and} \quad 0 \le \frac{\partial X_1}{\partial s} \le \frac{c}{c_1}\theta^2. \tag{3.3}$$

Therefore, if $m > n$, we obtain

$$\begin{aligned}&|X_m(r,\varphi,a_{-m}) - X_n(r,\varphi,a_{-n})| \\ &\quad = |X_1(r_{-1},\varphi_{-1}, X_{m-1}(r_{-m},\varphi_{-m},a_{-m})) - X_1(r_{-1},\varphi_{-1}, X_{n-1}(r_{-n},\varphi_{-n},a_{-n}))| \\ &\quad \le \frac{c_{-1}}{c}\theta^2 |X_{m-1}(r_{-m},\varphi_{-m},a_{-m}) - X_{n-1}(r_{-n},\varphi_{-n},a_{-n})| \\ &\quad \le \frac{c_{-(n-1)}}{c}\theta^{2n} |X_{m-n}(r_{-m},\varphi_{-m},a_{-m}) - a_{-n}| \\ &\quad \le \frac{c_{-n}}{c}\frac{2}{k_{\min}}\theta^{2n}\end{aligned}$$

by using the Mean Value Theorem repeatedly. Hence we have seen that the sequence of continuous functions $X_n(r(\varphi), \varphi, a_{-n})$ converges to a continuous function $X(\varphi)$ which is independent of the choice of $\{a_{-n}\}$ uniformly on any compact set in $(-\pi/2, \pi/2)$. Now we have proved $\gamma^u(x)$ is a K-increasing curve of class C^1 except for its end points.

In order to show $\gamma^u(x)$ is of class C^2 except for its end points, we need to prove $X(\varphi)$ is class C^1 in $(-\pi/2, \pi/2)$. Recall that $X(\varphi)$ is the uniform limit of the sequence of functions $\{X_n(r_{-n}, \varphi_{-n}, a)\}$ on any compact set in $(-\pi/2, \pi/2)$, where a is any number with $1/K_{\max} \le a \le 1/k_{\min}$. From the equations (3.1) and (3.2) combining with the formula (2.1), it is clear that $X_n(r_{-n}, \varphi_{-n}, a)$ is of class C^1 in the variable φ if each scatterer has C^3 boundary. Thus for $n \ge 2$ we have

$$\begin{aligned}
&\frac{d}{d\varphi}(X_n(r_{-n}, \varphi_{-n}, a)) \\
&\quad = \sum_{k=1}^{n} \left(\prod_{i=2}^{k} \frac{\partial X_1}{\partial s}(T^{-(i-1)}y, X_{n-i+1}(T^{-n}y, a)) \right) \times \\
&\quad\quad \times \left(\frac{\partial X_1}{\partial r_{-k}}(T^{-k}y, X_{n-k}(T^{-n}y, a)) \frac{dr_{-k}}{d\varphi_{-k}} + \frac{\partial X_1}{\partial \varphi_{-k}}(T^{-k}y, X_{n-k}(T^{-n}y, a)) \right),
\end{aligned}$$

where we denote (r_{-i}, φ_{-i}) by $T^{-i}y$ for $i = 1, 2, \ldots, n$ and we regard $\prod_{i=2}^{1} \cdot$ as 1 for convenience.

Let us assume that $\varphi \in [-\alpha, \alpha]$ for some $0 < \alpha < \pi/2$. The no eclipse condition (H.2) implies $|\varphi_i| \ge \varphi_0$ for some $\varphi_0 \in (0, \pi/2)$ depending only on Q for any $i = -1, -2, \ldots$. Thus there is a positive number $C(\alpha)$ depending only on Q and α such that

$$\begin{aligned}
&\left| \prod_{i=2}^{k} \frac{\partial X_1}{\partial s}(T^{-(i-1)}y, X_{n-i+1}(T^{-n}y, a)) \right| \le \frac{1}{\cos\alpha} \theta^{2(k-1)}, \\
&\left| \frac{\partial X_1}{\partial r_{-k}}(T^{-k}y, X_{n-k}(T^{-n}y, a)) \frac{dr_{-k}}{d\varphi_{-k}} + \frac{\partial X_1}{\partial \varphi_{-k}}(T^{-k}y, X_{n-k}(T^{-n}y, a)) \right| \le C(\alpha), \qquad (3.4) \\
&\left| \frac{d\varphi_{-k}}{d\varphi} \right| \le \theta^k
\end{aligned}$$

for any $y = (r, \varphi) \in \gamma^u(x)$ with $\varphi \in [-\alpha, \alpha]$. Note that the first inequality follows from (3.3) and we have used the fact that T^{-i}, $i = 1, 2, \ldots$ are all K-increasing. Since $X_{n-i+1}(T^{-n}y, a)$ converges uniformly on $[-\alpha, \alpha]$ as $n \to \infty$, so does $(dX_n/d\varphi)(T^{-n}y, a)$ in virtue of (3.4) above. Hence we have seen that $r = r(\varphi)$ is of class C^2. □

The facts in Remark 3.1 below will be used frequently in our argument.

Remark 3.1. If we just want to prove $\gamma^s(x)$ and $\gamma^u(x)$ are of class C^1 in Theorem 3.1, it is enough to assume that the boundary of each scatterer is of class C^2. The following are consequences of C^3 assumption of the boundary.

(1) T is a C^2 diffeomorphism from $\operatorname{int}\mathscr{D}_1$ onto $\operatorname{int}\mathscr{D}_{-1}$ in virtue of the expression (2.1) of the Jacobi matrix of T.

(2) In Lemma 2.1, if we assume that the curve γ is of class C^2, then the image curves $T\gamma$ and $T^{-1}\gamma$ are of class C^2 from (1) above.

(3) In Theorem 3.2, the local stable curve and the local unstable curve of each point in Ω^+ are shown to be of class C^2 except for their end points. As we noticed, if x

is an element in $\mathscr{D}_1 \cap \mathscr{D}_{-1}$, then we can find φ_0 with $0 < \varphi_0 < \pi/2$ depending only on Q such that $|\varphi(x)| < \varphi_0$ holds in virtue of the no eclipse condition (H.2). Thus

$$\inf_{x \in \mathscr{D}_1 \cap \mathscr{D}_{-1}} \cos \varphi(x) > C_4 \tag{3.5}$$

holds for some $C_4 > 0$ depending only on Q. Thus if we look at the proof of Theorem 3.2 carefully, it is not hard to see that there exists a positive number C_5 depending only on Q such that

$$\sup_{x \in \gamma^u \cap \mathscr{D}_1} \left| \frac{d^2 r}{d\varphi^2}(\varphi(x)) \right| \le C_5 \tag{3.6}$$

holds for any local unstable curve γ^u expressed as $r = r(\varphi)$, $-\pi/2 < \varphi < \pi/2$.

4. Construction of a K-stable foliation for (Ω^+, T).

The purpose of this section is to construct a K-stable foliation for the set Ω^+. A K-stable foliation for Ω^+ in this paper is a foliation $\mathscr{F}$ supported on $\mathscr{D}_1$ satisfying the following conditions:

($\mathscr{F}$.1) Each leaf of $\mathscr{F}$ is a K-decreasing curve.

($\mathscr{F}$.2) For any $x \in \Omega^+$, the leaf $\mathscr{F}(x)$ containing x coincides with the local stable curve $\gamma^s(x)$.

($\mathscr{F}$.3) For any point $x \in \mathscr{D}_2$, $T\mathscr{F}(x) \subset \mathscr{F}(Tx)$ holds.

($\mathscr{F}$.4) $\mathscr{F}$ is a Lipschitz continuous foliation on $\mathscr{D}_1$ with respect to the Euclidean distance in the (r, φ)-coordinates.

The construction is divided into several steps. Except for the proof of Lipschitz continuity, we follow the argument in Palis and Takens [**11**, Chapter 2]. Now let us begin the first step (see Figure 4.1 throughout the construction).

Step 1. We start with giving an initial foliation $\mathscr{F}_1$ supported on $\mathscr{D}_1$. Recall that $\mathscr{D}_1$ is written as

$$\mathscr{D}_1 = \bigcup_{ij \in \mathscr{W}_2} \mathscr{D}_1(ij).$$

For each j, we can choose the base point in ∂Q_j so that with respect to the corresponding (r, φ)-coordinates $\mathscr{D}_1(ji)$ and $\mathscr{D}_{-1}(ij)$ can be identified with quadrilaterals in the (r, φ)-plane for all $i \neq j$. In the other words such a choice of the base points enable us to carry out our construction as if $\mathscr{D}_1(ji)$ and $\mathscr{D}_{-1}(ij)$ themselves are quadrilaterals in the (r, φ)-plane. In this sense, each $\mathscr{D}_1(ij)$ is a quadrilateral whose boundary $\partial\mathscr{D}_1(ij)$ consists of two curves parallel to the r-axis, say $\gamma_t(ij)$ and $\gamma_b(ij)$ and two K-decreasing curves, say $\gamma_l(ij)$ and $\gamma_r(ij)$. Similarly, each $\mathscr{D}_{-1}(ij)$ is a quadrilateral whose boundary $\partial\mathscr{D}_{-1}(ij)$ consists of two curves parallel to the r-axis, say $\delta_t(ij)$ and $\delta_b(ij)$ and two K-increasing curves, say $\delta_l(ij)$ and $\delta_r(ij)$.

Our initial foliation $\mathscr{F}_1$ is chosen to be a Lipschitz continuous foliation on $\mathscr{D}_1$ satisfying the following.

($\mathscr{F}_1$.1) $\mathscr{F}_1$ restricted to $\mathscr{D}_1 \backslash (\partial\mathscr{D}_1 \cup \partial\mathscr{D}_2)$ is a C^2 foliation.

($\mathscr{F}_1$.2) All leaves of $\mathscr{F}_1$ are K-decreasing.

($\mathscr{F}_1$.3) $\gamma_l(ij), \gamma_r(ij), \gamma_l(ijp)$, and $\gamma_r(ijp)$ are leaves of $\mathscr{F}_1$ for any $ij \in \mathscr{W}_2$ and for any $ijp \in \mathscr{W}_3$.

We can choose such a foliation since $\gamma_l(ij), \gamma_r(ij), \gamma_l(ijp)$, and $\gamma_r(ijp)$ are K-decreasing curves of class C^1 and of class C^2 except for their end points (Recall the equation of the curves in TS and Remark 3.1 (1) and (2)).

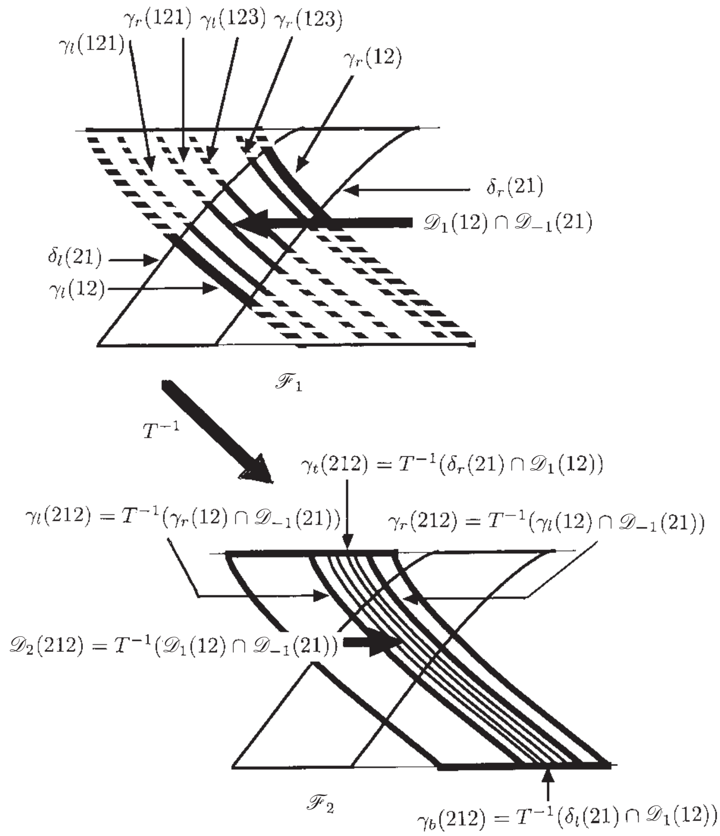

Figure 4.1.

Step 2. Inductively, we can construct a sequence of foliations $\mathscr{F}_n$ $(n \geq 2)$ on $\mathscr{D}_1$ with the following properties:

($\mathscr{F}_n$.1) $\mathscr{F}_n$ restricted to the set $\mathscr{D}_1 \setminus \bigcup_{k=1}^{n+1} \partial\mathscr{D}_k$ is a C^2 foliation.

($\mathscr{F}_n$.2) All leaves on $\mathscr{D}_1$ are K-decreasing.

($\mathscr{F}_n$.3) On the set $\mathscr{D}_k \setminus \mathscr{D}_{k+1}$, we have $\mathscr{F}_k|_{\mathscr{D}_k \setminus \mathscr{D}_{k+1}} = \mathscr{F}_n|_{\mathscr{D}_k \setminus \mathscr{D}_{k+1}}$ for each $1 \leq k \leq n-1$.

($\mathscr{F}_n$.4) Leaves of $\mathscr{F}_n$ on the set $\mathscr{D}_n$ are T^{-1}-images of leaves of $\mathscr{F}_{n-1}$ on the set $\mathscr{D}_{-1} \cap \mathscr{D}_{n-1}$.

If we are given a sequence of foliations $\mathscr{F}_1, \ldots, \mathscr{F}_n$ with initial foliation $\mathscr{F}_1$ for some $n \geq 1$, then we have to define $\mathscr{F}_{n+1}$ as follows because of the desired properties ($\mathscr{F}_{n+1}$.3) and ($\mathscr{F}_{n+1}$.4). First, we employ the leaves of $\mathscr{F}_n$ on $\mathscr{D}_1 \setminus \mathscr{D}_{n+1}$ as those of $\mathscr{F}_{n+1}$. Next, on the set $\mathscr{D}_{n+1}$, we employ the T^{-1}-images of leaves of $\mathscr{F}_n$ restricted to $\mathscr{D}_{-1} \cap \mathscr{D}_n$. Then it is not hard to see that the foliation $\mathscr{F}_{n+1}$ obtained above satisfies ($\mathscr{F}_{n+1}$.1) and ($\mathscr{F}_{n+1}$.2) in virtue of the following facts.

$\bigcup_{k=1}^{n} \partial\mathscr{D}_k$ consists of leaves $\gamma_l(w)$ and $\gamma_r(w)$ with $w \in \mathscr{W}_{k+1}$, $1 \leq k \leq n$ and the curves $\gamma_t(ij)$ and $\gamma_b(ij)$ with $ij \in \mathscr{W}_2$, where $\gamma_l(w)$ and (resp. $\gamma_r(w)$) denotes the K-decreasing curve appearing as the left (resp. right) side of the quadrilateral $\mathscr{D}_k(w)$ in terms of the

(r, φ)-coordinates for each $w \in \mathscr{W}_{k+1}$. The (r, φ)-coordinates being used in the above are the same as we choose in Step 1. Let $j_0 j_1 \cdots j_{n+1}$ be any element in $\mathscr{W}_{n+2}$. Then we have

$$T^{-1}(\gamma_l(j_1 \cdots j_{n+1}) \cap \mathscr{D}_{-1}(j_0 j_1)) = \gamma_r(j_0 \cdots j_{n+1})$$

$$T^{-1}(\gamma_r(j_1 \cdots j_{n+1}) \cap \mathscr{D}_{-1}(j_0 j_1)) = \gamma_l(j_0 \cdots j_{n+1})$$

$$T^{-1}(\delta_l(j_0 j_1) \cap \mathscr{D}_n(j_1 \cdots j_{n+1})) = \gamma_b(j_0 \cdots j_{n+1})$$

$$T^{-1}(\delta_r(j_0 j_1) \cap \mathscr{D}_n(j_1 \cdots j_{n+1})) = \gamma_t(j_0 \cdots j_{n+1}).$$

In addition, T^{-1} is a C^2 diffeomorphism from $\operatorname{int}(\mathscr{D}_n(j_1 \cdots j_{n+1}) \cap \mathscr{D}_{-1}(j_0 j_1))$ onto $\operatorname{int} \mathscr{D}_{n+1}(j_0 \cdots j_{n+1})$ as well as a homeomorphism from $\mathscr{D}_n(j_1 \cdots j_{n+1}) \cap \mathscr{D}_{-1}(j_0 j_1)$ onto $\mathscr{D}_{n+1}(j_0 \cdots j_{n+1})$.

Step 3. We define $\mathscr{F}$ on $\mathscr{D}_1$ as follows.

For each positive integer n, $\mathscr{F}$ restricted to $\mathscr{D}_n \backslash \mathscr{D}_{n+1}$ is defined so that $\mathscr{F}|_{\mathscr{D}_n \backslash \mathscr{D}_{n+1}} = \mathscr{F}_n|_{\mathscr{D}_n \backslash \mathscr{D}_{n+1}}$ holds. Then it remains to define $\mathscr{F}$ on the set

$$\mathscr{D}_1 \backslash \left(\bigcup_{n=1}^{\infty} (\mathscr{D}_n \backslash \mathscr{D}_{n+1}) \right) = \bigcap_{n=1}^{\infty} \mathscr{D}_n = \bigcup_{x \in \Omega^+} \gamma^u(x).$$

Therefore the leaf $\mathscr{F}(x)$ passing through $x \in \Omega^+$ must be defined so that $\mathscr{F}(x) = \gamma^s(x)$. Then it is obvious by definition that $\mathscr{F}$ satisfies ($\mathscr{F}$.1), ($\mathscr{F}$.2), and ($\mathscr{F}$.3). We note that for positive integers n and k with $k \le n$, we see that leaves of $\mathscr{F}$ restricted to $\mathscr{D}_n \backslash \mathscr{D}_{n+1}$ are T^{-k}-image of leaves of the foliation $\mathscr{F}$ restricted to $(\mathscr{D}_{n-k} \backslash \mathscr{D}_{n+1-k}) \cap \mathscr{D}_{-k}$.

We prove that $\mathscr{F}$ restricted to the set $\mathscr{D}_1$ satisfies the Lipschitz continuity ($\mathscr{F}$.4) in the next section.

5. Proof of Lipschitz continuity.

In what follows, $\mathscr{F}$ restricted to the set $\mathscr{D}_1$ will be denoted by the same notation $\mathscr{F}$. We fix the base points which are chosen in Step 1 of the construction of $\mathscr{F}$. Namely, to each j, we assign the (r, φ)-plane P_j and we treat $\mathscr{D}_1(ji)$ and $\mathscr{D}_{-1}(ij)$ as if they are quadrilaterals in P_j by means of the fixed (r, φ)-coordinates.

Let γ be a decreasing curve of class C^1 expressed as $r = r(\varphi)$, $\alpha \le \varphi \le \beta$ with $-\pi/2 < \alpha < \beta < \pi/2$. We consider a nonnegative valued function $u = u_0$ in φ which is continuous on $[\alpha, \beta]$ and C^1 in (α, β). Assume that T^n can be defined on γ for a positive integer n. Define u_i, $i = 0, 1, \ldots, n$ inductively by

$$u_{i+1} = k_{i+1} + \cfrac{c_{i+1}}{t_i^+ + \cfrac{c_i}{u_i + k_i}},$$

where the abbreviation $k_i = k(T^i(r(\varphi), \varphi))$, $t_i^+ = t_i^+((T^i(r(\varphi), \varphi))$ etc. are the same as before.

First of all we evaluate $(d/d\varphi) \log(u_n + k_n)$.

LEMMA 5.1. *Let γ be a C^1 curve on which T^{n+1} is defined and $T^n\gamma$ is K-decreasing. Let $u = u_0$ be a nonnegative valued function which is of class C^1 along the curve γ. Let u_i, $i = 0, 1, \ldots, n$ be defined as above. Then there are constant C_6 and C_7 depending only on the domain Q such that*

$$\left|\frac{d}{d\varphi}\log(u_n + k_n)\right| \le C_6\theta^n + C_7\theta^{2n}\left|\frac{d}{d\varphi}\log(u_0 + k_0)\right|$$

holds. In particular, if T^{-1} is also defined on γ, then we can put $C_7 = 1$.

PROOF. Put

$$b_{i+1} = \frac{2k_{i+1}}{c_{i+1}}\left(t_i^+ + \frac{c_i}{a_i + k_i}\right)$$

for $i = 0, 1, \ldots, n-1$. Since

$$u_{i+1} + k_{i+1} = 2k_{i+1} + \frac{c_{i+1}}{t_i^+ + \dfrac{c_i}{u_i + k_i}}$$

holds, we can obtain

$$\begin{aligned}\frac{(u_{i+1} + k_{i+1})\left(t_i^+ + \dfrac{c_i}{u_i + k_i}\right)}{c_{i+1}} &= \frac{2k_{i+1}}{c_{i+1}}\left(t_i^+ + \frac{c_i}{u_i + k_i}\right) + 1 \\ &= b_{i+1} + 1 \ge 1 + 2k_{\min}t_{\min}. \end{aligned} \tag{5.1}$$

For each integer $i = 0, 1, \ldots, n-1$, we have

$$\begin{aligned}&\frac{d}{d\varphi_{i+1}}\log(u_{i+1} + k_{i+1}) \\ &\quad= \frac{1}{u_{i+1} + k_{i+1}}\frac{d}{d\varphi_{i+1}}\left(2k_{i+1} + \frac{c_{i+1}}{t_i^+ + \dfrac{c_i}{u_i + k_i}}\right) \\ &\quad= \frac{2}{u_{i+1} + k_{i+1}}\frac{dk_{i+1}}{dr_{i+1}}\frac{dr_{i+1}}{d\varphi_{i+1}} - \frac{s_{i+1}}{u_{i+1} + k_{i+1}}\frac{1}{t_i^+ + \dfrac{c_i}{u_i + k_i}} \\ &\qquad - \frac{1}{u_{i+1} + k_{i+1}}\frac{c_{i+1}}{\left(t_i^+ + \dfrac{c_i}{u_i + k_i}\right)^2}\left(\frac{dt_i^+}{dr_i}\frac{dr_i}{d\varphi_i} - \frac{s_i}{u_i + k_i}\right)\frac{d\varphi_i}{d\varphi_{i+1}} \\ &\qquad + \frac{c_{i+1}}{(u_{i+1} + k_{i+1})\left(t_i^+ + \dfrac{c_i}{u_i + k_i}\right)^2}\frac{c_i}{u_i + k_i}\frac{d\varphi_i}{d\varphi_{i+1}}\frac{d}{d\varphi_i}\log(u_i + k_i) \\ &\quad= A_{i+1}(\varphi_{i+1}) + \frac{1}{(1 + b_{i+1})\left(1 + \dfrac{t_i^+(a_i + k_i)}{c_i}\right)}\frac{d\varphi_i}{d\varphi_{i+1}}\frac{d}{d\varphi_i}\log(u_i + k_i), \end{aligned} \tag{5.2}$$

where s_i denotes $\sin\varphi_i$ as c_i does $\cos\varphi_i$ and $A_{i+1}(\varphi_{i+1})$ denotes the sum of the three terms after the second "=". Therefore we have

$$\begin{aligned}\frac{d}{d\varphi}&\log(u_n+k_n)\\ &= A_n(\varphi_n)\frac{d\varphi_n}{d\varphi} + \frac{1}{(1+b_n)\left(1+\frac{t_{n-1}^{+}(u_{n-1}+k_{n-1})}{c_{n-1}}\right)}\frac{d}{d\varphi}\log(u_{n-1}+k_{n-1}).\end{aligned} \tag{5.3}$$

Since γ is contained in the set $\mathscr{D}_{n+1}$, we see that $c_i \geq C_4$, $i=1,2,\dots,n$ holds in virtue of the inequality (3.5) which is a consequence of no eclipse condition (H.2). But in the case when $i=0$, we can not use the condition (H.2). We have to proceed with the proof being careful about this difference. Recall the following equalities in Lemma 2.1.

$$\frac{d\varphi_i}{d\varphi_{i+1}} = k_i\frac{c_{i+1}}{c_i}\frac{dr_{i+1}}{d\varphi_{i+1}} - \left(1-\frac{t_{i+1}^{-}k_{i+1}}{c_i}\right)\left(1-k_{i+1}\frac{dr_{i+1}}{d\varphi_{i+1}}\right). \tag{5.4}$$

Since $T^i\gamma$ is a K-decreasing curve for $i=0,\dots,n$ and $c_i \geq C_4$ holds for $i=1,2,\dots,n$, we can see that for each $i=1,2,\dots,n-1$

$$-\theta^{-1} \geq \frac{d\varphi_i}{d\varphi_{i+1}} \geq -C_8 \tag{5.5}$$

for some $C_8 \geq 1$ depending only on Q. In fact, the second inequality in (5.5) follows from the equality in (5.4) if we set

$$C_8 = \frac{K_{\max}}{C_4k_{\min}} + \left(1+\frac{\operatorname{diam}(\partial Q)K_{\max}}{C_4k_{\min}}\right)\left(1+\frac{K_{\max}}{C_4k_{\min}}\right)$$

and the first inequality in (5.5) follows from Lemma 2.2. Hence we can find a positive number C_9 depending only on Q such that

$$|A_{i+1}| \leq C_9$$

holds for each $i=1,2,\dots,n-1$.

On the other hand, by the inequality (5.1), it is obvious that

$$(1+b_{i+1})\left(1+\frac{t_i^{+}(u_i+k_i)}{c_i}\right) \geq (1+k_{\min}t_{\min})^2 = \theta^{-2}$$

holds for $i=0,1,2,\dots,n$. In addition, since $T^i\gamma$ is decreasing, we have

$$0 > \frac{d\varphi_i}{d\varphi} > -\theta^i$$

holds for $i=1,2,\dots,n$ by Lemma 2.2. Thus if we use the equation (5.3) repeatedly, we can reach the inequality

$$\left|\frac{d}{d\varphi}\log(u_n+k_n)\right| \leq C_{10}\theta^n + \theta^{2(n-1)}\left|\frac{d}{d\varphi}\log(u_1+k_1)\right| \tag{5.6}$$

with $C_{10} = C_9/(1-\theta)$.

Now we consider the case when $i = 0$ in the formula (5.3). Since we do not have a lower bound for $\cos\varphi$ independent of φ, we can not have the estimate (5.5) for $|d\varphi/d\varphi_1|$ if T^{-1} is not defined on γ. Consequently, it seems hard to obtain an upper bound for $|A_1(\varphi_1)|$ depending only on Q. But it is easy to see from the explicit form (5.2) that there is a poitive number C_{11} depending only on Q such that

$$\left|A_1(\varphi_1)\frac{d\varphi_1}{d\varphi}\right| \le C_{11}.$$

Therefore we have

$$\left|\frac{d}{d\varphi}\log(u_1 + k_1)\right| \le C_{11} + \theta^2\left|\frac{d}{d\varphi}\log(u_0 + k_0)\right|. \tag{5.7}$$

Combining (5.6) with (5.7), we arrive at the desired result.

Finally, if T^{-1} is defined on γ, we see that $c_0 = \cos\varphi > C_4$ follows from the condition (H.2). Thus (5.5) is true for $i = 0$. This implies that we can proceed to one step further in (5.6). Now the proof of Lemma 5.1 is complete. □

Next we consider the following situation: Let γ and γ' be K-decreasing curves of class C^1 contained in the same connected component of $\mathscr{D}_1$, say $\mathscr{D}_1(ij)$ for some $ij \in \mathscr{W}_2$. Note that we are working on the direct sum of the (r, φ)-planes $P_1, \dots, P_J$ as mentioned in the beginning of the present section. Therefore we regard $\mathscr{D}_1(ij)$ as a quadrilateral in P_i. Assume that γ and γ' do not intersect each other. Let $\bar{x}$ and $\hat{x}$ be points on γ and $\bar{y}$ and $\hat{y}$ be points on γ'. We assume that the line segment $\bar{\gamma}$ joining $\bar{x}$ and $\bar{y}$ is parallel to the r-axis. We also assume that the line segment $\hat{\gamma}$ joining $\hat{x}$ and $\hat{y}$ is parallel to the r-axis. Let $\mathscr{E}$ be the quadrilateral enclosed by $\gamma, \gamma', \bar{\gamma}$, and $\hat{\gamma}$ whose vertices are labelled as $\bar{x}, \hat{x}, \hat{y}$, and $\bar{y}$ in counterclockwise order (see Figure 5.1). Then we can show:

Lemma 5.2. *Let $\mathscr{E}$ be the quadrilateral as above. Further we assume that T^{n+1} is defined on $\mathscr{E}$ for some $n \ge 1$. Then there exists a constant $C_{12} \ge 1$ depending only on the domain Q such that*

$$\frac{|r(\hat{x}) - r(\hat{y})|}{|r(\bar{x}) - r(\bar{y})|} \in [C_{12}^{-1}, C_{12}]\frac{\Theta(T^n\hat{\gamma})}{\Theta(T^n\bar{\gamma})}, \tag{5.8}$$

where $a \in [b, c]d$ means $bd \le a \le cd$.

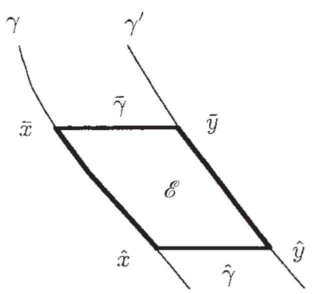

Figure 5.1.

Proof. In order to avoid trifling confusions, we put $^{-}$ and $\hat{}$ on variables corresponding to $\bar{\gamma}$ and $\hat{\gamma}$, respectively. For example, if $x \in \bar{\gamma}$, $\bar{c}_i = \cos\varphi(T^i x)$, $\bar{k}_i = k(T^i x)$ and so on.

First of all we note that $T\bar{\gamma}$ and $T\hat{\gamma}$ are in $\mathscr{D}_{-1} \cap \mathscr{D}_n$. Therefore $T^i\bar{\gamma}$ and $T^i\hat{\gamma}$ are all K-increasing and

$$\bar{c}_i \geq C_4, \quad \text{and} \quad \hat{c}_i \geq C_4 \tag{5.9}$$

for $i = 1, 2, \ldots, n$ by Lemma 2.2 and the inequality (3.5). In addition, these curves are of class C^2 except for their end points at worst since T^i is a C^2 diffeomorphism from $\mathrm{int}(\mathscr{D}_i)$ onto $\mathrm{int}(\mathscr{D}_{-i})$.

Next we show that there is a number $C_{13} \geq 1$ depending only on Q such that

$$|r(\bar{x}) - r(\bar{y})| \in [C_{13}^{-1}, C_{13}]\Theta(T\bar{\gamma}) \quad \text{and} \quad |r(\hat{x}) - r(\hat{y})| \in [C_{13}^{-1}, C_{13}]\Theta(T\hat{\gamma}). \tag{5.10}$$

Indeed, we have

$$\frac{d\bar{\varphi}_1}{d\bar{r}} = -\bar{k} - \frac{1}{\bar{c}_1}(\bar{c} + \bar{t}^+\bar{k})\bar{k}_1$$

from Lemma 2.1. Thus we can easily see that (5.10) holds for $\bar{\gamma}$ by the inequality (5.9). By exactly the same reason we obtain (5.10) for $\hat{\gamma}$.

Now we show that there exists a constant $C_{14} \geq 1$ depending only on Q such that

$$\begin{aligned} \Theta(T\bar{\gamma}) &\in [C_{14}^{-1}, C_{14}]\frac{d\bar{\varphi}_1}{d\bar{\varphi}_n}(\varphi(T^n\bar{x}))\Theta(T^n\bar{\gamma}) \\ \Theta(T\hat{\gamma}) &\in [C_{14}^{-1}, C_{14}]\frac{d\hat{\varphi}_1}{d\hat{\varphi}_n}(\varphi(T^n\hat{x}))\Theta(T^n\hat{\gamma}). \end{aligned} \tag{5.11}$$

Take any $\bar{x}(i) \in T^i\bar{\gamma}$ for $i = 1, \ldots, n$. Applying the formula in Lemma 2.1 to $\bar{\varphi}_i/\bar{\varphi}_{i+1}$ with $i = 1, 2, \ldots, n-1$, we have

$$\frac{d\varphi_i}{d\varphi_{i+1}} = k_i \frac{c_{i+1}}{c_i}\frac{dr_{i+1}}{d\varphi_{i+1}} - \left(1 - \frac{t_{i+1}^- k_{i+1}}{c_i}\right)\left(1 - k_{i+1}\frac{dr_{i+1}}{d\varphi_{i+1}}\right).$$

Then it is not hard to see that

$$\left|\frac{d\bar{\varphi}_i}{d\bar{\varphi}_{i+1}}(\varphi(T^{i+1}\bar{x})) - \frac{d\bar{\varphi}_i}{d\bar{\varphi}_{i+1}}(\varphi(\bar{x}(i+1)))\right| \leq C_{15}\Theta(T^{i+1}\bar{\gamma}) \leq C_{16}\theta^{n-i}$$

holds for $i = 1, \ldots, n-1$ in virtue of the Mean Value Theorem and the estimate (3.6), where C_{15} and C_{16} are positive numbers depending only on Q. Since $T^i\bar{\gamma}$ is K-increasing for each $i = 1, \ldots, n-1$, we have

$$-C_8 \leq \frac{d\bar{\varphi}_{i+1}}{d\bar{\varphi}_i} \leq -\theta^{-1}$$

by the same reason as (5.4) from the equation

$$\frac{d\varphi_{i+1}}{d\varphi_i} = -k_{i+1}\frac{c_i}{c_{i+1}}\frac{dr_i}{d\varphi_i} - \left(1 + \frac{t_i^+ k_i}{c_{i+1}}\right)\left(1 + k_i\frac{dr_i}{d\varphi_i}\right)$$

in Lemma 2.1. Hence we obtain for $i = 1, \dots, n-1$

$$\left| \frac{\dfrac{d\bar{\varphi}_i}{d\bar{\varphi}_{i+1}}(T^{i+1}\bar{x})}{\dfrac{d\bar{\varphi}_i}{d\bar{\varphi}_{i+1}}(\bar{x}(i+1))} - 1 \right| \le C_{17}\theta^{n-i}$$

for any $\bar{x}(i+1) \in T^{i+1}\bar{\gamma}$. Consequently, we arrive at the inequality

$$\left| \frac{d\bar{\varphi}_1}{d\bar{\varphi}_n}(\varphi(\bar{x}(n))) \right| \in [C_{18}^{-1}, C_{18}] \left| \frac{d\bar{\varphi}_1}{d\bar{\varphi}_n}(\varphi(T^n\bar{x})) \right|, \tag{5.12}$$

where C_{17} and C_{18} are positive constants depending only on Q.

Denote by $I(T\bar{\gamma})$ the interval in the φ-axis corresponding to the curve $T\bar{\gamma}$. Similarly, $I(T^n\bar{\gamma})$ denotes the interval corresponding to $T^n\bar{\gamma}$. Combining the formula

$$\Theta(T\bar{\gamma}) = \int_{I(T\bar{\gamma})} |d\bar{\varphi}_1| = \int_{I(T^n\bar{\gamma})} \left| \frac{d\bar{\varphi}_1}{d\bar{\varphi}_n} \right| |d\varphi_n|$$

with the inequality (5.12), we have the desired inequality (5.11) for $\bar{\gamma}$. (5.11) for $\hat{\gamma}$ is obtained in exactly the same way.

Finally we prove that

$$\left| \frac{d\hat{\varphi}_1}{d\hat{\varphi}_n}(T^n\hat{x}) \right| \in [C_{19}^{-1}, C_{19}] \left| \frac{d\bar{\varphi}_1}{d\bar{\varphi}_n}(T^n\bar{x}) \right| \tag{5.13}$$

holds for some C_{19} depending only on Q. Combining (5.10), (5.11), and (5.13), we can easily obtain the desired estimate (5.8).

Since (5.13) is trivial in the case when $n = 1$, we assume $n \ge 2$ in the sequel. Let γ be expressed as $r = r(\varphi)$, $\alpha \le \varphi \le \beta$. Without loss of generality, we may assume that $\hat{x} = (r(\alpha), \alpha)$ and $\bar{x} = (r(\beta), \beta)$. Consider a function $u = u_0$ identically 0 on γ. Note that the function u_i is defined so that

$$u_i(\varphi(T^i\bar{x})) = \frac{d\bar{\varphi}_i}{d\bar{r}_i}(r(T^i\bar{x})) \quad \text{and} \quad u_i(\varphi(T^i\hat{x})) = \frac{d\hat{\varphi}_i}{d\hat{r}_i}(r(T^i\hat{x})) \tag{5.14}$$

can hold for $i = 0, 1 \dots, n$. It is clear that γ is a K-decreasing curve lying in $\mathscr{D}_n$ and the function u satisfy the assumptions in Lemma 5.1. Therefore we have

$$\left| \frac{d}{d\varphi} \log(u_i + k_i) \right| \le C_6\theta^i + C_7\theta^{2i} \left| \frac{d}{d\varphi} \log k_0 \right| = C_6\theta^i + C_7\theta^{2i} \left| \frac{1}{k_0} \frac{dk}{dr} \frac{dr}{d\varphi} \right| \tag{5.15}$$

for each $i = 0, 1, \dots, n$. Thus from (5.14) and (5.15), it is not hard to show that there exists a positive number C_{20} depending only on Q such that

$$-C_{20}\theta^i - |\hat{k}_i - \bar{k}_i| \le \left| \frac{d\bar{\varphi}_i}{d\bar{r}_i}(r(T^i\bar{x})) - \frac{d\hat{\varphi}_i}{d\hat{r}_i}(r(T^i\hat{x})) \right| \le C_{20}\theta^i + |\hat{k}_i - \bar{k}_i| \tag{5.16}$$

holds for each $i = 0, 1, 2, \dots, n$, where $k_0 = k(x)$, $\bar{k}_i = k(T^i\bar{x})$, and $\hat{k}_i = k(T^i\hat{x})$ with $x = (r(\varphi), \varphi)$. Since the curvature of ∂Q is of class C^1, we have

$$k(T^i\bar{x}) - k(T^i\hat{x}) = k(r_i(\varphi_i(\bar{x}))) - k(r_i(\varphi_i(\hat{x}))) = \frac{dk}{dr}(r_i(\varphi_{i,0})) \frac{dr_i}{d\varphi_i}(\varphi_{i,0})(\varphi_i(\bar{x}) - \varphi_i(\hat{x}))$$

for some $\varphi_{i,0}$ by the Mean Value Theorem. Thus using the fact that $T^i\gamma$ is K-decreasing for each $i = 0, 1, \dots n$, we arrive at the inequality

$$|k(T^i\bar{x}) - k(T^i\hat{x})| \le \max_{r\in\partial Q}\left|\frac{dk}{dr}(r)\right|\frac{\theta^i\pi}{k_{\min}}.$$

This and the inequality (5.16) yield

$$\left|\frac{d\hat{\varphi}_i}{d\hat{r}_i}(r(T^i\hat{x})) - \frac{d\bar{\varphi}_i}{d\bar{r}_i}(r(T^i\bar{x}))\right| \le C_{21}\theta^i \tag{5.17}$$

for each $i = 0, 1, \dots n$ for some C_{21} depending only on Q.

We have to estimate the difference

$$\frac{d\bar{\varphi}_i}{d\bar{\varphi}_{i+1}} = \bar{k}_i\frac{\bar{c}_{i+1}}{\bar{c}_i}\frac{d\bar{r}_{i+1}}{d\bar{\varphi}_{i+1}} - \left(1 - \frac{\bar{t}_{i+1}^-\bar{k}_{i+1}}{\bar{c}_i}\right)\left(1 - \bar{k}_{i+1}\frac{d\bar{r}_{i+1}}{d\bar{\varphi}_{i+1}}\right)$$

from

$$\frac{d\hat{\varphi}_i}{d\hat{\varphi}_{i+1}} = \hat{k}_i\frac{\hat{c}_{i+1}}{\hat{c}_i}\frac{d\hat{r}_{i+1}}{d\hat{\varphi}_{i+1}} - \left(1 - \frac{\hat{t}_{i+1}^-\hat{k}_{i+1}}{\hat{c}_i}\right)\left(1 - \hat{k}_{i+1}\frac{d\hat{r}_{i+1}}{d\hat{\varphi}_{i+1}}\right)$$

for $i = 1, 2, \dots, n-1$. The functions $\bar{c}_i = \cos\bar{\varphi}_i$, $\bar{k}_i = k(\bar{r}_i)$, and $\bar{t}^- = t^-(\bar{r}_i, \bar{\varphi}_i)$ are obtained by substituting $(\bar{r}_i, \bar{\varphi}_i)$ for (r, φ) in the C^1 functions $\cos\varphi, k(r)$, and t^-. Similarly, $\hat{c}_i = \cos\hat{\varphi}_i$, $\hat{k}_i = k(\hat{r}_i)$, and $\hat{t}^- = t^-(\hat{r}_i, \hat{\varphi}_i)$ are obtained by substituting $(\hat{r}_i, \hat{\varphi}_i)$ for (r, φ) in the C^1 functions $\cos\varphi$, $k(r)$, and t^-. Therefore, we can easily estimate the differences $\bar{c}_i$ from $\hat{c}_i$, $\bar{k}_i$ from $\hat{k}_i$, and $\bar{t}^-$ from $\hat{t}^-$ directly. On the other hand, $(d\bar{r}_{i+1}/d\bar{\varphi}_{i+1})$ and $(d\hat{r}_{i+1}/d\hat{\varphi}_{i+1})$ are defined on $T^{i+1}\bar{\gamma}$ and $T^{i+1}\hat{\gamma}$. So at first sight, it seems hard to estimate their difference. But we have already established the estimate (5.17) of the difference $(d\bar{r}_{i+1}/d\bar{\varphi}_{i+1})$ from $(d\hat{r}_{i+1}/d\hat{\varphi}_{i+1})$ for $i = 1, 2, \dots, n-1$. It is easy to see that all the functions in the right hand sides of the equations expressing $d\bar{\varphi}_i/d\bar{\varphi}_{i+1}$ and $d\hat{\varphi}_i/d\hat{\varphi}_{i+1}$ are bounded from above by positive constants depending only on Q. In particular, $\bar{c}_i$ and $\hat{c}_i$ are also bounded from below by C_4 for $i = 1, 2, \dots, n-1$ in virtue of (3.5). Thus if we make use of the elementary inequality of type $|ab - cd| \le |a - c|\,|b| + |c|\,|b - d|$ repeatedly, we can arrive at the inequality

$$\left|\frac{d\hat{\varphi}_i}{d\hat{\varphi}_{i+1}} - \frac{d\bar{\varphi}_i}{d\bar{\varphi}_{i+1}}\right| \le C_{22}\theta^i$$

for each $i = 1, 2, \dots, n-1$ for some constant C_{22} depending only on Q. Since we can show that $|d\bar{\varphi}_i/d\bar{\varphi}_{i+1}| \ge C_8^{-1}$ for $i = 1, 2, \dots, n-1$ by the same way as (5.5), we reach

$$\left|\frac{\dfrac{d\hat{\varphi}_i}{d\hat{\varphi}_{i+1}}(T^{i+1}\hat{x})}{\dfrac{d\bar{\varphi}_i}{d\bar{\varphi}_{i+1}}(T^{i+1}\bar{x})} - 1\right| \le C_{23}\theta^i$$

for each $i = 1, 2, \dots, n-1$. Consequently, we conclude that (5.13) is valid. □

Now we are in a position to prove the main result in this section. The Lipschitz continuity ($\mathscr{F}$.4) turns out to be its easy consequence.

Theorem 5.1. *Let $\bar{x}$ and $\bar{y}$ be distinct points contained in the same connected component of $\mathscr{D}_1$ such that the line segment joining $\bar{x}$ and $\bar{y}$ is parallel to the r-axis. Choose any points $\hat{x} \in \mathscr{F}(\bar{x})$ and $\hat{y} \in \mathscr{F}(\bar{y})$ such that the line segment joining $\hat{x}$ and $\hat{y}$ is parallel to the r-axis. Then*

$$\frac{|r(\hat{x}) - r(\hat{y})|}{|r(\bar{x}) - r(\bar{y})|} \in [C_{24}^{-1}, C_{24}]$$

holds for some constant $C_{24} > 1$ depending only on Q.

Proof. Before we get into the body of the proof, we observe the structure of the quadrilateral $\mathscr{D}_n$ for our convenience (see Figure 5.2 which illustrates the case when $J = 3$ and $w_n = 3$). The quadrilateral $\mathscr{D}_n$ consists of $J(J-1)^{n-1}$ connected components $\mathscr{D}_n(w)$ with $w \in \mathscr{W}_{n+1}$, where $\mathscr{W}_{n+1}$ is the totality of admissible words of length $n+1$. For each $w = w_0 w_1 \cdots w_n \in \mathscr{W}_{n+1}$, $\gamma_l(w)$ (resp. $\gamma_r(w)$) denotes the K-decreasing curve lying on the left (resp. right) side boundary of $\mathscr{D}_n(w)$. Each $\mathscr{D}_n(w)$ contains $J-1$ components $\mathscr{D}_{n+1}(wj)$ of $\mathscr{D}_{n+1}$ and $\mathscr{D}_n(w)$ is divided into $2J-1$ quadrilaterals by $2(J-1)$ curves $\gamma_l(wj)$ and $\gamma_r(wj)$ with $j \neq w_n$, where wj denotes the concatenation of the words w and j. Since we regard $\mathscr{D}_n(w)$ as a quadrilateral in the (r, φ)-plane P_{w_0}, we can give indices to these $2J-1$ quadrilaterals as $\mathscr{E}_n(w)_1, \mathscr{E}_n(w)_2, \ldots, \mathscr{E}_n(w)_{2J-1}$ from the left to the right. Then each connected component of $\mathscr{D}_n \backslash \mathscr{D}_{n+1}$ contained in $\mathscr{D}_n(w)$ conincides with one of the quadrilaterals $\mathscr{E}_n(w)_1, \mathscr{E}_n(w)_3, \ldots, \mathscr{E}_n(w)_{2J-1}$ with odd indices up to boundary curves.

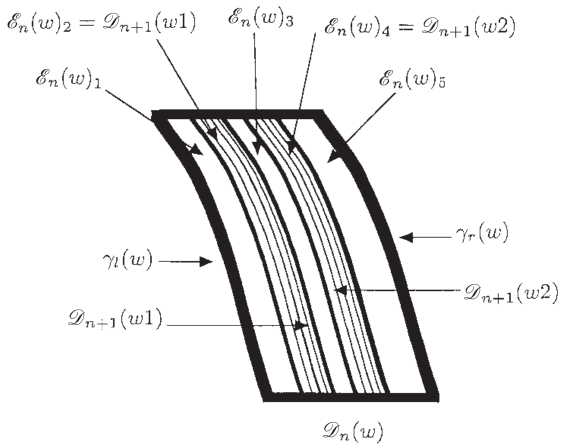

Figure 5.2.

In what follows $\bar{y}$ (resp. $\hat{y}$) denotes the line segment joining $\bar{x}$ and $\bar{y}$ (resp. $\hat{x}$ and $\hat{y}$). There are four possibilities.

Case (i) There exists an integer $N \geq 1$ such that $\bar{x}, \bar{y} \in \mathscr{D}_N$, $\xi_i(\bar{x}) = \xi_i(\bar{y})$ for each i with $0 \leq i \leq N-1$ but $\xi_N(\bar{x}) \neq \xi_N(\bar{y})$.

Case (ii) There exists an integer $N \geq 1$ such that $\bar{x}, \bar{y} \in \mathscr{D}_N \backslash \mathscr{D}_{N+1}$, $\xi_i(\bar{x}) = \xi_i(\bar{y})$ for each i with $0 \leq i \leq N$, and $\bar{x}$ and $\bar{y}$ are contained in distinct connected components of $\mathscr{D}_N \backslash \mathscr{D}_{N+1}$.

Case (iii) There exists an integer $N \geq 1$ such that $\bar{x}, \bar{y} \in \mathscr{D}_N \backslash \mathscr{D}_{N+1}$, $\xi_i(\bar{x}) = \xi_i(\bar{y})$ for each i with $0 \leq i \leq N$, and $\bar{x}$ and $\bar{y}$ are contained in the same connected component of $\mathscr{D}_N \backslash \mathscr{D}_{N+1}$.

Case (iv) There exists an integer $N \geq 1$ such that $\bar{x} \in \mathscr{D}_{N+1}$ (resp. $\bar{y} \in \mathscr{D}_{N+1}$), $\bar{y} \in \mathscr{D}_N \backslash \mathscr{D}_{N+1}$ (resp. $\bar{x} \in \mathscr{D}_N \backslash \mathscr{D}_{N+1}$), and $\xi_i(\bar{x}) = \xi_i(\bar{y})$ for each i with $0 \leq i \leq N$.

Case (i) and Case (ii). First of all, we consider Case (i) and Case (ii). Let Γ be the totality of K-decreasing curves which appear as sides of quadrilaterals $\mathscr{D}_1(w)$ with $w \in \mathscr{W}_2$ or $\mathscr{D}_2(w)$ with $w \in \mathscr{W}_3$. Namely, we can write as

$$\Gamma = \{\gamma_l(w), \gamma_r(w) : w \in \mathscr{W}_2 \cup \mathscr{W}_3\}.$$

We note that for each $ij \in \mathscr{W}_2$, the quadrilateral $\mathscr{D}_1(ij)$ contains $2J$ elements in Γ. Precisely, $\mathscr{D}_1(ij)$ has $\gamma_l(ij)$ and $\gamma_r(ij)$ as its sides and it is divided into $2J-1$ quadrilaterals by $2(J-1)$ curves $\gamma_l(ijk)$ and $\gamma_r(ijk)$ with $j \neq k$. Put

$$\Delta = \min\{\mathrm{dist}(\gamma, \gamma') : \gamma, \gamma' \in \Gamma, \gamma \neq \gamma', \gamma, \gamma' \subset \mathscr{D}_1(w) \text{ for some } w \in \mathscr{W}_2\},$$

where $\mathrm{dist}(\gamma, \gamma')$ denotes the Euclidean distance between the curves γ and γ' with respect to the fixed (r, φ)-coordinates. Clearly Δ is positive.

If $N = 1$, both $\bar{\gamma}$ and $\hat{\gamma}$ join distinct elements in Γ. Therefore we have $\Delta \leq |r(\bar{x}) - r(\bar{y})| \leq L$ and $\Delta \leq |r(\hat{x}) - r(\hat{y})| \leq L$ hold for both Case (i) and Case (ii), where

$$L = \max_{1 \leq j \leq J} \{\text{the perimeter of } \partial Q_j\}.$$

Therefore we obtain

$$\frac{|r(\hat{x}) - r(\hat{y})|}{|r(\bar{x}) - r(\bar{y})|} \in \left[\left(\frac{L}{\Delta}\right)^{-1}, \frac{L}{\Delta}\right]. \tag{5.18}$$

If $N \geq 2$, both $T^{N-1}\bar{\gamma}$ and $T^{N-1}\hat{\gamma}$ join distinct elements in Γ. Thus we see that $\Delta \leq |r(T^{N-1}\bar{x}) - r(T^{N-1}\bar{y})| \leq L$ and $\Delta \leq |r(T^{N-1}\hat{x}) - r(T^{N-1}\hat{y})| \leq L$. In addition, $T^{N-1}\bar{\gamma}$ and $T^{N-1}\hat{\gamma}$ are K-increasing. Therefore it follows that

$$\Delta k_{\min} \leq \Theta(T^{N-1}\bar{\gamma}) \leq \pi \quad \text{and} \quad \Delta k_{\min} \leq \Theta(T^{N-1}\hat{\gamma}) \leq \pi.$$

Then we can easily see that we can apply Lemma 5.2 with setting $\gamma = \mathscr{F}(\bar{x})$, $\gamma' = \mathscr{F}(\bar{y})$, and $n = N-1$. Consequently we have

$$\frac{|r(\hat{x}) - r(\hat{y})|}{|r(\bar{x}) - r(\bar{y})|} \in \left[\left(\frac{C_{12}\pi}{\Delta k_{\min}}\right)^{-1}, \frac{C_{12}\pi}{\Delta k_{\min}}\right]. \tag{5.19}$$

Case (iii). Next we consider Case (iii). If $N = 1$, we use the fact that the leaves of $\mathscr{F}$ on the set $\mathscr{D}_1 \backslash \mathscr{D}_2$ are those of $\mathscr{F}_1$. Obviously $\mathscr{F}_1$ is Lipschitz continuous. In fact, we can easily find a number C_{25} depending only on $\mathscr{F}_1$ (i.e. depending only on Q and the choice of the initial foliation $\mathscr{F}_0$) such that

$$\frac{|r(\hat{x}) - r(\hat{y})|}{|r(\bar{x}) - r(\bar{y})|} \in [C_{25}^{-1}, C_{25}]. \tag{5.20}$$

Now we consider the case $N \geq 2$. Let $\mathscr{E}$ denote the quadrilateral enclosed by the leaves $\mathscr{F}(\bar{x})$ and $\mathscr{F}(\bar{y})$ and the curves $\bar{\gamma}$ and $\hat{\gamma}$. Then it is easy to see that we can apply Lemma 5.2 to $\mathscr{E}$ with $\gamma = \mathscr{F}(\bar{x})$, $\gamma' = \mathscr{F}(\bar{y})$, and $n = N-1$. Thus (5.8) holds with $n = N-1$.

$$\frac{|r(\hat{x}) - r(\hat{y})|}{|r(\bar{x}) - r(\bar{y})|} \in [C_{12}^{-1}, C_{12}] \frac{\Theta(T^{N-1}\hat{\gamma})}{\Theta(T^{N-1}\bar{\gamma})}. \tag{5.21}$$

We note that the following fact which is an easy exercise. Let γ_1 and γ_2 be mutually disjoint K-decreasing curves and let δ be a K-increasing curve intersecting both of them. Without loss of generality we may assume that γ_1 is lying on the left hand side of γ_2. Consider any line l intersecting δ and parallel to the r-axis. Let x_1 and x_2 be the points where l intersects γ_1 and γ_2, respectively. Then we can easily show that

$$r(x_2) - r(x_1) \in \left[\frac{2}{K_{\max}}, \frac{2}{k_{\min}}\right] \Theta(\delta) \tag{5.22}$$

holds.

Now we can apply this fact to the case when $\gamma_1 = \mathscr{F}(T^{N-1}\bar{x})$, $\gamma_2 = \mathscr{F}(T^{N-1}\bar{y})$, and $\delta = T^{N-1}\bar{\gamma}$ and the case when $\gamma_1 = \mathscr{F}(T^{N-1}\bar{x})$, $\gamma_2 = \mathscr{F}(T^{N-1}\bar{y})$, and $\delta = T^{N-1}\hat{\gamma}$. On the other hand, in virtue of Step 4 of the construction of $\mathscr{F}$, the leaves of $\mathscr{F}$ on a connected component of $\mathscr{D}_N \backslash \mathscr{D}_{N+1}$ are mapped by T^{N-1} onto the leaves of $\mathscr{F}$ on the corresponding connected component of $(\mathscr{D}_1 \backslash \mathscr{D}_2) \cap \mathscr{D}_{-(N-1)}$. Thus we can apply (5.20) to any curves joining $\mathscr{F}(T^{N-1}\bar{x})$ and $\mathscr{F}(T^{N-1}\bar{y})$ parallel to the r-axis. Hence we arrive at

$$\frac{\Theta(T^{N-1}\bar{\gamma})}{\Theta(T^{N-1}\hat{\gamma})} \in \left[\left(\frac{K_{\max} C_{25}}{k_{\min}}\right)^{-1}, \frac{K_{\max} C_{25}}{k_{\min}}\right] \tag{5.23}$$

in virtue of (5.20) and (5.22). Combining (5.21) with (5.23) we obtain

$$\frac{|r(\hat{x}) - r(\hat{y})|}{|r(\bar{x}) - r(\bar{y})|} \in \left[\left(\frac{K_{\max} C_{12} C_{25}}{k_{\min}}\right)^{-1}, \frac{K_{\max} C_{12} C_{25}}{k_{\min}}\right]. \tag{5.24}$$

This completes the proof for Case (iii).

Case (iv). We set

$$C_{26} = \max\left\{\frac{L}{\Delta}, \frac{C_{12}\pi}{\Delta k_{\min}}, C_{25}, \frac{K_{\max} C_{12} C_{25}}{k_{\min}}\right\}.$$

Then the argument above implies that

$$\frac{|r(\hat{x}) - r(\hat{y})|}{|r(\bar{x}) - r(\bar{y})|} \in [C_{26}^{-1}, C_{26}] \tag{5.25}$$

is valid for Case (i), Case (ii), and Case (iii). It remains to prove (5.25) for Case (iv).

We may assume that $\bar{x} \in \mathscr{D}_{N+1}$, $\bar{y} \in \mathscr{D}_N \backslash \mathscr{D}_{N+1}$, and $r(\bar{x}) < r(\bar{y})$. The other cases are dealt with in the same way. Let $w_0 w_1 \cdots w_N w_{N+1}$ be the element in $\mathscr{W}_{N+2}$ such that $\bar{x} \in \mathscr{D}_{N+1}(w_0 w_1 \cdots w_N w_{N+1})$ and $\bar{y} \in \mathscr{D}_N(w_0 w_1 \cdots w_N)$. For simplicity we denote $w_0 w_1 \cdots w_N w_{N+1}$ as the concatenation $w w_{N+1}$ of $w = w_0 w_1 \cdots w_N$ and w_{N+1}. From our assumption, we see that $\bar{y}$ and $\hat{y}$ intersect $\gamma_r(ww_{N+1})$. Let $\bar{z}$ and $\hat{z}$ be the points at which $\bar{y}$ and $\hat{y}$ intersect $\gamma_r(ww_{N+1})$, respectively. We note that $\gamma_r(ww_{N+1}) = \mathscr{F}(\bar{z}) = \mathscr{F}(\hat{z})$ holds. We can show that

$$\frac{r(\hat{y}) - r(\hat{z})}{r(\bar{y}) - r(\bar{z})} \in [C_{26}^{-1}, C_{26}]. \tag{5.26}$$

In fact, any $\bar{u} \in \bar{y}$ with the difference $r(\bar{u}) - r(\bar{z}) > 0$ being small enough is contained in $\mathscr{D}_N(w) \backslash \mathscr{D}_{N+1}$. If $\hat{u}$ is the point in $\mathscr{F}(\bar{u}) \cap \hat{y}$, the same assertion is also true. Thus the argument for Case (ii) or that for Case (iii) can be applied to $\bar{u}, \bar{y}, \hat{u}$, and $\hat{y}$. Hence we have

$$\frac{r(\hat{y}) - r(\hat{u})}{r(\bar{y}) - r(\bar{u})} \in [C_{26}^{-1}, C_{26}].$$

Letting $\bar{u} \to \bar{z}$ we obtain (5.26).

Note that there are the following possibilities.

(iv-1) $\bar{x} \in \mathscr{D}_{N+2}$.

(iv-2) $\bar{x} \in \mathscr{D}_{N+1} \backslash \mathscr{D}_{N+2}$.

If (iv-1) occurs, then the line segment $\bar{y}(\bar{x}\bar{z})$ joining $\bar{x}$ and $\bar{z}$ (resp. $\hat{y}(\hat{x}\hat{z})$ joining $\hat{x}$ and $\hat{z}$) is contained in $\mathscr{D}_{N+1}(ww_{N+1})$ and crosses at least one connected component $\mathscr{E}_{N+1}(ww_{N+1})_i$ of $\mathscr{D}_{N+1} \backslash \mathscr{D}_{N+2}$. Therefore, it is easy to see that $\varDelta < |r(T^N\bar{z}) - r(T^N\bar{x})| < L$ (resp. $\varDelta < |r(T^N\hat{z}) - r(T^N\hat{x})| < L$). Since $T^i\bar{y}(\bar{x}\bar{z})$ and $T^i\hat{y}(\hat{x}\hat{z})$ are K-increasing curves for $i = 1, 2, \ldots, N$, we can apply Lemma 5.2 to our situation with $\gamma = \mathscr{F}(\bar{x})$, $\gamma' = \mathscr{F}(\bar{z})$, and $n = N$. Therefore we obtain

$$\frac{r(\hat{x}) - r(\hat{z})}{r(\bar{x}) - r(\bar{z})} \in \left[\left(\frac{C_{12}\pi}{\varDelta k_{\min}}\right)^{-1}, \frac{C_{12}\pi}{\varDelta k_{\min}}\right] \tag{5.27}$$

in the same way as (5.18).

Next we assume that (iv-2) occurs. If $\bar{x}$ coincides with $\bar{z}$ we do not need to prove any more in virtue of (5.26). Therefore we may assume that $\bar{x}$ does not coincide with $\bar{z}$. Then we can apply the same argument for Case (ii) or Case (iii) according as $\bar{x} \in \mathscr{E}_{N+1}(ww_{N+1})_{2i-1}$ with $i = 1, \ldots, J-1$ or $\bar{x} \in \mathscr{E}_{N+1}(ww_{N+1})_{2J-1}$. Hence we can conclude that

$$\frac{r(\hat{x}) - r(\hat{z})}{r(\bar{x}) - r(\bar{z})} \in [C_{26}^{-1}, C_{26}] \tag{5.28}$$

holds in both cases (iv-1) and (iv-2). Combining (5.26) with (5.28) we arrive at the desired inequality

$$\frac{r(\hat{x}) - r(\hat{y})}{r(\bar{x}) - r(\bar{y})} \in [C_{26}^{-1}, C_{26}].$$

This completes the proof of Theorem 5.1. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. We have only to verify the property ($\mathscr{F}$.4). As before we regard $\mathscr{D}_1(ij)$ as a quadrilateral in the (r,φ)-plane P_i for each $ij \in \mathscr{W}_2$. Fix $ij \in \mathscr{W}_2$. Let $a = a(ij) = \gamma_t(ij) \cap \gamma_l(ij)$, $b = b(ij) = \gamma_l(ij) \cap \gamma_b(ij)$, $c = c(ij) = \gamma_b(ij) \cap \gamma_r(ij)$, and $d = d(ij) = \gamma_r(ij) \cap \gamma_t(ij)$ be the vertices of the quadrilateral $\mathscr{D}_1(ij)$. Our goal is to construct a homeomorphism

$$\Phi = \Phi_{ij} : [r(a), r(d)] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathscr{D}_1(ij)$$

with the following properties.

(1) Φ is Lipschitz continuous with respect to the usual Euclidean distance on $[r(a), r(d)] \times [-\pi/2, \pi/2]$ and the Euclidean distance on the (r,φ)-plane.

(2) Φ^{-1} is Lipschitz continuous with respect to the Euclidean distance on the (r,φ)-plane and the usual Euclidean distance on $[r(a), r(d)] \times [-\pi/2, \pi/2]$.

(3) For each $r \in [r(a), r(d)]$, Φ maps $\{r\} \times [-\pi/2, \pi/2]$ homeomorphically to a leaf of $\mathscr{F}$.

For $(r,\varphi) \in [r(a), r(d)] \times [-\pi/2, \pi/2]$, there exists a unique point $x_r \in \gamma_t(ij)$ such that $r(x_r) = r$ (clearly, $\varphi(x) = \pi/2$). The point $\Phi(r,\varphi)$ is defined to be a unique point x on $\mathscr{F}(x_r)$ satisfying $\varphi(x) = \varphi$. Then property (3) is obviously satisfied.

The property (1) is verified as follows. Let (r_1, φ_1) and (r_2, φ_2) be any points in $[r(a), r(d)] \times [-\pi/2, \pi/2]$. Then we have

$$\begin{aligned}
|\Phi(r_1,\varphi_1) - \Phi(r_2,\varphi_2)| &\le |\Phi(r_1,\varphi_1) - \Phi(r_1,\varphi_2)| + |\Phi(r_1,\varphi_2) - \Phi(r_2,\varphi_2)| \\
&\le \sqrt{1 + \left(\frac{1}{k_{\min}}\right)^2}\,|\varphi_1 - \varphi_2| + C_{24}|r_1 - r_2| \\
&\le \sqrt{1 + \left(\frac{1}{k_{\min}}\right)^2 + C_{24}^2}\,|(r_1,\varphi_1) - (r_2,\varphi_2)|.
\end{aligned}$$

In the above, we have used the property ($\mathscr{F}$.1) and Theorem 5.1 to estimate the first term and the second term of the first line, respectively.

Next we verify the property (3). Let $x = (r(x), \varphi(x))$ and $x = (r(y), \varphi(y))$ be any points in $\mathscr{D}_1(ij)$. We can write $\Phi^{-1}x = (r_1, \varphi(x))$ and $\Phi^{-1}y = (r_2, \varphi(y))$ for some r_1 and r_2 in $[r(a), r(d)]$ by definition. Let z be the unique point in $\mathscr{F}(x)$ with $\varphi(z) = \varphi(y)$. Since we can write as $\Phi^{-1}z = (r_1, \varphi(y))$, we obtain

$$\begin{aligned}
|\Phi^{-1}x - \Phi^{-1}y| &\le |\Phi^{-1}x - \Phi^{-1}z| + |\Phi^{-1}z - \Phi^{-1}y| \\
&\le |\varphi(x) - \varphi(y)| + |r_1 - r_2| \\
&\le |\varphi(x) - \varphi(y)| + C_{24}|r(x) - r(z)|.
\end{aligned}$$

Note that the second inequality above is a consequence of Theorem 5.1. On the other hand the property ($\mathscr{F}$.1) yields

$$|r(x) - r(z)| = |z - y| \le |z - x| + |x - y| \le \sqrt{1 + \left(\frac{1}{k_{\min}}\right)^2}\,|\varphi(x) - \varphi(y)| + |x - y|.$$

Therefore we obtain

$$|\Phi^{-1}x - \Phi^{-1}y| \le \left(1 + C_{24}\left(1 + \sqrt{1 + \left(\frac{1}{k_{\min}}\right)^2}\right)\right)|x - y|.$$

Now (3) is verified. □

References

[1] L. Bunimovich, Ya. G. Sinai and N. Chernov, Markov partitions for two-dimensional billiards, Russian Math. Surveys, **45** (1990), 105–152.

[2] L. Bunimovich, Ya. G. Sinai and N. Chernov, Statistical properties of two dimensional hyperbolic billiards, Russian Math. Surveys, **46** (1991), 47–106.

[3] M. Ikawa, Decay of solutions of wave equations in the exterior of several strictly convex bodies, Ann. Inst. Fourier (Grenoble), **38** (1988), 113–146.

[4] I. Kubo, Perturbed billiard systems I, Nagoya Math. J., **61** (1976), 1–57.

[5] A. Lopes and R. Markarian, Open billiards: Cantor sets, invariant and conditionally invariant probabilities, SIAM J. Appl. Math., **56** (1996), 651–680.

[6] T. Morita, The symbolic representation of billiards without boundary conditions, Trans. Amer. Math. Soc., **325** (1991), 819–828.

[7] T. Morita, Billiards without boundary and their zeta functions, In: Zeta Functions in Geometry, Tokyo, 1990, (eds. N. Kurokawa and T. Sunada), Adv. Stud. Pure Math., **21**, Math. Soc. Japan, 1992, 173–179.

[8] T. Morita, Meromorphic extensions of a class of zeta functions for two dimensional billiards without eclipse, Preprint Series in Mathematics, **90**, Tokyo Institute of Technology, 1999.

[9] T. Morita, Meromorphic extensions of a class of zeta functions for two dimensional billiards without eclipse II, Preprint Series in Mathematics, **101**, Tokyo Institute of Technology, 2000.

[10] T. Morita, Meromorphic extensions of a class of dynamical zeta functions and their special values at $s = 0$, Preprint Series in Mathematics, **102**, Tokyo Institute of Technology, 2000.

[11] J. Palis and F. Takens, Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations, Cambridge Stud. Adv. Math., **35**, Cambridge Univ. Press, London-New York, 1993.

[12] W. Parry and M. Pollicott, An analogue of the prime number theorem for closed orbits of Axiom A flows, Ann. of Math., **118** (1983), 573–591.

[13] D. Ruelle, The thermodynamic formalism of expanding maps, Comm. Math. Phys., **125** (1989), 239–262.

[14] Ya. G. Sinai, Dynamical systems with elastic reflections, Russian Math. Surveys, **25** (1970), 137–189.

[15] L. Stoyanov, Exponential instability and entropy for a class of dispersing billiards, Ergodic Theory Dynam. Systems, **19** (1999), 201–226.

Takehiko Morita

Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8526
Japan