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Fourier-Ehrenpreis integral formula for harmonic functions

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Abstract. We give a Fourier-Ehrenpreis integral representation formula that expresses a harmonic function in a ball with a prescribed boundary value by superposition of harmonic exponentials.

1. Introduction.

The exponential $e^{-i\langle z,t\rangle}$ is harmonic in $t=(t_1,\dots,t_n)$ if $z=(z_1,\dots,z_n)$ satisfies $z\in V=\{z\in \boldsymbol{C}^n; z^2=\sum_{j=1}^n z_j^2=0\}$. According to the Ehrenpreis fundamental principle, harmonic functions are represented as integrals over this kind of harmonic exponentials with respect to some measures supported on V. The original proof was a very abstract argument based on the Hahn-Banach theorem and gave no explicit construction of such a measure.

On the other hand, integral formulas in Several Complex Variables led to explicit versions of the fundamental principle; see [**2**] and the references in [**1**] and [**4**].

The power of [**2**] resides in its generality. When applied to the particular case of the Laplacian, it has some redundancy: it involves not only the Dirichlet boundary value but also some other data. We have given a formula free from such superfluous data in the case $n=3$ in [**4**]. In the present paper we give a result for an arbitrary n in a different formulation.

Let $B_n=\{t\in \boldsymbol{R}^n; |t|=(\sum_{j=1}^n t_j^2)^{1/2}<1\}$ be the open unit ball of $\boldsymbol{R}_t^n$ and $u(t)\in \mathscr{C}^0(\bar{B}_n)$ be harmonic in B_n. We denote its Dirichlet boundary value by $f\in\mathscr{C}^0(S^{n-1})$. Let $\int_V$ be the $(1,1)$-current of integration along $V\backslash\{0\}$, which is the smooth locus of V and has a natural orientation as a complex manifold. For a $(n-1,n-1)$-form (possibly with singularities) ω on $\boldsymbol{C}^n$, we have $\int_V\omega=\int_{V\backslash\{0\}}\Phi^*(\omega)$, where $\Phi: V\backslash\{0\}\to\boldsymbol{C}^n$ is the natural embedding.

Set $x_j=\operatorname{Re} z_j$, $y_j=\operatorname{Im} z_j$ $(j=1,\dots,n)$ and $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$. Put $dx\wedge dy=\sum_{j=1}^n dx_j\wedge dy_j$. We will prove that in B_n $(n\geq 3)$ we have the Fourier-Ehrenpreis integral representation formula:

$$u(t)=\frac{1}{2(2\pi)^{n-1}}\int_V\left(1-\frac{n-2}{2|y|}\right)f(y/|y|)e^{-i\langle z,t\rangle}e^{-|y|}\left(\frac{dx\wedge dy}{|y|}\right)^{n-1}.$$

If $n=2$ we have a slightly different formula:

$$u(t)=\frac{1}{4\pi}\int_V f(y/|y|)e^{-|y|}\left(e^{-i\langle z,t\rangle}-\frac{1}{2}\right)\frac{dx\wedge dy}{|y|}.$$

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2. Geometry.

The equation $z^2 = |x|^2 - |y|^2 + 2i\langle x, y\rangle = 0$ is satisfied if and only if $|x| - |y| = \langle x, y\rangle = 0$. For a fixed value of $s = y/|y| \in S^{n-1}$, $r = |x| = |y|$ can take an arbitrary positive value and x can be any vector which is orthogonal to s. Therefore there is a diffeomorphism

$$V\backslash\{0\} \to TS^{n-1}\backslash S^{n-1}, \quad (x, y) \mapsto (y/|y|, x),$$

where TS^{n-1} is the tangent bundle of S^{n-1} whose fiber at s is $H(s) = \{x \in \boldsymbol{R}^n; \langle s, x\rangle = 0\}$. Here S^{n-1} is identified with the zero section $S^{n-1} \times \{0\} \subset TS^{n-1}$. The inverse mapping is

$$TS^{n-1}\backslash S^{n-1} \to V\backslash\{0\}, \quad (s, x) \mapsto (x, |x|s).$$

For $y \in \boldsymbol{R}^n\backslash\{0\}$, set $H(y) = H(y/|y|) = \{x \in \boldsymbol{R}^n; \langle y, x\rangle = 0\}$, which is the boundary of $\{x; \langle y, x\rangle < 0\}$ and is oriented accordingly. Let $\omega_y \in \Omega^{n-1}(H(y))$ be the surface-area element of $H(y)$ and π be the orthogonal projection from $\boldsymbol{R}^n$ to $H(y)$. Put $w_y(x) = \pi^*\omega_y$, $y \cdot dx = \sum_{j=1}^n y_j\, dx_j$. Then

$$(y \cdot dx) \wedge w_y(x) = |y|\, dx_1 \wedge \cdots \wedge dx_n. \tag{2.1}$$

Set $v(y) = \sum_{j=1}^n (-1)^{j-1} y_j\, dy_1 \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_n$, $y \cdot dy = \sum_{j=1}^n y_j\, dy_j$. Then

$$(y \cdot dy) \wedge v(y) = |y|^2\, dy_1 \wedge \cdots \wedge dy_n. \tag{2.2}$$

At each point on $\{x \cdot y = 0\}$, we have

$$(x \cdot dy) \wedge v(y) = (x \cdot y)\, dy_1 \wedge \cdots \wedge dy_n = 0. \tag{2.3}$$

Lemma 2.1. *At each point on* $\{x \cdot y = |y| - |x| = 0\}$, *we have*

$$d(x \cdot y) \wedge d(|y|^2 - |x|^2) \wedge v(y) \wedge w_y(x) = 2(-1)^n |x|^3\, dy_1 \wedge \cdots \wedge dy_n \wedge dx_1 \wedge \cdots \wedge dx_n.$$

Proof.

$$\begin{aligned}
&\frac{1}{2} d(x \cdot y) \wedge d(|y|^2 - |x|^2) \wedge v(y) \wedge w_y(x) \\
&\quad = (y \cdot dx + x \cdot dy) \wedge (y \cdot dy - x \cdot dx) \wedge v(y) \wedge w_y(x) \\
&\quad = (y \cdot dx) \wedge (y \cdot dy) \wedge v(y) \wedge w_y(x) - (x \cdot dy) \wedge (x \cdot dx) \wedge v(y) \wedge w_y(x) \\
&\qquad \text{because } (n+1)\text{-forms in } x \text{ or } y \text{ are } 0 \\
&\quad = (-1)^n |y|^3\, dy_1 \wedge \cdots \wedge dy_n \wedge dx_1 \wedge \cdots \wedge dx_n \quad \text{by (2.1), (2.2) and (2.3).}
\end{aligned}$$

□

Lemma 2.2. *At each point on* $\{|y| - |x| = 0\}$, *we have*

$$d(x \cdot y) \wedge d(|y|^2 - |x|^2) \wedge (dx \wedge dy)^{n-1} = 4(-1)^{n(n+1)/2} |x|^2\, dy_1 \wedge \cdots \wedge dy_n \wedge dx_1 \wedge \cdots \wedge dx_n.$$

Proof. First we can see easily that $(dx \wedge dy)^{n-1} = \sum_{\ell=1}^n \prod_{k \neq \ell} dx_k \wedge dy_k$, which implies that $(dx_p \wedge dy_q) \wedge (dx \wedge dy)^{n-1} = 0$ if $p \neq q$. Therefore

$$(y \cdot dx) \wedge (y \cdot dy) \wedge (dx \wedge dy)^{n-1} = \left(\sum_{j=1}^{n} y_j^2 \, dx_j \wedge dy_j\right) \wedge \sum_{\ell=1}^{n} \prod_{k \neq \ell} dx_k \wedge dy_k$$

$$= |y|^2 (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n).$$

In a similar way, we are led to

$$\frac{1}{2} d(x \cdot y) \wedge d(|y|^2 - |x|^2) \wedge (dx \wedge dy)^{n-1}$$

$$= (y \cdot dx + x \cdot dy) \wedge (y \cdot dy - x \cdot dx) \wedge (dx \wedge dy)^{n-1}$$

$$= (|x|^2 + |y|^2)(dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n). \qquad \square$$

On account of Lemmas 2.1 and 2.2, we arrive at

Lemma 2.3. *At each point on* $V \setminus \{0\}$, *we have*

$$d(x \cdot y) \wedge d(|y|^2 - |x|^2) \wedge (dx \wedge dy)^{n-1}$$

$$= 2(-1)^{n(n-1)/2} d(x \cdot y) \wedge d(|y|^2 - |x|^2) \wedge |x|^{n-1} \sigma(y/|y|) \wedge w_y(x),$$

where σ *is the surface-area element of* S^{n-1} *and* $\sigma(y/|y|) = |y|^{-n} v(y)$ *is its pullback by the projection* $\boldsymbol{R}^n \setminus \{0\} \to S^{n-1}$, $y \mapsto y/|y|$.

Lemma 2.4. *Let* f_1 *and* f_2 *be* $\mathscr{C}^\infty$*-functions on an m-dimensional manifold M such that* $df_1 \wedge df_2 \neq 0$ *near* $N = \{f_1 = f_2 = 0\}$. *If an* $(m-2)$*-form* ω *satisfies* $\omega \wedge df_1 \wedge df_2 = 0$ *at each point on the submanifold N, then* $\phi^* \omega = 0$ *where* $\phi : N \to M$ *is the embedding.*

Proof. Choose a local coordinate system $x = (x_1, \dots, x_m)$ with $x_j = f_j$ $(j = 1, 2)$. An $(m-2)$-form ω can be written in the form

$$\omega(x) = \eta(x) \, dx_3 \wedge \cdots \wedge dx_n + dx_1 \wedge \sum_{j_1 < \cdots < j_{m-3}} \eta_{j_1, \dots, j_{m-3}}(x) \, dx_{j_1} \wedge \cdots \wedge dx_{j_{m-3}}$$

$$+ dx_2 \wedge \sum_{j_1 < \cdots < j_{m-3}} \eta'_{j_1, \dots, j_{m-3}}(x) \, dx_{j_1} \wedge \cdots \wedge dx_{j_{m-3}}.$$

Then at each point on N we get

$$\omega \wedge df_1 \wedge df_2 = \omega \wedge dx_1 \wedge dx_2 = \eta(0, 0, x_3, \dots, x_n) \, dx_1 \wedge \cdots \wedge dx_n.$$

On the other hand, we have $\phi^* \omega = \eta(0, 0, x_3, \dots, x_n) \, dx_3 \wedge \cdots \wedge dx_n$. $\square$

By virtue of Lemmas 2.3 and 2.4 (with $f_1 = x \cdot y$, $f_2 = |y|^2 - |x|^2$) we deduce

Proposition 2.5.

$$\Phi^*((dx \wedge dy)^{n-1}) = \Phi^*(2(-1)^{n(n-1)/2} |x|^{n-1} \sigma(y/|y|) \wedge w_y(x)).$$

3. Integrals.

In this section, we will show Lemma 3.2 below which will be used in the proof of Proposition 4.1.

LEMMA 3.1. *Put* $K_n = \int_0^\pi \sin^n x/(a + ip\cos x)^{n+1}\,dx$ *for* $a > 0$, $p \in \boldsymbol{R}$, $n \in \boldsymbol{N}_0 = \{0, 1, 2, \dots\}$. *Then we have* $K_n = \gamma_n (a^2 + p^2)^{-(n+1)/2}$, *where* $\gamma_n = \int_0^\pi \sin^n x\,dx$.

PROOF. It is well-known that

$$\gamma_n = \frac{2(n-1)!!}{n!!}\ (\text{if } n \text{ is odd}), \quad \gamma_n = \frac{\pi(n-1)!!}{n!!}\ (\text{if } n \ge 2 \text{ is even}), \quad \gamma_0 = \pi.$$

We have

$$\frac{\partial K_n}{\partial p} = -i(n+1)\int_0^\pi \frac{\sin^n x\cos x}{(a + ip\cos x)^{n+2}}\,dx,$$

$$\frac{\partial^2 K_{n-2}}{\partial a^2} = n(n-1)\int_0^\pi \frac{\sin^{n-2} x}{(a + ip\cos x)^{n+1}}\,dx \quad (n \ge 2).$$

By integrating $K_n = \int_0^\pi (-\cos x)' \sin^{n-1} x/(a + ip\cos x)^{n+1}\,dx$ by parts, we obtain

$$\left(p\frac{\partial}{\partial p} + n\right)K_n = \frac{1}{n}\frac{\partial^2}{\partial a^2}K_{n-2} \quad (n \ge 2).$$

Note that $p\partial/\partial p + n$ is injective on the set of power series in p. □

LEMMA 3.2. *If* $n \ge 3$, $a > 0$ *and* $(b_1, \dots, b_{n-1}) \in \boldsymbol{R}^{n-1}$, *then*

$$I_n := \int_{S^{n-2}} \frac{dv'(X)}{(a + i\sum_{\ell=1}^{n-1} b_\ell X_\ell)^{n-2}} = \frac{C_{n-2}}{(a^2 + \sum_{\ell=1}^{n-1} b_\ell^2)^{(n-2)/2}},$$

$$J_n := \int_{S^{n-2}} \frac{dv'(X)}{(a + i\sum_{\ell=1}^{n-1} b_\ell X_\ell)^{n-1}} = \frac{C_{n-2}a}{(a^2 + \sum_{\ell=1}^{n-1} b_\ell^2)^{n/2}},$$

where C_m $(m \ge 1)$ *is the surface-area of* S^m *and* v' *is the surface-area measure of* S^{n-2}.

PROOF. It is well-known that $C_m = 2\gamma_0 \cdots \gamma_{m-1}$ and that

$$C_m = \frac{(2\pi)^{(m+1)/2}}{(m-1)!!}\ (\text{if } m \text{ is odd}), \quad \frac{2(2\pi)^{m/2}}{(m-1)!!}\ (\text{if } m \text{ is even}).$$

We have only to calculate I_n because $J_n = (2-n)^{-1}\partial I_n/\partial a$.

The group $SO(n-1)$ acts on S^{n-2} transitively. So we may replace $(b_1, \dots, b_{n-1})$ by $(p, 0, \dots, 0)$ with $p = (\sum_{\ell=1}^{n-1} b_\ell^2)^{1/2}$. By using the polar coordinates we find that

$$\int_{S^{n-2}} \frac{dv'(X)}{(a + ipX_1)^{n-2}} = \int_0^{2\pi} d\theta_{n-2}\left(\prod_{j=2}^{n-3}\int_0^\pi \sin^{n-2-j}\theta_j\,d\theta_j\right)\int_0^\pi \frac{\sin^{n-3}\theta_1\,d\theta_1}{(a + ip\cos\theta_1)^{n-2}}$$

$$= 2\gamma_0 \cdots \gamma_{n-4}K_{n-3}.$$

By using Lemma 3.1, we get the formula for I_n. □

4. Main result.

Put $r = |x| = |y|$, $s = y/r$, $\xi = x/r$ on $V\backslash\{0\}$. We see that ξ is an element of $S(s) = \{x \in H(s); |x| = 1\} \simeq S^{n-2}$ for each fixed $s \in S^{n-1}$. Let μ, m and v be the surface-area measures of $S^{n-1} \subset \boldsymbol{R}^n$, $H(s) \simeq \boldsymbol{R}^{n-1}$ and $S(s)$ respectively.

Since $(z_1, \dots, \widehat{z_\ell}, \dots, z_n)$ is a holomorphic coordinate system of $V\setminus\{0\}$, the $2(n-1)$-form $\Phi^*(\prod_{k\neq\ell} dx_k \wedge dy_k)$ $(\ell = 1, \dots, n)$ is positive with respect to its natural orientation and so are the forms in Proposition 2.5.

For a function F on $V\setminus\{0\}$ we have

$$\begin{aligned}\int_{V\setminus\{0\}} F\Phi^*((dx\wedge dy)^{n-1}) &= \int_{V\setminus\{0\}} F\Phi^*(2(-1)^{n(n-1)/2}|x|^{n-1}\sigma(y/|y|)\wedge w_y(x)) \\ &= 2\int_{S^{n-1}} d\mu(s) \int_{H(s)} r^{n-1} F\, dm(x) \\ &= 2\int_{S^{n-1}} d\mu(s) \int_{S(s)} dv(\xi) \int_0^\infty r^{2n-3} F\, dr.\end{aligned}$$

Proposition 4.1. *Assume $n \geq 3$. Put $F_j = r^{-(n-1+j)} e^{-\{(1-\langle s,t\rangle)r + i\langle x,t\rangle\}} f(s)$ $(j = 0, 1)$ on $V\setminus\{0\}$ where $f \in \mathscr{C}^0(S^{n-1})$. Then if $|t| < 1$, we have*

$$\int_{V\setminus\{0\}} F_0\Phi^*((dx\wedge dy)^{n-1}) = 2(n-2)!C_{n-2}\int_{S^{n-1}} \frac{(1-\langle s,t\rangle)f(s)}{|s-t|^n}\, d\mu(s), \tag{4.1}$$

$$\int_{V\setminus\{0\}} F_1\Phi^*((dx\wedge dy)^{n-1}) = 2(n-3)!C_{n-2}\int_{S^{n-1}} \frac{f(s)}{|s-t|^{n-2}}\, d\mu(s). \tag{4.2}$$

Proof. Put $a = 1 - \langle s,t\rangle$, then $a > 0$ because $|s| = 1$, $|t| < 1$. It implies the convergence of the following integral:

$$\begin{aligned}\int_{V\setminus\{0\}} F_j\Phi^*((dx\wedge dy)^{n-1}) &= 2\int_{S^{n-1}} f(s)\, d\mu(s) \int_{S(s)} dv(\xi) \int_0^\infty r^{n-2-j} e^{-r(a+i\langle\xi,t\rangle)}\, dr \\ &= 2(n-2-j)!\int_{S^{n-1}} f(s)\, d\mu(s) \int_{S(s)} \frac{dv(\xi)}{(a+i\langle\xi,t\rangle)^{n-1-j}}.\end{aligned} \tag{4.3}$$

Let $\{u_1, \dots, u_{n-1}\}$ be an orthonormal system of the linear subspace $H(s)$ of $\boldsymbol{R}^n$. Then each $\xi = x/r \in S(s) \subset H(s)$ is expressed as $\xi = \sum_{\ell=1}^{n-1} X_\ell u_\ell$, $\sum_{\ell=1}^{n-1} X_\ell^2 = 1$. Set $b_\ell = \langle u_\ell, t\rangle$, then $\langle\xi, t\rangle = \sum_{\ell=1}^{n-1} b_\ell X_\ell$. We have

$$\int_{S(s)} \frac{dv(\xi)}{(a+i\langle\xi,t\rangle)^{n-1-j}} = \int_{S^{n-2}} \frac{dv'(X)}{(a+i\sum_{\ell=1}^{n-1} b_\ell X_\ell)^{n-1-j}},$$

where $X = (X_1, \dots, X_{n-1})$.

Recall that $r = |y|$, $s = y/r$. Since $\{s, u_1, \dots, u_{n-1}\}$ is an orthonormal system of $\boldsymbol{R}^n$, $b_\ell = \langle u_\ell, t\rangle$ satisfies $\langle s,t\rangle^2 + \sum_{\ell=1}^{n-1} b_\ell^2 = |t|^2$. Hence $a^2 + \sum_{\ell=1}^{n-1} b_\ell^2 = 1 - 2\langle s,t\rangle + |t|^2 = |s-t|^2$. By using Lemma 3.2 we complete the proof of Proposition 4.1. □

We introduce two integral operators $Q_0, Q_1 : \mathscr{C}^0(S^{n-1}) \to \mathscr{C}^\infty(B_n)$ by

$$Q_0[f](t) = \int_V f(y/|y|) e^{-i\langle z,t\rangle} e^{-|y|} \left(\frac{dx\wedge dy}{|y|}\right)^{n-1},$$

$$Q_1[f](t) = \int_V \frac{f(y/|y|)}{|y|} e^{-i\langle z,t\rangle} e^{-|y|} \left(\frac{dx\wedge dy}{|y|}\right)^{n-1}.$$

On V we have $-i\langle z,t\rangle - |y| = \langle y,t\rangle - |y| - i\langle x,t\rangle = -\{(1-\langle s,t\rangle)r + i\langle x,t\rangle\}$. We can use Proposition 4.1 to calculate Q_j. The result is

$$Q_0[f](t) = 2(n-2)!C_{n-2}\int_{S^{n-1}} \frac{(1-\langle s,t\rangle)f(s)}{|s-t|^n}\,d\mu(s),$$

$$Q_1[f](t) = 2(n-3)!C_{n-2}\int_{S^{n-1}} \frac{f(s)}{|s-t|^{n-2}}\,d\mu(s).$$

We find that $\{1/(2C_{n-2}C_{n-1})\}\{2Q_0[f](t)/(n-2)! - Q_1[f](t)/(n-3)!\}$ is nothing but the Poisson integral of f. By using $C_{n-1}C_{n-2} = 2(2\pi)^{n-1}/(n-2)!$, we obtain our main result:

THEOREM 4.2. *Assume that* $u(t) \in \mathscr{C}^0(\bar{B}_n)$ $(n \geq 3)$ *is harmonic in* B_n *and let* $f = u|_{\partial B_n} \in \mathscr{C}^0(S^{n-1})$ *be its Dirichlet boundary value. Then in* B_n, *we have*

$$u(t) = \frac{1}{2(2\pi)^{n-1}}\int_V \left(1 - \frac{n-2}{2|y|}\right) f(y/|y|) e^{-i\langle z,t\rangle} e^{-|y|} \left(\frac{dx \wedge dy}{|y|}\right)^{n-1}.$$

In particular, $u(t)$ *is given by superposition of the exponentials* $\exp(-i\langle z,t\rangle)$ *with* $z^2 = \sum_{j=1}^n z_j^2 = 0$ *and* $y/|y| \in \operatorname{supp} f$.

5. 2-dimensional case.

Only (4.1) in Proposition 4.1 holds if $n = 2$. (The left hand side of (4.2) is divergent.) Here we set $C_0 = 2$. We have

$$\int_{V\setminus\{0\}} F_0 \Phi^*(dx \wedge dy) = 4\int_{S^1} \frac{1-\langle s,t\rangle}{|s-t|^2} f(s)\,d\mu(s).$$

In the same way as in the previous section, define $Q_0 : \mathscr{C}^0(S^1) \to \mathscr{C}^\infty(B_2)$ by

$$Q_0[f](t) = \int_V f(y/|y|) e^{-i\langle z,t\rangle} e^{-|y|} \frac{dx \wedge dy}{|y|}.$$

Then $Q_0[f](t) = 4\int_{S^1}((1-\langle s,t\rangle)/|s-t|^2) f(s)\,d\mu(s)$. Hence $(8\pi)^{-1}(2Q_0[f](t) - Q_0[f](0))$ equals the Poisson integral of f.

THEOREM 5.1. *If* $u(t) \in \mathscr{C}^0(\bar{B}_2)$ *is harmonic in* B_2 *and* $f \in \mathscr{C}^0(S^1)$ *is its Dirichlet boundary value, then in* B_2, *we have*

$$u(t) = \frac{1}{4\pi}\int_V f(y/|y|) e^{-|y|} \left(e^{-i\langle z,t\rangle} - \frac{1}{2}\right) \frac{dx \wedge dy}{|y|}.$$

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