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A characterization of the group association scheme of A_n

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Abstract. In this paper, we investigate a certain fusion scheme $\tilde{\mathscr{X}}(A_n)$ of the group association scheme $\mathscr{X}(A_n)$ of the alternating group of arbitrary degree n. In particular, under some extra assumption on 'geometry of maximal cliques', we characterize $\tilde{\mathscr{X}}(A_n)$ by parameters.

1. Introduction.

Let X be a finite set, and let R_i $(i = 0, 1, \ldots, d)$ be relations on X, i.e., subsets of $X \times X$. Then $\mathscr{X} = (X, \{R_i\}_{0 \le i \le d})$ is an *association scheme of d classes* if the following conditions hold.

(1) $R_0 = \{(x, x) \mid x \in X\}$.

(2) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$, and $R_i \cap R_j = \phi$ if $i \neq j$.

(3) ${}^tR_i = R_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$, where ${}^tR_i = \{(x, y) \mid (y, x) \in R_i\}$.

(4) For $i, j, k \in \{0, 1, \ldots, d\}$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant, $p_{i,j}^k$, whenever $(x, y) \in R_k$.

(The relation R_0 mentioned in (1) above is called the *diagonal relation*.)

An association scheme $\mathscr{X}$ is called *commutative* if the condition

(5) $p_{i,j}^k = p_{j,i}^k$ for all $i, j, k \in \{0, 1, \ldots, d\}$

holds, and *symmetric* if the condition

(6) ${}^tR_i = R_i$ for all $i \in \{0, 1, \ldots, d\}$

holds. Note that a symmetric association scheme is also commutative, but that the converse does not necessarily hold.

The non-negative integers $\{p_{i,j}^k\}_{0 \le i,j,k \le d}$ are called the *intersection numbers* or *parameters* of $\mathscr{X}$.

The reader is referred to [**2**] and [**3**] for the general theory of association schemes and related terminologies.

Let G be a finite group. Let $C_0(= \{\mathrm{id}\})$, $C_1, \ldots, C_d$ be the conjugacy classes of G. Define relations R_i $(i = 0, 1, \ldots, d)$ on G by $R_i = \{(x, y) \mid yx^{-1} \in C_i\}$. Then $\mathscr{X}(G) = (G, \{R_i\}_{0 \le i \le d})$ is a commutative association scheme of d classes called the *group association scheme* of G. (See Example II.2.1(2) of [**2**].)

By S_n and A_n, we denote the symmetric group and the alternating group of degree n, respectively.

Among various problems around an association scheme, it seems one of the most important problems is to classify association schemes having the same set of parameters.

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In particular, when it is the group association scheme $\mathcal{X}(G)$ of some group G, the problem can be regarded as "the combinatorial version" of the classification problem of groups having a given character table. There are several works for the latter problem in the field of group theory. The list contains characterizations of S_n and A_n. On the other hand, the former problem is in the field of algebraic combinatorics (association schemes), and in order to try this, we need delicate 'structure analysis' of association schemes, which is the subject of this paper. (We cannot use group theory!)

For example, it is shown in [**4**] that when the set of parameters of $\mathcal{X}(S_4)$ is given, there exist exactly two association schemes (except $\mathcal{X}(S_4)$) having it, and each of them is not the group association scheme of any group.

In case of $\mathcal{X}(S_n)$ with $n \geq 5$, it is shown in [**5**] and [**6**] that it is the unique one for given parameters. Also, for $\mathcal{X}(A_5)$, uniqueness is shown in [**4**].

In this paper, we attempt to solve this problem for A_n with $n \geq 6$. However, particularly in this paper, we focus on one fusion scheme of $\mathcal{X}(A_n)$, which is mentioned in the following.

It is well known that the conjugacy classes of S_n depend on its cycle-shapes. Let $\Lambda(n)$ be the set of all partitions of n, or, equivalently, the set of all unordered m-tuples $(i_1, i_2, \ldots, i_m)$ such that $1 \leq m \leq n$, $\sum_{j=1}^{m} i_j = n$, and i_j is a positive integer for $1 \leq j \leq m$. (For example, we identify $(1,2,1,3,2) \in \Lambda(9)$ with $(1,1,2,2,3)$.) For $\lambda = (1, \ldots, 1, i_1, \ldots, i_m) \in \Lambda(n)$ ($i_s \geq 2$ for $1 \leq s \leq m$), we sometimes write $\lambda = (i_1, \ldots, i_m)$ and $(1) = (1, \ldots, 1)$. (For example, $(3) = (1, \ldots, 1, 3) \in \Lambda(n)$.) For $\lambda \in \Lambda(n)$, let $C_\lambda = \{x \in S_n \mid x$ has the cycle-shape $\lambda\}$. Then we see that $\{C_\lambda\}_{\lambda \in \Lambda(n)}$ are the family of all conjugacy classes of S_n, and $\mathcal{X}(S_n) = (S_n, \{R_\lambda^*\}_{\lambda \in \Lambda(n)})$ is a symmetric association scheme of $|\Lambda(n)| - 1$ classes, where for $\lambda \in \Lambda(n)$,

$$R_\lambda^* = \{(x, y) \in S_n \times S_n \mid yx^{-1} \in C_\lambda\}.$$

(Note that $R_{(1)}^*$ is the diagonal relation.)

For $\lambda = (i_1, \ldots, i_m) \in \Lambda(n)$, let $\varphi(\lambda) = \sum_{s=1}^{m} (i_s - 1)$. Let $\Lambda^e(n)$ denote the subset of $\Lambda(n)$ as follows;

$$\Lambda^e(n) = \{\lambda \in \Lambda(n) \mid \varphi(\lambda)\text{: even integer}\}.$$

We write $\Lambda^o(n) = \Lambda(n) \backslash \Lambda^e(n)$.

Let us consider A_n as the subset of vertex set of $\mathcal{X}(S_n)$. Then it is easy to see;

$$A_n = \bigcup_{\lambda \in \Lambda^e(n)} C_\lambda.$$

The leading object of this paper is the following configuration;

$$\tilde{\mathcal{X}}(A_n) = (A_n, \{R_\lambda^* \cap (A_n \times A_n)\}_{\lambda \in \Lambda^e(n)}).$$

It is easy to see the following:

Proposition 1.1. *$\tilde{\mathcal{X}}(A_n)$ is a symmetric association scheme of $|\Lambda^e(n)| - 1$ classes. Moreover, $\tilde{\mathcal{X}}(A_n)$ is a fusion scheme of the symmetrization of $\mathcal{X}(A_n)$.*

In this paper, we approach the characterization problem of $\tilde{\mathcal{X}}(A_n)$. The author regards this problem as the first step of the characterization problem of $\mathcal{X}(A_n)$. Indeed, by general theory of fusion schemes, which is seen in, for example, [**1**], we have the following:

Proposition 1.2. *Let $\mathscr{X}$ be an association scheme having the same set of parameters as of $\mathscr{X}(A_n)$. Then there exists a fusion scheme $\tilde{\mathscr{X}}$ of $\mathscr{X}$ having the same set of parameters as of $\tilde{\mathscr{X}}(A_n)$.*

The main assertion of this paper is that $\tilde{\mathscr{X}}(A_n)$ is characterized by parameters under certain assumptions on "geometry of maximal cliques". In order to describe such geometry, let us prepare some notation on a graph.

In the following, we sometimes regard a notation Γ as the vertex set of a graph Γ. For example, if x is a vertex in Γ, then we write $x \in \Gamma$.

For two vertices x and y in Γ, we write $x \sim_\Gamma y$ or $x \sim y$ if x is connected to y with an edge.

By a triangle xyz, we mean the set of three vertices x, y and z with $x \sim y \sim z \sim x$.

For a graph Γ, let $\partial = \partial_\Gamma$ denote the distance function in Γ.

Let $d = d_\Gamma$ be the *diameter* of Γ.

For $x \in \Gamma$ and for an integer i with $0 \le i \le d$, let $\Gamma_i(x) = \{y \in \Gamma \mid \partial(x, y) = i\}$. Let $\Gamma(x) = \Gamma_1(x)$.

For $Y = \{y_1, \dots, y_i\} \subset \Gamma$, write $\bigcap_{1 \le j \le i} \Gamma(y_j)$ by $\Gamma(Y)$ or $\Gamma(y_1, \dots, y_i)$.

For a subset Y and Z of Γ, we denote by $e(Y, Z)$ the number of edges crossing between Y and Z. If $Y = \{y\}$ for $y \in \Gamma$, we denote $e(y, Z) = e(Y, Z)$.

We call a *clique* for a subgraph which forms a complete graph.

In the rest of this paper, we denote by Γ^* the relation graph $(A_n, R^*_{(3)})$ of $\tilde{\mathscr{X}}(A_n)$. Note that Γ^* is just the same as $\mathrm{Cay}_{C_{(3)}}(A_n)$, the *Cayley graph of A_n based on $C_{(3)}$*.

The following is a more precise description of "geometry of maximal cliques".

Observation 1.3. *Let Γ be a graph. If Γ is isomorphic to Γ^*, then Γ satisfies the conditions $(M1)$–$(M4)$ as follows*:

$(M1)$ *The size of any maximal clique is* 3 *or* $n-1$.

$(M2)$ *For any triangle xyz in Γ, there exists a unique maximal clique containing xyz.*

$(M3)$ *For any pair of vertices (x, y) with $x \sim_\Gamma y$, there exists a unique element $M \in \mathscr{M}_1$ containing xy, where $\mathscr{M}_1$ is the set of all maximal cliques of size* 3 *in Γ.*

$(M4)$ *Let $M \in \mathscr{M} = \mathscr{M}_1 \cup \mathscr{M}_2$, where $\mathscr{M}_2$ is the set of all maximal cliques of size $n-1$, and let $y \in \Gamma$ with $M \cap R'_{(2,2)}(y) \ne \emptyset$, where*;

$$R'_{(2,2)}(y) = \{z \in \Gamma \mid \partial(y, z) = 2, |\Gamma(y, z)| = 8\}.$$

Then $e(y, M) \ne 1$.

Remark that if a graph Γ satisfies both (M2) and (M4), then it also holds that;

$(M5)$ Let $M \in \mathscr{M}$, and let $y \in \Gamma$ with $M \cap R'_{(2,2)}(y) \ne \emptyset$. Then $e(y, M) \in \{0, 2\}$.

The following is the main theorem of this paper:

Theorem 1.4. *Let $\mathscr{X} = (X, \{R_\lambda\}_{\lambda \in \Lambda^e(n)})$ be an association scheme having the same set of parameters as of $\tilde{\mathscr{X}}(A_n)$, and let Γ be the relation graph $(X, R_{(3)})$. In addition, assume that Γ satisfies conditions $(M1)$–$(M4)$ as in Observation* 1.3.

Then $\mathscr{X}$ is isomorphic to $\tilde{\mathscr{X}}(A_n)$.

In the next section, we prepare two important preliminary propositions, Propositions 2.5 and 2.6. The former claims that characterizing problem of $\tilde{\mathscr{X}}(A_n)$ can be reduced to the characterization of one relation graph, which corresponds to the conjugacy class of 3-cycles in A_n. The latter is on spherical representation, which is a key tool in many places of this paper.

In Sections 3 and 4, we determine local structure of the relation graph corresponding to 3-cycles in A_n by information of parameters and the geometry of maximal cliques as seen in Observation 1.3. There, we can observe that the geometry 'controls' many important properties on local structure used for the determination of global structure.

In Section 5, we analyze local structure of the image of spherical representation.

In Section 6, we complete the proof of Theorem 1.4 by using properties of local structure obtained in Sections 3 and 4 and very simple properties of the image of spherical representation obtained in Section 5.

2. Notation and preliminaries.

In the rest of the paper, for an integer $i \le n$, let N_i denote the family of ordered sets of i distinct integers with $1 \le i \le n$. Let $\overline{N_i}$ denote the family of unordered sets $\{n_1, \dots, n_i\}$ with $(n_1, \dots, n_i) \in N_i$. Write the set $\{1, 2, \dots, n\}$ by N. Note that;

$$C_{(3)} = \{(i\ j\ k) \in A_n \mid (i, j, k) \in N_3\}.$$

Note that for $(i, j, k) \in N_3$, $(i\ j\ k) = (j\ k\ i) = (k\ i\ j) \neq (i\ k\ j)$.

An induced subgraph Γ' of a graph Γ is called *geodetically closed* if for any pair of vertices (x, y) in Γ', all shortest paths in Γ from x to y is also contained in Γ'. For a subset Y of Γ, let $\Phi(Y)$ be the smallest geodetically closed subgraph whose vertex set contains Y. It is clear that $\Phi(Y)$ is unique for any given subset Y.

Let $\mathscr{X} = (X, \{R_\lambda\}_{\lambda \in \Lambda})$ be an association scheme with index set Λ. Then for $\lambda \in \Lambda$ and for $x \in X$, let $R_\lambda(x) = \{y \in X \mid (x, y) \in R_\lambda\}$. For $\lambda_1, \dots, \lambda_m \in \Lambda$, write $\bigcup_{j=1}^{m} R_{\lambda_j}(x)$ by $R_{\lambda_1, \dots, \lambda_m}(x)$.

In the rest of this paper, we denote by Γ the relation graph $(X, R_{(3)})$ of $\mathscr{X} = (X, \{R_\lambda\}_{\lambda \in \Lambda^e(n)})$, where $\mathscr{X}$ is an association scheme having the same set of parameters as of $\tilde{\mathscr{X}}(A_n)$.

In [**5**, Lemma 3.4], we have observed that every relation of $\mathscr{X}(S_n)$ can be characterized by 'some characteristics' of the relation graph $(S_n, R_{(2)})$ corresponding to transpositions. In fact, a general proposition is used there, which is as follows:

Proposition 2.1. *Let $\mathscr{X}^1 = (X^1, \{R_i^1\}_{i=0,1,\dots,d})$ and $\mathscr{X}^2 = (X^2, \{R_i^2\}_{i=0,1,\dots,d})$ be association schemes with the same set of parameters. Let Γ^i $(i = 1, 2)$ be the relation graph (X^i, R_1^i), and assume that Γ^1 is connected. Let $\delta : \{0, 1, \dots, d\} \to \mathbf{Z}_{\ge 0}$ be a function such that for a pair $(x, y) \in R_i^1$, $\partial_{\Gamma^1}(x, y) = \delta(i)$.*

Suppose that Γ^1 and Γ^2 are isomorphic as graphs, and, in addition, for any pair (i, j) in $\{0, 1, 2, \dots, d\}$ with $\delta(i) = \delta(j)$, either of the following holds:

(i) $C(i) \neq C(j)$,

(ii) $C(i) = C(j)$, *and* $p_{1,l}^i \neq p_{1,l}^j$ *for some* $l \in C(i)$,

where $C(i) = \{k \in \{0, 1, 2, \dots, d\} \mid p_{1,k}^i \neq 0$ with $\delta(k) = \delta(i) - 1\}$.

Then $\mathscr{X}^1$ and $\mathscr{X}^2$ are isomorphic as association schemes.

In the following, let us observe that the similar situation holds between $\mathscr{X}$ and Γ.

For $\lambda = (\lambda_1, \dots, \lambda_i)$, $\mu = (\mu_1, \dots, \mu_j) \in \bigcup_{t=1}^{n} \Lambda(t)$ with $i + j \le n$, we denote (λ, μ) for the partition $(\lambda_1, \dots, \lambda_i, \mu_1, \dots, \mu_j) \in \bigcup_{t=1}^{n} \Lambda(t)$. We sometimes write $(\lambda, \mu) = (\lambda_1, \dots, \lambda_i, \mu)$.

Now we define a function $\psi : \bigcup_{t=1}^{n} \Lambda^e(t) \to \boldsymbol{Z}_{\ge 0}$ satisfying the following:

(a) For $\lambda^{(1)} \in \Lambda^e(i)$ and $\lambda^{(2)} \in \Lambda^e(j)$ with $i + j \le n$, $\psi((\lambda^{(1)}, \lambda^{(2)})) = \psi(\lambda^{(1)}) + \psi(\lambda^{(2)})$.

(b) $\psi((2i+1)) = i$ with $1 \le 2i + 1 \le n$.

(c) $\psi((2i, 2j)) = i + j$ for $i, j \ge 1$ with $2i + 2j \le n$.

Note that $(2i) \notin \bigcup_{t=1}^{n} \Lambda^e(t)$ and $(2i+1), (2i, 2j) \in \bigcup_{t=1}^{n} \Lambda^e(t)$.

For $\lambda \in \Lambda^e(n)$ and $g \in R_\lambda^*(\mathrm{id})$ $(= C_\lambda)$, set

$$T_\lambda(g) := \{\mu \in \Lambda^e(n) \mid g \cdot h \in R_\mu^*(\mathrm{id}) \text{ for } h \in C_{(3)}\}.$$

In addition, let $T_\lambda = \{\mu \in \Lambda^e(n) \mid p_{(3),\mu}^\lambda \ne 0\}$. Then we easily have

$$T_\lambda(g) = T_\lambda \quad \text{for any } g \in R_\lambda^*(\mathrm{id}).$$

It is clear that $T_{(1)} = \{(3)\}$. Next, let us determine $T_{(3)}$. Let $g = (1\ 2\ 3)$ $(\in C_{(3)})$ and $h \in (i\ j\ k)$ $(\in C_{(3)})$. Then we easily see that;

$$
\begin{aligned}
&(g, h) \in R_{(1)}^* && \text{if } h = g,\\
&(g, h) \in R_{(3)}^* && \text{if } h = g^{-1},\\
&(g, h) \in R_{(3),(2,2)}^* && \text{if } |\{1,2,3\} \cap \{i,j,k\}| = 2,\\
&(g, h) \in R_{(5)}^* && \text{if } |\{1,2,3\} \cap \{i,j,k\}| = 1,\\
&(g, h) \in R_{(3,3)}^* && \text{if } |\{1,2,3\} \cap \{i,j,k\}| = 0.
\end{aligned}
$$

Note that, for example, if $h = (1\ 2\ 4)$ (resp., $h = (2\ 1\ 4)$), then $(g, h) \in R_{(3)}^*$ (resp., $(g, h) \in R_{(2,2)}^*$). Thus we have $T_{(3)} = \{(1), (3), (2,2), (5), (3,3)\}$. Moreover, we also see that for any pair (x, y) in Γ^*,

$$
\begin{aligned}
&\partial_{\Gamma^*}(x, y) = 0 \quad \text{if and only if } (x, y) \in R_{(1)}^*,\\
&\partial_{\Gamma^*}(x, y) = 1 \quad \text{if and only if } (x, y) \in R_{(3)}^*,\\
&\partial_{\Gamma^*}(x, y) = 2 \quad \text{if and only if } (x, y) \in R_{(2,2),(5),(3,3)}^*.
\end{aligned}
$$

Note that $(\psi((1)), \psi((3)), \psi((2,2)), \psi((5)), \psi((3,3))) = (0, 1, 2, 2, 2)$.

By such argument, by induction on $\psi(\lambda)$ $(\lambda \in \Lambda^e(n))$, we have the following:

Lemma 2.2. *For ψ, the following hold.*

(1) *Let $\lambda, \mu \in \Lambda^e(n)$ with $p_{(3),\mu}^\lambda \ne 0$. Then $\psi(\lambda) \in \{\psi(\mu) - 1, \psi(\mu), \psi(\mu) + 1\}$.*

(2) *Let $\lambda, \mu \in \Lambda^e(n)$ with $\psi(\lambda) = \psi(\mu) + 1$. Then $p_{(3),\mu}^\lambda \ne 0$ if and only if one of the following holds*:

(i) *$\lambda = (i + j + k, \lambda')$ and $\mu = (i, j, k, \lambda')$, where i, j and k are odd positive integers, and where $\lambda' \in \Lambda^e(n - i - j - k)$.*

(ii) $\lambda = (i + j + k, \lambda')$ *and* $\mu = (i, j, k, \lambda')$, *where* i *and* j *are odd positive integers*, k *is an even positive integer*, *and where* $\lambda' \in \Lambda^e(n - i - j - k)$.

(iii) $\lambda = (i, j, \lambda')$ *and* $\mu = (i + k, j - k, \lambda')$, *where* i *and* j *are even positive integers*, k *is an odd positive integer with* $1 \le k \le j$, *and where* $\lambda' \in \Lambda^e(n - i - j)$.

(3) *Let* $x, y \in \Gamma$ *with* $(x, y) \in R_\lambda$ $(\lambda \in \Lambda^e(n))$. *Then* $\partial(x, y) = \psi(\lambda)$.

Remarks. (1) The author recommends the readers to see [**5**, Lemma 3.1].

(2) For λ and μ in Lemma 2.2 (2) (i) or (ii), if at most one of $\{i, j, k\}$ is odd, then $\psi(\lambda) = \psi(\mu)$.

(3) For λ and μ in Lemma 2.2 (2) (iii), if all of $\{i, j, k\}$ are odd or all are even, then $\psi(\lambda) = \psi(\mu)$.

Let us prepare some notation for $\lambda \in \Lambda^e(n)$.

Let $|\lambda|$ be the length of λ, that is, the number of entries of λ. For example, if $\lambda = (1, 1, 2, 2, 3)$ $(\in \Lambda^e(9))$, then $|\lambda| = 5$.

We sometimes reorder entries so that $\lambda = (\lambda^+, \lambda^-)$, where any entry of λ^+ (resp. λ^-) is even (resp. odd). For example, if $\lambda = (1, 1, 2, 2, 3)$ $(\in \Lambda^e(9))$, then $\lambda^+ = (2, 2)$ and $\lambda^- = (1, 1, 3)$. Note that $|\lambda^+|$ is even, i.e., $\lambda^+ \in \bigcup_{t=1}^n \Lambda^e(t)$.

If $\lambda = (1, 1, 3)$ and $\mu = (2, 2)$, for example, then we write $\lambda^+ = \mu^- = 0$.

If $\mu \in \Lambda^e(t)$ for some t with $1 \le t \le n$, we write $\sum \mu = t$. For example, for $\lambda \in \Lambda^e(n)$, $\sum \lambda^+ + \sum \lambda^- = n$.

By $\max(\lambda)$, we denote the maximum of entries of λ.

Let $\Psi(\lambda) = \{\mu \in \Lambda^e(n) \mid \psi(\mu) = \psi(\lambda) - 1, p^\lambda_{(3),\mu} \ne 0\}$. Note that $\Psi((3)) = \{(1)\}$ and $\Psi(\lambda) = \{(3)\}$ for $\lambda \in \{(2, 2), (5), (3, 3)\}$. Note also that if $\psi(\mu) \ne \psi(\lambda)$ for $\mu \in \Lambda^e(n)$, then $\Psi(\lambda) \cap \Psi(\mu) = \varnothing$.

By Lemma 2.2 (2), we easily have the following.

Lemma 2.3. *For* $\lambda \in \Lambda^e(n)$, *the following hold.*

(1) *For any* $\mu \in \Psi(\lambda)$, $|\mu^-| = |\lambda^-| + 2$.

(2) *For* $\mu \in \Psi(\lambda)$, $|\mu^+| \in \{|\lambda^+|, |\lambda^+| - 2\}$.

(3) *If* $|\lambda^+| \ge 2$, *then there exists* $\mu \in \Psi(\lambda)$ *such that* $|\mu^+| = |\lambda^+| - 2$.

(4) *Let* $\mu \in \Psi(\lambda)$ *with* $\mu^+ = \lambda^+$. *Then* $\sum \mu^- = \sum \lambda^-$.

(5) *If* $\max(\lambda^-) \ge 3$, *then there exists* $\mu \in \Psi(\lambda)$ *such that* $\mu^+ = \lambda^+$.

(6) *If* $\max(\lambda^+) \ge 4$, *then there exists* $\mu \in \Psi(\lambda)$ *such that* $|\mu^+| = |\lambda^+|$.

(7) *Assume* $|\lambda^+| = s \ge 2$, *and write* $\lambda^+ = (\lambda_1, \dots, \lambda_s)$ *with* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_s$. *Pick any* $\mu \in \Psi(\lambda)$ *with* $|\mu^+| = s$, *and let* $\mu^+ = (\mu_1, \dots, \mu_s)$ *with* $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_s$. *Then* $\lambda_i \ge \mu_i$ *for* $1 \le i \le s$.

(8) *Let* $\lambda = \lambda^- = (\lambda_1, \dots, \lambda_s)$ *with* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_s$, *and let* $\mu = (\mu_1, \dots, \mu_{s+2}) \in \Psi(\lambda)$ *with* $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{s+2}$. *Then* $\mu = \mu^-$ *and* $\lambda_i \ge \mu_i$ $(1 \le i \le s)$.

Proof. Straightforward. □

Lemma 2.4. *Let* $\lambda, \pi \in \Lambda^e(n)$ *with* $\lambda \ne \pi$ *and* $\psi(\lambda) = \psi(\pi) \ge 3$. *Assume* $\{\lambda, \pi\} \notin \{\{(7), (3, 5)\}, \{(9), (3, 7)\}\}$. *Then* $\Psi(\lambda) \ne \Psi(\pi)$.

Proof. At first, we only assume that $\lambda \ne \pi$ and $\psi(\lambda) = \psi(\pi) \ge 3$ for $\lambda, \pi \in \Lambda^e(n)$. Then it suffices to consider only 8 cases as follows:

Case (I): $|\lambda^+| > |\pi^+| \geq 2$.
Case (II_1): $|\lambda^+| > |\pi^+| = 0$ with $|\lambda^+| \geq 4$.
Case (II_2): $|\lambda^+| > |\pi^+| = 0$ with $\max(\lambda^+) \geq 4$.
Case (II_3): $|\lambda^+| > |\pi^+| = 0$ with $\lambda^+ = (2,2)$.
Case (III_1): $|\lambda^+| = |\pi^+| \geq 4$ with $\lambda^+ \neq \pi^+$.
Case (III_2): $|\lambda^+| = |\pi^+| = 2$ with $\lambda^+ \neq \pi^+$.
Case (IV): $\lambda^+ = \pi^+ \neq 0$.
Case (V): $\lambda^+ = \pi^+ = 0$.

Let us consider Case (I). Then by Lemma 2.3 (3), we can find $\mu \in \Psi(\pi)$ such that $|\mu^+| = |\pi^+| - 2$. However, by Lemma 2.3 (2), it cannot occur that $\mu \in \Psi(\lambda)$. Thus we have;

(2.4.1): In Case (I), $\Psi(\lambda) \neq \Psi(\pi)$.

Next, consider Case (II_1). Then by Lemma 2.3 (3), there exists $\mu \in \Psi(\lambda)$ such that $|\mu^+| = |\lambda^+| - 2 > |\pi^+|$. Thus by Lemma 2.3 (2), we have;

(2.4.2): In Case (II_1), $\Psi(\lambda) \neq \Psi(\pi)$.

Consider Case (II_2). Then by Lemma 2.3 (6), there exists $\mu \in \Psi(\lambda)$ such that $|\mu^+| = |\lambda^+| > |\pi^+|$. Thus by Lemma 2.3 (2),

(2.4.3): In Case (II_2), $\Psi(\lambda) \neq \Psi(\pi)$.

Consider Case (II_3). Then by the assumption $\psi(\lambda) = \psi(\pi) \geq 3$, $\max(\lambda^-) \geq 3$. Hence by Lemma 2.3 (5), there exists $\mu \in \Psi(\lambda)$ such that $|\mu^+| = |\lambda^+| > |\pi^+|$. Thus by Lemma 2.3 (2),

(2.4.4): In Case (II_3), $\Psi(\lambda) \neq \Psi(\pi)$.

Consider Case (III_1) and (III_2) together. In both cases, we may assume that

$$\lambda^+ = (\lambda_1^+, \dots, \lambda_t^+, \lambda_{t+1}^+, \dots, \lambda_s^+),$$

$$\pi^+ = (\lambda_1^+, \dots, \lambda_t^+, \pi_{t+1}^+, \dots, \pi_s^+),$$

where $s \geq t+1 \geq 1$, $\lambda_1^+ \geq \cdots \geq \lambda_s^+$ and $\lambda_{t+1}^+ > \pi_{t+1}^+ \geq \cdots \geq \pi_s^+$. Note that $\lambda_{t+1}^+ \geq 4$. At first, assume $\max(\lambda^-) \geq 3$. Then by Lemma 2.3 (5), there exists $\mu \in \Psi(\lambda)$ such that $\mu^+ = \lambda^+$. However, by Lemma 2.3 (7), it is impossible that $\mu \in \Psi(\pi)$. Next, assume that $\max(\lambda^-) = 1$ or $\lambda^- = 0$. If $t \neq 0$ and $\lambda_t^+ > \lambda_{t+1}^+$, then there exists $\mu \in \Psi(\lambda)$ such that

$$\mu^+ = (\lambda_1^+, \dots, \lambda_{t-1}^+, \lambda_t^+ - 2, \lambda_{t+1}^+, \dots, \lambda_s^+)$$

with $\lambda_t^+ - 2 \geq \lambda_{t+1}^+$. However, by Lemma 2.3 (7), $\mu \notin \Psi(\pi)$. Suppose $t \neq 0$ with $\lambda_t^+ = \lambda_{t+1}^+$. Then there exists $\mu \in \Psi(\pi)$ such that

$$\mu^+ = (\lambda_1^+, \dots, \lambda_{t-1}^+, \lambda_t^+ - 2, \pi_{t+1}^+, \dots, \pi_s^+).$$

However, we also see that $\mu \notin \Psi(\lambda)$. (Why? For example, observe that $\Psi((4,4,1)) \not\ni (2,2,3,11) \in \Psi((4,2,3))$.) Suppose $t = 0$. In the case (III_1), it immediately follows that $s \geq 4$, and we easily find μ with $\Psi(\lambda) \ni \mu \notin \Psi(\pi)$. (Indeed, for example, if $\lambda = (6,2,2,2)$ and $\pi^+ = (4,4,2,2)$, then by Lemma 2.3 (7), $\Psi(\lambda) \ni (6,2,3,1) \notin \Psi(\pi)$.) Thus we have;

(2.4.5): In Case (III_1), $\Psi(\lambda) \neq \Psi(\pi)$.

It remains to consider the case (III_2). As seen above, we may assume $\lambda = (\lambda_1, \lambda_2, 1, \dots, 1)$ and $\pi^+ = (\pi_1^+, \pi_2^+)$ with $\lambda_1 > 4$ and $\lambda_1 > \pi_1^+ \geq \pi_2^+ \geq 2$. Assume $\lambda_2 = 2$. Then we easily find $\mu = (\lambda_1, \lambda_2 - 2, 1, \dots, 1) \in \Psi(\lambda)$, and moreover, by Lemma 2.3 (7), $\mu \notin \Psi(\pi)$. Next, assume $\lambda_2 = 2$. If $\lambda_1 - \pi_1 \geq 4$, then by Lemma 2.3 (7), $\Psi(\lambda) \ni (\lambda_1 - 2, 2, 1, \dots, 1) \notin \Psi(\pi)$, so that we assume that $\pi_1 = \lambda_1 - 2$. Moreover, since $\psi(\lambda) = \psi(\pi)$, $\pi = (\lambda_1 - 2, 4, 1, \dots, 1)$ with $\lambda_1 \geq 6$ or $\pi = (\lambda_1 - 2, 2, 3, 1, \dots, 1)$ with $\lambda_1 \geq 4$. In the former case, we have $\Psi(\lambda) \ni (2, 2, \lambda_1 - 3, 1, \dots, 1) \notin \Psi(\pi)$, and in the latter case, $\Psi(\lambda) \ni (\lambda_1 + 1, 1, \dots, 1) \notin \Psi(\pi)$. Thus we have;

(2.4.6): In Case (III_2), $\Psi(\lambda) \neq \Psi(\pi)$.

Consider Case (IV). Note that $\lambda^- \neq \pi^-$. At first, assume $\max(\lambda^+) = \max(\pi^+) = a \geq 4$. Let $\lambda^+ = \pi^+ = (\eta, i, j)$ with $\eta \in \Lambda^e(|\lambda^+| - a)$. Then there exists $\mu \in \Psi(\lambda)$ such that $\mu^+ = (a - 2, \eta)$ and $\mu^- = (1, 1, \lambda^-)$. However, since $\pi^- \neq \lambda^-$, it must hold that $\mu \notin \Psi(\pi)$. Next, assume $\max(\lambda^+) = \max(\pi^+) = 2$. Then we may assume that $\lambda^+ = \pi^+ = (2, 2, \eta)$ with $\eta \in \Lambda^e(|\lambda^+| - 4)$, and moreover, we have $\Psi(\lambda) \ni (\eta, 3, 1, \lambda^-) \notin \Psi(\pi)$. Thus we have;

(2.4.7): In Case (IV), $\Psi(\lambda) \neq \Psi(\pi)$.

Finally, consider Case (V). Here, we shall show;

(2.4.8): In Case (V), $\{\lambda, \pi\} \in \{\{(7), (3, 5)\}, \{(9), (3, 7)\}\}$.

In this case, we may assume that

$$\lambda = \lambda^- = (\lambda_1, \dots, \lambda_s, \lambda_{s+1}, \dots, \lambda_{s+t}, 1, \dots, 1),$$

$$\pi = \pi^- = (\lambda_1, \dots, \lambda_s, \pi_{s+1}, \dots, \lambda_{s+t'}, 1, \dots, 1),$$

where $s \geq 0$, $t \geq 1$, $t' \geq 2$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{s+t} \geq 3$ and where $\lambda_{s+1} > \pi_{s+1} \geq \cdots \geq \pi_{s+t'} \geq 3$. Note that $\lambda_{s+1} \geq 5$.

At first, assume $t \geq 2$. Then there exists $\mu \in \Psi(\lambda)$ such that

$$\mu = \mu^- = (\lambda_1, \dots, \lambda_{s+t-1}, \lambda_{s+t} - 2, 1, \dots, 1).$$

However, it follows from Lemma 2.3 (8) that $\mu \notin \Psi(\pi)$.

Next, assume that $t = 1$. If $s \geq 1$, then there exists $\mu \in \Psi(\pi)$ such that

$$\mu = \mu^- = (\lambda_1, \dots, \lambda_{s-1}, \lambda_s - 2, \pi_{s+1}, \dots, \lambda_{s+t'}, 1, \dots, 1).$$

However, we also see $\mu \notin \Psi(\lambda)$. (Why? For example, observe $\Psi((5, 5, 1)) \not\ni (3, 3, 3, 1, 1) \in \Psi((5, 3, 3))$.) Suppose $s = 0$, i.e., $\lambda = (a, 1, \dots, 1)$ for some odd integer a. (Note that $a \geq 7$ by the assumption $\psi(\lambda) \geq 3$.) If $t' \geq 3$, then we see that $\Psi(\lambda) \ni (a - 2, 1, \dots, 1) \notin \Psi(\pi)$.

It remains to consider the case $\lambda = (a, 1, \dots, 1)$ and $\pi = (b, c, 1, \dots, 1)$ with $a \geq 7$ and $a > b \geq c \geq 3$. If $a - b \geq 4$, then by Lemma 2.3 (8), we have $\Psi(\lambda) \ni (a - 2, 1, \dots, 1) \notin \Psi(\pi)$. Assume $b = a - 2$ (≥ 5). Then by the assumption $\psi(\lambda) = \psi(\pi)$, $b = 3$. Suppose $a \geq 11$. Then there exist odd integers $a_1, a_2 \geq 5$ such that $a_1 + a_2 + 1 = a$. Hence we see that $\Psi(\lambda) \ni (a_1, a_2, 1, \dots, 1) \notin \Psi(\pi)$. Thus we have the assertion of (2.4.8).

By (2.4.1)–(2.4.8), we complete the proof. □

Remarks. (1) $\Psi((7)) = \Psi((5, 3)) = \{(5), (3, 3)\}$.

(2) $\Psi((9)) = \Psi((7, 3)) = \{(7), (5, 3), (3, 3, 3)\}$.

By the previous lemma, we have the following.

Proposition 2.5. *If Γ is isomorphic to Γ^* as a graph, then $\mathscr{X}$ is isomorphic to $\tilde{\mathscr{X}}(A_n)$ as an association scheme.*

Proof. We easily have that;

$$(p^{(2,2)}_{(3),(3)}, p^{(5)}_{(3),(3)}, p^{(3,3)}_{(3),(3)}) = (8,5,2),$$

$$(p^{(7)}_{(3),(5)}, p^{(5,3)}_{(3),(5)}) = (7,1),$$

$$(p^{(9)}_{(3),(7)}, p^{(7,3)}_{(3),(7)}) = (9,1).$$

Thus by Proposition 2.1 and Lemma 2.4, we have the assertion. (Correspond δ and $C(i)$ in Proposition 2.1 to ψ and $\Psi(\lambda)$, respectively.) □

As seen in [**5**], theory of spherical representation of an commutative association scheme acts an impotant role also in this paper. In particular, we use the following.

Proposition 2.6. *Let n be an arbitrary positive integer, and let $\mathscr{X} = (X, \{R_\lambda\}_{\lambda \in \Lambda^e(n)})$ be an association scheme which has the same set of intersection numbers as of $\tilde{\mathscr{X}}(A_n)$. Let $\boldsymbol{V}$ be the Euclidean space of dimention $(n-1)^2$ with the standard inner product $\langle\ \rangle_V$.*
Then there exists an injection $f: X \to \boldsymbol{V}$ satisfying the following;
(i) *$f(X)$ spans $\boldsymbol{V}$,*
(ii) *for any pair (x,y) in R_λ $(\lambda \in \Lambda^e(n))$, it holds that*

$$\langle f(x), f(y)\rangle_V = \chi(\lambda),$$

where for $\lambda = (i_1, \dots, i_m) \in \Lambda^e(n)$ $(i_1, \dots, i_m \geq 2)$,

$$\chi(\lambda) = n - 1 - \sum_{s=1}^{m} i_s.$$

In this paper, we write, for example, $\chi(3)$ and $\chi(2,2)$ for $\chi(\lambda)$ $(\lambda \in \{(3),(2,2)\})$.
We write $|\boldsymbol{v}|_V^2$ for $\langle \boldsymbol{v}, \boldsymbol{v}\rangle_V$.
For a subset A of $\boldsymbol{V}$, we denote by $\mathrm{Span}(A)$ the subspace of $\boldsymbol{V}$ spanned by elements of A.
We can easily observe the following.

Lemma 2.7. *Let $\mathscr{X}$ and f be as in Proposition 2.6. Then for $x, y \in X$, the following hold.*
(1) *$\langle f(x), f(y)\rangle_V = n-1$ if and only if $x = y$.*
(2) *$\langle f(x), f(y)\rangle_V = n-4$ if and only if $(x,y) \in R_{(3)}$.*
(3) *$\langle f(x), f(y)\rangle_V = n-5$ if and only if $(x,y) \in R_{(2,2)}$.*
(4) *$\langle f(x), f(y)\rangle_V = n-6$ if and only if $(x,y) \in R_{(5)}$.*
(5) *$\langle f(x), f(y)\rangle_V = n-7$ if and only if $(x,y) \in R_{(3,3),(2,4)}$.*
Moreover if there exists $z \in X$ such that $\langle f(x), f(z)\rangle_V = \langle f(y), f(z)\rangle_V = n-4$, then $(x,y) \in R_{(3,3)}$.

Proof. Straightforward. □

3. Determination of local structure (1).

In the rest of the paper, let n be an arbitrary positive integer with at least 5.

In this and the next sections, we particularly focus on the local structure of the relation graph Γ^*.

Information of parameters of $\tilde{\mathscr{X}}(A_n)$ implies the following, immediately:

Lemma 3.1. *Let $\mathscr{X} = (X, \{R_\lambda\}_{\lambda \in \Lambda^e(n)})$ be an association scheme having the same set of parameters as of $\tilde{\mathscr{X}}(A_n)$, and let Γ be the relation graph $(X, R_{(3)})$. Then for any vertex x in Γ, the following hold.*

(1) $|\Gamma(x)| = n(n-1)(n-2)/3$.

(2) $\Gamma_2(x) = R_{(2,2),(5),(3,3)}(x)$. *Moreover, for $\lambda \in \{(2,2),(5),(3,3)\}$, it holds that $R_\lambda = R'_\lambda$, where*

$$R'_{(2,2)} = \{(x,y) \in \Gamma \times \Gamma \mid \partial(x,y) = 2, |\Gamma(x,y)| = 8\},$$

$$R'_{(5)} = \{(x,y) \in \Gamma \times \Gamma \mid \partial(x,y) = 2, |\Gamma(x,y)| = 5\},$$

$$R'_{(3,3)} = \{(x,y) \in \Gamma \times \Gamma \mid \partial(x,y) = 2, |\Gamma(x,y)| = 2\}.$$

(3) *For any $y \in \Gamma(x)$,*

(a) $|\Gamma(x,y)| = p^{(3)}_{(3),(3)} = 3(n-3)+1$.

(b) $|\Gamma(x) \cap R_{(2,2)}(y)| = p^{(3)}_{(3),(2,2)} = 3(n-3)$.

(c) $|\Gamma(x) \cap R_{(5)}(y)| = p^{(3)}_{(3),(5)} = 3(n-3)(n-4)$.

(d) $|\Gamma(x) \cap R_{(3,3)}(y)| = p^{(3)}_{(3),(3,3)} = (n-3)(n-4)(n-5)/3$.

(4) *There exists no pair of vertices (y,z) such that $y \in R_{(2,2)}(x)$, $z \in R_{(3,3)}(x)$ and $y \sim z$.*

(5) $|R_{(2,2)}(x)| = 3\binom{n}{4}$, $|R_{(5)}(x)| = 24\binom{n}{5}$ *and* $|R_{(3,3)}(x)| = 40\binom{n}{6}$.

Proof. Straightforward. □

The main purpose in this section is to show the following:

Proposition 3.2. *Let $\mathscr{X} = (X, \{R_\lambda\}_{\lambda \in \Lambda^e(n)})$ be an association scheme having the same set of parameters as of $\tilde{\mathscr{X}}(A_n)$, and let Γ be the relation graph $(X, R_{(3)})$. In addition, assume that Γ satisfies conditions $(M1)$–$(M4)$ as in Observation 1.3.*

Then for any $x \in \Gamma$, the induced subgraph of Γ with respect to $\Gamma(x)$ is isomorphic to the induced subgraph of Γ^* with respect to $\Gamma^*(g)$ for any $g \in \Gamma^*$.

In the rest of this section, assume that Γ is a graph as in the previous proposition.

Let x, y be vertices in Γ with $x \sim y$. Then by ${}^x y$, we mean the vertex such that $\{x, y, {}^x y\} \in \mathscr{M}_1$. (By $(M3)$, such a vertex is unique.)

Lemma 3.3. *For all $x, y \in \Gamma$ with $\partial(x,y) = 2$, $\Gamma(x,y)$ contains no triangle.*

Proof. Immediate from $(M2)$. □

Lemma 3.4. *For any pair of vertices (x,y) in Γ with $x \sim y$, the following hold.*

(1) $|\{M \in \mathscr{M}_1 \mid \{x,y\} \subset M\}| = 1$.

(2) $|\{M \in \mathscr{M}_2 \mid \{x,y\} \subset M\}| = 3$.

Proof. (1) Immediate from (M3).

(2) By (1), $(M1)$ and $(M2)$, there exist maximal cliques $M_1, \dots, M_s \in \mathscr{M}_2$ such that;

(3.4.1) $\Gamma(x,y) = (\bigcup_{i=1}^{s} (M_i \backslash \{x,y\})) \cup \{{}^x y\}$,

and,

(3.4.2) $M_i \cap M_j = \{x,y\}$ if $i \neq j$.

By Lemma 3.1 (3a), it follows that $s = 3$, which is desired. □

Lemma 3.5. *Pick any vertex $x \in \Gamma$, and let $\Delta = \Delta_x$ be the induced subgraph of Γ whose vertex set is $\Gamma(x)$. Then the following hold.*

(1) *Δ contains no subgraph which is isomorphic to $K_{2,1,1}$.*

(2) *For any vertex $y \in \Delta$, $\Delta(y)$ forms the disjoint union of one point and three cliques of size $n-3$. Moreover, for $z \in \Delta(y)$, z is not contained in any of three cliques of size $n-3$ if and only if $\{x,y,z\} \in \mathscr{M}_1$.*

Proof. Immediate from Lemmas 3.3 and 3.4. □

Lemma 3.6. *Let $\{x,y,z\} \in \mathscr{M}_1$. Then*

$$\Gamma(x,y)\backslash\{z\} = \Gamma(x) \cap R_{(2,2)}(z) = \Gamma(y) \cap R_{(2,2)}(z).$$

Proof. By $(M5)$, we easily have:

(3.6.1) $\Gamma(x) \cap R_{(2,2)}(z) \subset \Gamma(x,y)\backslash\{z\}$, $\Gamma(y) \cap R_{(2,2)}(z) \subset \Gamma(x,y)\backslash\{z\}$.

By Lemma 3.1 (3a),(3b), we have:

(3.6.2) $|\Gamma(x) \cap R_{(2,2)}(z)| = |\Gamma(y) \cap R_{(2,2)}(z)| = 3(n-3)$,

and;

(3.6.3) $|\Gamma(x,y)\backslash\{z\}| = 3(n-3)$.

By (3.6.1)–(3.6.3), we have the assertion. □

Lemma 3.7. *Let $M = \{x,y,z\}$ be an element of $\mathscr{M}_1$, and let u be a vertex in $\Gamma(x)$ with $u \notin M$. Then $y \sim u$ if and only if $(z,u) \in R_{(2,2)}$.*

Proof. Immediate from Lemma 3.6. □

Lemma 3.8. *Pick any $x \in \Gamma$, and pick $y,z \in \Gamma(x)$ with $y \neq z$ and $y \not\sim z$. Then $|\Gamma(x,y,z)| \leq 4$.*

Proof. Suppose $|\Gamma(x,y,z)| \geq 5$. Then by Lemma 3.5 (2), there exist $u,v \in \Gamma(x,y,z)$ such that $u \sim v$, which contradicts Lemma 3.5 (1). Thus we have the assertion. □

For a clique $M \in \mathscr{M}_2$, and for a vertex $x \in M$, assume

$${}^x M = \{x\} \cup \{{}^x y \mid y \in M\backslash\{x\}\}$$

Lemma 3.9. *Let $M \in \mathscr{M}_2$ and $x \in M$.*
Then the following hold.

(1) *${}^x M \in \mathscr{M}_2$. Moreover, $M \cap {}^x M = \{x\}$ and ${}^x({}^x M) = M$.*

(2) *Let $y \in {}^x M\backslash\{x\}$. Then $e(y,M) = 2$ and $M \subset R_{(3),(2,2)}(y)$.*

Proof. (1) By $(M3)$ and Lemma 3.7, we can easily see that ${}^xM$ forms a clique of size $n-1$. The latter two claims are clear.

(2) Immediate from $(M5)$ and Lemma 3.7. □

Lemma 3.10. *Let $x, y \in \Gamma$ with $\partial(x, y) = 2$. Then the following are equivalent.*

(i) $(x, y) \in R_{(2,2)}$.

(ii) $|\Gamma(x, y, z)| = 4$ *for any vertex* $z \in \Gamma(x, y)$.

Proof. At first we assume (i), and pick any $z \in \Gamma(x, y)$. Consider the graph $\Delta = \Delta_z$ as in Lemma 3.5. Note that $x, y \in \Delta$. Write $\Delta(x) = \{{}^zx\} \cup M_1 \cup M_2 \cup M_3$, where M_1, M_2, M_3 are cliques of size $n-3$. (See Lemma 3.5 (2).) Note that $M_i \cup \{z, x\} \in \mathscr{M}_2$ for $i \in \{1, 2, 3\}$. Hence by $(M5)$, $e(y, M_1) = e(y, M_2) = e(y, M_3) = 1$. Moreover, by Lemma 3.7, $y \sim {}^zx$. Thus we have (i) ⇒ (ii). It follows from Lemmas 3.1 (2) and 3.3 that (ii) ⇒ (i). (Clearly it cannot hold that $(x, y) \in R_{(3,3)}$. If $(x, y) \in R_{(5)}$, then $\Gamma(x, y)$ becomes a clique of size 5.) Now we complete the proof. □

Lemma 3.11. *Let x and y be vertices with (x, y) in $R_{(2,2)}$, then the following hold.*

(1) *The induced subgraph whose vertex set is $\Gamma(x, y)$ forms $K_{4,4}$.*

(2) *If $u, v \in \Gamma(x, y)$ with $u \not\sim v$, then $(u, v) \in R_{(2,2)}$.*

(3) *$\Gamma(x, y) \cup \{x\}$ contains exactly 4 cliques in $\mathscr{M}_1$.*

Proof. By Lemmas 3.1 (2) and 3.7, we may assume that;

$$\Gamma(x, y) = \{z_1, z_2, z_3, z_4, {}^xz_1, {}^xz_2, {}^xz_3, {}^xz_4\},$$

which implies (3). By Lemmas 3.3 and 3.10, we may assume that $z_i \sim {}^xz_j$, $z_i \not\sim z_l$ and that ${}^xz_i \not\sim {}^xz_l$ for $i, j, l \in \{1, 2, 3, 4\}$ with $i \neq l$, which imply (1). (2) follows from Lemma 3.7. Now we complete the proof. □

Lemma 3.12. *Let x, y, u and v be distinct vertices with $x \sim y \sim u \sim v \sim x$, $x \not\sim u$ and $y \not\sim v$. Then $(x, u) \in R_{(2,2)}$ if and only if $(y, v) \in R_{(2,2)}$.*

Proof. Immediate from Lemma 3.11 (2). □

Lemma 3.13. *For any pair $(x, y) \in R_{(2,2)}$, $\Phi(x, y)$ is isomorphic to $K_{4,4,4}$. Moreover, for any pair of distinct vertices (u, v) in $\Phi(x, y)$, $(u, v) \in R_{(3),(2,2)}$.*

Proof. Let $\Gamma(x, y) = \{z_1, z_2, z_3, z_4, {}^xz_1, {}^xz_2, {}^xz_3, {}^xz_4\}$ be as in the proof of Lemma 3.11. For $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$, let $v_1^{(i,j)}, v_2^{(i,j)}, w_1^{(i,j)}, w_2^{(i,j)}$ be vertices satisfying

$$\{v_1^{(i,j)}, v_2^{(i,j)}\} = \Gamma(z_i, z_j) \backslash \{x, y, {}^xz_1, {}^xz_2, {}^xz_3, {}^xz_4\},$$

$$\{w_1^{(i,j)}, w_2^{(i,j)}\} = \Gamma({}^xz_i, {}^xz_j) \backslash \{x, y, z_1, z_2, z_3, z_4\}.$$

(Note that $|\Gamma(z_i, z_j)| = |\Gamma({}^xz_i, {}^xz_j)| = 8$.)

Now we can observe that

(3.13.1) $\{v_1^{(i,j)}, v_2^{(i,j)}\} = \{w_1^{(i',j')}, w_2^{(i',j')}\}$ for $i, j, i', j' \in \{1, 2, 3, 4\}$ with $i \neq j$ and $i' \neq j'$.

Indeed, by Lemma 3.11 (1), we have $v_1^{(i,j)} \sim {}^xz_k \sim v_2^{(i,j)}$ for $k \in \{1, 2, 3, 4\}$, so that $\Gamma({}^xz_{i'}, {}^xz_{j'}) \ni v_1^{(i,j)}, v_2^{(i,j)}$.

This easily implies that;

(3.13.2) $\{v_1^{(i,j)}, v_2^{(i,j)}\} = \{v_1^{(i',j')}, v_2^{(i',j')}\}$ for $i, j, i', j' \in \{1,2,3,4\}$ with $i \neq j$ and $i' \neq j'$.

(3.13.3) $\{w_1^{(i,j)}, w_2^{(i,j)}\} = \{w_1^{(i',j')}, w_2^{(i',j')}\}$ for $i, j, i', j' \in \{1,2,3,4\}$ with $i \neq j$ and $i' \neq j'$.

Moreover, by Lemma 3.11 (2);

(3.13.4) $(v_1^{(i,j)}, v_2^{(i,j)}), (w_1^{(i,j)}, w_2^{(i,j)}) \in R_{(2,2)}$ for $i, j \in \{1,2,3,4\}$ with $i \neq j$.

By (3.13.1)–(3.13.4), we see that there exist two vertices v_1, v_2 such that

$$\{x, y, v_1, v_2, z_1, z_2, z_3, z_4, {}^x z_1, {}^x z_2, {}^x z_3, {}^x z_4\}$$

forms the minimal geodetically closed subgraph containing x, y which is isomorphic to $K_{4,4,4}$. □

Lemma 3.14. *Pick any $x, y \in \Gamma$ with $x \sim y$. Let $\Delta = \Delta_x$ be the induced subgraph in Γ whose vertex set is $\Gamma(x)$. Assume*:

$$A = \Gamma(x, y) \backslash \{{}^x y\},$$

$$B = \Gamma(x, {}^x y) \backslash \{y\},$$

$$C = \Delta_2(y) \cap \Delta_2({}^x y).$$

Then the following hold.

(1) *$\Delta(y) \cap \Delta({}^x y) = \emptyset$.*

(2) *$A = \Gamma(x) \cap R_{(2,2)}({}^x y)$, $B = \Gamma(x) \cap R_{(2,2)}(y)$.*

(3) *Each of A and B is the disjoint union of 3 cliques of size $n-3$.*

(4) *For any vertex $u \in A$ (resp. $\in B$), $e(u, B) = 3$ (resp. $e(u, A) = 3$). Moreover, if $\{v_1, v_2, v_3\} = \Gamma(u) \cap B$ (resp. $= \Gamma(u) \cap A$), then $(v_1, v_2), (v_2, v_3), (v_3, v_1) \in R_{(2,2)}$.*

(5) *If $u \in A$ (resp. $\in B$), then ${}^x u \in B$ (resp. $\in A$).*

(6) *Let $u \in A$ and $v \in B$ with $u \sim v$. Let $\Gamma(u) \cap B = \{v = v_1, v_2, v_3\}$ and $\Gamma(v) \cap A = \{u = u_1, u_2, u_3\}$. Then for any $i, j \in \{1,2,3\}$, $u_i \sim v_j$, that is, $\{u_i, v_j \mid i, j \in \{1,2,3\}\}$ forms $K_{3,3}$.*

(7) *$\Delta_2(y) = B \cup C$ and $\Delta_2({}^x y) = A \cup C$, i.e., $\Delta_2(y) \cap \Delta_3({}^x y) = \Delta_3(y) \cap \Delta_2({}^x y) = \emptyset$.*

(8) *For any vertex $u \in A \cup B$, $e(u, C) = 2(n-4)$.*

(9) *Let $u \in A$ (resp. $u \in B$) and $v \in C$ with $u \sim v$. Then there exists a unique vertex $w \in B$ (resp. $w \in A$) such that $u \sim w \sim v$.*

(10) *Let $u \in C$ such that $e(u, A) \geq 2$ (resp. $e(u, B) \geq 2$). Then $(y, u) \in R_{(5)}$ (resp. $({}^x y, u) \in R_{(5)}$). Moreover, for distinct vertices $v, w \in \Gamma(u) \cap A$, $(v, w) \in R_{(5)}$.*

(11) *For $u \in C$, $e(u, A) = e(u, B) \leq 3$.*

(12) *For $u \in \Delta \cap R_{(5)}(y)$, $e(u, A) = e(u, B) \leq 3$.*

Proof. (1) Since $\{x, y, {}^x y\}$ is a maximal clique, (1) holds.

(2) Immediate from Lemma 3.6.

(3) From (1) and Lemma 3.5 (2), (3) holds.

(4) Assume $u \in A$. Then by Lemma 3.10, we see that $|\Gamma(x, {}^x y, u)| = 4$. By (1) and (2), it holds that $\Gamma(x, {}^x y, u) \backslash \{y\} \subset B$, which is desired.

The latter claim follows from Lemma 3.11 (2).

(5) Immediate from Lemma 3.7.

(6) Note that $\Phi(y, v_1) = \Phi({}^x y, u_1) \ni x$. It follows from Lemma 3.11 that $\{y, {}^x y, u_1, u_2, u_3, v_1, v_2, v_3\}$ forms $K_{4,4}$, which implies the assertion of (6).

(7), (8) Assume $u \in A$. By Lemma 3.5 (2), assume $\Delta(u) = D_1 \cup D_2 \cup D_3 \cup \{{}^x u\}$, where D_1, D_2 and D_3 are cliques of size $n-3$. Then by (1), (5) and (6), we may assume that $D_1 \subset \{y\} \cup A$ and $D_2, D_3 \subset B \cup C$, and, moreover, $|D_2 \cap C| = |D_3 \cap C| = n-4$. Thus we have (7) and (8).

(9) Immediate from the proof of (7) and (8).

(10) Assume $e(u, A) \geq 2$. Then we see that $|\Gamma(y, u)| \geq 3$, so that $(y, u) \in R_{(2,2),(5)}$. Thus by (2), we have the former assertion. By Lemma 3.5 (1), we also have the latter.

(11) By (9), we may assume $e(u, A) \geq 2$. Pick distinct vertices $v, w \in \Gamma(u) \cap A$. By (9), we can find $v' \in \Gamma(u, v) \cap B$ and $w' \in \Gamma(u, w) \cap B$. Note that by (10), $v \not\sim w$. Suppose $v' = w'$. Then $\{u, v', v, w\}$ forms $K_{2,1,1}$, which contradicts Lemma 3.5 (1). Hence we have $v' \neq w'$. Thus we have $e(u, A) \leq e(u, B)$. Similarly, we also have the invert inequality, and we have $e(u, A) = e(u, B)$. It immediately follows from (3) and Lemma 3.5 (1) that $e(u, A) \leq 3$. Thus we have the assertion.

(12) Immediate from (2), (4) and (11). □

Lemma 3.15. *Pick any vertices x, y and z such that $y \sim x \sim z$ and $(y, z) \in R_{(5)}$. Then $|\Gamma(x, y, z)| \leq 2$.*

Proof. In this proof, let x, y, A, B and C be as in Lemma 3.14. Let $D_1, D_2, D_3, D_1', D_2', D_3'$ be cliques of size $n-3$ which are disjoint with each other such that $A = D_1 \cup D_2 \cup D_3$, $B = D_1' \cup D_2' \cup D_3'$, and that;

(3.15.1): $\{x, {}^x y\} \cup D_i' = {}^x(\{x, y\} \cup D_i)$ for $i \in \{1, 2, 3\}$.

(See Lemmas 3.9 (1) and 3.14 (3).) Note that $\{x, y\} \cup D_i, \{x, {}^x y\} \cup D_i' \in \mathscr{M}_2$ for $i \in \{1, 2, 3\}$. By Lemma 3.14 (4), we also see that;

(3.15.2): For $u \in A$ (resp. $v \in B$) and $i \in \{1, 2, 3\}$, $e(u, D_i') = 1$ (resp. $e(v, D_i) = 1$).

Suppose that there exists $z \in \Gamma(x) \cap R_{(5)}(y)$ such that $|\Gamma(x, y, z)| \geq 3$. Note that by Lemma 3.14 (7), $z \in C$, and by Lemma 3.14 (11), $e(z, A) = e(z, B) = 3$. Let $\Gamma(z) \cap A = \{u_1, u_2, u_3\}$ and $\Gamma(z) \cap B = \{v_1, v_2, v_3\}$. Then by Lemma 3.14 (10), we may assume that $u_i \in D_i$ and $v_i \in D_i'$ $(i \in \{1, 2, 3\})$ with;

(3.15.3): $(u_i, u_j), (v_i, v_j) \in R_{(5)}$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$.

Note that by (3.15.1),

(3.15.4): ${}^x u_i \in D_i'$ and ${}^x v_i \in D_i$ for $i \in \{1, 2, 3\}$.

Suppose $u_1 \sim v_1$. Then it follows from (3.15.2) and (3.15.4) that $v_1 = {}^x u_1$. However, since $z \in \Gamma(x, u_1, v_1)$, this is a contradiction. Thus we have;

(3.15.5): $u_i \not\sim v_i$ for $i \in \{1, 2, 3\}$.

By the above argument, we also have;

(3.15.6): $u_i \neq {}^x v_i$ and $v_i \neq {}^x u_i$ for $i \in \{1, 2, 3\}$.

It follows from Lemma 3.7 that;

(3.15.7): $(u_i, v_i) \in R_{(2,2)}$ for $i \in \{1, 2, 3\}$.

Thus by Lemma 3.14 (9), we may assume that;

(3.15.8): $v_2 \sim u_1 \not\sim v_3$, $v_3 \sim u_2 \not\sim v_1$, $v_1 \sim u_3 \not\sim v_2$.

By (3.15.2), there exists $w \in D_3'$ such that $\Gamma(u_1, v_3) \supset \{x, z, w\}$, that is, $|\Gamma(u_1, v_3)| \geq 3$. Hence by Lemma 3.1 (2), we have $(u_1, v_3) \notin R_{(3,3)}$. Thus by Lemma 3.7, we have;

(3.15.9): $(u_1, v_3), (u_2, v_1), (u_3, v_2) \in R_{(5)}$.

From now on, by using the spherical representation f as in Proposition 2.6, we shall show that there cannot exists such a set

$$\{x, y, {}^x y, z, u_1, u_2, u_3, v_1, v_2, v_3\}.$$

Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}$ and $\boldsymbol{b}$ be vectors in $\boldsymbol{V}$ as follows;

$$\boldsymbol{a}_1 = f(x) + f(y) + f(z) - f(u_1) - f(u_2) - f(u_3),$$

$$\boldsymbol{a}_2 = f(x) + f({}^x y) + f(z) - f(v_1) - f(v_2) - f(v_3),$$

$$\boldsymbol{a} = \boldsymbol{a}_1 + \boldsymbol{a}_2,$$

$$\boldsymbol{b} = f(x) - f(z).$$

At first, we shall observe;
(3.15.10): $|\boldsymbol{a}_1|_{\boldsymbol{V}}^2 = |\boldsymbol{a}_2|_{\boldsymbol{V}}^2 = 2.$
Indeed, by Lemma 3.1 and (3.15.3), we can calculate as follows;

$$\begin{aligned}|\boldsymbol{a}_1|_{\boldsymbol{V}}^2 &= 6 \cdot \chi(1) - 18 \cdot \chi(3) + 4 \cdot \chi(3) + 8 \cdot \chi(5)\\ &= 6(n-1) - 14(n-4) + 8(n-6)\\ &= 2,\end{aligned}$$

so is $|\boldsymbol{a}_2|_{\boldsymbol{V}}^2$.
By the similar calculation, we have;
(3.15.11): $\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle_{\boldsymbol{V}} = -1,$
(3.15.12): $|\boldsymbol{b}|_{\boldsymbol{V}}^2 = 6,$
and;
(3.15.13): $\langle \boldsymbol{a}_1, \boldsymbol{b} \rangle_{\boldsymbol{V}} = \langle \boldsymbol{a}_2, \boldsymbol{b} \rangle_{\boldsymbol{V}} = 2.$
It follows from (3.15.10) and (3.15.11) that;
(3.15.14): $|\boldsymbol{a}|_{\boldsymbol{V}}^2 = 2.$
It follows from (3.15.13) that;
(3.15.15): $\langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\boldsymbol{V}} = 4.$
On the other hand, by the Cauthy-Schwartz inequality, we have;

$$|\boldsymbol{a}|_{\boldsymbol{V}}^2 \cdot |\boldsymbol{b}|_{\boldsymbol{V}}^2 \geq (\langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\boldsymbol{V}})^2.$$

However, this contradicts (3.15.12), (3.15.14) and (3.15.15).
Thus we have the assertion. □

Corollary 3.16. *Let* $x, y \in \Gamma$ *with* $\partial(x, y) = 2$. *Then the following are equivalent.*
(i) $(x, y) \in R_{(2,2)}$.
(ii) $|\Gamma(x, y, u)| \geq 3$ *for some vertex* $u \in \Gamma(x, y)$.

Proof. In Lemma 3.10, we have already shown that (i) ⇒ (ii). It immediately follows from Lemmas 3.1 (2) and 3.15 that (ii) ⇒ (i). Thus we have the assertion. □

Lemma 3.17. *Let* x, y, Δ, A, B, C *be as in Lemma* 3.14. *Then the following hold.*
(1) *For any vertex* $u \in C$, $e(u, A) = e(u, B) = 2$.
(2) $\Gamma(x) \cap R_{(5)}(y) = \Gamma(x) \cap R_{(5)}({}^x y) = C$.

(3) *Let* $u, v, w \in \Delta$ *with* $u \sim v \sim w$ *and* $(u, w) \in R_{(5)}$. *Then there exists a unique vertex* $z \in \Delta(u, w)$ *such that* $(v, z) \in R_{(5)}$.

(4) *Let* $u \in C$. *Then* $\Delta(u)$ *contains exactly two cliques* M_1, M_2 *of size* $n - 3$ *such that* $|M_i \cap A| = |M_i \cap B| = 1$ *and* $|M_i \cap C| = n - 5$ $(i \in \{1, 2\})$.

Proof. (1),(2) By Lemmas 3.1 (4) and 3.14 (2), we have;

(3.17.1) $C \subset \Gamma(x) \cap R_{(5)}(y)$,

and;

(3.17.2) $C \subset \Gamma(x) \cap R_{(5)}({}^x y)$.

It follows from Lemma 3.1 (3c) that;

(3.17.3) $|C| \leq 3(n-3)(n-4)$.

By Lemma 3.15, we have;

(3.17.4) For $u \in C$, $e(u, A) \leq 2$ and $e(u, B) \leq 2$.

(3.17.3) and (3.17.4) imply;

(3.17.5) $e(A, C) \leq 6(n-3)(n-4)$,

and;

(3.17.6) $e(B, C) \leq 6(n-3)(n-4)$.

On the other hand, by Lemmas 3.14 (2), (8) and 3.1 (3b), we have;

(3.17.7) $e(A, C) = e(B, C) = 6(n-3)(n-4)$.

Thus by (3.17.1)–(3.17.7), we have (1) and (2).

(3) Immediate from (1), (2) and Lemma 3.14 (10).

(4) Immediate from (1) and Lemma 3.14 (5),(8),(9). □

Lemma 3.18. *Let* x, y *be vertices with* $\partial(x, y) = 2$. *Then the following are equivalent.*

(i) $(x, y) \in R_{(3,3)}$.

(ii) $|\Gamma(x, y, u)| = 0$ *for some* $u \in \Gamma(x, y)$.

Proof. Immediate from Lemma 3.1 (2), Corollary 3.16, Lemma 3.17 (1) and (2). □

Lemma 3.19. *Let* x, y *be vertices with* $(x, y) \in R_{(5)}$. *Then* $\Gamma(x, y)$ *forms a pentagon. Moreover, for* $u, v \in \Gamma(x, y)$, *if* $u \not\sim v$, *then* $(u, v) \in R_{(5)}$.

Proof. Immediate from Lemma 3.17 (1) and (3). □

Lemma 3.20. *Let* x, y, u *and* v *be distinct vertices with* $x \sim y \sim u \sim v \sim x$, $x \not\sim u$ *and* $y \not\sim v$. *Then for* $\lambda \in \{(2,2), (5), (3,3)\}$, $(x, u) \in R_\lambda$ *if and only if* $(y, v) \in R_\lambda$.

Proof. Immediate from Lemmas 3.12 and 3.19. □

Lemma 3.21. *Let* $M = \{x, y, z\}$ *be an element of* $\mathscr{M}_1$, *and let* u *be a vertex in* $\Gamma(x)$ *with* $u \notin M$.

Then for $\lambda \in \{(5), (3,3)\}$,

$$(y, u) \in R_\lambda \quad \text{if and only if} \quad (z, u) \in R_\lambda.$$

Proof. By Lemmas 3.7 and 3.17 (2), the assertion follows. (Note that, if $(y, u) \in R_{(5)}$, then there exists a pair (w_1, w_2) such that ${}^x w_1 = w_2$, $y \sim w_1 \sim u$ and $z \sim w_2 \sim v$.) □

For $s \in \{0,1,2,3\}$ and $(i,j,k) \subset N_3$, denote by $C_{(3)}^{3-s}\{i,j,k\}$ the subset of $C_{(3)}$ as follows:

$$C_{(3)}^{3-s}\{i,j,k\} = \{(l\ \ m\ \ n) \in C_{(3)} \mid (l,m,n) \in N_3, |\{i,j,k\} \cap \{l,m,n\}| = s\}.$$

Note that $C_{(3)}^{0}\{i,j,k\} = \{(i\ \ j\ \ k),(j\ \ i\ \ k)\}$.

Let $C_{(3)}^{\leq s}\{i,j,k\} = \bigcup_{t=0}^{s} C_{(3)}^{t}\{i,j,k\}$. Note that $C_{(3)} = C_{(3)}^{\leq 3}\{i,j,k\}$ for any $(i,j,k) \subset N_3$.

Lemma 3.22. *Let x and y be vertices with $x \sim y$, and let A, B be as in Lemma* 3.14. *Then there exists a bijection*

$$v_1 : C_{(3)}^{\leq 1}\{1,2,3\} \to \{y, {}^x y\} \cup A \cup B$$

satisfying;

(i) $(1\ \ 2\ \ 3)^{v_1} = y$, $(1\ \ 3\ \ 2)^{v_1} = {}^x y$,

(ii) $\{A_1, A_2, A_3\}$ *is a partition of A, where*

$$A_1 = \{(1\ \ 2\ \ i)^{v_1} \mid 4 \leq i \leq n\},$$

$$A_2 = \{(2\ \ 3\ \ i)^{v_1} \mid 4 \leq i \leq n\},$$

$$A_3 = \{(3\ \ 1\ \ i)^{v_1} \mid 4 \leq i \leq n\}.$$

(iii) $\{B_1, B_2, B_3\}$ *is a partition of B, where*

$$B_1 = \{(2\ \ 1\ \ i)^{v_1} \mid 4 \leq i \leq n\},$$

$$B_2 = \{(3\ \ 2\ \ i)^{v_1} \mid 4 \leq i \leq n\},$$

$$B_3 = \{(1\ \ 3\ \ i)^{v_1} \mid 4 \leq i \leq n\}.$$

(iv) $A_j \cup \{x,y\}$, $B_j \cup \{x, {}^x y\} \in \mathscr{M}_2$ *for* $j \in \{1,2,3\}$.

(v) $(2\ \ 1\ \ i)^{v_1} = {}^x((1\ \ 2\ \ i)^{v_1})$, $(3\ \ 2\ \ i)^{v_1} = {}^x((2\ \ 3\ \ i)^{v_1})$ *and* $(1\ \ 3\ \ i)^{v_1} = {}^x((3\ \ 1\ \ i)^{v_1})$ *for* $i \in \{4,\dots,n\}$.

(vi) *For* $g,h \in C_{(3)}^{\leq 1}\{1,2,3\}$ *and* $\lambda \in \{(1),(3),(2,2),(5)\}$,

$$(g,h) \in R_\lambda^* \quad \text{if and only if} \quad (g^{v_1}, h^{v_1}) \in R_\lambda.$$

Proof. By Lemma 3.9 (1), Lemma 3.14 (3),(4),(5) and (6), there exists a bijection v_1 satisfying (i)–(v), and satisfying: (3.22.1)–(3.22.4) as follows:

(3.22.1) For $i \in \{4,\dots,n\}$,

$$\Gamma((1\ \ 2\ \ i)^{v_1}) \cap B = \Gamma((2\ \ 3\ \ i)^{v_1}) \cap B = \Gamma((3\ \ 1\ \ i)^{v_1}) \cap B$$
$$= \{(2\ \ 1\ \ i)^{v_1}, (3\ \ 2\ \ i)^{v_1}, (1\ \ 3\ \ i)^{v_1}\}.$$

(3.22.2) For $i \in \{4,\dots,n\}$,

$$\Gamma((2\ \ 1\ \ i)^{v_1}) \cap A = \Gamma((3\ \ 2\ \ i)^{v_1}) \cap A = \Gamma((1\ \ 3\ \ i)^{v_1}) \cap A$$
$$= \{(1\ \ 2\ \ i)^{v_1}, (2\ \ 3\ \ i)^{v_1}, (3\ \ 1\ \ i)^{v_1}\}.$$

(3.22.3) For $i \in \{4,\dots,n\}$, $((1\ \ 2\ \ i)^{v_1}, (2\ \ 3\ \ i)^{v_1})$, $((2\ \ 3\ \ i)^{v_1}, (3\ \ 1\ \ i)^{v_1})$, $((3\ \ 1\ \ i)^{v_1}, (1\ \ 2\ \ i)^{v_1})$, $((2\ \ 1\ \ i)^{v_1}, (3\ \ 2\ \ i)^{v_1})$, $((3\ \ 2\ \ i)^{v_1}, (1\ \ 3\ \ i)^{v_1})$, $((1\ \ 3\ \ i)^{v_1}, (2\ \ 1\ \ i)^{v_1}) \in R_{(2,2)}$.

(3.22.4) ${}^x(B_j \cup \{x, {}^x y\}) = A_j \cup \{x,y\}$ for $j \in \{1,2,3\}$.

In the following, we shall show that (vi) occurs.

Note that by the above assumption;

(3.22.5) For $g, h \in C_{(3)}^{\leq 1}\{1,2,3\}$, $(g,h) \in R_{(3)}^*$ if and only if $(g^{\nu_1}, h^{\nu_1}) \in R_{(3)}$.

By (3.22.4) and Lemma 3.9 (2), we have;

(3.22.6) For $i, j \in \{4, \ldots, n\}$ with $i \neq j$, $((1\ 2\ i)^{\nu_1}, (2\ 1\ j)^{\nu_1})$, $((2\ 3\ i)^{\nu_1}, (3\ 2\ j)^{\nu_1})$, $((3\ 1\ i)^{\nu_1}, (1\ 3\ j)^{\nu_1}) \in R_{(2,2)}$.

Next we shall show;

(3.22.7) For $i, j \in \{4, \ldots, n\}$ with $i \neq j$, $((1\ 2\ i)^{\nu_1}, (2\ 3\ j)^{\nu_1})$, $((2\ 3\ i)^{\nu_1}, (3\ 1\ j)^{\nu_1})$, $((3\ 1\ i)^{\nu_1}, (1\ 2\ j)^{\nu_1})$, $((2\ 1\ i)^{\nu_1}, (3\ 2\ j)^{\nu_1})$, $((3\ 2\ i)^{\nu_1}, (1\ 3\ j)^{\nu_1})$, $((1\ 3\ i)^{\nu_1}, (2\ 1\ j)^{\nu_1}) \in R_{(5)}$.

For example, focus on the pair $((1\ 2\ 4)^{\nu_1}, (2\ 3\ 5)^{\nu_1})$. By (3.22.5), we see that $\partial((1\ 2\ 4)^{\nu_1}, (2\ 3\ 5)^{\nu_1}) = 2$. Since $\Gamma(x, (1\ 2\ 4)^{\nu_1}, (2\ 3\ 5)^{\nu_1}) \ni (1\ 2\ 3)^{\nu_1}$, it follows from Lemma 3.18 that $((1\ 2\ 4)^{\nu_1}, (2\ 3\ 5)^{\nu_1}) \in R_{(2,2),(5)}$. Since $(2\ 1\ 4)^{\nu_1} \not\sim (2\ 3\ 5)^{\nu_1}$ (by (3.22.5)), it follows from Lemma 3.6 that $((1\ 2\ 4)^{\nu_1}, (2\ 3\ 5)^{\nu_1}) \notin R_{(2,2)}$, so that we have $((1\ 2\ 4)^{\nu_1}, (2\ 3\ 5)^{\nu_1}) \in R_{(5)}$. By such argument, we have the assertion of (3.22.7).

By (v), (3.22.7) and Lemma 3.21, we have;

(3.22.8) For $i, j \in \{4, \ldots, n\}$ with $i \neq j$, $((1\ 2\ i)^{\nu_1}, (3\ 2\ j)^{\nu_1})$, $((2\ 3\ i)^{\nu_1}, (1\ 3\ j)^{\nu_1})$, $((3\ 1\ i)^{\nu_1}, (2\ 1\ j)^{\nu_1})$, $((2\ 1\ i)^{\nu_1}, (2\ 3\ j)^{\nu_1})$, $((3\ 2\ i)^{\nu_1}, (3\ 1\ j)^{\nu_1})$, $((1\ 3\ i)^{\nu_1}, (1\ 2\ j)^{\nu_1}) \in R_{(5)}$.

Thus by (3.22.5)–(3.22.8), we can show that (vi) is satisfied. □

Lemma 3.23. *Let x, y, A, B and C be as in Lemma 3.14. Pick any vertex $u \in C$, and take $v \in \Gamma(u) \cap A$, $w \in \Gamma(u) \cap B$ with $v \not\sim w$. Then $(v, w) \in R_{(2,2)}$.*

Proof. Since we can take two vertices $v' \in \Gamma(v, w) \cap A$ and $w' \in \Gamma(v, w) \cap B$, Corollary 3.16 and Lemma 3.18 imply that $(v, w) \notin R_{(5),(3,3)}$. Thus we have the assertion. □

Lemma 3.24. *Let x, y, A, B and C be as in Lemma 3.14, and let ν_1 be a mapping as in Lemma 3.22. Then there exists a unique bijection*

$$\nu_2 : C_{(3)}^{\leq 2}\{1,2,3\} \to \{y, {}^x y\} \cup A \cup B \cup C$$

satisfying;

(i) $\nu_2|_{C_{(3)}^{\leq 1}\{1,2,3\}} = \nu_1$,

(ii) $C = (C_{(3)}^{2}\{1,2,3\})^{\nu_2}$,

(iii) *For $k \in \{1,2,3\}$ and $i \in \{4, \ldots, n\}$, $\{x\} \cup \{(k\ i\ l)^{\nu_2} \mid l \in N \backslash \{k, i\}\} \in \mathscr{M}_2$,*

(iv) *For $k \in \{1,2,3\}$ and $i, j \in \{4, \ldots, n\}$ with $i \neq j$, $(k\ j\ i)^{\nu_2} = {}^x((k\ i\ j)^{\nu_2})$,*

(v) *For $g, h \in C_{(3)}^{\leq 2}\{1,2,3\}$ and $\lambda \in \{(1), (3), (2,2), (5)\}$, if $(g,h) \in R_\lambda^*$, then $(g^{\nu_2}, h^{\nu_2}) \in R_\lambda$.*

Proof. At first, for $(i\ j\ k) \in C_{(3)}^{\leq 1}\{1,2,3\}$, we write the vertex $(i\ j\ k)^{\nu_1} \in \{y, {}^x y\} \cup A \cup B$ by $(i\ j\ k)^{\nu_2}$. By Lemma 3.17 (3), we have;

(3.24.1): For $(k, l_1) \in \{(1,2), (2,3), (3,1)\}$, $(k, l_2) \in \{(1,3), (2,1), (3,2)\}$, and $i, j \in N \backslash \{1,2,3\}$ with $i \neq j$, there exists a unique vertex $u_{(k\ i\ j)}^1$ in C such that

$$\Gamma(u_{(k\ i\ j)}^1) \cap A = \{(k\ l_1\ j)^{\nu_2}, (k\ i\ l_2)^{\nu_2}\},$$

and;

(3.24.2): For $(k, l_1) \in \{(1,2),(2,3),(3,1)\}$, $(k, l_2) \in \{(1,3),(2,1),(3,2)\}$, and $i, j \in N\backslash\{1,2,3\}$ with $i \neq j$, there exists a unique vertex $u^2_{(k\ i\ j)}$ in C such that

$$\Gamma(u^2_{(k\ i\ j)}) \cap B = \{(k\ \ l_2\ \ j)^{v_2}, (k\ \ i\ \ l_1)^{v_2}\}.$$

We want to show that;

(3.24.3): For $k \in \{1,2,3\}$ and $i, j \in \{4, \ldots, n\}$ with $i \neq j$,

(a) $u^1_{(k\ i\ j)} = u^2_{(k\ i\ j)}$,

(b) ${}^x(u^1_{(k\ i\ j)}) = u^1_{(k\ j\ i)}$.

If (3.24.3a) holds, we shall write $(k\ i\ j)^{v_2} = u^1_{(k\ i\ j)} = u^2_{(k\ i\ j)} \in C$. Then by Lemma 3.17 (1), $v_2|_{C^2_{(3)}\{1,2,3\}}$ can be regarded as a bijection to C.

Now we assume;

(3.24.4): $A' = \Gamma(x, (1\ 2\ 4)^{v_2})\backslash\{(1\ 4\ 2)^{v_2}\}$, $B' = \Gamma(x, (1\ 4\ 2)^{v_2})\backslash\{(1\ 2\ 4)^{v_2}\}$.

Note that;

(3.24.5): $(1\ 2\ 3)^{v_2}, (1\ 2\ 5)^{v_2}, (2\ 4\ 3)^{v_2}, (4\ 1\ 3)^{v_2} \in A', (1\ 3\ 2)^{v_2}, (1\ 5\ 2)^{v_2}, (4\ 2\ 3)^{v_2}, (1\ 4\ 3)^{v_2} \in B'$.

By Lemmas 3.9 (1) and 3.14 (3), we may assume;

(3.24.6): $\{A'_1, A'_2, A'_3\}$ is a partition of A' such that, for $j \in \{1,2,3\}$, $A'_j \cup \{x, (1\ 2\ 4)^{v_2}\} \in \mathscr{M}_2$,

(3.24.7): $\{B'_1, B'_2, B'_3\}$ is a partition of B' such that, for $j \in \{1,2,3\}$, $B'_j \cup \{x, (1\ 4\ 2)^{v_2}\} \in \mathscr{M}_2$,

(3.24.8): ${}^x(A'_i \cup \{x, (1\ 2\ 4)^{v_2}\}) = B'_i \cup \{x, (1\ 4\ 2)^{v_2}\}$.

Moreover, since $(1\ 2\ 3)^{v_2} \sim (1\ 2\ 5)^{v_2}$, $(1\ 3\ 2)^{v_2} \sim (1\ 5\ 2)^{v_2}$, $(1\ 5\ 2)^{v_2} = {}^x((1\ 2\ 5)^{v_2})$, $(4\ 2\ 3)^{v_2} = {}^x((2\ 4\ 3)^{v_2})$, $(1\ 4\ 3)^{v_2} = {}^x((4\ 1\ 3)^{v_2})$, and since both $\{(1\ 2\ 5)^{v_2}, (2\ 4\ 3)^{v_2}, (4\ 1\ 3)^{v_2}\}$ and $\{(1\ 5\ 2)^{v_2}, (4\ 2\ 3)^{v_2}, (1\ 4\ 3)^{v_2}\}$ are cocliques, we may assume that;

(3.24.9): $(1\ 2\ 3)^{v_2}, (1\ 2\ 5)^{v_2} \in A'_1$, $(2\ 4\ 3)^{v_2} \in A'_2$, $(4\ 1\ 3)^{v_2} \in A'_3$, $(1\ 3\ 2)^{v_2}, (1\ 5\ 2)^{v_2} \in B'_1$, $(4\ 2\ 3)^{v_2} \in B'_2$, $(1\ 4\ 3)^{v_2} \in B'_3$.

By Lemma 3.14 (5),(6), (3.24.8) and (3.24.9), we see that;

(3.24.10): there exists a unique pair (u_1, u_2) with $u_1 \in A'_2$ and $u_2 \in A'_3$ such that;

(a) ${}^xu_1 \in B'_2$, ${}^xu_2 \in B'_3$,

(b) $\Gamma((1\ 2\ 5)^{v_2}) \cap B' = \Gamma(u_1) \cap B' = \Gamma(u_2) \cap B' = \{(1\ 5\ 2)^{v_2}, {}^xu_1, {}^xu_2\}$,

(c) $\Gamma((1\ 5\ 2)^{v_2}) \cap A' = \Gamma({}^xu_1) \cap A' = \Gamma({}^xu_2) \cap A' = \{(1\ 2\ 5)^{v_2}, u_1, u_2\}$.

Since $(2\ 4\ 3)^{v_2} \nsim (1\ 5\ 2)^{v_2} \nsim (4\ 1\ 3)^{v_2}$ and $(4\ 2\ 3)^{v_2} \nsim (1\ 2\ 5)^{v_2} \nsim (1\ 4\ 3)^{v_2}$, we see that;

(3.24.11): $u_1 \neq (2\ 4\ 3)^{v_2}$, $u_2 \neq (4\ 1\ 3)^{v_2}$, ${}^xu_1 \neq (4\ 2\ 3)^{v_2}$, ${}^xu_2 \neq (1\ 4\ 3)^{v_2}$,

so that by (3.24.6), (3.24.7) and (3.24.9),

(3.24.12): $u_1 \sim (2\ 4\ 3)^{v_2}$, $u_2 \sim (4\ 1\ 3)^{v_2}$, ${}^xu_1 \sim (4\ 2\ 3)^{v_2}$, ${}^xu_2 \sim (1\ 4\ 3)^{v_2}$.

Since $u_1, u_2, {}^xu_1, {}^xu_2 \in \Delta_2((1\ 2\ 3)^{v_2}, (1\ 3\ 2)^{v_2})$, we have;

(3.24.13): $u_1, u_2, {}^xu_1, {}^xu_2 \in C$.

Hence by (3.24.10), (3.24.12) and (3.24.13), we have;

(3.24.14): $u_1 = u^2_{(2\ 4\ 5)}$, $u_2 = u^2_{(1\ 5\ 4)}$, ${}^xu_1 = u^1_{(2\ 5\ 4)}$, ${}^xu_2 = u^1_{(1\ 4\ 5)}$,

and;

(3.24.15): $u^2_{(2\ 4\ 5)} \sim (1\ 2\ 4)^{v_2} \sim u^2_{(1\ 5\ 4)}$, $u^1_{(2\ 5\ 4)} \sim (1\ 4\ 2)^{v_2} \sim u^1_{(1\ 4\ 5)}$.

Next, for example, for the vertex $u^2_{(2\ 4\ 5)}$, we shall observe;

(3.24.16): $u^2_{(2\ 4\ 5)} \sim (2\ 3\ 5)^{\nu_2}$,
which implies by (3.24.1) and (3.24.15);

(3.24.17): $u^1_{(2\ 4\ 5)} = u^2_{(2\ 4\ 5)}$.

Review that $(1\ 2\ 4)^{\nu_2} \in \Gamma(u^2_{(2\ 4\ 5)}) \cap A$, $\{(2\ 1\ 5)^{\nu_2}, (3\ 2\ 4)^{\nu_2}\} = \Gamma(u^2_{(2\ 4\ 5)}) \cap B$ and $(1\ 2\ 4)^{\nu_2} \sim (3\ 2\ 4)^{\nu_2}$. Since $\Gamma((2\ 1\ 5)^{\nu_2}) \cap A = \{(1\ 2\ 5)^{\nu_2}, (2\ 3\ 5)^{\nu_2}, (1\ 5\ 3)^{\nu_2}\}$, it follows from Lemma 3.14 (9) that;

$$|\Gamma(u^2_{(2\ 4\ 5)}) \cap \{(1\ 2\ 5)^{\nu_2}, (2\ 3\ 5)^{\nu_2}, (1\ 5\ 3)^{\nu_2}\}| = 1.$$

By Lemma 3.7, it holds that $u^2_{(2\ 4\ 5)} \not\sim (1\ 2\ 5)^{\nu_2}$. Hence, since $((2\ 3\ 5)^{\nu_2}, (3\ 2\ 4)^{\nu_2}) \in R_{(2,2)}$ and $((1\ 5\ 3)^{\nu_2}, (3\ 2\ 4)^{\nu_2}) \in R_{(5)}$, (3.24.16) follows from Lemma 3.23. Thus we have the assertion of (3.24.3a), so in the following, we shall use the notation $(k\ i\ j)^{\nu_2}$ for $k \in \{1,2,3\}$ and $i,j \in \{4,\dots,n\}$ with $i \neq j$. Note that by (3.24.14), we see that, for example, ${}^x((2\ 4\ 5)^{\nu_2}) = (2\ 5\ 4)^{\nu_2}$. Thus we also have (3.24.3b). Now we have just defined the bijection ν_2 satisfying (i), (ii) and (iv) in the statement of the lemma.

Consider the set $(C^{\leq 1}_{(3)}\{1,2,4\})^{\nu_2}$. Note that $A'_j, B'_j \subset (C^{\leq 1}_{(3)}\{1,2,4\})^{\nu_2}$ for $j \in \{1,2,3\}$, where A'_j, B'_j satisfy (3.24.6)–(3.24.10). Then we see that;

(3.24.18):

$$A'_1 = \{(1\ 2\ i)^{\nu_2} \mid i \in \{1,\dots,n\} \backslash \{1,2,4\}\},$$

$$A'_2 = \{(2\ 4\ i)^{\nu_2} \mid i \in \{1,\dots,n\} \backslash \{1,2,4\}\},$$

$$A'_3 = \{(4\ 1\ i)^{\nu_2} \mid i \in \{1,\dots,n\} \backslash \{1,2,4\}\},$$

$$B'_1 = \{(2\ 1\ i)^{\nu_2} \mid i \in \{1,\dots,n\} \backslash \{1,2,4\}\},$$

$$B'_2 = \{(4\ 2\ i)^{\nu_2} \mid i \in \{1,\dots,n\} \backslash \{1,2,4\}\},$$

$$B'_3 = \{(1\ 4\ i)^{\nu_2} \mid i \in \{1,\dots,n\} \backslash \{1,2,4\}\}.$$

Hence, by the same argument as in the proof of Lemma 3.22, we have;

(3.24.19) $(k\ 4\ i)^{\nu_2} \sim (k\ 4\ j)^{\nu_2}$, $(k\ i\ 4)^{\nu_2} \sim (k\ 4\ i)^{\nu_2}$ for $k \in \{1,2\}$ and $i,j \in \{1,\dots,n\} \backslash \{1,2,4\}$ with $i \neq j$,

(3.24.20) $(k\ 4\ i)^{\nu_2} \sim (l\ 4\ i)^{\nu_2}$ for $\{k,l\} = \{1,2\}$ and $i \in \{1,\dots,n\} \backslash \{1,2,4\}$,

(3.24.21) $((k\ i\ 4)^{\nu_2}, (k\ 4\ j)^{\nu_2})$, $((k\ l\ i)^{\nu_2}, (l\ 4\ i)^{\nu_2})$, $((k\ l\ i)^{\nu_2}, (k\ i\ 4)^{\nu_2})$, $((k\ 4\ i)^{\nu_2}, (l\ i\ 4)^{\nu_2}) \in R_{(2,2)}$ for $\{k,l\} = \{1,2\}$ and $i,j \in \{1,\dots,n\} \backslash \{1,2,4\}$ with $i \neq j$,
and;

(3.24.22) $((k\ l\ i)^{\nu_2}, (l\ 4\ j)^{\nu_2})$, $((k\ l\ i)^{\nu_2}, (k\ j\ 4)^{\nu_2})$, $((k\ l\ i)^{\nu_2}, (k\ 4\ j)^{\nu_2})$, $((k\ l\ i)^{\nu_2}, (l\ j\ 4)^{\nu_2})$, $((k\ i\ 4)^{\nu_2}, (l\ j\ 4)^{\nu_2})$, $((k\ i\ 4)^{\nu_2}, (l\ 4\ j)^{\nu_2}) \in R_{(5)}$ for $\{k,l\} = \{1,2\}$ and $i \in \{1,\dots,n\} \backslash \{1,2,4\}$ with $i \neq j$.

By such argument, we have (iii) and (v) of the lemma, and we complete the proof. □

Lemma 3.25. *Let x, y and C be as in Lemma 3.14, and let $D = \Delta_3(y) \cap \Delta_3({}^x y)$. Then the following hold.*

(1) $D = \Gamma(x) \cap R_{(3,3)}(y) = \Gamma(x) \cap R_{(3,3)}({}^x y)$.

(2) *Pick any $u \in C$, and take $v, w \in D \cap \Gamma(u)$ with $v \neq w$. Then $v \sim w$.*

Proof. (1) Immediate from Lemmas 3.14 (2) and 3.17.

(2) Immediate from Lemmas 3.5 (2) and 3.24 (iii), (iv). □

Lemma 3.26. *Let v_2 be as in Lemma 3.24. Then for $g, h \in C^{\leq 2}_{(3)}\{1,2,3\}$ and $\lambda \in \{(1),(3),(2,2),(5),(3,3)\}$, $(g,h) \in R^*_\lambda$ if and only if $(g^{v_2}, h^{v_2}) \in R_\lambda$.*

Proof. By (v) of Lemma 3.24, it suffices to show;

(3.26.1): For $g \in C^{\leq 2}_{(3)}\{1,2,3\}$ and $h \in C^2_{(3)}\{1,2,3\}$, if $(g,h) \in R^*_{(3,3)}$, then $(g^{v_2}, h^{v_2}) \in R_{(3,3)}$.

By Lemmas 3.17 (1) and Lemma 3.24 (v), we easily see that;

(3.26.2): For $g \in C^1_{(3)}\{1,2,3\}$ and $h \in C^2_{(3)}\{1,2,3\}$, if $(g,h) \in R^*_{(3,3)}$, then $(g^{v_2}, h^{v_2}) \notin R_{(3)}$.

Next, we shall show;

(3.26.3): For $g, h \in C^2_{(3)}\{1,2,3\}$, if $(g,h) \in R^*_{(3,3)}$, then $(g^{v_2}, h^{v_2}) \notin R_{(3)}$.

For example, suppose that $((1\ 4\ 5)^{v_2}, (2\ 6\ 7)^{v_2}) \in R_{(3)}$, i.e., $(1\ 4\ 5)^{v_2} \sim (2\ 6\ 7)^{v_2}$. Let $C' = \Delta_2((1\ 2\ 4)^{v_2}, (1\ 4\ 2)^{v_2})$, and consider sets A' and B' as in the proof of Lemma 3.24. Then we see that $(2\ 6\ 7)^{v_2} \in C'$ and $\Gamma((2\ 6\ 7)^{v_2}) \cap B' \ni (1\ 4\ 5)^{v_2}$, $(2\ 6\ 4)^{v_2}, (2\ 1\ 7)^{v_2}$, which contradicts Lemma 3.17 (1). Thus by such argument, we have (3.26.3).

Suppose $((1\ 2\ 4)^{v_2}, (3\ 5\ 6)^{v_2}) \in R_{(2,2)}$. Then by Lemma 3.7, it must hold that $((1\ 4\ 2)^{v_2}, (3\ 5\ 6)^{v_2}) \in R_{(3)}$, which contradicts that $\Delta((3\ 5\ 2)^{v_2}) \cap B = \{(3\ 2\ 6)^{v_2}, (1\ 3\ 5)^{v_2}\}$. (Note that ${}^x((1\ 2\ 4)^{v_2}) = (1\ 4\ 2)^{v_2}$.) Next, suppose $((1\ 4\ 5)^{v_2}, (2\ 6\ 7)^{v_2}) \in R_{(2,2)}$. Then by Lemma 3.7, it must hold that $((1\ 6\ 5)^{v_2}, (2\ 7\ 8)^{v_2}) \in R_{(3)}$, which contradicts (3.26.3). Thus, combined with (3.26.2) and (3.26.3), we have;

(3.26.4): For $g, h \in C^{\leq 2}_{(3)}\{1,2,3\}$, if $(g,h) \in R^*_{(3,3)}$, then $(g^{v_2}, h^{v_2}) \notin R_{(3),(2,2)}$.

Suppose $((1\ 2\ 4)^{v_2}, (3\ 5\ 6)^{v_2}) \in R_{(5)}$. Then, since $\Gamma((3\ 5\ 6)^{v_2}) \cap A = \{(3\ 5\ 2)^{v_2}, (3\ 1\ 6)^{v_2}\}$, $\Gamma((3\ 5\ 6)^{v_2}) \cap B = \{(3\ 2\ 6)^{v_2}, (3\ 5\ 1)^{v_2}\}$, and since $(3\ 5\ 2)^{v_2}, (3\ 1\ 6)^{v_2}, (3\ 2\ 6)^{v_2}, (3\ 5\ 1)^{v_2} \notin \Gamma((1\ 2\ 4)^{v_2})$, we see that $\Gamma(x, (1\ 2\ 4)^{v_2}, (3\ 5\ 6)^{v_2}) \subset C$. However, by (3.26.3), this is also impossible. Hence by (3.26.4), it must hold that $((1\ 2\ 4)^{v_2}, (3\ 5\ 6)^{v_2}) \in R_{(3,3)}$.

Thus we have;

(3.26.5): For $g \in C^1_{(3)}\{1,2,3\}$ and $h \in C^2_{(3)}\{1,2,3\}$, if $(g,h) \in R^*_{(3,3)}$, then $(g^{v_2}, h^{v_2}) \in R_{(3,3)}$.

Suppose $((1\ 4\ 5)^{v_2}, (2\ 6\ 7)^{v_2}) \in R_{(5)}$. Then by Lemma 3.17, there exist exactly two vertices u_1 and u_2 in $\Gamma(x)$ such that $u_1, u_2 \in \Gamma((1\ 4\ 5)^{v_2}, (2\ 6\ 7)^{v_2})$. Note that by Lemma 3.14 (10), $u_1 \not\sim u_2$. On the other hand, by Lemma 3.24 (v) and (3.26.3), it must hold that $u_1, u_2 \in D = \Delta_3(y) \cap \Delta_3(z)$. However, it follows from Lemma 3.25 that $u_1 \sim u_2$, which is a contradiction. Hence by (3.26.4), we have $((1\ 4\ 5)^{v_2}, (2\ 6\ 7)^{v_2}) \in R_{(3,3)}$.

Thus we have;

(3.26.6): For $g, h \in C^2_{(3)}\{1,2,3\}$, if $(g,h) \in R^*_{(3,3)}$, then $(g^{v_2}, h^{v_2}) \in R_{(3,3)}$.

Therefore by (3.26.5) and (3.26.6), we have the assertion of (3.26.1). Now we complete the proof. □

We define two families $\mathscr{U}_1, \mathscr{U}_2$ of 3-element-subset of $(C^2_{(3)}\{1,2,3\})^{v_2}$ in Lemma 3.24 as follows:

$$\mathscr{U}_1 = \{\{(1\ i\ j)^{v_2}, (2\ i\ j)^{v_2}, (3\ i\ j)^{v_2}\} \mid i, j \in N \setminus \{1,2,3\}, i \neq j\},$$

$$\mathscr{U}_2 = \{\{(l\ i\ j)^{v_2}, (l\ j\ k)^{v_2}, (l\ k\ i)^{v_2}\} \mid l \in \{1,2,3\}, (i,j,k) \in N_3, \{i,j,k\} \cap \{1,2,3\} = \varnothing\}.$$

Lemma 3.27. *Let* x, y, Δ, A, B *and* C *be as in Lemma* 3.14, D *as in Lemma* 3.25, *and* v_2 *as in Lemma* 3.24. *Then the following hold.*

(1) *For any* $u \in C$, *there exists a unique element* U *in* $\mathscr{U}_1$ *containing* u.

(2) *For any* $U \in \mathscr{U}_1$, *there exists a unique maximal clique* M *of size* $n-3$ *in* Δ *such that* $M \supset U$, $M \cap A = M \cap B = \emptyset$, $|M \cap C| = 3$ *and* $|M \cap D| = n-3$.

(3) *For any* $u \in C$ *and* $v \in D$ *with* $u \sim v$, *it holds that* $\Delta(v) \supset U$, *where* U *is the element of* $\mathscr{U}_1$ *containing* u.

In particular, if $n = 5$, *then* $D = \emptyset$.

(4) *For any* $v \in D$, $e(v, C) \in \{3, 6, 9\}$.

Proof. Straightforward. In particular, see Lemmas 3.5 (2) and 3.24. □

Next, we want to show the following.

Lemma 3.28. *Let* x, y, Δ, A, B *and* C *be as in Lemma* 3.14, D *as in Lemma* 3.25, *and* v_2 *as in Lemma* 3.24. *Then the following hold.*

(1) *For any element* U *in* $\mathscr{U}_2$, *there exists a unique vertex* $v \in D$ *such that* $\Delta(v) \supset U$.

(2) *Let* $U = \{u_1, u_2, u_3\}$ *and* $U' = \{u_1', u_2', u_3'\}$ *be elements in* $\mathscr{U}_2$ *such that* $u_i' = {}^x u_i$ $(i \in \{1, 2, 3\})$, *and let* $v, v' \in D$ *with* $\Delta(v) \supset U$ *and* $\Delta(v') \supset U'$. *Then* $v' = {}^x v$.

In order to prove this, we prepare two lemmas as follows.

Lemma 3.29. *Pick any* $x \in \Gamma$, *and pick* $u, v, w \in \Gamma(x)$ *with* $(u, v), (v, w), (w, u) \in R_{(2,2)}$. *Then there exists a unique vertex* $z \in \Gamma(x, u, v, w)$ *such that* $z \notin \{{}^x u, {}^x v, {}^x w\}$.

Proof. Immediate from observing the structure of $(C_{(3)}^{\leq 2}\{1, 2, 3\})^{v_2}$. □

Lemma 3.30. *Let* x, u, v, w *and* z *be vertices as in Lemma* 3.29. *Then the following hold.*

(1) $({}^x u, {}^x v), ({}^x v, {}^x w), ({}^x w, {}^x u) \in R_{(2,2)}$.

(2) *Let* $z' \in \Gamma(x, {}^x u, {}^x v, {}^x w)$ *with* $z' \notin \{{}^x u, {}^x v, {}^x w\}$ (*as in Lemma* 3.29). *Then* $z' = {}^x z$.

Proof. (1) Immediate from Lemma 3.7.

(2) Immediate from observing the structure of $(C_{(3)}^{\leq 2}\{1, 2, 3\})^{v_2}$. □

Proof of Lemma 3.28. (1) By Lemma 3.29, for any $U = \{u_1, u_2, u_3\} \in \mathscr{U}_2$, there exists a unique vertex $v \in \Delta$ such that $\Delta(v) \supset U$ and $v \notin \{{}^x u_1, {}^x u_2, {}^x u_3\}$. Moreover, by Lemma 3.24 (v), it is impossible that $v \in (C_{(3)}^{\leq 2}\{1, 2, 3\})^{v_2}$. Hence we have $v \in D$, which is desired.

(2) Immediate from Lemma 3.30. □

Lemma 3.31. *Let* x, y, A, B *and* C *be as in Lemma* 3.14, D *as in Lemma* 3.25, *and let* v_2 *be a mapping as in Lemma* 3.24.

Then there exists a unique bijection

$$v : C_{(3)} = C_{(3)}^{\leq 3}\{1, 2, 3\} \to \Delta$$

satisfying;

(i) $v|_{C_{(3)}^{\le 2}\{1,2,3\}} = v_2$,
(ii) $D = (C_{(3)}^{3}\{1,2,3\})^{v}$,
(iii) *For* $i, j \in \{4, \dots, n\}$ *with* $i \neq j$, $\{x\} \cup \{(k\ i\ j)^{v} \mid k \in N \backslash \{i, j\}\} \in \mathscr{M}_2$,
(iv) *For* $\{i, j, k\} \in C_{(3)}$, $(i\ k\ j)^{v} = {}^{x}((i\ j\ k)^{v})$,
(v) *For* $g \in C_{(3)}^{2}$ *and* $h \in C_{(3)}^{3}$, *if* $(g, h) \in R_{(3)}^{*}$, *then* $(g^{v}, h^{v}) \in R_{(3)}$.

Proof. At first, for $\{i,j,k\} \in C_{(3)}^{\le 2}\{1,2,3\}$, we rewrite the vertex $(i\ j\ k)^{v_2}$ by $(i\ j\ k)^{v}$.

For $\{i, j, k\} \in C_{(3)}^{3}\{1,2,3\}$, we denote by $U_{(i\ j\ k)}$ the 9-element-subset of C as follows:

$$U_{(i\ j\ k)} = \{(l_1\ \ l_2\ \ l_3)^{v} \in C \mid l_1 \in \{1,2,3\}, (l_2\ \ l_3) \in \{(i,j),(j,k),(k,i)\}\}.$$

Then by Lemmas 3.27 (3) and 3.28 (1), there exists a unique vertex in $D \cap \Gamma(U_{(i\ j\ k)})$. We write this vertex by $(i\ j\ k)^{v}$. Now let us regard v as a mapping from $C_{(3)}$ to Δ. Then we easily see that v satisfies (i) and (v) of this lemma. By Lemmas 3.24 and 3.28 (2), v also satisfies (iv). Moreover, by Lemma 3.27 (4), we also see that $v|_{C_{(3)}^{3}\{1,2,3\}}$ is bijective, so is v. Since $|\Delta| = |C_{(3)}|$, (ii) follows immediately. (iii) follows from Lemmas 3.25 (2) and 3.27 (2).

Thus we have the assertion. □

Lemma 3.32. *Let v be a bijection as in Lemma* 3.31.
Then for $g, h \in C_{(3)}$ *and* $\mu \in \{(3),(2,2),(5),(3,3)\}$,

$$(g,h) \in R_{\mu}^{*} \quad \text{if and only if} \quad (g^{v}, h^{v}) \in R_{\mu}.$$

Proof. By Lemma 3.31 (iii), (iv) and 3.5 (2), we easily have;
(3.32.1): For $g, h \in C_{(3)}$, $(g, h) \in R_{(3)}^{*}$ if and only if $(g^{v}, h^{v}) \in R_{(3)}$.
This means that the structure of edges in Δ is completely determined. Hence we have the assertion by analysing the relation between the graph structure of Δ and relations $\{R_{\lambda} \cap (\Delta \times \Delta)\}_{\lambda \in \{(3),(2,2),(5),(3,3)\}}$. □

Proof of Proposition 3.2. Immediate from Lemmas 3.31 and 3.32. □

4. Determination of local structure (2).

In this section, let Γ be a graph as in Proposition 3.2.
We define the following families of sets in $\Gamma(x)$ for any $x \in \Gamma$:

$$\mathscr{D}_x = \{\{x_1, \dots x_8\} \subset \Gamma(x) : |\Gamma(x_1, \dots, x_8) \cap R_{(2,2)}(x)| = 3\},$$

$$\mathscr{D}_x' = \{\{x_1, \dots x_8\} \subset \Gamma(x) : |\Gamma(x_1, \dots, x_8) \cap R_{(2,2)}(x)| \ge 1\},$$

$$\mathscr{E}_x = \{\{x_1, \dots x_5\} \subset \Gamma(x) : |\Gamma(x_1, \dots, x_5) \cap R_{(5)}(x)| = 1\},$$

$$\mathscr{E}_x' = \{\{x_1, \dots x_5\} \subset \Gamma(x) : |\Gamma(x_1, \dots, x_5) \cap R_{(5)}(x)| \ge 1\}.$$

Pick any $Y \in \mathscr{D}_x$. Then by Lemma 3.11 (1) and (2), the induced subgraph on Y is isomorphic to $K_{4,4}$, and for $y, z \in Y$, $\partial(y,z) = 2$ if and only if $(y,z) \in R_{(2,2)}$.

By $\mathscr{D}_x''$, we denote the family of the minimal geodetically closed subgraphs of Δ containing $y, z \in \Gamma(x)$ for all pairs (y,z)'s in $R_{(2,2)}$, where $\Delta = \Delta_x$ means the induced subgraph of Γ with vertex set $\Gamma(x)$ $(x \in \Gamma)$. Note that by Lemma 3.11 (1) and (2), $\mathscr{D}_x \subset \mathscr{D}'_x \subset \mathscr{D}''_x$.

We denote by $\mathscr{E}''_x$ the family of 5-points subsets in $\Gamma(x)$ consisting of $x_1, \dots, x_5$ such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_5 \sim x_1$ and $(x_1, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_1), (x_5, x_2) \in R_{(5)}$. Note that by Lemma 3.19, $\mathscr{E}_x \subset \mathscr{E}'_x \subset \mathscr{E}''_x$.

LEMMA 4.1. *For any $x \in \Gamma$, the following hold.*

(1) $\mathscr{D}_x = \mathscr{D}'_x = \mathscr{D}''_x$.

(2) $\mathscr{E}_x = \mathscr{E}'_x = \mathscr{E}''_x$.

(3) *Let $y, z \in \Gamma(x)$ such that $(y, z) \in R_{(5)}$. Then there exist exactly 2 elements $Y, Y' \in \mathscr{E}_x$ containing y and z. Moreover, if we pick $u \in \Gamma(x)$ with $y \sim u \sim z$, then $Y \ni u \notin Y'$ or $Y \not\ni v \in Y'$.*

PROOF. (1) Immediate from Lemmas 3.1 (5) and 3.13.

(2), (3) At first, let us observe the structure of $(C_{(3)})^\nu$ as in Lemma 3.31. Focus on $(1\ 2\ 3)^\nu$ and $(3\ 4\ 5)^\nu$. Note that $((1\ 2\ 3)^\nu, (3\ 4\ 5)^\nu) \in R_{(5)}$, and that $\Delta((1\ 2\ 3)^\nu, (3\ 4\ 5)^\nu) = \{(2\ 3\ 4)^\nu, (3\ 1\ 5)^\nu\}$. Then we see that;

$$Y = \{(1\ 2\ 3)^\nu, (2\ 3\ 4)^\nu, (3\ 4\ 5)^\nu, (4\ 5\ 1)^\nu, (5\ 1\ 2)^\nu\},$$

$$Y' = \{(1\ 2\ 3)^\nu = (3\ 1\ 2)^\nu, (5\ 3\ 1)^\nu, (3\ 4\ 5)^\nu = (4\ 5\ 3)^\nu, (2\ 4\ 5)^\nu, (1\ 2\ 4)^\nu\}$$

are unique sets in $\mathscr{E}_x$ containing $\{(1\ 2\ 3)^\nu, (2\ 3\ 4)^\nu, (3\ 4\ 5)^\nu\}$ and $\{(3\ 1\ 2)^\nu, (5\ 3\ 1)^\nu, (4\ 5\ 3)^\nu\}$, respectively. Thus we have (3), and, moreover, we also obtain that $|\mathscr{E}''_x| = 24\binom{n}{5}$. Therefore by Lemma 3.1 (5), we have the assertion. □

For any $x \in \Gamma$ and any $Y \in \mathscr{E}''_x$, we denote by $[Y]$ the minimal subset of $\Gamma(x)$ containing Y such that the following hold:

(i) for any $u, v \in [Y]$ with $u \not\sim v$, $\Delta(u, v) \subset Y$,

(ii) for any $u, v \in [Y]$ with $(u, v) \in R_{(5)}$, and for any $Y' \in \mathscr{E}''_x$ containing u and v, $Y' \subset [Y]$.

Let $\mathscr{L}_x = \{[Y] \mid Y \in \mathscr{E}''_x\}$.

Fix any vertex $x \in \Gamma$, and let $\nu : C_{(3)} \to \Gamma(x)$ be any bijection as in Lemma 3.31.

For $s \in \{4, 5\}$ and $I \in \overline{N_s}$, let $C_s(I)$ denote the subset of $C_{(3)}$ consisting of $(i\ j\ k)$ with $(i, j, k) \in N_3$ and $\{i, j, k\} \subset I$. Let $\mathscr{C}_s$ denote the family of all $C_s(I)$'s with $I \in \overline{N_s}$ $(s \in \{4, 5\})$. Note that the following hold.

(1) For $I \in N_4$, $|C_4(I)| = 8$,

(2) For $I \in N_5$, $|C_5(I)| = 20$,

(3) $|\mathscr{C}_s| = |\overline{N_s}| = \binom{n}{s}$ for $s \in \{4, 5\}$.

Denote

$$(C_s(I))^\nu = \{(i\ j\ k)^\nu \mid (i, j, k) \in N_3, \{i, j, k\} \subset I\}$$

for $I \in \overline{N_s}$ $(s \in \{4, 5\})$, and

$$(\mathscr{C}_s)^\nu = \{(C_s(I))^\nu \mid I \in \overline{N_s}\}.$$

LEMMA 4.2. *Pick any $x \in \Gamma$, and let $\nu : C_{(3)} \to \Gamma(x)$ be any bijection as in Lemma 3.32. Then the following hold.*

(1) $\mathscr{D}_x = (\mathscr{C}_4)^v$.
(2) $\mathscr{L}_x = (\mathscr{C}_5)^v$.
(3) *For any $L \in \mathscr{L}_x$, there exist exactly 5 elements of $\mathscr{D}_x$ which are subsets of L.*
(4) *For any $D \in \mathscr{D}_x$, there exist exactly $(n-4)$ elements of $\mathscr{L}_x$ containing D.*

Proof. Straightforward. □

Pick any vertex $x \in \Gamma$, and pick any $Y = \{y_1, y_2, y_3, y_4, {}^xy_1, {}^xy_2, {}^xy_3, {}^xy_4\} \in \mathscr{D}_x$ such that $y_i \sim {}^xy_j$, $(y_i, y_k), ({}^xy_i, {}^xy_k) \in R_{(2,2)}$ $(i, j, k \in \{1,2,3,4\}, i \neq k)$. Then by Lemma 4.1 (1), there exist three vertices $z_1, z_2, z_3 \in \Gamma(Y)\backslash\{x\}$. Note that $(z_1, z_2), (z_2, z_3), (z_3, z_1) \in R_{(2,2)}$. By Lemma 3.11 (3), for $k \in \{1,2,3\}$, there exists a permutation σ_k on $\{1,2,3,4\}$ such that ${}^{z_k}y_i = {}^xy_{\sigma_k(i)}$. By the condition (M3) as in Observation 1.3, we have;

(a) for $k \in \{1,2,3\}$, σ_k does not fix any element, that is, σ_k $(\in S_4)$ is of type (4) or (2,2).

(b) $\sigma_k(i) \neq \sigma_{k'}(i)$ for $i \in \{1,2,3,4\}$ and $k, k' \in \{1,2,3\}$ with $k \neq k'$.

Assume that there exists an element of (4)-type in $\{\sigma_1, \sigma_2, \sigma_3\}$. Suppose that $\sigma_1 = (1\ 2\ 3\ 4)$. Then by (a) and (b), it must hold that

$$\{\sigma_2, \sigma_3\} = \{(1\ 3)(2\ 4), (1\ 4\ 3\ 2)\},$$

that is, $\langle \sigma_1, \sigma_2, \sigma_3 \rangle \simeq C_4$. On the other hand, if there exists no element of (4)-type, then we see that

$$\{\sigma_1, \sigma_2, \sigma_3\} = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\},$$

that is, $\langle \sigma_1, \sigma_2, \sigma_3 \rangle \simeq D_4$. (Note that in each case there exists an element of (2,2)-type.)

Thus, without loss of generality, we may assume;

$$(I)\colon \begin{pmatrix} \sigma_1(1) & \sigma_1(2) & \sigma_1(3) & \sigma_1(4) \\ \sigma_2(1) & \sigma_2(2) & \sigma_3(3) & \sigma_4(4) \\ \sigma_3(1) & \sigma_3(2) & \sigma_3(3) & \sigma_4(4) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

or;

$$(II)\colon \begin{pmatrix} \sigma_1(1) & \sigma_1(2) & \sigma_1(3) & \sigma_1(4) \\ \sigma_2(1) & \sigma_2(2) & \sigma_3(3) & \sigma_4(4) \\ \sigma_3(1) & \sigma_3(2) & \sigma_3(3) & \sigma_4(4) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{pmatrix}.$$

We need to show the following:

Lemma 4.3. *Let x, Y and $\{z_1, z_2, z_3\}$ be as above. Then the condition (II) cannot occur.*

In order to prove this, we prepare one lemma as follows.

Lemma 4.4. *Let x_1, x_2, x_3, x_4 and x_5 be 5 vertices in Γ satisfying the following;*
(i) $x_2, x_3 \in \Gamma(x_1)$,
(ii) $x_4, x_5 \in R_{(2,2)}(x_1)$,
(iii) $\{x_2, x_3, x_4, x_5\}$ *forms a clique of size* 4.
Then ${}^{x_4}x_5 \in R_{(2,2)}(x_1)$.

Proof. Let $u_1 = {}^{x_2}x_4$, $u_2 = {}^{x_3}x_4$, $v_1 = {}^{x_2}x_5$ and $v_2 = {}^{x_3}x_5$. Then since $x_2 \neq x_3$ and $x_4 \neq x_5$, we see that;

(4.4.1): $u_1 \neq u_2,\ u_1 \neq v_1,\ v_1 \neq v_2,\ u_2 \neq v_2$.

Note that by Lemma 3.7, we have

(4.4.2): $u_1, u_2 \in \Gamma(x_1, x_4)$,

(4.4.3): $v_1, v_2 \in \Gamma(x_1, x_5)$.

Since $x_2, x_3, u_1, u_2 \in \Gamma(x_1, x_4)$ with $x_2 \sim x_3$, $x_2 \sim u_1$ and $x_3 \sim u_2$, it follows from Lemma 3.11 that;

(4.4.4): $u_1 \sim u_2$,

(4.4.5): $(x_3, u_1), (x_2, u_2) \in R_{(2,2)}$.

Similarly, we have

(4.4.6): $v_1 \sim v_2$,

(4.4.7): $(x_3, v_1), (x_2, v_2) \in R_{(2,2)}$.

Since $x_2 \sim x_5 \sim x_4$ and $x_3 \sim x_4 \sim x_5$, it follows from Lemma 3.7 that;

(4.4.8): $(u_1, x_5), (v_1, x_4) \in R_{(2,2)}$.

Similarly, we have

(4.4.9): $(u_2, x_5), (v_2, x_4) \in R_{(2,2)}$.

Note that by (4.4.2), (4.4.3) and (4.4.8), we have $u_1 \neq v_2$ and $u_2 \neq v_1$, so that by (4.4.1),

(4.4.10): u_1, u_2, v_1 and v_2 are distinct with each other.

Since $v_1 \sim x_2$ and $(v_1, x_4) \in R_{(2,2)}$, it follows from Lemma 3.7 that;

(4.4.11): $u_1 \sim v_1$.

Similarly, we have;

(4.4.12): $u_2 \sim v_2$.

Lemma 3.7 also implies;

(4.4.13): $u_1, u_2, v_1, v_2 \in \Gamma(x_1, {}^{x_4}x_5)$.

Note that by the assumption (i) and Lemma 3.7, we have $\partial(x_1, {}^{x_4}x_5) \geq 2$, so that by (4.4.13), $(x_1, {}^{x_4}x_5) \in R_{(2,2),(5),(3,3)}$. Clearly by (4.4.10), we have $(x_1, {}^{x_4}x_5) \notin R_{(3,3)}$. Also, by (4.4.4), (4.4.6), (4.4.11) and (4.4.12), it follows from Lemma 3.19 and (4.4.13) that $(x_1, {}^{x_4}x_5) \notin R_{(5)}$. Thus we have the assertion. □

Proof of Lemma 4.3. Let $\nu : C_{(3)} \to \Gamma(x)$ be the bijection as above. Then without loss of generality, we may assume that $y_1 = (1\ 2\ 3)^\nu$, $y_2 = (1\ 4\ 2)^\nu$, $y_3 = (1\ 3\ 4)^\nu$ and $y_4 = (2\ 4\ 3)^\nu$. Note that ${}^{x}y_1 = (1\ 3\ 2)^\nu$, ${}^{x}y_2 = (1\ 2\ 4)^\nu$, ${}^{x}y_3 = (1\ 4\ 3)^\nu$, ${}^{x}y_4 = (2\ 3\ 4)^\nu$, so that $Y = (C_4(\{1,2,3,4\}))^\nu$.

Suppose that the condition (II) is satisfied for $\{x, z_1, z_2, z_3\} \cup Y$. Note that;

(4.3.1): $\{z_2, (1\ 2\ 3)^\nu, (1\ 4\ 3)^\nu\}, \{z_2, (1\ 3\ 2)^\nu, (2\ 4\ 3)^\nu\} \in \mathscr{M}_1$.

Focus on the vertex $(1\ 5\ 3)^\nu$. Since $(1\ 2\ 3)^\nu \sim (1\ 5\ 3)^\nu \sim (1\ 4\ 3)^\nu$, it follows from Lemma 3.7 and (4.3.1) that;

(4.3.2): $((1\ 5\ 3)^\nu, z_2) \in R_{(2,2)}$.

Next, focus on $(5\ 1\ 3)^\nu$. Since $(1\ 3\ 2)^\nu \sim (5\ 1\ 3)^\nu$ and $((2\ 4\ 3)^\nu, (5\ 1\ 3)^\nu) \in R_{(5)}$, it follows from Lemma 3.21 and (4.3.1) that;

(4.3.3): $((5\ 1\ 3)^\nu, z_2) \in R_{(5)}$.

Now rename $z_2, (1\ 2\ 3)^\nu, (1\ 4\ 3)^\nu, x$ and $(1\ 5\ 3)^\nu$ by x_1, x_2, x_3, x_4 and x_5, respectively. Then by (4.3.1) and (4.3.2), we see that the 5-tuple $(x_1, x_2, x_3, x_4, x_5)$ satisfies (i),

(ii) and (iii) of Lemma 4.4. However, since ${}^{x_4}x_5 = (5\ 1\ 3)^{\nu}$, (4.3.3) contradicts Lemma 4.4. Thus we have the assertion. □

What does Lemma 4.3 mean? To answer it, let us observe local structure of the graph $\Gamma^* = \mathrm{Cay}_{C_{(3)}}(A_n)$.

Clearly we see that

$$\Gamma^*(C_4(\{1,2,3,4\}))\backslash\{\mathrm{id}\} = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}.$$

Moreover, we can observe the following:

Observation 4.5. *Let $\mathscr{M}_1^*$ be the set of all maximal cliques of size 3 in Γ^*. Then the following hold.*

(1) $\{(1\ 2)(3\ 4), (1\ 2\ 3), (1\ 2\ 4)\}, \{(1\ 2)(3\ 4), (1\ 4\ 2), (1\ 3\ 2)\}, \{(1\ 2)(3\ 4), (1\ 3\ 4), (2\ 3\ 4)\}, \{(1\ 2)(3\ 4), (2\ 4\ 3), (1\ 4\ 3)\} \in \mathscr{M}_1^*$.

(2) $\{(1\ 3)(2\ 4), (1\ 2\ 3), (1\ 4\ 3)\}, \{(1\ 3)(2\ 4), (1\ 4\ 2), (2\ 3\ 4)\}, \{(1\ 3)(2\ 4), (1\ 3\ 4), (1\ 3\ 2)\}, \{(1\ 3)(2\ 4), (2\ 4\ 3), (1\ 2\ 4)\} \in \mathscr{M}_1^*$.

(3) $\{(1\ 4)(2\ 3), (1\ 2\ 3), (2\ 3\ 4)\}, \{(1\ 4)(2\ 3), (1\ 4\ 2), (1\ 4\ 3)\}, \{(1\ 4)(2\ 3), (1\ 3\ 4), (1\ 2\ 4)\}, \{(1\ 4)(2\ 3), (2\ 4\ 3), (1\ 3\ 2)\} \in \mathscr{M}_1^*$.

The following lemma, which follows from Lemma 4.3, says that the above observation is the property which does not depend on 'how to label elements of Γ^* (or Γ)'.

Lemma 4.6. *Pick any $x \in \Gamma$ and $Y \in \mathscr{D}_x$. Then the following hold.*

(1) *There exists a bijection $\nu_Y : C_4(\{1,2,3,4\}) \to Y$ satisfying;*

(i) *for $g, h \in C_4(\{1,2,3,4\})$ and $\lambda \in \{(1),(3),(2,2)\}$,*

$$(g^{\nu_Y}, h^{\nu_Y}) \in R_\lambda \quad \text{if and only if} \quad (g,h) \in R_\lambda^*,$$

(ii) *for $g, h \in C_4(\{1,2,3,4\})$,*

$$\{x, g^{\nu_Y}, h^{\nu_Y}\} \in \mathscr{M}_1 \quad \text{if and only if} \quad \{\mathrm{id}, g, h\} \in \mathscr{M}_1^*.$$

(2) *For ν_Y as above, there exists a unique bijection*

$$\nu_Y \uparrow : \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \to \Gamma(Y)\backslash\{x\}$$

such that for $g \in \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ and $h, h' \in \Gamma^*(\mathrm{id})$,

(i) *$g^{\nu_Y\uparrow} \sim h^{\nu_Y}$ if and only if $h \in C_4(\{1,2,3,4\})$,*

(ii) *$\{g^{\nu_Y\uparrow}, h^{\nu_Y}, (h')^{\nu_Y}\} \in \mathscr{M}_1$ if and only if $\{g, h, h'\} \in \mathscr{M}_1^*$.*

Proof. (1) Immediate from Proposition 3.2.

(2) Immediate from Lemma 4.3. □

The following is also the property which does not depend on how to label vertices of $\Gamma(x)$ $(x \in \Gamma)$.

Lemma 4.7. *Pick any $x \in \Gamma$, $L \in \mathscr{L}_x$ and $Y \in \mathscr{D}_x$ with $L \supset Y$. Let $\nu_Y : C_4(\{1,2,3,4\}) \to Y$ be a bijection as in Lemma 4.6.*

Then there exists a unique bijection $\nu_{(L,Y)} : C_5(\{1,2,3,4,5\}) \to L$ satisfying;

(i) $\nu_{(L,Y)}|_{C_4(\{1,2,3,4\})} = \nu_Y$,

(ii) *for* $g, h \in C_5(\{1,2,3,4,5\})$ *and* $\lambda \in \{(1),(3),(2,2),(5)\}$,

$$(g^{v_{(L,Y)}}, h^{v_{(L,Y)}}) \in R_\lambda \quad \text{if and only if} \quad (g,h) \in R^*_\lambda,$$

(iii) *for* $g, h \in C_5(\{1,2,3,4,5\})$,

$$\{x, g^{v_{(L,Y)}}, h^{v_{(L,Y)}}\} \in \mathscr{M}_1 \quad \text{if and only if} \quad \{\mathrm{id}, g, h\} \in \mathscr{M}^*_1.$$

PROOF. We have the assertion by analyzing the relation between $C_4(\{1,2,3,4\})$ and $C_5(\{1,2,3,4,5\})\backslash C_4(\{1,2,3,4\})$. For example, $(1\ 2\ 5)$ can be characterized as a unique element h in $C_5(\{1,2,3,4,5\})\backslash C_4(\{1,2,3,4\})$ such that $(1\ 2\ 3) \sim_{\Gamma^*} h \sim_{\Gamma^*} (1\ 2\ 4)$. $\square$

Next, let us observe the relation between $(1\ 2)(3\ 4)$ and $C_5(\{1,2,3,4,5\})$.

OBSERVATION 4.8. *For* $g = (1\ 2)(3\ 4) \in A_n$ *and* $I = C_4(\{1,2,3,4\})$, *the following hold.*

(1) $(g, (i\ j\ k)) \in R^*_{(3)}$ *for* $(i\ j\ k) \in C_4(I)$.
(2) $(g, (i\ j\ 5)) \in R^*_{(2,2)}$ *for* $\{i,j\} \in \{\{1,2\},\{3,4\}\}$.
(3) $(g, (i\ j\ 5)) \in R^*_{(5)}$ *for* $\{i,j\} \in \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$.

Pick any $x \in \Gamma$ and $L \in \mathscr{L}_x$. The following lemma claims that we can uniquely determine all relations between 20 vertices in L and 3 vertices in $\Gamma(Y)\backslash\{x\}$ for any $Y \in \mathscr{D}_x$ with $Y \subset L$.

LEMMA 4.9. *Pick any* $x \in \Gamma$, $L \in \mathscr{L}_x$ *and* $Y \in \mathscr{D}_x$ *with* $L \supset Y$. *For* v_Y *as in Lemma* 4.6, *let* $\bar{v}$ *be a bijection from* $C_5(\{1,2,3,4,5\}) \cup \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ *to* $L \cup (\Gamma(Y)\backslash\{x\})$ *such that* $\bar{v}|_{\{(1\ 2)(3\ 4),(1\ 3)(2\ 4),(1\ 4)(2\ 3)\}} = v_Y\uparrow$ *and* $\bar{v}|_{C_5(\{1,2,3,4,5\})} = v_{(L,Y)}$, *where* $v_Y\uparrow$ *and* $v_{(L,Y)}$ *are unique bijections as in Lemmas* 4.6 *and* 4.7, *respectively.*

Then for $g \in \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, $h \in C_5(\{1,2,3,4,5\})$ *and* $\lambda \in \{(1), (3),(2,2),(5)\}$,

$$(g^{\bar{v}}, h^{\bar{v}}) \in R_\lambda \quad \text{if and only if} \quad (g,h) \in R^*_\lambda.$$

PROOF. Lemmas 3.8 and 3.22 are keys for proof. For example, for $g = (1\ 2)(3\ 4)$, we have $(g^{\bar{v}}, (1\ 2\ 5)^{\bar{v}}) \in R_{(2,2)}$ since $\{g^{\bar{v}}, (1\ 2\ 3)^{\bar{v}}, (1\ 2\ 4)^{\bar{v}}\} \in \mathscr{M}_1$, $((1\ 2\ 3)^{\bar{v}}, (1\ 2\ 5)^{\bar{v}}) \in R_{(3)}$ and $((1\ 2\ 4)^{\bar{v}}, (1\ 2\ 5)^{\bar{v}}) \in R_{(3)}$. $\square$

By Lemmas 4.6, 4.7 and 4.9, we have the following, which is one goal of this section.

PROPOSITION 4.10. *Pick any* $g \in \Gamma^*$ *and* $x \in \Gamma$. *Let* $Y^* \in \mathscr{D}^*_g$ *(resp.,* $Y \in \mathscr{D}_x$*) and* $L^* \in \mathscr{L}^*_g$ *(resp.,* $L \in \mathscr{L}_x$*) with* $L^* \supset Y^*$ *(resp.,* $L \supset Y$*). Let* $v : L^* \to L$ *be a bijection satisfying;*

(i) $v(Y^*) = Y$,
(ii) *for* $h_1, h_2 \in L^*$ *and* $\lambda \in \{(1),(3),(2,2)\}$,

$$((h_1)^v, (h_2)^v) \in R_\lambda \quad \text{if and only if} \quad (h_1, h_2) \in R^*_\lambda,$$

(iii) *for* $h_1, h_2 \in L^*$,

$$\{x, (h_1)^v, (h_2)^v\} \in \mathscr{M}_1 \quad \text{if and only if} \quad \{g, h_1, h_2\} \in \mathscr{M}^*_1.$$

Then there exists a unique bijection

$$\bar{\nu} : L^* \cup (\Gamma^*(Y^*) \backslash \{g\}) \to L \cup (\Gamma(Y) \backslash \{x\})$$

such that;

(iv) $\bar{\nu}|_{L^*} = \nu$,

(v) *for* $h_1, h_2 \in L^*$ *and* $h \in \Gamma^*(Y^*) \backslash \{g\}$,

$$\{h^{\bar{\nu}}, (h_1)^{\bar{\nu}}, (h_2)^{\bar{\nu}}\} \in \mathscr{M}_1 \text{ if and only if } \{h, h_1, h_2\} \in \mathscr{M}_1^* \quad \text{with } h_1, h_2 \in Y^*.$$

(vi) *for* $h_1 \in L^*$, $h \in \Gamma^*(Y^*) \backslash \{g\}$ *and* $\lambda \in \{(1), (3), (2,2), (5)\}$,

$$(h^{\bar{\nu}}, (h_1)^{\bar{\nu}}) \in R_\lambda \quad \text{if and only if} \quad (h, h_1) \in R_\lambda^*.$$

Next, focus on the element $(1\ 2\ 3\ 4\ 5) \in R_{(5)}^*(\mathrm{id})$. Note that

$$\Gamma^*(\mathrm{id}, (1\ 2\ 3\ 4\ 5)) = \{(1\ 2\ 3), (2\ 3\ 4), (3\ 4\ 5), (4\ 5\ 1), (5\ 1\ 2)\} =: Y^* \in \mathscr{E}_{\mathrm{id}},$$

and that $[Y^*] = C_5(\{1,2,3,4,5\})$.

We easily see the following:

Observation 4.11. *For* $g = (1\ 2\ 3\ 4\ 5)$ *and* Y^* *as above, the following hold.*

(1) $(g, h^2) \in R_{(5)}^*$ *for* $h \in Y^*$.

(2) $(g, h) \in R_{(2,2)}^*$ *for* $h \in \{(1\ 2\ 4), (2\ 3\ 5), (3\ 4\ 1), (4\ 5\ 2), (5\ 1\ 3)\}$.

(3) $(g, h^2) \in R_{(5)}^*$ *for* h *as in* (2).

The following is another goal in this section:

Proposition 4.12. *Pick any* $g \in \Gamma^*$ *and* $x \in \Gamma$. *Let* $Y^* \in \mathscr{E}_g^*$ *(resp.,* $Y \in \mathscr{E}_x$*) and* $L^* = [Y^*]$ *(resp.,* $L = [Y]$*). Let* $h \in \Gamma^*$ *(resp.,* $z \in \Gamma$*) with* $\{h\} = \Gamma^*(Y^*) \backslash \{g\}$ *(resp.,* $\{z\} = \Gamma(Y) \backslash \{x\}$*). Let* $\nu : L^* \to L$ *be a bijection satisfying*;

(i) $\nu(Y^*) = Y$,

(ii) *for* $h_1, h_2 \in L^*$ *and* $\lambda \in \{(1), (3), (2,2), (5)\}$,

$$((h_1)^\nu, (h_2)^\nu) \in R_\lambda \quad \text{if and only if} \quad (h_1, h_2) \in R_\lambda^*.$$

Then for $h' \in L^*$ *and* $\lambda \in \{(1), (3), (2,2), (5)\}$,

$$(z, (h')^\nu) \in R_\lambda \quad \text{if and only if} \quad (h, h') \in R_\lambda^*.$$

Before observing it, we prepare the following lemma.

Lemma 4.13. *Let* $\{x_1, x_2, x_3, x_4\}$ *be a clique of size* 4 *in* Γ. *Let* y *be a vertex in* Γ *such that* $(y, x_1) \in R_{(3)}$, $(y, x_2) \in R_{(3)}$, $(y, x_3) \in R_{(5)}$ *and* $(y, x_4) \in R_{(2,2)}$, *and let* z *be a vertex in* Γ *such that* $(z, x_1) \in R_{(3)}$, $(z, x_2) \in R_{(5)}$, $(z, x_3) \in R_{(3)}$ *and* $(y, z) \in R_{(3)}$.
Then $(z, x_4) \in R_{(2,2)}$.

Proof. Let $x = x_1$, and let $\nu : C_{(3)} \to \Gamma(x)$ be a bijection as in Lemma 3.32. Then without loss of generality, we may assume that $x_2 = (1\ 2\ 3)^\nu$, $x_3 = (1\ 2\ 5)^\nu$, $x_4 = (1\ 2\ 4)^\nu$ and $y = (2\ 3\ 4)^\nu$. Then by the assumption on z, it must hold that $z = (2\ 5\ 4)^\nu$. Hence it must also hold that $((2\ 5\ 4)^\nu, (1\ 2\ 4)^\nu) \in R_{(2,2)}$, which is desired. □

Now we shall observe Proposition 4.12. Note that the existense of v is guaranteed in the previous section. Moreover, by definition of h and z, it holds that for $h' \in L^*$,

$$(z, (h')^v) \in R_{(3)} \quad \text{if and only if} \quad h' \in Y^*.$$

Assume that

$$Y^* = \{(1\ 2\ 3), (2\ 3\ 4), (3\ 4\ 5), (4\ 5\ 1), (5\ 1\ 2)\}.$$

Then we see that $h = (1\ 2\ 3\ 4\ 5)$ ($\in R^*_{(5)}(\mathrm{id})$), and we can identify $(1\ 2\ 3)^v, x, (5\ 1\ 2)^v, (1\ 2\ 4)^v, (2\ 3\ 4)^v$ and z with x_1, x_2, x_3, x_4, y and z in Lemma 4.13, respectively. Hence by Lemma 4.13, we have $((1\ 2\ 4)^v, z) \in R_{(2,2)}$. (Note that $((1\ 2\ 4), h) \in R^*_{(2,2)}$ by Observation 4.11 (2).)

We also see that $z, (2\ 1\ 4)^v \in \Gamma((4\ 5\ 1)^v, (2\ 3\ 4)^v)$ with $((4\ 5\ 1)^v, (2\ 3\ 4)^v) \in R_{(5)}$ and $z \not\sim (2\ 1\ 4)^v$. Hence by Lemma 3.20, we have $((2\ 1\ 4)^v, z) \in R_{(5)}$. (See Observation 4.11 (3).)

Thus we have the assertion of Proposition 4.12.

5. Analysis of $f^*(\langle C_{(3)} \cap A_5 \rangle)$.

In this section, we only consider $\tilde{\mathcal{X}}(A_n)$ $(n \geq 5)$ and f^*, where f^* is an injection from A_n to $\boldsymbol{V}$ as in Proposition 2.6.

In this section, we write $E = C_5(\{1, 2, 3, 4, 5\})$. Note that $|E| = 20$, and that $\langle E \rangle \simeq A_5$.

The main claim in this section is as follows:

Proposition 5.1. $\mathrm{Span}(f^*(\langle E \rangle)) = \mathrm{Span}(f^*(E \cup \{\mathrm{id}\}))$.

At first, we shall observe the following:

Lemma 5.2. (1) *If* $n \geq 6$, *then* $\dim(\mathrm{Span}(f^*(\langle E \rangle))) = 17$.
(2) *If* $n = 5$, *then* $\dim(\mathrm{Span}(f^*(\langle E \rangle))) = 16$.

Proof. Let R be the Gram matrix with respect to $f^*(\langle E \rangle)$. Then we easily see that R is contained in the Bose-Mesner algebra of the group association scheme $\mathcal{X}(A_5) = (A_5, \{R'_\lambda\}_{\lambda \in \Lambda})$, where $\Lambda = \{(1), (3), (2,2), (5)_1, (5)_2\}$. (The relations $R'_{(5)_1}$ and $R'_{(5)_2}$ correspond to the conjugacy classes of A_5 containing $(1\ 2\ 3\ 4\ 5)$ and $(1\ 2\ 3\ 5\ 4)$, respectively.) More precisely, R can be represented as follows:

$$R = (n-1)L_{(1)} + (n-4)L_{(3)} + (n-5)L_{(2,2)} + (n-6)(L_{(5)_1} + L_{(5)_2}),$$

where L_λ $(\lambda \in \Lambda)$ is the adjacency matrix of R'_λ. We easily see that the Bose-Mesner algebra of $\mathcal{X}(A_5)$ is symmetric, and that the first eigenmatrix of $\mathcal{X}(A_5)$ is as follows:

$$P = \begin{pmatrix} 1 & 20 & 15 & 12 & 12 \\ 1 & 5 & 0 & -3 & -3 \\ 1 & -4 & 3 & 0 & 0 \\ 1 & 0 & -5 & 2+2\sqrt{5} & 2-2\sqrt{5} \\ 1 & 0 & -5 & 2-2\sqrt{5} & 2+2\sqrt{5} \end{pmatrix}.$$

(For how to calculate P, see [2].) Note that each row of P corresponds to the common eigenspace of L_λ's with multiplicity $1, 16, 25, 9$ and 9, respectively. Thus we can easily calculate the spectrum of R as follows;

$$\mathrm{Spec}(R) = \begin{pmatrix} 60(n-5) & 15 & 0 \\ 1 & 16 & 43 \end{pmatrix}.$$

Hence we have $\mathrm{rank}(R) = 17$ (resp. 16) for $n \geq 6$ (resp. $n = 5$), which is desired. □

Remark. The above caluculation on $\mathrm{Spec}(R)$ is related to the fact that;

$$\chi_1|_{A_5} = (n-5)\chi_0' + \chi_1',$$

where χ_1 is the irreducible character of A_n with $\chi_1(\mathrm{id}) = n-1$, χ_0' is the identity character of A_5, and where χ_1' is the irreducible character of A_5 with $\chi_1'(\mathrm{id}) = 4$. Note that $\chi_{perm} = \chi_0 + \chi_1$ and $\chi_{perm}' = \chi_0' + \chi_1'$, where χ_{perm} and χ_{perm}' are the permutation characters of degree n and 5, respectively.

Next, we shall show;

Lemma 5.3. $\dim(\mathrm{Span}(f^*(E))) = 16$.

In order to show this, let us consider the structure of E. Define five relations on E as follows:

$$R_0 = \{(g, g) \in E \times E\},$$

$$R_1 = \{(g, g^2) \in E \times E\},$$

$$R_2 = \{((i\ j\ k), (i\ j\ l)) \in E \times E \mid k \neq l\},$$

$$R_3 = \{((i\ j\ k), (j\ i\ l)) \in E \times E \mid k \neq l\},$$

$$R_4 = \{((i\ j\ k), (i\ l\ m)) \in E \times E \mid \{j, k\} \cap \{l, m\} = \emptyset\}.$$

Then we have the following.

Lemma 5.4. *$\mathscr{Y} = (E, \{R_0, R_1, R_2, R_3, R_4\})$ forms a symmetric association scheme having the first eigenmatrix as follows;*

$$P = \begin{pmatrix} 1 & 1 & 6 & 6 & 6 \\ 1 & 1 & 1 & 1 & -4 \\ 1 & 1 & -2 & -2 & 2 \\ 1 & -1 & 2 & -2 & 0 \\ 1 & -1 & -3 & 3 & 0 \end{pmatrix},$$

where each row corresponds to the eigenspace with multiplicity $1, 4, 5, 6, 4$, *respectively.*

Proof. We easily see that $\mathscr{Y}$ is a group-case association scheme by the natural transitive action of S_5 on E. Let $G = S_5$, and let H be the stabilizer of $(1\ 2\ 3)$ $(\in E)$. Then we have $H = \langle (1\ 2\ 3), (4\ 5) \rangle \simeq C_3 \times C_2$.

Let 1_H^G be the permutation character of the above. The following are character values of 1_H^G and $\chi_0, \dots, \chi_6$, irreducible characters of S_5:

	(1)	(2)	(3)	(4)	(2,2)	(2,3)	(5)
1_H^G	20	2	2	0	0	2	0
χ_0	1	1	1	1	1	1	1
χ_1	4	2	1	0	0	−1	−1
χ_2	5	1	−1	−1	1	1	0
χ_3	6	0	0	0	−2	0	1
χ_4	4	−2	1	0	0	1	−1
χ_5	5	−1	−1	1	1	−1	0
χ_6	1	−1	1	−1	1	−1	1

By calculation of inner product of characters, we have

$$1_H^G = \chi_0 + \chi_1 + \chi_2 + \chi_3 + \chi_4.$$

On the other hand, we have the double coset decomposition of G by H as follows:

$$G = \sum_{i=0}^{4} Ha_iH,$$

where $a_0 = \mathrm{id}$, $a_1 = (1\ 2)$, $a_2 = (1\ 4)$, $a_3 = (1\ 2\ 4)$ and $a_4 = (1\ 4)(2\ 5)$.

Thus we can calculate the above matrix P by the following formula, which is written in Corollary 11.7 (ii) of Chapter II in [**2**],

$$P_{i'}(j)(= P_i(j)) = \frac{1}{|H|} \sum_{x \in Ha_iH} \chi_j(x)$$

$(i, j \in \{0, 1, 2, 3, 4\})$.

By caluculating the second eigenmatrix $Q = 20P^{-1}$, we have multiplicity corresponding to each row of P. (See [**2**].) □

Proof of Lemma 5.3. Let S be the Gram matrix with respect to $f^*(E)$. Then we easily have;

$$S = (n-1)L_0 + (n-4)(L_1 + L_2) + (n-5)L_3 + (n-6)L_4,$$

where L_i $(i \in \{0, 1, 2, 3, 4\})$ is the adjacency matrix of R_i as in the above lemma. Hence, similar to the proof of Lemma 5.2, we can calculate the spectrum of S as follows;

$$\mathrm{Spec}(S) = \begin{pmatrix} 20n - 95 & 10 & 1 & 5 & 0 \\ 1 & 4 & 5 & 6 & 4 \end{pmatrix}.$$

Now we have $\mathrm{rank}(S) = 16$, as desired. □

We also need;

LEMMA 5.5. *If $n \neq 5$, then $f^*(\mathrm{id}) \notin \mathrm{Span}(f^*(E))$.*

PROOF. Here, we write $\bar{g} = f^*(g)$ for $g \in E \cup \{\mathrm{id}\}$.

Suppose $\overline{\mathrm{id}} = \sum_{g \in E} \lambda_g \bar{g}$. Write $\lambda = \sum_{g \in E} \lambda_g$. Since the group $\langle E \rangle$ acts on E by conjugation, we easily see that for any $h \in \langle E \rangle$,

(5.5.1): $\overline{\mathrm{id}} = \sum_{g \in E} \lambda_g \bar{g} = \sum_{g \in E} \lambda_{g^h} \bar{g}$.

Hence we have;

(5.5.2): $|\langle E \rangle| \cdot \overline{\mathrm{id}} = \sum_{h \in \langle E \rangle} (\sum_{g \in E} \lambda_{g^h} \bar{g}) = \sum_{g \in E} (\sum_{h \in \langle E \rangle} \lambda_{g^h}) \bar{g}$.

On the other hand, for any $g \in E$, we easily have;

(5.5.3): $\sum_{h \in \langle E \rangle} \lambda_{g^h} = 3\lambda$.

Therefore by (5.5.2) and (5.5.3), we have;

(5.5.4): $\overline{\mathrm{id}} = \dfrac{\lambda}{20} \sum_{g \in E} \bar{g}$.

Since $\langle \overline{\mathrm{id}}, \overline{\mathrm{id}} \rangle_V = n - 1$ and $\langle \overline{\mathrm{id}}, \bar{g} \rangle_V = n - 4$ $(g \in E)$, it follows from (5.5.4),

$$n - 1 = \frac{\lambda}{20} \cdot 20(n - 4),$$

so that;

(5.5.5): $\lambda = \dfrac{n-1}{n-4}$.

Similarly, since $\langle \overline{\mathrm{id}}, \overline{(1\ 2\ 3)} \rangle_V = n - 4$,

$$n - 4 = \frac{\lambda}{20}(n - 1 + 7(n - 4) + 6(n - 5) + 6(n - 6)) = \frac{\lambda}{20}(20n - 95),$$

so that;

(5.5.6): $\lambda = \dfrac{20(n-4)}{20n - 95}$.

By (5.5.5) and (5.5.6), we have;

$$20(n - 4)^2 = (n - 1)(20n - 95),$$

so that $n = 5$, a contradiction.

Thus we have the assertion. □

PROOF OF PROPOSITION 5.1. Immediate from Lemmas 5.2, 5.3 and 5.5. □

Proposition 5.1 implies that $f^*(C_{(2,2)} \cup C_{(5)}) \subset \mathrm{Span}(f^*(\{\mathrm{id}\} \cup C_{(3)}))$. In fact, it also holds that $f^*(C_{(3,3)}) \subset \mathrm{Span}(f^*(\{\mathrm{id}\} \cup C_{(3)}))$. Pick any $g \in C_{(3,3)}$. Then there exists a unique pair (h_1, h_2) in $C_{(3)}$ such that $(h_1, h_2) \in R^*_{(3,3)}$ and $g = h_1 \cdot h_2$. Moreover, by Proposition 2.6, we have

$$|f^*(g) + f^*(\mathrm{id}) - f^*(h_1) - f^*(h_2)|_V^2 = 4(n - 1) - 8(n - 4) + 4(n - 7) = 0,$$

so that

$$f^*(g) = f^*(h_1) + f^*(h_2) - f^*(\mathrm{id}) \in \mathrm{Span}(f^*(\{\mathrm{id}\} \cup C_{(3)})).$$

Thus, by inductive argument, we have;

COROLLARY 5.6. $V = \mathrm{Span}(f^*(\{\mathrm{id}\} \cup C_{(3)}))$.

6. Determination of global structure.

From now on, we shall prove Theorem 1.4.

Let us consider the structure of $f(X)$ and $f^*(A_n)$, where f and f^* are bijections as in Proposition 2.6. By Γ^f (resp. Γ^{f^*}), we mean the graph whose vertex set is $f(X)$ (resp. $f^*(A_n)$) such that for $a, b \in f(X)$ (resp. $\in f^*(A_n)$), a and b are adjacent if and only if $\langle a, b\rangle_V = n - 4$.

Lemma 6.1. *Γ^f (resp. Γ^{f^*}) is isomorphic to Γ (resp. Γ^*).*

Proof. Immediate from Lemma 2.7 (2). □

We can see that the local structure of $f(X)$ is the same as of $f^*(A_n)$ under the assumption as in Theorem 1.4. Indeed, we have the following.

Lemma 6.2. *Pick any $x \in \Gamma$ and $g \in \Gamma^*$. Then the following hold.*

(1) *There exists a bijection $\theta : f^*(\{g\} \cup \Gamma^*(g)) \to f(\{x\} \cup \Gamma(x))$ satisfying the following*:

(i) $\theta(f^*(g)) = f(x)$,

(ii) *for $a, b \in f^*(\{g\} \cup \Gamma^*(g))$,*

$$\langle a, b\rangle_V = \langle \theta(a), \theta(b)\rangle_V.$$

(2) *Let θ be as in* (1). *Then θ can be extended into a unique element $\bar{\theta}$ of $O(\boldsymbol{V})$, the orthogonal group on $\boldsymbol{V}$.*

(3) *Let θ be as in* (1). *Then the mapping*

$$\kappa := f^{-1}\theta f^* : \{g\} \cup \Gamma^*(g) \to \{x\} \cup \Gamma(x)$$

forms a bijection such that;

(iii) $\kappa(g) = x$,

(iv) *for $h_1, h_2 \in \Gamma^*(g)$ and $\lambda \in \{(1), (3), (2,2), (5), (3,3)\}$,*

$$(\kappa(h_1), \kappa(h_2)) \in R_\lambda \quad \text{if and only if} \quad (h_1, h_2) \in R_\lambda^*,$$

(v) *for $h_1, h_2 \in \Gamma^*(g)$,*

$$\{x, \kappa(h_1), \kappa(h_2)\} \in \mathscr{M}_1 \quad \text{if and only if} \quad \{g, h_1, h_2\} \in \mathscr{M}_1^*.$$

Proof. (1) and (3) are clear. (2) follows from Corollary 5.6. □

By $\mathscr{D}^f_{f(x)}$, we denote the family of the minimal geodetically closed subgraphs of $\Delta^f_{f(x)}$ containing a, b for all pairs (a, b)'s in $\Gamma^f(f(x))$ with $\langle a, b\rangle_V = n - 5$, where $\Delta^f_{f(x)}$ means the induced subgraph of Γ^f with vertex set $\Gamma^f(f(x))$ $(x \in \Gamma)$.

We denote by $\mathscr{E}^f_{f(x)}$ the family of 5-point-subsets in $\Gamma^f(f(x))$ consisting of $a_1, \dots, a_5$ such that $a_1 \sim_{\Gamma^f} a_2 \sim_{\Gamma^f} a_3 \sim_{\Gamma^f} a_4 \sim_{\Gamma^f} a_5 \sim_{\Gamma^f} a_1$, and $\langle a_1, a_3\rangle_V = \langle a_2, a_4\rangle_V = \langle a_3, a_5\rangle_V = \langle a_4, a_1\rangle_V = \langle a_5, a_2\rangle_V = n - 6$.

For any $x \in \Gamma$ and any $Y \in \mathscr{E}^f_{f(x)}$, we denote by $[Y]$ the minimal subset of $\Gamma^f(f(x))$ containing Y such that the following hold:

(i) for any $a, b \in [Y]$ with $a \not\sim_{\Gamma^f} b$, $\Delta^f_{f(x)}(a,b) \subset Y$,

(ii) for any $a, b \in [Y]$ with $\langle a, b \rangle_V = n-6$, and for any $Y' \in \mathscr{E}^f_{f(x)}$ containing a and b, $Y' \subset [Y]$.

Let $\mathscr{L}^f_{f(x)} = \{[Y] \mid Y \in \mathscr{E}^f_{f(x)}\}$.

Lemma 6.3. *For $x \in \Gamma$, the following hold.*

(1) $\mathscr{D}^f_{f(x)} = f(\mathscr{D}_x)$.

(2) $\mathscr{E}^f_{f(x)} = f(\mathscr{E}_x)$.

(3) $\mathscr{L}^f_{f(x)} = f(\mathscr{L}_x)$.

Proof. Immediate from Lemmas 2.7 and 4.1. □

Our strategy for the proof of Theorem 1.4 can be described as follows:

Lemma 6.4. *Let x, g and θ be as in Lemma 6.2. Then $f(X) = \bar{\theta}(f^*(A_n))$.*

The following is the key lemma for the proof of the previous lemma.

Lemma 6.5. *Let x, g and θ be as in Lemma 6.2.*
Then $f(\Gamma_2(x)) = \bar{\theta}(f^*(\Gamma_2^*(g)))$.

Proof. Let κ be a bijection as in Lemma 6.2 (3). Note that;

(6.5.1): $\mathscr{D}_x = \kappa(\mathscr{D}^*_g)$,

(6.5.2): $\mathscr{E}_x = \kappa(\mathscr{E}^*_g)$,

(6.5.3): $\mathscr{L}_x = \kappa(\mathscr{L}^*_g)$,

and,

(6.5.4): for $h_1, h_2 \in \Gamma^*(g)$,

$$\{x, \kappa(h_1), \kappa(h_2)\} \in \mathscr{M}_1 \quad \text{if and only if} \quad \{g, h_1, h_2\} \in \mathscr{M}_1^*.$$

We know that $\Gamma_2(x) = R_{(2,2),(5),(3,3)}(x)$ and $\Gamma_2^*(g) = R^*_{(2,2),(5),(3,3)}(g)$. At first, we shall show the following:

(6.5.5): $f(R_{(2,2)}(x)) = \bar{\theta}(f^*(R^*_{(2,2)}(g)))$.

Pick any $Y \in \mathscr{D}^{f^*}_{f^*(g)}$. Note that;

(6.5.6): $Y_{\Gamma^*} = (f^*)^{-1}(Y) \in \mathscr{D}^*_g$,

and,

(6.5.7): $Y_\Gamma = \kappa(f^*)^{-1}(Y) \in \mathscr{D}_x$.

Then by Lemma 6.3 (1), there exist 3 vertices h_1, h_2, h_3 in Γ^* such that;

(6.5.8): $\{h_1, h_2, h_3\} = \Gamma^*(Y_{\Gamma^*}) \backslash \{g\}$,

and there exist 3 vertices y_1, y_2, y_3 in Γ such that

(6.5.9): $\{y_1, y_2, y_3\} = \Gamma^*(Y_\Gamma) \backslash \{x\}$.

Next, pick any $L \in \mathscr{L}^{f^*}_{f^*(g)}$ containing Y. Note that;

(6.5.10): $L_{\Gamma^*} = (f^*)^{-1}(L) \in \mathscr{L}^*_g$,

(6.5.11): $L_\Gamma = \kappa(f^*)^{-1}(L) \in \mathscr{L}_x$,

(6.5.12): $L_{\Gamma^*} \supset D_{\Gamma^*}$,

and,

(6.5.13): $L_\Gamma \supset D_\Gamma$.

Then by Proposition 4.10, there exists a bijection

$$\kappa' := \overline{\kappa|_{L_{\Gamma^*}}} : L_{\Gamma^*} \cup \{h_1, h_2, h_3\} \to L_{\Gamma} \cup \{y_1, y_2, y_3\}$$

satisfying;
(6.5.14): $\kappa'|_{L_{\Gamma^*}} = \kappa|_{L_{\Gamma^*}}$,
(6.5.15): (without loss of generality,) $\kappa'(h_i) = y_i$ $(i \in \{1, 2, 3\})$,
and,
(6.5.16): for $i \in \{1, 2, 3\}$, $h \in L_{\Gamma^*}$ and $\lambda \in \{(3), (2, 2), (5), (3, 3)\}$,

$$(\kappa'(h_i), \kappa'(h)) \in R_\lambda \quad \text{if and only if} \quad (h_i, h) \in R^*_\lambda.$$

(6.5.16) implies that;
(6.5.17): for $i \in \{1, 2, 3\}$ and $h \in L_{\Gamma^*}$,

$$\langle f\kappa'(h_i), f\kappa'(h)\rangle_V = \langle f^*(h_i), f^*(h)\rangle_V.$$

By Proposition 5.1, there exists some linear equation such that;
(6.5.18): $f^*(h_i) = \sum_{h \in L_{\Gamma^*} \cup \{g\}} \lambda_h f^*(h)$,
so that by (6.5.14), (6.5.17) and (6.5.18),

$$\begin{aligned}
0 &= \left| f^*(h_i) - \sum_{h \in L_{\Gamma^*} \cup \{g\}} \lambda_h f^*(h) \right|_V^2 \\
&= \left| f(y_i) - \sum_{h \in L_{\Gamma^*} \cup \{g\}} \lambda_h \cdot f\kappa(h) \right|_V^2 \\
&= \left| f(y_i) - \sum_{h \in L_{\Gamma^*} \cup \{g\}} \lambda_h \cdot \theta f^*(h) \right|_V^2 \\
&= \left| f(y_i) - \sum_{h \in L_{\Gamma^*} \cup \{g\}} \lambda_h \cdot \bar{\theta} f^*(h) \right|_V^2 \\
&= \left| f(y_i) - \bar{\theta}\left(\sum_{h \in L_{\Gamma^*} \cup \{g\}} \lambda_h f^*(h) \right) \right|_V^2 \\
&= |f(y_i) - \bar{\theta} f^*(h_i)|_V^2.
\end{aligned}$$

Hence we have $f(y_i) = \bar{\theta} f^*(h_i)$. (Note that this claim itself does not depend on 'the choice of L'.) Thus we have the assertion of (6.5.5).

By the same argument as above, it follows from Proposition 4.12 that;
(6.5.19): $f(R_{(5)}(x)) = \bar{\theta}(f^*(R^*_{(5)}(g)))$.
Finally, we shall observe that;
(6.5.20): $f(R_{(3,3)}(x)) = \bar{\theta}(f^*(R^*_{(3,3)}(g)))$.

Pick any $h \in R^*_{(3,3)}(g)$. Then there exists a pair $(h_1, h_2) \in R^*_{(3,3)}$ in $\Gamma^*(g)$. On the other hand, there exists a vertex $y \in R_{(3,3)}(x)$ such that $\{y\} = \Gamma(\kappa(h_1), \kappa(h_2)) \backslash \{x\}$. Thus we have;

$$\begin{aligned} f(y) &= f(\kappa(h_1)) + f(\kappa(h_2)) - f(x) \\ &= \bar{\theta}f^*(h_1) + \bar{\theta}f^*(h_1) - \bar{\theta}f^*(g) \\ &= \bar{\theta}(f^*(h_1) + f^*(h_2) - f^*(g)) \\ &= \bar{\theta}(f^*(h)). \end{aligned}$$

Thus we have the assertion of (6.5.20).

Now by (6.5.5), (6.5.19) and (6.5.20), we complete the proof. □

Proof of Lemma 6.4. Pick any vertex $x \in \Gamma$, and let θ be a bijection from $f^*(\{\mathrm{id}\} \cup \Gamma^*(\mathrm{id}))$ to $f(\{x\} \cup \Gamma(x))$ as in Lemma 6.2. Then by applying induction on the distance (on Γ^*) from id, we can show that for any $g \in \Gamma^*$, there exists a vertex $y \in \Gamma$ such that $f(y) = \bar{\theta}f^*(g)$. (Of course, we apply Lemma 6.5 here. Note that the uniqueness of $\bar{\theta}$ mentioned in Lemma 6.2 (2) is important.) Thus we have the assertion. □

Proof of Theorem 1.4. Immediate from Lemmas 6.1, 6.4 and 2.5. □

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